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VIRTUAL STABILITY WITHOUT CONTINUITY

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The concept of virtual stability is generalized to discontinuous maps and a sufficient condition that guarantees the fixed point set of a virtually stable map to be a retract of its convergence set is given. A sufficient condition that makes the domain of a given map the convergence set is also given.

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CONTENTS

CHAPTER I
INTRODUCTION & PRELIMINARY

Virtual stability was firstly introduced in [2]. It is proved to be useful to relate some topological properties between the fixed point set and the convergence set via a retraction. However, the original definition requires continuity while there are still some interesting maps in fixed point theory that are not continuous. Therefore, in this work, we generalize virtual stability to include some discontinuous maps and investigate the topology of the convergence set for some discontinuous maps.

Let us recall some basic concepts in topology, fixed point theory and virtual stability which will be used in this thesis. (See [2], [3], [5] for more details.)

Definition 1.1. A **topological space** is a set X together with τ, a collection of subsets of X, satisfying the following :-

1. The empty set and X are in τ.
2. The union of any collection of sets in τ is also in τ.
3. The intersection of any pair of sets in τ is also in τ.

The collection τ is called a **topology** on X and every set in τ is called an **open set**.

Definition 1.2. Let X be a topological space and $x \in X$. A **neighbourhood** of x is any open set that contains x.

Definition 1.3. A subset of a topological space X is a G_δ**-set** if it is a countable intersection of open sets.

Definition 1.4. The topological space X is said to be

1. A **Hausdorff space** if for any distinct points $x, y \in X$, there exist a neighbourhood G of x and a neighbourhood H of y such that $G \cap H = \varnothing$.

2. A **regular space** if every singleton is closed and for each $x \in X$, each neighbourhood G of x, there exists a neighbourhood H of x such that $\overline{H} \subseteq G$.

Definition 1.5. Let X and Y be topological spaces. A map $f : X \to Y$ is **continuous** at $x \in X$ if for each neighbourhood G of $f(x)$, there exists a neighbourhood H of x such that $f(H) \subseteq G$. We simply say that f is **continuous** if it is continuous at every $x \in X$.

Definition 1.6. Let X be a topological space and A be a subspace of X. A continuous map $r : X \to A$ is a **retraction** if $r(a) = a$ for all $a \in A$. In this case, a subspace A is called a **retract** of X.

Definition 1.7. Let $X,\ Y$ be topological spaces and $f,\ g : X \to Y$ be continuous maps. A **homotopy** from f to g is a continuous map $H : X \times [0, 1] \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$. In this case, we say that f is **homotopic** to g.

It is easy to verify that being homotopic is an equivalence relation on the set of all continuous functions from X to Y.

Definition 1.8. A topological space X is **contractible** if the identity map on X is homotopic to a constant map.

Theorem 1.9. *Let X be a topological space and A be a retract subspace of X. If X is contractible, A is contractible.*

Definition 1.10. A **metric** on a set X is a map $d : X \times X \to [0, \infty)$ such that the following holds for any $x,\ y,\ z \in X$:-

1. $d(x, y) = 0$ iff $x = y$.

2. $d(x, y) = d(y, x)$.

3. $d(x, z) \leq d(x, y) + d(y, z)$.

A **metric space** is an ordered pair (X, d) where X is a non-empty set and d is a metric on X.

Definition 1.11. Let (X, d) be a metric space. A sequence (x_n) in X is called a **Cauchy sequence** if for every $\epsilon > 0$ there is $N \in \mathbb{N}$ such that $d(x_m, x_n) < \epsilon$ for all natural numbers $m,\ n \geq N$.

Definition 1.12. A metric space X is said to be **complete** if every Cauchy sequence in X converges in X.

Definition 1.13. Let X be a vector space. A set $C \subseteq X$ is said to be

1. **Convex** if $(1-t)x + ty \in C$ for all $x,\ y \in C$ and $t \in [0, 1]$.

2. **Star-convex** if there exists $x \in C$ such that $(1-t)x + ty \in C$ for all $y \in C$ and $t \in [0, 1]$.

It is easy to see that a nonempty convex set is star-convex.

Definition 1.14. A **norm** on a vector space V is a map $p : V \to [0, \infty)$ such that the following holds for any $x,\ y \in X$ and $a \in \mathbb{R}$:-

1. $p(x) = 0$ iff $x = 0$

2. $p(ax) = |a|p(x)$

3. $p(x + y) \leq p(x) + p(y)$

Definition 1.15. A **Banach space** is a vector space X with a norm $\|\ \|$ such that X is complete with respect to the metric $d(x, y) = \|x - y\|$.

Example 1.16. *The space ℓ_∞ of all bounded sequences in $\mathbb{R}$ with respect to the supremum norm $\|x\| = \sup_n |x_n|$ is a Banach space.*

Definition 1.17. Let $X,\ Y$ be metric spaces and $\mathcal{F}$ be a family of functions from X to Y. The family $\mathcal{F}$ is **equicontinuous** at $x \in X$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that $d(f(x), f(t)) < \epsilon$ for all $f \in \mathcal{F}$ and $t \in X$ with $d(x, t) < \delta$. The family is **equicontinuous** if it is equicontinuous at each point in X.

