CHAPTER III

GENERALIZED TRANSFORMATION SEMIGROUPS HAVING PROPER DENSE SUBSEMIGROUPS

Generalized transformation semigroups defined in Chapter I are natural and extensive generalizations of usual transformation semigroups. Then any results of such generalized transformation semigroups extending from some results of usual transformation semigroups should be interesting. Higgins has characterized the three standard transformation semigroups having proper dense subsemigroups in [2] as follows:

> "If X is a set and $\mathcal{S}$ denotes any one of $\mathcal{T}_X$, $\mathcal{PT}_X$ or $\mathcal{I}_X$, then $\mathcal{S}$ has a proper dense subsemigroup if and only if X is infinite."

For convenience, this result will be referred as "Higgins Theorem".

The main purpose of this research is to introduce a generalization of Higgins Theorem in terms of generalized transformation semigroups as follows:

> "If X and Y are sets, $\mathcal{S}(X,Y)$ denotes any one of $\mathcal{T}(X,Y)$, $\mathcal{PT}(X,Y)$ or $\mathcal{I}(X,Y)$ and $\theta \in \mathcal{S}(Y,X)$, then $(\mathcal{S}(X,Y),\theta)$ has a proper dense subsemigroup if and only if X and Y are both infinite and $|\nabla\theta| = \min\{|X|,|Y|\}$."

Higgins has proved Higgins Theorem by using the Zigzag Theorem, Theorem 1.1, Theorem 1.5 and Theorem 1.6 as lemmas. In his proof, he separated the proof for $\mathcal{S} = \mathcal{T}_X$, the proof for $\mathcal{S} = \mathcal{PT}_X$ and that for $\mathcal{S} = \mathcal{I}_X$ completely.

However, the proof for $\mathcal{S} = \mathcal{I}_X$ was given briefly by referring to the previous proofs for $\mathcal{S} = \mathcal{T}_X$ and $\mathcal{S} = \mathcal{PT}_X$.

Included in this chapter, a new proof for Higgins Theorem is given. Theorem 1.1 and Theorem 1.5 are not required in our proof. The proof of Theorem 1.5 given in [1] is quite complicated. Moreover, our proof is not separated for the different types of $\mathcal{S}$. However, the Zigzag Theorem, Theorem 1.6, Proposition 2.1, Proposition 2.3 and Proposition 2.4 are required for our proof. Remind that the proofs of these propositions require only elementary concepts of mappings and cardinalities of sets. In proving our main theorem of this chapter, we use our technique of proving Higgins Theorem, presented as follows:

Assume that X is finite and let $\mathcal{S}$ be any one of $\mathcal{T}_X$, $\mathcal{PT}_X$ or $\mathcal{I}_X$. Then the symmetric group on X, $\mathcal{G}_X$, is a finite group. Let U be a dense subsemigroup of $\mathcal{S}$. To show that $U \cap \mathcal{G}_X \neq \varnothing$, let $\alpha \in \mathcal{G}_X$. Since U is dense in $\mathcal{S}$, we have by the Zigzag Theorem that $\alpha = \beta\gamma$ for some $\beta \in U, \gamma \in \mathcal{S}$. Then $|X| = |\nabla\alpha| \leq |\nabla\beta|$, so $\Delta\beta = \nabla\beta = X$ since X is finite. Thus $\beta \in U \cap \mathcal{G}_X$, so $U \cap \mathcal{G}_X \neq \varnothing$. Now we have that $U \cap \mathcal{G}_X$ is a subgroup of $\mathcal{G}_X$. Since X is finite and $|\nabla\alpha\beta| \leq \min\{|\nabla\alpha|, |\nabla\beta|\}$ for all $\alpha, \beta \in \mathcal{PT}_X$, we have that for all $\alpha, \beta \in \mathcal{S}, \alpha\beta \in \mathcal{G}_X$ implies $\alpha, \beta \in \mathcal{G}_X$. From this fact and the Zigzag Theorem, it follows that $U \cap \mathcal{G}_X$ is dense in $\mathcal{G}_X$, that is, $\mathrm{Dom}(U \cap \mathcal{G}_X, \mathcal{G}_X) = \mathcal{G}_X$. By Theorem 1.6, $U \cap \mathcal{G}_X = \mathcal{G}_X$, so $\mathcal{G}_X \subseteq U$.

To show that $U = \mathcal{S}$, suppose not. Since $\mathcal{S}$ is finite, there exists $\alpha \in \mathcal{S} \setminus U$ such that $|\nabla\alpha| = \max\{|\nabla\beta| \mid \beta \in \mathcal{S} \setminus U\}$. Then $|\nabla\alpha| < |X|$ since $\mathcal{G}_X \subseteq U$. But U is dense in $\mathcal{S}$, so by Corollary 1.4 $\alpha = \lambda\gamma = (\beta\mu)\gamma$ where $\lambda, \mu \in U, \beta, \gamma \in \mathcal{S} \setminus U$ and $\lambda = \beta\mu$. From $\alpha = \lambda\gamma = \beta\mu\gamma$, we have $|\nabla\alpha| \leq |\nabla\beta|, |\nabla\alpha| \leq |\nabla\gamma|$ and $|\nabla\alpha| \leq |\nabla\lambda|$, and from $\lambda = \beta\mu$, we have $|\nabla\lambda| \leq |\nabla\beta|$. By the property of α, we have that $|\nabla\alpha| \geq |\nabla\beta|$ and $|\nabla\alpha| \geq |\nabla\gamma|$. These inequalities yield that $|\nabla\alpha| = |\nabla\beta| = |\nabla\gamma| = |\nabla\lambda|$. Then $|\nabla\alpha| = |\nabla\lambda\gamma| = |\nabla\gamma|$. Since $\nabla\lambda\gamma \subseteq \nabla\gamma$ and $|\nabla\gamma| < \infty$, we get that $\nabla\lambda\gamma = \nabla\gamma$. Now we have $\nabla\lambda\gamma = \nabla\gamma$ and $|\nabla\lambda| < |X|$, so by Proposition 2.1(i), $\lambda\eta = \lambda\gamma$ for some $\eta \in \mathcal{S}$ with $|\nabla\eta| > |\nabla\gamma|$. Then $|\nabla\eta| > |\nabla\alpha|$ which implies by the property of α that $\eta \in U$. Thus $\alpha = \lambda\gamma = \lambda\eta \in U$ since $\lambda, \eta \in U$. But $\alpha \notin U$, so we have a contradiction.This proves that $\mathcal{S}$ has no proper dense subsemigroup.

