Chapter 2

FORMULATION OF THE TWO-NODE CATENARY ELEMENT

Basic Equations

In this study the two-node catenary element proposed by Jayaraman and Knudson (13) is employed. The material is assumed to be linear elastic undergoing small strain deformation through large displacements. For the elastic cable element stretched between points I and J in vertical plane YZ under the action of uniform gravity loads as shown in figure 2.1, the resulting deformed shape is given by the well known catenary equations :

$$L^2 = V^2 + \frac{H^2 \sinh^2 \lambda}{\lambda^2} \tag{2.1}$$

$$\lambda = \frac{w\left|H\right|}{2\left|F_1\right|} \tag{2.2}$$

where L is the actual stressed length of cable element, V and H are the vertical and horizontal components of vector IJ, w is the weight per unit unstretched length of the cable element and F_1 is the horizontal component of the cable force

The vertical component of the cable force, F_2, is related to the external load, w, by

$$F_2 = \frac{w}{2}\left[-\frac{V\cosh\lambda}{\sinh\lambda} + L\right] \quad (2.3)$$

The horizontal and vertical projections of the cable element are related to the cable forces and self weight through the elastic catenary equations, in which elastic deformation is included (15). Thus,

$$H = -F_1\left[\frac{L_u}{EA} + \frac{1}{w}\log_e\frac{F_4 + T_J}{T_I - F_2}\right] \quad (2.4)$$

$$V = \frac{1}{2EAw}\left[T_J^2 - T_I^2\right] + \frac{\left[T_J - T_I\right]}{w} \quad (2.5)$$

$$L = L_u + \frac{1}{2EAw}\left[F_4T_J + F_2T_I + F_1^2\log_e\frac{F_4 + T_J}{T_I - F_2}\right] \quad (2.6)$$

in which L_u is the unstretched length of the cable element, E is the modulus of elasticity, A is the unstressed cross sectional area, F_4 is the vertical component of the cable force at end J, and T_I and T_J are the tensions at origin I and end J, respectively.

The variables F_1, F_2, F_3, F_4, T_I and T_J are related by simple static equilibrium conditions as follows :

$$F_3 = -F_1 \quad (2.7a)$$

$$F_4 = -F_2 + w\,L_u \quad (2.7b)$$

$$T_I \quad = \quad (F_1 + F_2)^{1/2} \tag{2.7c}$$

$$T_J \quad = \quad (F_3 + F_4)^{1/2} \tag{2.7d}$$

Flexibility Iteration of Elastic Catenary Element

Equations (2.1) to (2.6) relate the force quantities with geometry in the equilibrium state. Since the relations are nonlinear, one cannot readily solve for the internal forces given a set of end positions. Iterative procedures are inevitably required. To do this one needs to know the relationship between small changes in cable forces and end displacements.

By virtue of eqns. (2.4), (2.5) and (2.7a) to (2.7d), H and V can be written as functions of F_1 and F_2 only. Small changes in H and V (i.e. δH^1 and δV^1 in figure 2.2) can thus be written in terms of δF_1 and δF_2 by using the chain rule of differentiation. Thus,

$$\begin{Bmatrix} \delta H^1 \\ \delta V^1 \end{Bmatrix} = \begin{bmatrix} \xi_1^1 & \xi_2^1 \\ \xi_3^1 & \xi_4^1 \end{bmatrix} \begin{Bmatrix} \delta F_1^1 \\ \delta F_2^1 \end{Bmatrix} \tag{2.8}$$

where
$$\xi_1^1 \quad = \quad (\partial H/\partial F_1)^1 \tag{2.9a}$$

$$\xi_2^1 \quad = \quad (\partial H/\partial F_2)^1 \tag{2.9b}$$

$$\xi_3^1 \quad = \quad (\partial V/\partial F_1)^1 \tag{2.9c}$$

$$\xi_4^i = (\partial V/\partial F_2)^i \qquad (2.9d)$$

and i refers to the i^{th} iteration

Evaluation of the relations (2.9a),(2.9b),(2.9c) and (2.9d), in view of eqns.(2.4) and (2.5), leads to

$$\xi_1^i = \frac{H^i}{F_1^i} + \frac{1}{w}\left[\frac{F_4^i}{T_J^i} + \frac{F_2^i}{T_I^i}\right] \qquad (2.10a)$$

$$\xi_2^i = \frac{F_1^i}{w}\left[\frac{1}{T_J^i} - \frac{1}{T_I^i}\right] \qquad (2.10b)$$

$$\xi_3^i = \frac{F_1^i}{w}\left[\frac{1}{T_J^i} - \frac{1}{T_I^i}\right] \qquad (2.10c)$$

$$\xi_4^i = -\frac{L_u}{EA} - \frac{1}{w}\left[\frac{F_4^i}{T_J^i} + \frac{F_2^i}{T_I^i}\right] \qquad (2.10d)$$

One may note that ξ_2^i is equal to ξ_3^i due to symmetry requirement.