Let X be a Hausdorff space and $f : X \to X$. Define $f^0 = id_X$ and $f^n = f \circ f^{n-1}$ for each $n \in \mathbb{N}$. The fixed point set and the convergence set are respectively defined to be $F(f) = \{x \in X|\ f(x) = x\}$ and $C(f) = \left\{x \in X|\ \lim_n f^n(x) \text{ exists}\right\}$. It is easy to see that $F(f) \subseteq C(f)$ and we will assume that $F(f) \neq \varnothing$. Moreover, The map $f^\infty : C(f) \to X$ is defined by $f^\infty(x) = \lim_n f^n(x)$. Note that $F(f) \subseteq f^\infty(C(f))$ in general, and $F(f) = f^\infty(C(f))$ if f is continuous.

Definition 1.18. Let (X, d) be a metric space. A map f is said to be

1. A **nonexpansive map** if $d(f(x), f(y)) \leq d(x, y)$ for all $x,\ y \in X$.

2. An **asymptotically nonexpansive map** if there is a sequence (a_n) in $\mathbb{R}^+$ converging to 1 such that $d(f^n(x), f^n(y)) \leq a_n d(x, y)$ for all $x,\ y \in X$ and $n \in \mathbb{N}$.

3. A **quasi-nonexpansive map** if $d(f(x), p) \leq d(x, p)$ for all $x \in X$ and $p \in F(f)$.

4. An **asymptotically quasi-nonexpansive map** if there is a sequence (a_n) in $\mathbb{R}^+$ converging to 1 such that $d(f^n(x), p) \leq a_n d(x, p)$ for all $x \in X,\ p \in F(f)$ and $n \in \mathbb{N}$.

5. A **virtually nonexpansive map** [1] if it is continuous and the collection $\{f^n \mid n \in \mathbb{N}\}$ is equicontinuous on $F(f)$.

6. A **Kannan map** [4] if there is $r \in [0, \frac{1}{2})$ such that for every $x, y \in X$, $d(f(x), f(y)) \leq rd(x, f(x)) + rd(y, f(y))$.

7. A **Suzuki generalized nonexpansive map** [6] if $d(f(x), f(y)) \leq d(x, y)$ for all $x, y \in X$ satisfying $\frac{1}{2}d(x, f(x)) \leq d(x, y)$.

8. An **asymptotically regular map** if $\lim_n d(f^{n+1}(x), f^n(x)) = 0$ for all $x \in X$.

It is easy to see that nonexpansive maps are asymptotically nonexpansive and quasi-nonexpansive. Asymptotically nonexpansive maps and quasi-nonexpansive maps are asymptotically quasi-nonexpansive. Moreover, by [1], continuous asymptotically quasi-nonexpansive maps are virtually nonexpansive, and f^∞ is continuous if f is virtually nonexpansive. Therefore, the fixed point set of a virtually nonexpansive map is always a retract of its convergence set.

We now recall the original definition and properties of virtual stability from [2].

Definition 1.19. Let f be a continuous map and $p \in F(f)$. We say that

1. p is **virtually stable** if for each neighbourhood G of p, there are a neighbourhood H of p and a strictly increasing sequence (k_n) in $\mathbb{N}$ such that $f^{k_n}(H) \subseteq G$ for all $n \in \mathbb{N}$.
2. p is **uniformly virtually stable** with respect to a strictly increasing sequence (k_n) in $\mathbb{N}$ if for each neighbourhood G of p, there is a neighbourhood H of p such that $f^{k_n}(H) \subseteq G$ for all $n \in \mathbb{N}$.
3. f is **virtually stable** if each fixed point of f is virtually stable.
4. f is **uniformly virtually stable** if each fixed point of f is uniformly virtually stable with respect to the same sequence.

Theorem 1.20. *Suppose X is a regular space. If f is virtually stable, then f^∞ is continuous and hence $F(f)$ is a retract of $C(f)$.*

It is easy to see that virtually nonexpansive maps are uniformly virtually stable with respect to a sequence (n). Therefore, its fixed point set is always a retract of its convergence set by the previous theorem.

Theorem 1.21. *Let X be a complete metric space. If f is uniformly virtually stable with respect to a sequence (nh) for some $h \in \mathbb{N}$, then $C(f)$ is a G_δ-set.*

CHAPTER II
VIRTUAL STABILITY WITHOUT CONTINUITY

In this chapter, we introduce the concept of virtual stability without continuity and give a sufficient condition eusuring that the fixed point set of a virtually stable map is a retract of its convergence set.

Let X be a Hausdorff space and $f : X \to X$ with $F(f) \neq \varnothing$.

Definition 2.1. Let $p \in F(f)$. We say that

1. p is **virtually stable** if for each neighbourhood G of p, there are a neighbourhood H of p and a strictly increasing sequence (k_n) in $\mathbb{N}$ such that $f^{k_n}(H) \subseteq G$ for all $n \in \mathbb{N}$.

2. p is **uniformly virtually stable** with respect to a strictly increasing sequence (k_n) in $\mathbb{N}$ if for each neighbourhood G of p, there is a neighbourhood H of p such that $f^{k_n}(H) \subseteq G$ for all $n \in \mathbb{N}$.

3. f is **virtually stable** if each fixed point of f is virtually stable.

4. f is **uniformly virtually stable** if each fixed point of f is uniformly virtually stable with respect to the same sequence.