Conversely, assume that X is infinite. Let $A \subseteq X$ be such that A is infinite and $|X \setminus A| = |X|$. Set.

$$U = \left\{ \alpha \in \mathcal{S} \;\middle|\; |A\alpha \cap (X \setminus A)| < |A| \right\}.$$

We have by Proposition 2.3 that U is a proper subsemigroup of $\mathcal{S}$.

To show that U is dense in $\mathcal{S}$, let $\alpha \in \mathcal{S}$. Then $\Delta\alpha = \bigcup_{x \in \nabla\alpha} x\alpha^{-1}$, $|\{x\alpha^{-1} | x \in \nabla\alpha \text{ and } x\alpha^{-1} \cap A \neq \varnothing\}| \leq |A|$ and $|\{x\alpha^{-1} | x \in \nabla\alpha \text{ and } x\alpha^{-1} \cap A = \varnothing\}| \leq |\nabla\alpha| \leq |X| = |X \setminus A|$, so there exists a 1–1 mapping δ such that $\Delta\delta = \{x\alpha^{-1} \mid x \in \nabla\alpha\}$, $(x\alpha^{-1})\delta \in A$ if $x\alpha^{-1} \cap A \neq \varnothing$ and $(x\alpha^{-1})\delta \in X \setminus A$ if $x\alpha^{-1} \cap A = \varnothing$. Define $\lambda: \Delta\alpha \to X$ by $(x\alpha^{-1})\lambda = \{(x\alpha^{-1})\delta\}$ for all $x \in \nabla\alpha$. Then $\lambda \in \mathcal{S}$ and by the definition of δ, we have $A\lambda \subseteq A$. By the definition of λ, we have that for each $x \in \nabla\alpha, |(x\alpha^{-1})\lambda| = 1$, so $(x\alpha^{-1})\lambda = \{y\lambda\}$ for all $x \in \nabla\alpha$ and $y \in x\alpha^{-1}$. Since δ is 1–1, it implies that $(y\lambda)\lambda^{-1} = x\alpha^{-1}$ for all $x \in \nabla\alpha$ and $y \in x\alpha^{-1}$. If $x \in \Delta\lambda (= \Delta\alpha)$, then $x\alpha \in \nabla\alpha$ and $x \in (x\alpha)\alpha^{-1}$, and hence $(x\lambda)\lambda^{-1} = (x\alpha)\alpha^{-1}$. This implies that $(x\lambda)\lambda^{-1} = (x\alpha)\alpha^{-1}$ for all $x \in \Delta\lambda$. By Proposition 2.4(i), there exists $\gamma \in \mathcal{S}$ such that $\alpha = \lambda\gamma$. Since $|\nabla\lambda| \leq |X| = |X \setminus A|$, there exists $\eta_1 \in \mathcal{I}_X$ with $\Delta\eta_1 \subseteq X \setminus A$ and $\nabla\eta_1 = \nabla\lambda$.

Let $\eta_2: X \to X$ be such that $\eta_2|_{\Delta\eta_1} = \eta_1$ and $|A\eta_2| < |A|$. Define $\eta = \eta_1$ if $\mathcal{S} = \mathcal{PT}_X$ or $\mathcal{I}_X$ and $\eta = \eta_2$ if $\mathcal{S} = \mathcal{T}_X$. Then $\eta \in \mathcal{S}$. Since $A\lambda \subseteq A$ and $|A\eta| < |A|$, it follows that $\lambda, \eta \in U$. From defining λ and η, we have $\nabla\lambda \subseteq \nabla\eta$, so by Proposition 2.4(ii), $\beta\eta = \lambda$ for some $\beta \in \mathcal{S}$. Since $|A\eta| < |A|$, we get that $|A\eta\gamma| < |A|$. This implies that $\eta\gamma \in U$. Now we have the following zigzag in $\mathcal{S}$ over U with value α:

$$\begin{aligned} \alpha &= \lambda\gamma \quad , \lambda \in U \quad , \gamma \in \mathcal{S}, \\ &= (\beta\eta)\gamma \quad , \eta \in U \quad , \beta \in \mathcal{S}, \quad \lambda = \beta\eta, \\ &= \beta(\eta\gamma) \quad , \eta\gamma \in U. \end{aligned}$$

Then $\alpha \in \mathrm{Dom}(U, \mathcal{S})$ by the Zigzag Theorem. This proves that U is dense in $\mathcal{S}$, as required.#

We give a remark that a proper dense subsemigroup of $\mathcal{S}$ where X is infinite, given by Higgins, was $\{\alpha \in \mathcal{S} \mid |X \smallsetminus \nabla\alpha| = \infty\} \cup \{i_X\}$. Our constructed proper dense subsemigroup U of $\mathcal{S}$ is more complicated and depends on our given set A. It can be shown by our construction of U that $\mathcal{S}$ has infinitely many proper dense subsemigroups. In fact, we can show that there is a collection of cardinality $|X|$ of proper dense subsemigroups of $\mathcal{S}$. Moreover, our constructed U given here is also a guide for our construction of proper dense subsemigroups in generalized transformation semigroups. To show that there exists a collection $\mathcal{C}$ of proper dense subsemigroups of $\mathcal{S}$ such that $|\mathcal{C}| = |X|$ where X is infinite, consider the set X and $X \times X$. We have that $|X \times X| = |X|$. Then there exists a bijection $\varphi: X \times X \to X$. For each $x \in X$, let $A_x = \{(x,t)\varphi \mid t \in X\}$. Then