If the misclosure vector J^iJ at iteration step i (figure 2.2) exceeds the prescribed tolerance value, the process will be repeated with the new values of F_1 and F_2 for the next step i+1. The end forces F_1 and F_2 at the step i+1 are simply

$$F_1^{i+1} \quad = \quad F_1^{i} + \delta F_1^{i} \tag{2.11a}$$

$$F_2^{i+1} \quad = \quad F_2^{i} + \delta F_2^{i} \tag{2.11b}$$

The correcting forces δF_1^i and δF_2^i are easily determined by inverting eqn.(2.8). Thus,

$$\begin{Bmatrix} \delta F_1^i \\ \delta F_2^i \end{Bmatrix} = \begin{bmatrix} \alpha_1^i & \alpha_2^i \\ \alpha_3^i & \alpha_4^i \end{bmatrix} \begin{Bmatrix} \delta H^i \\ \delta V^i \end{Bmatrix} \tag{2.12}$$

in which $\alpha_1^i = \xi_4^i/\Delta_i$, $\alpha_2^i = -\xi_3^i/\Delta_i$, $\alpha_3^i = -\xi_2^i/\Delta_i$, $\alpha_4^i = \xi_1^i/\Delta_i$ and $\Delta_i = \xi_1^i\,\xi_4^i - \xi_2^i\,\xi_3^i$

Good starting values for F_1 and F_2 are needed in order to start the flexibility iteration and to get good convergence. The appropriate starting values for F_1 and F_2 can be evaluated by first estimating λ from the relation

$$\lambda \quad = \quad \left[3\left(\frac{L_u^2 - V^2}{H^2} - 1\right)\right]^{1/2} \tag{2.13}$$

in which L_u is replaced by L in eqn. (2.1) and keeping only the first term of a series expansion of $(\sinh^2 \lambda)/(\lambda^2)$.

Values of F_1 and F_2 then follow from eqns. (2.2) and (2.3) by noting that the sign of F_1 is always opposite to that of H. Thus,

$$F_1 = \frac{-\,w\,H}{2\,\lambda} \tag{2.14}$$

$$F_2 = \frac{w}{2}\left[\frac{-\,V\cosh\lambda}{\sinh\lambda} + L_u\right] \tag{2.15}$$

Again the unstretched length is used instead of the actual one. In case where the unstretched length L_u is shorter than the chord length, then eqn. (2.13) cannot be solved. Then, the value of λ may be estimated to be 0.2 as suggested by Peyrot and Goulois (12).

<u>Formulation of Tagent Stiffness Matrix and Element Nodal Forces</u>

The local tangent stiffness matrix is obtained by combining the stiffness in the direction perpendicular to the element plane to the inplane stiffness. Thus,

$$\begin{Bmatrix} \delta F_x^I \\ \delta F_y^I \\ \delta F_z^I \\ \delta F_x^J \\ \delta F_y^J \\ \delta F_z^J \end{Bmatrix} = \begin{bmatrix} \frac{-F_1}{H} & 0 & 0 & \frac{F_1}{H} & 0 & 0 \\ & -\alpha_1 & -\alpha_2 & 0 & \alpha_1 & \alpha_2 \\ & & -\alpha_4 & 0 & \alpha_2 & \alpha_1 \\ & & & \frac{-F_1}{H} & 0 & 0 \\ & \text{symmetry} & & & -\alpha_1 & -\alpha_2 \\ & & & & & -\alpha_4 \end{bmatrix} \begin{Bmatrix} \delta u^I \\ \delta v^I \\ \delta w^I \\ \delta u^J \\ \delta v^J \\ \delta w^J \end{Bmatrix} \tag{2.16}$$

This tangent stiffness matrix is then transformed to the global coordinate system by appling the usual transformation matrix [a],

$$[a] = \begin{bmatrix} m & -l & 0 & 0 & 0 & 0 \\ l & m & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & m & -l & 0 \\ 0 & 0 & 0 & l & m & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad (2.17)$$

where l and m are sin β and cos β respectively, and β is the angle between the element plane and the global YZ plane. The tangent stiffness matrix after transformation becomes

$$[S] = \begin{bmatrix} [\ C] & [-C] \\ [-C] & [\ C] \end{bmatrix} \qquad (2.18)$$

in which

$$[C] = \begin{bmatrix} \frac{-F_1}{H} m^2 - \alpha_1 l^2 & \frac{-F_1}{H} lm - \alpha_1 lm & -\alpha_2 l \\ & \frac{-F_1}{H} l^2 - \alpha_1 m^2 & -\alpha_2 m \\ \text{symmetry} & & -\alpha_4 \end{bmatrix} \qquad (2.19)$$

The element nodal forces after transformation can be estimated by the standard procedure. Thus,

$$\{R\} = \begin{Bmatrix} -F_1 l \\ -F_1 m \\ -F_2 \\ -F_3 l \\ -F_3 m \\ -F_4 \end{Bmatrix} \qquad (2.20)$$

Flow Chart for Flexibility Iterative Process