Example 2.2. *Let $f : [0,1] \to [0,1]$ be defined by* $f(x) = \begin{cases} x^2 & ;\ x \neq \frac{1}{2} \\ 1 & ;\ x = \frac{1}{2} \end{cases}$
It is easy to see that 0 *is a virtually stable fixed point but* 1 *is not.*

Proposition 2.3. *If f is continuous and uniformly virtually stable with respect to a sequence (n), then it is virtually nonexpansive.*

Proof. To show that f is virtually nonexpansive, let $p \in F(f)$ and $\epsilon > 0$. There is a neighbourhood H of p such that $f^n(H) \subseteq B(p;\epsilon)$ for all $n \in \mathbb{N}$. Let $\delta > 0$ be such that $B(p;\delta) \subseteq H$. Then $f^n(B(p;\delta)) \subseteq f^n(H) \subseteq B(p;\epsilon)$ for all $n \in \mathbb{N}$. □

The following propositions show that the class of virtually stable maps contains many well-known maps in fixed point theory.

Proposition 2.4. *If f is a Kannan map, then it is uniformly virtually stable with respect to a sequence (n).*

Proof. Let (X, d) be a metric space and f be a Kannan map. Then there is $r \in [0, \frac{1}{2})$ such that $d(f(x), f(y)) \leq rd(x, f(x)) + rd(y, f(y))$ for all $x,\ y \in X$. Let $p \in F(f),\ \epsilon > 0,\ n \in \mathbb{N}$ and $x \in B(p; \epsilon)$. Thus

$$\begin{aligned} d(f^n(x), p) &= d(f(f^{n-1}(x)), f(f^{n-1}(p))) \\ &\leq rd(f^{n-1}(x), f^n(x)) \leq rd(f^{n-1}(x), p) + rd(p, f^n(x)). \end{aligned}$$

Then

$$d(f^n(x), p) \leq \frac{r}{1-r} d(f^{n-1}(x), p) \leq d(f^{n-1}(x), p) \leq \ldots \leq d(x, p) < \epsilon.$$

Therefore $f^n(x) \in B(p; \epsilon)$ and hence f is uniformly virtually stable with respect to a sequence (n). □

Proposition 2.5. *If f is an asymptotically quasi-nonexpansive map, then it is uniformly virtually stable.*

Proof. Let (X, d) be a metric space and f be an asymptotically quasi-nonexpansive map. Then there is a sequence (a_n) in $\mathbb{R}^+$ converging to 1 such that $d(f^n(x), p) \leq a_n d(x, p)$ for all $x \in X,\ p \in F(f)$ and $n \in \mathbb{N}$. Because $\lim_n a_n = 1$, there is $N \in \mathbb{N}$ such that $0 < a_n < 2$ for all $n \geq N$. Let $p \in F(f),\ \epsilon > 0,\ n \in \mathbb{N}$ and $x \in B(p; \frac{\epsilon}{2})$. Therefore

$$d(f^{N+n}(x), p) \leq a_{N+n} d(x, p) < 2(\frac{\epsilon}{2}) = \epsilon.$$

Then $f^{N+n}(x) \in B(p; \epsilon)$ and hence f is uniformly virtually stable. □

Now, let us introduce new classes of maps that are motivated by the definition of Suzuki generalized nonexpansive maps.

Definition 2.6. Let (X, d) be a metric space. A map f is said to be

1. a **conditionally lipschitzian map** if there are $r,\ L > 0$ such that if $x, y \in X$ satisfying $rd(x, f(x)) \leq d(x, y)$, then $d(f(x), f(y)) \leq Ld(x, y)$.

2. a **conditionally uniformly lipschitzian map** if there are $r,\ L > 0$ such that $d(f^n(x), f^n(y)) \leq Ld(x, y)$ for all $n \in \mathbb{N}$ and $x,\ y \in X$ satisfying $rd(x, f(x)) \leq d(x, y)$.

It is easy to see that conditionally uniformly lipschitzian maps and Suzuki generalized nonexpansive maps are conditionally lipschitzian. However, Suzuki generalized nonexpansive maps may not conditionally uniformly lipschitzian.

Proposition 2.7. *If f is a conditionally lipschitzian map with $L \leq 1$, then it is uniformly virtually stable with respect to a sequence (n).*

Proof. Let (X, d) be a metric space and f be a conditionally lipschitzian map. Then there is $r > 0$ such that $d(f(x), f(y)) \leq Ld(x, y)$ for all $x,\ y \in X$ satisfying $rd(x, f(x)) \leq d(x, y)$. Let $p \in F(f),\ \epsilon > 0,\ n \in \mathbb{N}$ and $x \in B(p; \epsilon)$. Since $d(f(x), f(p)) \leq d(x, p)$ for all $p \in F(f)$ and $x \in X$, then

$$d(f^n(x), p) = d(f^n(x), f^n(p)) \leq Ld(f^{n-1}(x), f^{n-1}(p)) \leq \ldots \leq L^n d(x, p) < \epsilon.$$

Therefore $f^n(x) \in B(p; \epsilon)$ and hence f is uniformly virtually stable with respect to a sequence (n). □

Proposition 2.8. *If f is a conditionally uniformly lipschitzian map, then it is uniformly virtually stable with respect to a sequence (n).*

Proof. Let (X, d) be a metric space and f be a conditionally uniformly lipschitzian map. Then there are $r,\ L > 0$ such that $d(f^n(x), f^n(y)) \leq Ld(x, y)$ for all $n \in \mathbb{N}$ and $x,\ y \in X$ satisfying $rd(x, f(x)) \leq d(x, y)$. Let $p \in F(f)$. Now, for each $\epsilon > 0,\ n \in \mathbb{N}$ and $x \in B(p; \frac{\epsilon}{L})$. Since $d(f^n(x), f^n(p)) \leq Ld(x, p)$ for all $n \in \mathbb{N},\ p \in F(f)$ and $x \in X$, then

$$d(f^n(x), p) = d(f^n(x), f^n(p)) \leq Ld(x, p) < L(\frac{\epsilon}{L}) = \epsilon.$$