$X = \bigcup_{x\in X} A_x$, $A_x \cap A_y = \varnothing$ if $x \neq y$ and $|A_x| = |X|$ for every $x \in X$. Hence $|X \setminus A_x| = |X|$ for all $x \in X$. For each $x \in X$, let

$$U_x = \{\alpha \in \mathcal{S} \mid |A_x\alpha \cap (X \setminus A_x)| < |A_x|\}$$

and $\mathcal{C} = \{U_x \mid x \in X\}$. Then by the previous proof, U_x is a proper dense subsemigroup of $\mathcal{S}$. To show that $U_x \neq U_y$ if $x \neq y$ in X, let $x, y \in X$ be distinct. Then $A_x \cap A_y = \varnothing$. Since $|A_x \cup A_y| = |A_x|$, there exists $\alpha \in \mathcal{I}_X$ such that $\Delta\alpha = A_x \cup A_y$ and $\nabla\alpha = A_x$. Let $\beta \in \mathcal{T}_X$ be an extension of α and set $\gamma = \beta$ if $\mathcal{S} = \mathcal{T}_X$ and $\gamma = \alpha$ if $\mathcal{S} = \mathcal{PT}_X$ or $\mathcal{I}_X$. Then $\gamma \in \mathcal{S}$. Since $A_x\gamma = A_x\alpha \subseteq A_x$ and $A_y\gamma = A_y\alpha \subseteq A_x \subseteq X \setminus A_y$, it follows that $|A_x\gamma \cap (X \setminus A_x)| = 0$ and $|A_y\gamma \cap (X \setminus A_y)| = |A_y\gamma| = |A_y|$. Then $\gamma \in U_x$ but $\gamma \notin U_y$. Hence $U_x \neq U_y$. This proves that $|\mathcal{C}| = |X|$.

Our proof of the main result is quite complicated. To make the proof easier to follow, we shall separate the proof to be lemmas. There are five lemmas. The first lemma is required for the second one.

Lemma 3.1. *Let X and Y be sets, $\mathcal{S}(X,Y)$ denote any one of $\mathcal{T}(X,Y)$, $\mathcal{PT}(X,Y)$ or $\mathcal{I}(X,Y)$, $\theta \in \mathcal{S}(Y,X)$ and U a dense subsemigroup of $(\mathcal{S}(X,Y),\theta)$. Then $\{\alpha \in \mathcal{S}(X,Y) \mid |\nabla\alpha| > |\nabla\theta|\} \subseteq U$.*

Proof. If $\alpha \in \mathcal{S}(X,Y) \setminus U$, then by the Zigzag Theorem, $\alpha = \beta\theta\gamma$ for some $\beta \in U, \gamma \in \mathcal{S}(X,Y)$, which implies that $|\nabla\alpha| \leq |\nabla\theta|$. Hence we have that for $\alpha \in \mathcal{S}(X,Y)$, $|\nabla\alpha| > |\nabla\theta|$ implies that $\alpha \in U$, so the lemma is proved. #

Lemma 3.2. *Let X and Y be sets, $\mathcal{S}(X,Y)$ denote any one of $\mathcal{T}(X,Y)$, $\mathcal{PT}(X,Y)$ or $\mathcal{I}(X,Y)$, $\theta \in \mathcal{S}(Y,X)$ and U a dense subsemigroup of $(\mathcal{S}(X,Y),\theta)$. If $|\nabla\theta| < \min\{|X|,|Y|\}$, then $U = \mathcal{S}(X,Y)$.*

Proof. To prove that $U = \mathcal{S}(X,Y)$, suppose not. Then $\mathcal{S}(X,Y) \setminus U \neq \varnothing$.

Case 1. X or Y is finite. Then $\min\{|X|,|Y|\} < \infty$. Since $|\nabla\alpha| \leq \min\{|X|,|Y|\}$ for all $\alpha \in \mathcal{S}(X,Y)$ and $\mathcal{S}(X,Y) \setminus U \neq \varnothing$, it follows that there exists $\beta \in \mathcal{S}(X,Y) \setminus U$ such that $|\nabla\beta| \geq |\nabla\alpha|$ for all $\alpha \in \mathcal{S}(X,Y) \setminus U$. Since U is dense in $(\mathcal{S}(X,Y),\theta)$, we have by Corollary 1.4 that $\beta = \lambda\theta\gamma$ for some $\lambda \in U, \gamma \in \mathcal{S}(X,Y) \setminus U$. By the property of β, we have $|\nabla\gamma| \leq |\nabla\beta|$. But $|\nabla\beta| = |\nabla(\lambda\theta\gamma)| \leq |\nabla\theta\gamma| \leq |\nabla\gamma|$, so we have $|\nabla\beta| = |\nabla\gamma| = |\nabla\theta\gamma| < \infty$. Since $\nabla\theta\gamma \subseteq \nabla\gamma$, we get that $\nabla\theta\gamma = \nabla\gamma$. By Proposition 2.1(i), there exists $\mu \in \mathcal{S}(X,Y)$ such that $|\nabla\mu| > |\nabla\gamma|$ and $\theta\mu = \theta\gamma$. Then $|\nabla\mu| > |\nabla\beta|$, so the property of β yields $\mu \in U$. Hence $\beta = \lambda\theta\gamma = \lambda\theta\mu$. Since $\lambda, \mu \in U$, it follows that $\beta \in U$, which is a contradiction.

Case2. X and Y are infinite. Let $\eta \in \mathcal{S}(X,Y) \setminus U$. Then by the Zigzag Theorem, $\eta = \lambda\theta\gamma$ for some $\lambda \in U, \gamma \in \mathcal{S}(X,Y)$. By Proposition 2.1(ii), $\theta\gamma = \theta\beta$ for some $\beta \in \mathcal{S}(X,Y)$ with $|\nabla\beta| = \min\{|X|,|Y|\}$. Since $|\nabla\beta| = \min\{|X|,|Y|\} > |\nabla\theta|$, by Lemma 3.1, we have $\beta \in U$, and hence $\eta = \lambda\theta\gamma = \lambda\theta\beta \in U$, which is a contradiction.

This proves that $U = \mathcal{S}(X,Y)$, as required. #

Lemma 3.3. *Let X and Y be sets, $\mathcal{S}(X,Y)$ denote any one of $\mathcal{T}(X,Y)$, $\mathcal{PT}(X,Y)$ or $\mathcal{I}(X,Y)$, $\theta \in \mathcal{S}(X,Y)$ and U a dense subsemigroup of $(\mathcal{S}(X,Y),\theta)$. If X is finite and $\nabla\theta = X$, then $U = \mathcal{S}(X,Y)$.*

Proof. Since $\nabla\theta = X$, there exists a 1–1 mapping $\gamma: X \to Y$ such that $\gamma\theta = i_X$. Then $\gamma \in \mathcal{S}(X,Y)$.

Next, we set

$$V = \{\alpha \in \mathcal{S}(X,Y) \mid \alpha\theta \in \mathcal{G}_X\}$$

and define a relation ρ on V by

$$\alpha\rho\beta \Longleftrightarrow \alpha\theta = \beta\theta \qquad (\alpha\beta \in V).$$

The following statements give an outline of our proof for the lemma.

(1) V is a subsemigroup of $(\mathcal{S}(X,Y),\theta)$.

(2) For $\alpha \in \mathcal{S}(X,Y)$, $\nabla\alpha\theta = X$ implies that $\alpha \in V$.

(3) For $\alpha,\beta \in \mathcal{S}(X,Y)$, $\alpha\theta\beta \in V$ implies that $\alpha,\beta \in V$.

(4) ρ is a congruence on V and $V/\rho \cong \mathcal{G}_X$.

(5) If $V \not\subseteq U$, then $U \cap V$ is a dense subsemigroup of V.

(6) $V \subseteq U$.

(7) $U = \mathcal{S}(X,Y)$.

We shall prove (1). Since $\gamma\theta = i_X \in \mathcal{G}_X$, $\gamma \in V$, so $V \neq \varnothing$. If $\alpha,\beta \in V$, then $\alpha\theta,\beta\theta \in \mathcal{G}_X$, which implies that $(\alpha\theta\beta)\theta \in \mathcal{G}_X$, and thus $\gamma\theta\beta \in V$. Hence V is a subsemigroup of $(\mathcal{S}(X,Y),\theta)$.

Let $\alpha \in \mathcal{S}(X,Y)$ be such that $\nabla\alpha\theta = X$. Then $\Delta\alpha\theta \subseteq X$ and $\nabla\alpha\theta = X$. The finiteness of X gives $\Delta\alpha\theta = \nabla\alpha\theta = X$ and so $\alpha\theta \in \mathcal{G}_X$. Then $\alpha \in V$. This proves (2).

To prove (3), let $\alpha,\beta \in \mathcal{S}(X,Y)$ be such that $\alpha\theta\beta \in V$. By the definition of V, we have $(\alpha\theta)(\beta\theta) = (\alpha\theta\beta)\theta \in \mathcal{G}_X$. It follows from the finiteness of X that $\alpha\theta,\beta\theta \in \mathcal{G}_X$. Hence $\alpha,\beta \in V$.

To prove (4), it is obvious that ρ is an equivalence relation on V. If $\alpha,\beta \in V$ are such that $\alpha\rho\beta$, then $\alpha\theta = \beta\theta$, so $(\gamma\theta\alpha)\theta = (\gamma\theta\beta)\theta$ and $(\alpha\theta\gamma)\theta = (\beta\theta\gamma)\theta$ for all $\gamma \in V$ and hence $(\gamma\theta\alpha)\rho(\gamma\theta\beta)$ and $(\alpha\theta\gamma)\rho(\beta\theta\gamma)$ for all $\gamma \in V$. Thus ρ is a congruence on V. Define $\varphi: V/\rho \to \mathcal{G}_X$ by $(\alpha\rho)\varphi = \alpha\theta$ $(\alpha \in V)$. By the definition of ρ, φ is well-defined and 1–1. For $\alpha,\beta \in V$, $((\alpha\rho) * (\beta\rho))\,\varphi =$

$((\alpha\theta\beta)\rho)\varphi = (\alpha\theta\beta)\theta = (\alpha\theta)(\beta\theta) = (\alpha\rho)\varphi(\beta\rho)\varphi$ where $*$ is the operation on V/ρ. To show that φ is onto, let $\beta \in \mathcal{G}_X$. Then $\beta\gamma \in \mathcal{S}(X,Y)$ and $(\beta\gamma)\theta = \beta(\gamma\theta) = \beta i_X = \beta \in \mathcal{G}_X$, which implies $\beta\gamma \in V$ and $((\beta\gamma)\rho)\varphi = (\beta\gamma)\theta = \beta$. This proves that φ is an isomorphism of V/ρ onto $\mathcal{G}_X$. Therefore $V/\rho \cong \mathcal{G}_X$.

To prove (5), assume that $V \not\subseteq U$. Let $\eta \in V \setminus U$. Since U is dense in $\mathcal{S}(X,Y)$, by the Zigzag Theorem, there exists a zigzag, say Z, in $\mathcal{S}(X,Y)$ over U with value η. Since $\eta \in V$, it follows by (3) that Z is a zigzag in V over $U \cap V$ with value η. This proves that $U \cap V$ is a dense subsemigroup of V.

Suppose that (6) does not hold. Let $\eta \in V \setminus U$. By (5) and Proposition 1.2, we have that $\{\alpha\rho \mid \alpha \in U \cap V\}$ is a dense subsemigroup of V/ρ. Since $\mathcal{G}_X$ is a finite group, by Corollary 1.7, $\mathcal{G}_X$ has no proper dense subsemigroup. This implies that $\{\alpha\rho \mid \alpha \in U \cap V\} = V/\rho$. Therefore we have by the definition of ρ that for each $\alpha \in V$, there exists $\beta \in U \cap V$ such that $\alpha\theta = \beta\theta$. Since $U \cap V$ is dense in V and $\eta \in V \setminus U$, by the Zigzag Theorem, $\eta = \alpha\theta\lambda$ for some $\alpha \in V, \lambda \in U \cap V$. Then $\alpha\theta = \alpha'\theta$ for some $\alpha' \in U \cap V$. Thus $\eta = \alpha'\theta\lambda \in U \cap V$, a contradiction. This proves that (6) holds.