Therefore $f^n(x) \in B(p; \epsilon)$ and hence f is uniformly virtually stable with respect to a sequence (n). □

Example 2.9. *Let* $f : [0,4] \to [0,4]$ *be defined by* $f(x) = \begin{cases} 0 \ ; \ x \neq 4 \\ 1 \ ; \ x = 4 \end{cases}$
Then f *is a Kannan and conditionally uniformly lipschitzian map with* $r = \frac{1}{3}$ *and* $L = 1$*. Note that* $F(f) = \{0\}$ *and* $C(f) = [0,4]$*.*

To show that f is a Kannan map, let $x < y = 4$. Then

$$|f(x) - f(y)| = 1 \leq \frac{x+3}{3} = \frac{1}{3}|x - f(x)| + \frac{1}{3}|y - f(y)|.$$

To show that f is a conditionally uniformly lipschitzian map, let $y < x = 4$ and $1 = \frac{1}{3}|x - f(x)| \leq |x - y| = 4 - y$. Then $|f^n(x) - f^n(y)| \leq 1 \leq 4 - y = |x - y|$ for all $n \in \mathbb{N}$. Next, let $x < y = 4$ and $\frac{x}{3} = \frac{1}{3}|x - f(x)| \leq |x - y| = 4 - x$. Then $1 \leq 4 - x$ and hence $|f^n(x) - f^n(y)| \leq 1 \leq 4 - x = |x - y|$ for all $n \in \mathbb{N}$.

Example 2.10. *[6] Let* $f : [0,3] \to [0,3]$ *be defined by* $f(x) = \begin{cases} 0 \ ; \ x \neq 3 \\ 1 \ ; \ x = 3 \end{cases}$
Then f *is a quasi-nonexpansive and Suzuki generalized nonexpansive map. Note that* $F(f) = \{0\}$ *and* $C(f) = [0,3]$*.*

It is easy to see that $f(x) \leq x$ for all $x \in [0,3]$. Then f is a quasi-nonexpansive map. By the similar argument in the previous example, f is a Suzuki generalized nonexpansive map.

Example 2.11. *Let* $X = \mathbb{R}^2$ *be equipped with the supremum norm and* $f : X \to X$ *be defined by* $f(x,y) = \begin{cases} (x, |x|) & ; \ (x,y) \neq (0,3) \\ (0,1) & ; \ (x,y) = (0,3) \end{cases}$
Then f *is a Suzuki generalized nonexpansive map,* $F(f) = \{(x, |x|) \mid x \in \mathbb{R}\}$ *and* $C(f) = \mathbb{R}^2$*. Notice that* f *is not continuous and hence it is not nonexpansive.*

Firstly, let $(a,b), \ (x,y) \neq (0,3)$. Then

$$\begin{aligned} \|f(a,b) - f(x,y)\| &= \max\{|a - x|, ||a| - |x||\} = |a - x| \\ &\leq \max\{|a - x|, |b - y|\} = \|(a,b) - (x,y)\|. \end{aligned}$$

Secondly, let (x, y) be such that $1 = \frac{1}{2}\|(0,3) - f(0,3)\| \leq \|(0,3) - (x,y)\|$.
If $|x| \leq 1$, $\|f(0,3) - f(x,y)\| = \max\{|x|, 1 - |x|\} \leq 1 \leq \|(0,3) - (x,y)\|$.
If $|x| > 1$,

$$\begin{aligned}\|f(0,3) - f(x,y)\| &= \max\{|x|, |x| - 1\} = |x| \\ &\leq \max\{|x|, |y-3|\} = \|(0,3) - (x,y)\|.\end{aligned}$$

Finally, let (x, y) be such that

$$\left|\frac{y - |x|}{2}\right| = \frac{1}{2}\|(x,y) - f(x,y)\| \leq \|(0,3) - (x,y)\| \tag{2.10.1}$$

Note that the case $1 \leq \|(0,3) - (x,y)\|$ has been proved in the previous paragraph, we may assume $\max\{|x|, |y-3|\} = \|(0,3) - (x,y)\| < 1$ and hence $\frac{y-|x|}{2} > 0$. So (2.10.1) becomes

$$\frac{y - |x|}{2} = \frac{1}{2}\|(x,y) - f(x,y)\| \leq \|(0,3) - (x,y)\| < 1 \tag{2.10.2}$$

If $3 \leq y < 4$, $1 < \frac{y-|x|}{2} \leq \|(0,3) - (x,y)\| < 1$ which is a contradiction.
If $2 < y < 3$ and $|x| \leq 3 - y$,

$$\frac{y - |x|}{2} \leq \|(0,3) - (x,y)\| = \max\{|x|, 3 - y\} = 3 - y$$

which is equivalent to $y \leq 2 + \frac{|x|}{3}$.
If $2 < y < 3$ and $3 - y < |x|$,

$$\frac{y - |x|}{2} \leq \|(0,3) - (x,y)\| = \max\{|x|, 3 - y\} = |x|$$

which is equivalent to $y \leq 3|x|$.
Thus, the solution set of (2.10.2) is

$$S := \left\{(x,y) \in (-1,1) \times (2,3) \mid y \leq 2 + \frac{|x|}{3} \text{ or } y \leq 3|x|\right\}$$

To see that $\|f(0,3) - f(x,y)\| \leq \|(0,3) - (x,y)\|$ for each $(x,y) \in S$, let $(x,y) \in S$.
If $y \leq 2 + \frac{|x|}{3}$, $1 - |x| \leq 1 - \frac{|x|}{3} \leq 3 - y$.
If $y \leq 3|x|$, $1 - |x| \leq 3 - 3|x| \leq 3 - y$.
Then $\|f(0,3) - f(x,y)\| = \max\{|x|, 1 - |x|\} \leq \max\{|x|, 3 - y\} = \|(0,3) - (x,y)\|$.