Finally, we shall prove (7). Suppose $U \neq \mathcal{S}(X,Y)$. Since $|\nabla\alpha\theta| \leq |X| < \infty$ for all $\alpha \in \mathcal{S}(X,Y)$, then there exists $\mu \in \mathcal{S}(X,Y) \setminus U$ such that $|\nabla\mu\theta| = \max\{|\nabla\alpha\theta| \mid \alpha \in \mathcal{S}(X,Y) \setminus U\}$. Since $\mu \notin U$ and $V \subseteq U$ (from(6)), it follows from (2) that $\nabla\mu\theta \subsetneq X$, and so $|\nabla\mu\theta| < |X|$ since X is finite. By Corollary 1.4, $\mu = \lambda_0\theta\beta = (\delta\theta\lambda_1)\theta\beta$ for some $\lambda_0, \lambda_1 \in U$, $\beta, \delta \in \mathcal{S}(X,Y) \setminus U$ such that $\lambda_0 = \delta\theta\lambda_1$. Then $\mu\theta = \delta\theta\lambda_1\theta\beta\theta$ and $\lambda_0\theta = \delta\theta\lambda_1\theta$, so we have that $|\nabla\mu\theta| \leq |\nabla\beta\theta|, |\nabla\mu\theta| \leq |\nabla\delta\theta|$ and $|\nabla\mu\theta| \leq |\nabla\lambda_0\theta| \leq |\nabla\delta\theta|$. Since $\beta, \delta \in \mathcal{S}(X,Y) \setminus U$, by the property of μ, we get

$$|\nabla\mu\theta| = |\nabla\beta\theta| = |\nabla\delta\theta| = |\nabla\lambda_0\theta| < |X| < \infty.$$

Let $x_0 \in X \setminus \nabla\beta\theta$ and $x_1 \in X \setminus \nabla\lambda_0\theta$. Since $\nabla\theta = X$ (by assumption), $x_0 \in \nabla\theta$. Let $y \in Y$ be such that $y\theta = x_0$. Let $\beta' \in \mathcal{PT}(X,Y)$ be such that

$$\Delta\beta' = \Delta\beta \cup \{x_1\},$$

$$x\beta' = x\beta \quad \text{for all } \alpha \in \Delta\beta \setminus \{x_1\} \text{ and}$$

$$x_1\beta' = y.$$

Claim that $\beta' \in \mathcal{S}(X,Y)$ for any case of $\mathcal{S}(X,Y)$. It is clear that $\beta' \in \mathcal{S}(X,Y)$ for the case that $\mathcal{S}(X,Y) = \mathcal{T}(X,Y)$ or $\mathcal{PT}(X,Y)$. If $y = x\beta'$ for some $x \in \Delta\beta \setminus \{x_1\}$, then $x_0 = y\theta = x\beta'\theta = x\beta\theta \in \nabla\beta\theta$. But $x_0 \notin \nabla\beta\theta$, so $x_1\beta' = y \neq x\beta'$ for all $x \in \Delta\beta \setminus \{x_1\}$. Thus β' is 1–1 at x_1. Therefore if $\beta \in \mathcal{I}(X,Y)$, then $\beta' \in \mathcal{I}(X,Y)$. Hence we have the claim. From the fact that $\beta'|_{\Delta\beta\setminus\{x_1\}} = \beta|_{\Delta\beta\setminus\{x_1\}}$ and $x_1 \notin \nabla\lambda_0\theta$, it follows that $\lambda_0\theta\beta' = \lambda_0\theta\beta = \mu$. From $\mu = \lambda_0\theta\beta$, we have $\nabla\mu\theta \subseteq \nabla\beta\theta$. But $|\nabla\mu\theta| = |\nabla\beta\theta| < \infty$, so $\nabla\mu\theta = \nabla\beta\theta$. Thus

$$\begin{aligned}
|\nabla\beta'\theta| &\geq |(\nabla\lambda_0\theta \cup \{x_1\})\beta'\theta| && (\text{ since } A\gamma = (A \cap \Delta\gamma)\gamma \subseteq \nabla\gamma \\
& && \quad \text{for all } \gamma \in \mathcal{PT}(X,Y), A \subseteq X) \\
&= |(\nabla\lambda_0\theta)\beta'\theta \cup \{x_1\beta'\theta\}| \\
&= |(\nabla\lambda_0\theta\beta')\theta \cup \{x_0\}| && (\text{ since } x_1\beta' = y \text{ and } y\theta = x_0) \\
&= |\nabla\mu\theta \cup \{x_0\}| && (\text{ since } \mu = \lambda_0\theta\beta') \\
&= |\nabla\beta\theta \cup \{x_0\}| && (\text{ since } \nabla\beta\theta = \nabla\mu\theta) \\
&= |\nabla\beta\theta| + 1 && (\text{ since } x_0 \notin \nabla\beta\theta) \\
&> |\nabla\beta\theta| \\
&= |\nabla\mu\theta|
\end{aligned}$$

It follows from the property of μ that $\beta' \in U$. Since $\lambda_0 \in U$ and $\mu = \lambda_0\theta\beta'$, we have $\mu \in U$ which is a contradiction since $\mu \in \mathcal{S}(X,Y) \setminus U$.

This proves that $U = \mathcal{S}(X,Y)$, as required. #

Lemma 3.4. *Let X and Y be sets, $\mathcal{S}(X,Y)$ denote any one of $\mathcal{T}(X,Y)$, $\mathcal{PT}(X,Y)$ or $\mathcal{I}(X,Y)$, $\theta \in \mathcal{S}(Y,X)$ and U a dense subsemigroup of $(\mathcal{S}(X,Y),\theta)$. If Y is finite and $\Delta\theta = Y$, then $U = \mathcal{S}(X,Y)$.*

Proof. Since $|\Delta\theta| = |Y| < \infty$ and $|\nabla\theta| \le |\Delta\theta|$, we have that $|\nabla\theta| < \infty$. By the fact that $|\nabla\theta| \le \min\{|X|,|Y|\}$, we have that $|\nabla\theta| < \min\{|X|,|Y|\}$, $|\nabla\theta| = |X|$ or $|\nabla\theta| = |Y|$. If $|\nabla\theta| < \min\{|X|,|Y|\}$, it follows by Lemma 3.2 that $U = \mathcal{S}(X,Y)$. If $|\nabla\theta| = |X|$, then X is finite and $\nabla\theta = X$, so by Lemma 3.3, $U = \mathcal{S}(X,Y)$.

Next we assume that $|\nabla\theta| = |Y|$. Then $|\Delta\theta| = |\nabla\theta| < \infty$ which implies that θ is 1–1, and hence $|Y| \le |X|$. Set

$$V = \{\alpha \in \mathcal{S}(X,Y) \mid \nabla\theta \subseteq \Delta\alpha \text{ and } (\nabla\theta)\alpha = Y\}$$

and define a relation ρ on V by

$$\alpha\rho\beta \Longleftrightarrow \alpha|_{\nabla\theta} = \beta|_{\nabla\theta} \qquad (\alpha,\beta \in V).$$

Since $|\nabla\theta| = |Y| < \infty$, it follows that for $\alpha \in \mathcal{S}(X,Y)$, $(\nabla\theta)\alpha = Y$ implies that $\nabla\theta \subseteq \Delta\alpha$. Then $V = \{\alpha \in \mathcal{S}(X,Y) \mid (\nabla\theta)\alpha = Y\}$. The following steps give an outline for our proof of this case:

(1) V is a subsemigroup of $(\mathcal{S}(X,Y),\theta)$.

(2) For $\alpha,\beta \in \mathcal{S}(X,Y)$, $\nabla\alpha\theta\beta = Y$ implies that $\beta \in V$.

(3) For $\alpha,\beta \in \mathcal{S}(X,Y)$, $\alpha\theta\beta \in V$ implies that $\alpha,\beta \in V$.

(4) ρ is a congruence on V and $V/\rho \cong \mathcal{G}_Y$.

(5) If $V \not\subseteq U$, then $U \cap V$ is a dense subsemigroup of V.

(6) $V \subseteq U$.

(7) For $\alpha \in \mathcal{S}(X,Y)$, $\nabla\alpha = Y$ implies that $\alpha \in U$.

(8) $U = \mathcal{S}(X,Y)$.

To prove (1), let $\lambda: X \to Y$ be such that $\lambda|_{\nabla\theta} = \theta^{-1}$. Define $\gamma = \theta^{-1}$ if $\mathcal{S}(X,Y) = \mathcal{PT}(X,Y)$ or $\mathcal{I}(X,Y)$ and $\gamma = \lambda$ if $\mathcal{S}(X,Y) = \mathcal{T}(X,Y)$. Then $\gamma \in \mathcal{S}(X,Y)$ and $(\nabla\theta)\gamma = (\nabla\theta)\theta^{-1} = \Delta\theta = Y$, so $\gamma \in V$. If $\alpha, \beta \in V$, then $(\nabla\theta)\alpha = Y = (\nabla\theta)\beta$ and hence $(\nabla\theta)(\alpha\theta\beta) = ((\nabla\theta)\alpha)\,\theta\beta = Y\theta\beta = (\nabla\theta)\beta = Y$ which implies that $\alpha\theta\beta \in V$. Thus, V is a subsemigroup of $(\mathcal{S}(X,Y),\theta)$.

If $\alpha, \beta \in \mathcal{S}(X,Y)$ are such that $\nabla\alpha\theta\beta = Y$, then $Y = \nabla\alpha\theta\beta \subseteq \nabla\theta\beta = (\nabla\theta)\beta \subseteq Y$ which implies that $(\nabla\theta)\beta = Y$, so $\beta \in V$ by the definition of V. Hence (2) is proved.

To prove (3), let $\alpha, \beta \in \mathcal{S}(X,Y)$ be such that $\alpha\theta\beta \in V$. Then $Y = (\nabla\theta)(\alpha\theta\beta)$. From $(\nabla\theta)(\alpha\theta\beta) = ((\nabla\theta)\alpha\theta)\,\beta \subseteq (\nabla\theta)\beta \subseteq Y$ and $|(\nabla\theta)(\alpha\theta\beta)| = |((\nabla\theta)\alpha)\,\theta\beta| \leq |(\nabla\theta)\alpha| \leq |\nabla\theta| = |Y| < \infty$, we get that $Y = (\nabla\theta)\beta$ and $Y = (\nabla\theta)\alpha$. Hence $\alpha, \beta \in V$.

Next, we shall prove (4). The relation ρ is obviously an equivalence relation on V. To show that ρ is an congruence on V, it suffices to show that for $\alpha, \beta \in V$, $(\alpha\theta\beta)|_{\nabla\theta} = (\alpha|_{\nabla\theta})\theta(\beta|_{\nabla\theta})$. Let $\alpha, \beta \in V$ and $x \in \nabla\theta$. Then $x\alpha|_{\nabla\theta} = x\alpha$ and $((x\alpha)\theta)\,\beta = ((x\alpha|_{\nabla\theta})\theta)\,(\beta|_{\nabla\theta})$ and hence $x(\alpha\theta\beta)|_{\nabla\theta} = x(\alpha\theta\beta) = ((x\alpha|_{\nabla\theta})\theta)\,(\beta|_{\nabla\theta}) = x\,((\alpha|_{\nabla\theta})\theta(\beta|_{\nabla\theta}))$. Therefore, ρ is a congruence on V. Since $|\nabla\theta| = |Y|$, $\mathcal{G}_{\nabla\theta} \cong \mathcal{G}_Y$. It can be seen by the definition of V that for every $\alpha \in V$, $\Delta(\alpha|_{\nabla\theta}\theta) = \nabla(\alpha|_{\nabla\theta}\theta) = \nabla\theta$. But $|\nabla\theta| < \infty$, so $\alpha|_{\nabla\theta}\theta \in \mathcal{G}_{\nabla\theta}$ for every $\alpha \in V$. To show $V/\rho \cong \mathcal{G}_{\nabla\theta}$, define $\varphi: V/\rho \to \mathcal{G}_{\nabla\theta}$ by $(\alpha\rho)\varphi = \alpha|_{\nabla\theta}\theta$ for all $\alpha \in V$. It is clear from the definition of ρ that φ is well-defined. Since θ is 1–1 and $\Delta\theta = Y$, it follows that $\theta\theta^{-1} = i_Y$. If $\alpha, \beta \in V$ are such that $\alpha|_{\nabla\theta}\theta = \beta|_{\nabla\theta}\theta$, then $\alpha|_{\nabla\theta} = \alpha|_{\nabla\theta}i_Y = (\alpha|_{\nabla\theta}\theta)\theta^{-1} = (\beta|_{\nabla\theta}\theta)\theta^{-1} = \beta|_{\nabla\theta}i_Y = \beta|_{\nabla\theta}$. Thus φ is