The next example shows that the fixed point set of a virtually stable map may not be a retract of its convergence set.

Example 2.12. *Let* $f : [0,1] \to [0,1]$ *be defined by* $f(x) = \begin{cases} x & ; \; x < \frac{1}{2} \\ 1 & ; \; x \geq \frac{1}{2} \end{cases}$

Observe that f *is uniformly virtually stable with respect to a sequence* (n) *and* $f^n = f$ *which is not continuous for all* $n \in \mathbb{N}$. *It is easy to see that* $F(f) = [0, \frac{1}{2}) \cup \{1\}$ *and* $C(f) = [0,1]$.

The next theorem generalize Theorem 1.20 to discontinuous maps.

Theorem 2.13. *Suppose* X *is a regular space and* $f^{\infty}(C(f)) \subseteq F(f)$. *If* f *is virtually stable and* f^N *is continuous for some* $N \in \mathbb{N}$, *then* f^{∞} *is continuous and hence* $F(f)$ *is a retract of* $C(f)$.

Proof. Let $x \in C(f)$ and G be a neighbourhood of $f^{\infty}(x)$ in $F(f)$. Then there is a neighbourhood U of $f^{\infty}(x)$ in X such that $G = U \cap F(f)$. Since X is regular, there is a neighbourhood V of $f^{\infty}(x)$ in X such that $V \subseteq \overline{V} \subseteq U$. Now, by virtual stability, there exist a neighbourhood W of $f^{\infty}(x)$ in X and a strictly increasing sequence $(k_n) \subseteq \mathbb{N}$ such that $f^{k_n}(W) \subseteq V$ for all $n \in \mathbb{N}$. Since W is a neighbourhood of $f^{\infty}(x)$, there is $n_{\circ} \in \mathbb{N}$ such that $f^n(x) \in W$ for all $n \geq n_{\circ}$. Let $H = f^{-Nn_{\circ}}(W) \cap C(f)$. Then H is a neighbourhood of x in $C(f)$. To show that $f^{\infty}(H) \subseteq G$, let $a \in H$. Note that

$$f^{\infty}(a) = \lim_n f^n(a) = \lim_n f^{k_n}(f^{Nn_{\circ}}(a)) \in \overline{V} \cap F(f) \subseteq U \cap F(f) = G.$$

Thus f^{∞} is continuous and $F(f)$ is a retract of $C(f)$. □

It is easy to verify that the maps in Example 2.9, 2.10 and 2.11 satisfy all the conditions in the previous theorem. So their fixed point sets are retracts of their convergence sets. Meanwhile, the map f^N in Example 2.12 is not continuous for all $N \in \mathbb{N}$.

The next example shows that the result of the previous theorem may still be true even when the continuity of f^N is not assumed.

Example 2.14. *Let* $X = \mathbb{R}^2$ *be equipped with the supremum norm and*

$f : X \to X$ *be defined by* $f(x,y) = \begin{cases} (0, 3^{n-1}) & ; \; (x,y) = (0, 3^n) \text{ for some } n \in \mathbb{N} \\ (x, |x|) & ; \; (x,y) \neq (0, 3^n) \text{ for all } n \in \mathbb{N} \end{cases}$

Then f is a Suzuki generalized nonexpansive map. (Note that f^N is not continuous for all $N \in \mathbb{N}$ but $f^\infty(x,y) = (x,|x|)$ for all $(x,y) \in X$ which is continuous.)

Notice that $\|f(z) - f(w)\| \le \|z - w\|$ for all $z,\ w \notin A := \{(0,3^n) \mid n \in \mathbb{N}\}$ and it is easy to see that $\|f(z) - f(w)\| \le \|z - w\|$ for all $z,\ w \in A$.

Next, let $n \in \mathbb{N}$ and $(x,y) \notin A$ be such that

$$3^{n-1} = \frac{1}{2}\|(0,3^n) - f(0,3^n)\| \le \|(0,3^n) - (x,y)\|.$$

If $|x| \le 3^{n-1}$,

$$\|f(0,3^n) - f(x,y)\| = \max\left\{|x|, 3^{n-1} - |x|\right\} \le 3^{n-1} \le \|(0,3^n) - (x,y)\|.$$

If $|x| > 3^{n-1}$,

$$\begin{aligned}\|f(0,3^n) - f(x,y)\| &= \max\left\{|x|, |x| - 3^{n-1}\right\} = |x| \\ &\le \max\{|x|, |y - 3^n|\} = \|(0,3^n) - (x,y)\|.\end{aligned}$$

Now, let $n \in \mathbb{N}$ and (x,y) be such that

$$\left|\frac{y - |x|}{2}\right| = \frac{1}{2}\|(x,y) - f(x,y)\| \le \|(0,3^n) - (x,y)\| \qquad (2.13.1)$$