1–1. To show φ is onto, let $\lambda \in \mathcal{G}_{\nabla\theta}$. Then $\lambda\theta^{-1} \in \mathcal{I}(X,Y)$ and $\Delta\lambda\theta^{-1} = \nabla\theta$. Let $\lambda_1 \in \mathcal{T}(X,Y)$ be such that $\lambda_1|_{\nabla\theta} = \lambda\theta^{-1}$ and then let $\lambda' = \lambda\theta^{-1}$ if $\mathcal{S}(X,Y) = \mathcal{PT}(X,Y)$ or $\mathcal{I}(X,Y)$ and $\lambda' = \lambda_1$ if $\mathcal{S}(X,Y) = \mathcal{T}(X,Y)$. Then $\lambda' \in \mathcal{S}(X,Y)$ and $\lambda'|_{\nabla\theta} = \lambda\theta^{-1}$. Since $(\nabla\theta)\lambda' = (\nabla\theta)\lambda\theta^{-1} = ((\nabla\theta)\lambda)\,\theta^{-1} = (\nabla\theta)\theta^{-1} = \Delta\theta = Y$, we have that $\lambda' \in V$ and $(\lambda'\rho)\varphi = \lambda'|_{\nabla\theta}\theta = (\lambda\theta^{-1})\theta = \lambda$. From the fact that $(\alpha\theta\beta)|_{\nabla\theta} = (\alpha|_{\nabla\theta})\theta(\beta|_{\nabla\theta})$ for all $\alpha, \beta \in V$, we have that $(\alpha\theta\beta)|_{\nabla\theta}\theta = (\alpha|_{\nabla\theta}\theta)(\beta|_{\nabla\theta}\theta)$ for all $\alpha, \beta \in V$ which implies that φ is a homomorphism. Hence $V/\rho \cong \mathcal{G}_{\nabla\theta} \cong \mathcal{G}_Y$.

To prove (5), assume that $V \not\subseteq U$. Let $\eta \in V \smallsetminus U$ and let Z be a zigzag in $\mathcal{S}(X,Y)$ over U with value η. It follows from (3) that Z is a zigzag in V over $U \cap V$ with value η. This prove that $U \cap V$ is a dense subsemigroup of V.

To prove (6), suppose that $V \not\subseteq U$. Let $\eta \in V \smallsetminus U$. By (5) and Proposition 1.2, $\{\alpha\rho \mid \alpha \in U \cap V\}$ is a dense subsemigroup of V/ρ. Since Y is finite and $\mathcal{G}_Y \cong V/\rho$, by Corollaly 1.7, V/ρ has no proper dense subsemigroup. Then $\{\alpha\rho \mid \alpha \in U \cap V\} = V/\rho$ and thus by the definition of ρ, we have that for each $\alpha \in V$ there exists $\beta \in U \cap V$ such that $\alpha|_{\nabla\theta} = \beta|_{\nabla\theta}$. Since $U \cap V$ is dense in V and $\eta \in V \smallsetminus U$, by the Zigzag Theorem, $\eta = \lambda\theta\beta$ for some $\lambda \in U \cap V$ and $\beta \in V$. Then $\beta|_{\nabla\theta} = \mu|_{\nabla\theta}$ for some $\mu \in U \cap V$. Thus $\theta\beta = \theta\mu$ and hence $\eta = \lambda\theta\beta = \lambda\theta\mu \in U$ since $\lambda, \mu \in U$. This is a contradiction since $\eta \in V \smallsetminus U$. This proves that $V \subseteq U$.

To prove (7), let $\alpha \in \mathcal{S}(X,Y)$ be such that $\nabla\alpha = Y$. Suppose $\alpha \notin U$. Then by the Zigzag Theorem, $\alpha = \beta\theta\gamma$ for some $\beta \in U$ and $\gamma \in \mathcal{S}(X,Y)$. Thus $Y = \nabla\alpha = \nabla\beta\theta\gamma$. By (2), we have that $\gamma \in V$. We have by (6) that $\gamma \in U$. It then follows that $\alpha = \beta\theta\gamma \in U$, a contradiction. Hence $\alpha \in U$.

Finally, we shall prove (8). Suppose that $U \neq \mathcal{S}(X,Y)$. Since for all $\alpha \in \mathcal{S}(X,Y)$, $|\nabla\alpha| \leq |Y| < \infty$, it follows that there exists $\mu \in \mathcal{S}(X,Y) \setminus U$ such that $|\nabla\mu| = \max\{|\nabla\alpha| \mid \alpha \in \mathcal{S}(X,Y) \setminus U\}$. This implies by (7) that $|\nabla\mu| < |Y|$. Since U is dense in $(\mathcal{S}(X,Y),\theta)$ and $\mu \in \mathcal{S}(X,Y) \setminus U$, by Corollary 1.4, $\mu = \lambda_0\theta\gamma = (\beta\theta\lambda_1)\theta\gamma$ for some $\lambda_0, \lambda_1 \in U$, $\beta, \gamma \in \mathcal{S}(X,Y) \setminus U$ such that $\lambda_0 = \beta\theta\lambda_1$. Then $|\nabla\mu| \leq |\nabla\lambda_0| \leq |\nabla\beta|$ and $|\nabla\mu| \leq |\nabla\gamma|$. By the property of μ, we have that $|\nabla\beta| \leq |\nabla\mu|$ and $|\nabla\gamma| \leq |\nabla\mu|$, hence $|\nabla\lambda_0| = |\nabla\beta| = |\nabla\gamma| = |\nabla\mu| < |Y| < \infty$. Then $\nabla\mu = \nabla\gamma$ since $\nabla\mu \subseteq \nabla\gamma$. The facts that $\mu = \lambda_0\theta\gamma$, $\mu \in \mathcal{S}(X,Y) \setminus U$ and $\lambda_0 \in U$ give $\Delta\lambda_0 \neq \varnothing$, and then $(\nabla\lambda_0)\theta \neq \varnothing$. Next, we claim that there exists $\theta' \in \mathcal{S}(Y,X)$ such that $\lambda_0\theta' = \lambda_0\theta$, $\nabla\theta'\gamma = \nabla\gamma$ and $|\nabla\theta'| < |Y|$. To prove the claim, let $\theta_0 : Y \to X$ be such that $\theta_0|_{\nabla\lambda_0} = \theta|_{\nabla\lambda_0}$ and $(Y \setminus \nabla\lambda_0)\theta_0 \subseteq (\nabla\lambda_0)\theta$. Then $\theta_0 \in \mathcal{T}(Y,X)$ and $\nabla\theta_0 = (\nabla\lambda_0)\theta$. Set $\theta' = \theta|_{\nabla\lambda_0}$ if $\mathcal{S}(X,Y) = \mathcal{PT}(X,Y)$ or $\mathcal{I}(X,Y)$ and $\theta' = \theta_0$ if $\mathcal{S}(X,Y) = \mathcal{T}(X,Y)$. Then $\theta' \in \mathcal{S}(Y,X), \lambda_0\theta' = \lambda_0\theta$ and $|\nabla\theta'| = |(\nabla\lambda_0)\theta| \leq |\nabla\lambda_0| < |Y|$. We also have that $\nabla\theta'\gamma = (\nabla\theta')\gamma = (\nabla\lambda_0\theta)\gamma = \nabla(\lambda_0\theta\gamma) = \nabla\mu = \nabla\gamma$. Hence the claim is obtained. Now we have $|\nabla\theta'| < \min\{|X|, |Y|\}$ (since $|Y| \leq |X|$ and $|\nabla\theta'| < |Y|$) and $\nabla\theta'\gamma = \nabla\gamma$. Then by Proposition 2.1(i), there exists $\eta \in \mathcal{S}(X,Y)$ such that $|\nabla\eta| > |\nabla\gamma|$ and $\theta'\eta = \theta'\gamma$. Therefore $|\nabla\eta| > |\nabla\mu|$, so $\eta \in U$ by the property of μ. From $\lambda_0\theta = \lambda_0\theta'$ and $\theta'\gamma = \theta'\eta$, we have $\mu = \lambda_0\theta\gamma = (\lambda_0\theta)\gamma = (\lambda_0\theta')\gamma = \lambda_0(\theta'\gamma) = \lambda_0(\theta'\eta) = (\lambda_0\theta')\eta = \lambda_0\theta\eta$. Since $\lambda_0, \eta \in U$, it follows that $\mu \in U$. This is a contradiction since $\mu \notin U$. This proves that $U = \mathcal{S}(X,Y)$, as required. #