Note that the case $3^{n-1} \le \|(0,3^n) - (x,y)\|$ has been proved in the previous paragraph, we may assume $\max\{|x|, |y - 3^n|\} = \|(0,3^n) - (x,y)\| < 3^{n-1}$ and hence $\frac{y-|x|}{2} > 0$. So (2.13.1) becomes

$$\frac{y - |x|}{2} = \frac{1}{2}\|(x,y) - f(x,y)\| \le \|(0,3^n) - (x,y)\| < 3^{n-1} \qquad (2.13.2)$$

If $3^n \le y < 3^n + 3^{n-1}$,

$$3^{n-1} < \frac{y - |x|}{2} \le \|(0,3^n) - (x,y)\| < 3^{n-1}$$

which is a contradiction.

If $2 \cdot 3^{n-1} = 3^n - 3^{n-1} < y < 3^n$ and $|x| \le 3^n - y$,

$$\frac{y - |x|}{2} \le \|(0,3^n) - (x,y)\| = \max\{|x|, 3^n - y\} = 3^n - y$$

which is equivalent to $y \leq 2 \cdot 3^{n-1} + \frac{|x|}{3}$.
If $2 \cdot 3^{n-1} = 3^n - 3^{n-1} < y < 3^n$ and $3^n - y < |x|$,

$$\frac{y - |x|}{2} \leq \|(0, 3^n) - (x, y)\| = \max\{|x|, 3^n - y\} = |x|$$

which is equivalent to $y \leq 3|x|$.
Thus, the solution set of (2.13.2) is

$$S := \left\{(x, y) \in (-3^{n-1}, 3^{n-1}) \times (2 \cdot 3^{n-1}, 3^n) \mid y \leq 2 \cdot 3^{n-1} + \frac{|x|}{3} \ \text{ or } \ y \leq 3|x|\right\}$$

To see that $\|f(0, 3^n) - f(x, y)\| \leq \|(0, 3^n) - (x, y)\|$ for each $(x, y) \in S$, let $(x, y) \in S$.
If $y \leq 2 \cdot 3^{n-1} + \frac{|x|}{3}$, $3^{n-1} - |x| \leq 3^{n-1} - \frac{|x|}{3} \leq 3^n - y$.
If $y \leq 3|x|$, $3^{n-1} - |x| \leq 3^n - 3|x| \leq 3^n - y$.
Then

$$\begin{aligned}\|f(0, 3^n) - f(x, y)\| &= \max\left\{|x|, 3^{n-1} - |x|\right\} \\ &\leq \max\{|x|, 3^n - y\} = \|(0, 3^n) - (x, y)\|.\end{aligned}$$

The following lemma refines Theorem 1.21 for discontinuous maps.

Lemma 2.15. *Suppose X is a topological space. If f is uniformly virtually stable with respect to a sequence (k_nM) such that $\sup\{k_{n+1} - k_n \mid n \in \mathbb{N}\} < \infty$ and f^M is continuous on $F(f)$ for some $M \in \mathbb{N}$, then f is uniformly virtually stable with respect to a sequence (nM).*

Proof. Let $h := \sup\{k_{n+1} - k_n \mid n \in \mathbb{N}\} \in \mathbb{N}$, $p \in F(f)$ and G be a neighbourhood of p. We have to show that there is a neighbourhood H of p such that $f^{nM}(H) \subseteq G$ for all $n \in \mathbb{N}$. Because f is uniformly virtually stable, there is a neighbourhood H' of p such that $f^{k_nM}(H') \subseteq G$ for all $n \in \mathbb{N}$. Because f^{iM} is continuous at p for all $i \in \mathbb{N} \cup \{0\}$, there is a neighbourhood H of p such that $f^{iM}(H) \subseteq H' \cap G$ for all $i = 0, ..., \max\{h, k_1\}$. Now let $n \in \mathbb{N}$.
If $n \leq k_1$,

$$f^{nM}(H) \subseteq H' \cap G \subseteq G.$$

If $n > k_1$, then there is $N \in \mathbb{N}$ such that $n = k_N + i$ for some $i = 0, ..., h$. So

$$f^{nM}(H) = f^{k_N M}(f^{iM}(H)) \subseteq f^{k_N M}(H' \cap G) \subseteq f^{k_N M}(H') \subseteq G.$$

□

Theorem 2.16. *Suppose X is a metric space. If f is continuous and uniformly virtually stable with respect to a sequence (k_n) such that* $\sup\{k_{n+1} - k_n \mid n \in \mathbb{N}\} < \infty$*, then f is virtually nonexpansive. In particular when X is complete, then $C(f)$ is a G_δ-set.*

Proof. Take $M = 1$ in the previous lemma, then f is uniformly virtually stable with respect to a sequence (n) and hence virtually nonexpansive by Proposition 2.3. When X is complete, by Theorem 1.21, $C(f)$ is a G_δ-set. □

Corollary 2.17. *Suppose X is a complete metric space. If f is uniformly virtually stable with respect to a sequence (nM) and f^M is continuous for some $M \in \mathbb{N}$, then $C(f^M)$ is a G_δ-set.*

Proof. Let $g = f^M$. It is easy to see that g is continuous and uniformly virtually stable with respect to a sequence (n). By the previous theorem, $C(f^M) = C(g)$ is a G_δ-set. □

CHAPTER III
SOME RESULTS ON CONVERGENCE SETS

The first corollary in this chapter gives a sufficient condition which makes the domain of a given map the convergence set. Then, the combination of such condition on a contractible domain and virtual stability yields the contractibility of a fixed point set.

Theorem 3.1. *Let (X, d) be a complete metric space, $f : X \to X$ and $x \in X$. Suppose that there is $c \in [0, 1)$ such that*

$$d(f^{n+2}(x), f^{n+1}(x)) \leq cd(f^{n+1}(x), f^n(x))$$

for all sufficiently large $n \in \mathbb{N}$, then $x \in C(f)$.

Proof. Let $\epsilon > 0$ and $N \in \mathbb{N}$ be such that $d(f^{n+2}(x), f^{n+1}(x)) \leq cd(f^{n+1}(x), f^n(x))$ for every $n \geq N$. Then $\frac{c^{n_\circ - N}}{1-c} d(f^{N+1}(x), f^N(x)) < \epsilon$ for some integer $n_\circ > N$. Let $m,\ n \in \mathbb{N}$ such that $m > n > n_\circ$. Therefore

$$\begin{aligned} d(f^m(x), f^n(x)) &\leq d(f^m(x), f^{m-1}(x)) + \ldots + d(f^{n+1}(x), f^n(x)) \\ &\leq (c^{m-1-N} + \ldots + c^{n-N}) d(f^{N+1}(x), f^N(x)) \\ &< (\sum_{i=n}^{\infty} c^{i-N}) d(f^{N+1}(x), f^N(x)) < \epsilon. \end{aligned}$$

Then $(f^n(x))$ is Cauchy and hence $x \in C(f)$ by the completeness of X. □

Corollary 3.2. *Let (X, d) be a complete metric space and $f : X \to X$. If for each $x \in X$, there is $c_x \in [0, 1)$ such that*

$$d(f^{n+2}(x), f^{n+1}(x)) \leq c_x d(f^{n+1}(x), f^n(x)) \tag{3.2.1}$$

for all sufficiently large $n \in \mathbb{N}$, then $C(f) = X$.

Corollary 3.3. *Let (X, d) be a complete metric space, $f : X \to X$ and $0 \le c < 1$. If for all $x \in X$,*

$$d(f^2(x), f(x)) \le cd(f(x), x) \tag{3.3.1}$$

then $C(f) = X$.

Clearly, Condition (3.3.1) in Corollary 3.3 holds for a contraction but not vice versa. The next example shows that a map satisfying this condition in Corollary 3.3 need not be even nonexpansive.

Example 3.4. *Let $f : [0, \infty) \to [0, \infty)$ be defined by*

$$f(x) = \begin{cases} \frac{x}{4} & ; \ x = 2^k \text{ for some } k \in \mathbb{Z} \\ \frac{x}{2} & ; \ x \ne 2^k \text{ for all } k \in \mathbb{Z} \end{cases}$$

It is easy to see that $|f^2(x) - f(x)| = \frac{1}{4}|f(x) - x|$ if $x = 2^k$ and $|f^2(x) - f(x)| = \frac{1}{2}|f(x) - x|$ if $x \ne 2^k$. By Corollary 3.3, $C(f) = [0, \infty)$. However, f is not nonexpansive because f is not continuous.

Remark 3.5. *Condition (3.3.1) in Corollary 3.3 implies Condition (3.2.1) in Corollary 3.2. Condition (3.2.1) in Corollary 3.2 implies asymptotic regularity.*

The next example shows that Condition (3.2.1) in Corollary 3.2 is weaker than Condition (3.3.1) in Corollary 3.3.

Example 3.6. *Let $H = \{re^{i\pi t} \in \mathbb{C} \mid r \ge 0 \text{ and } 0 \le t \le 1\}$ and $f : H \to H$ be defined by $f(re^{i\pi t}) = tre^{i\pi t}$; $r \ge 0$ and $0 \le t \le 1$*

Note that $f^n(re^{i\pi t}) = t^n re^{i\pi t}$ for all $n \in \mathbb{N}$, $r \ge 0$ and $0 \le t \le 1$.
If $t = 1$, $re^{i\pi t} = -r$ is a fixed point. And then

$$|f^{n+2}(re^{i\pi t}) - f^{n+1}(re^{i\pi t})| = 0 = |f^{n+1}(re^{i\pi t}) - f^n(re^{i\pi t})|$$

for all $n \in \mathbb{N}$. So we can choose c to be any number.
If $t < 1$, it is easy to verify that

$$|f^{n+2}(re^{i\pi t}) - f^{n+1}(re^{i\pi t})| = t|f^{n+1}(re^{i\pi t}) - f^n(re^{i\pi t})|$$

for all $n \in \mathbb{N},\ r \geq 0$. So we can choose $c = t$

By Corollary 3.2, $C(f) = H$. Observe that t can tend to 1, so f does not satisfy Corollary 3.3.

The next examples show that asymptotic regularity is weaker than Condition (3.2.1) in Corollary 3.2.

Example 3.7. *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by*

$$f(x) = \begin{cases} x + \frac{1}{n+1} & ;\ x = 1 + \frac{1}{2} + ... + \frac{1}{n} \text{ for some } n \in \mathbb{N} \\ x & ;\ x \neq 1 + \frac{1}{2} + ... + \frac{1}{n} \text{ for all } n \in \mathbb{N} \end{cases}$$

It is easy to see that $\lim_n |f^{n+1}(x) - f^n(x)| = 0$ for each $x \in \mathbb{R}$ but $1 \notin C(f)$.

Example 3.8. *Let $s_n = 1 + \frac{1}{2} + ... + \frac{1}{n}$ for each $n \in \mathbb{N}$ and let $f : [1, \infty) \to [1, \infty)$ be defined by $f(x) = (x - s_n)(\frac{n+1}{n+2}) + s_{n+1}$; $s_n \leq x < s_{n+1}$*

Note that $\lim_n |f^{n+1}(s_k) - f^n(s_k)| = 0$ and $f[s_k, s_{k+1}) = [s_{k+1}, s_{k+2})$ for all $k \in \mathbb{N}$. Hence if $s_k \leq x < s_{k+1}$,

$$\begin{aligned} \lim_n |f^{n+1}(x) - f^n(x)| \leq \lim_n |s_{k+n+2} - s_{k+n}| &= \lim_n |f^{n+2}(s_k) - f^n(s_k)| \\ &\leq \lim_n |f^{n+2}(s_k) - f^{n+1}(s_k)| + \lim_n |f^{n+1}(s_k) - f^n(s_k)| = 0 \end{aligned}$$

So $\lim_n |f^{n+1}(x) - f^n(x)| = 0$ for each $x \in [1, \infty)$ and $C(f) = \varnothing$. Also note that this map is indeed continuous.

Lemma 3.9. *Let X be a Hausdorff space, $x \in X$, $f : X \to X$ and $M,\ N \in \mathbb{N}$ such that $\gcd(M, N) = 1$. If $x \in F(f^M) \cap F(f^N)$, then $x \in F(f)$.*

Proof. Suppose $1 < M < N$. By the Euclidean algorithm, let $q_1, ..., q_{k+1} \in \mathbb{N} \cup \{0\}$, $r_1, ..., r_k \in \mathbb{N}$ be such that $1 < r_k < ... < r_1 < M$ and

$$N = q_1 M + r_1 \tag{3.9.1}$$

$$M = q_2 r_1 + r_2 \tag{3.9.2}$$

$$\vdots$$

$$r_{k-1} = q_{k+1} r_k + 1$$

By Equation (3.9.1), $x \in F(f^{r_1})$. And then by Equation (3.9.2), $x \in F(f^{r_2})$. Following these steps, $x \in F(f)$ by the last equation. □

The next theorem is the combination of Theorem 2.12 in Chapter II and Corollary 3.2 in Chapter III.

Theorem 3.10. *If (X, d) is a contractible complete metric space and $f : X \to X$ is a virtually stable map satisfying the following conditions :-*

1. *f^M and f^N are continuous for some $M,\ N \in \mathbb{N}$ such that $\gcd(M, N) = 1$.*
2. *For each $x \in X$, there is $c_x \in [0, 1)$ such that for all sufficiently large $n \in \mathbb{N}$, $d(f^{n+2}(x), f^{n+1}(x)) \leq c_x d(f^{n+1}(x), f^n(x))$.*

then $F(f)$ is contractible.

Proof. Observe that for each $x \in C(f)$, $f^\infty(x) = \lim_n f^{nM}(x) \in F(f^M)$ and $f^\infty(x) = \lim_n f^{nN}(x) \in F(f^N)$. By the previous lemma, $f^\infty : C(f) \to F(f)$. By Condition 2., $X = C(f)$ and hence $F(f)$ is contractible. □

Corollary 3.11. *If (X, d) be a contractible complete metric space, $0 \leq c < 1$ and $f : X \to X$ is a virtually stable map satisfying the following conditions :-*

1. *f^M and f^N are continuous for some $M,\ N \in \mathbb{N}$ such that $\gcd(M, N) = 1$.*
2. *$d(f^2(x), f(x)) \leq cd(f(x), x)$ for all $x \in X$.*

then $F(f)$ is contractible.

The followings are examples of the previous corollary.

Example 3.12. *Let f be the map in Example 2.9. It is easy to verify that $f^2 = f^3$ and they are continuous with $\|f^2(x) - f(x)\| \leq \frac{1}{2}\|f(x) - x\|$ for all $x \in \mathbb{R}^2$. So $F(f)$ is contractible by Corollary 3.11. Note that $F(f)$ is not convex.*

Example 3.13. *Let X be the closed unit ball in ℓ_∞ and $f : X \to X$ be defined by $f(x_1, x_2, ...) = (x_1, \frac{x_1^2}{2}, 0, 0, ...)$. Then f is nonexpansive and it is easy to see that $\|f^2(x) - f(x)\| \leq \frac{1}{2}\|f(x) - x\|$ for all $x \in X$. So $F(f)$ is contractible by Corollary 3.11. Note that $F(f)$ is not star-convex.*

To show that f is nonexpansive, let $x = (x_1, x_2, \ldots),\ y = (y_1, y_2, \ldots) \in X$. Then

$$\begin{aligned}\|f(x) - f(y)\| &= \sup\left\{|x_1 - y_1|, |\frac{x_1^2}{2} - \frac{y_1^2}{2}|\right\} \\ &= \sup\left\{|x_1 - y_1|, \frac{|x_1 - y_1||x_1 + y_1|}{2}\right\} = |x_1 - y_1| \\ &\leq \sup\{|x_1 - y_1|, |x_2 - y_2|, \ldots\} = \|x - y\|.\end{aligned}$$

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