Lemma 3.5. *Let X and Y be infinite sets, $\mathcal{S}(X,Y)$ denote any one of $\mathcal{T}(X,Y)$, $\mathcal{PT}(X,Y)$ or $\mathcal{I}(X,Y)$ and let $\theta \in \mathcal{S}(Y,X)$ be such that $|\nabla\theta| = \min\{|X|,|Y|\}$. Let*

$A \subseteq \nabla\theta$ be such that A is infinite and $|\nabla\theta \smallsetminus A| = |\nabla\theta|$. For each $a \in A$, let $y_a \in a\theta^{-1}$. Set

$$U = \{\alpha \in \mathcal{S}(X,Y) \mid |A\alpha \cap (Y \smallsetminus \{y_a \mid a \in A\})| < |A|\}.$$

Then U is a proper dense subsemigroup of $(\mathcal{S}(X,Y),\theta)$.

Proof. We have by Proposition 2.3 that U is a proper subsemigroup of $(\mathcal{S}(X,Y),\theta)$. To avoid confusing between the notations y_a's $(a \in A)$ in the assumption and the notation y which will be used as any element of Y, we shall denote y_a in the assumption by f_a $(a \in A)$. Then $f_a \in a\theta^{-1}$ for all $a \in A$ and

$$U = \left\{\alpha \in \mathcal{S}(X,Y) \;\middle|\; |A\alpha \cap (Y \smallsetminus \{f_a \mid a \in A\})| < |A|\right\}.$$

To show that U is dense in $(\mathcal{S}(X,Y),\theta)$, that is, $\mathrm{Dom}(U,\mathcal{S}(X,Y)) = \mathcal{S}(X,Y)$, let $\alpha \in \mathcal{S}(X,Y)$. To show that there exists a zigzag in $\mathcal{S}(X,Y)$ over U with value α, we first construct a mapping φ, depending on α as follows: For $y \in \nabla\alpha$, if $y\alpha^{-1} \cap A \neq \varnothing$, choose $a_y \in y\alpha^{-1} \cap A$. Then $\{a_y\theta^{-1} \mid y \in \nabla\alpha$ and $y\alpha^{-1} \cap A \neq \varnothing\} \subseteq \{x\theta^{-1} \mid x \in A\}$ and the mapping $y\alpha^{-1} \mapsto a_y\theta^{-1}$ gives a 1–1 correspondence between the sets $\{y\alpha^{-1} \mid y \in \nabla\alpha$ and $y\alpha^{-1} \cap A \neq \varnothing\}$ and $\{a_y\theta^{-1} \mid y \in \nabla\alpha$ and $y\alpha^{-1} \cap A \neq \varnothing\}$. We have by assumption that $\min\{|X|,|Y|\} = |\nabla\theta \smallsetminus A|$, so $|\{y\alpha^{-1} \mid y \in \nabla\alpha$ and $y\alpha^{-1} \cap A = \varnothing\}| \leq |\nabla\alpha| \leq \min\{|X|,|Y|\} = |\nabla\theta \smallsetminus A| = |\{x\theta^{-1} \mid x \in \nabla\theta \smallsetminus A\}|$. Now we have

$y\alpha^{-1} \mapsto a_y\theta^{-1}$ is a bijection of $\{y\alpha^{-1} \mid y \in \nabla\alpha$ and $y\alpha^{-1} \cap A \neq \varnothing\}$ onto $\{a_y\theta^{-1} \mid y \in \nabla\alpha$ and $y\alpha^{-1} \cap A \neq \varnothing\}$,

$\{a_y\theta^{-1} \mid y \in \nabla\alpha$ and $y\alpha^{-1} \cap A \neq \varnothing\} \subseteq \{x\theta^{-1} \mid x \in A\}$ and

$|\{y\alpha^{-1} \mid y \in \nabla\alpha$ and $y\alpha^{-1} \cap A = \varnothing\}| \leq |\{x\theta^{-1} \mid x \in \nabla\theta \smallsetminus A\}|$.

Then there exists a 1–1 mapping $\varphi\colon \{y\alpha^{-1} \mid y \in \nabla\alpha\} \to \{x\theta^{-1} \mid x \in \nabla\theta\}$ such that

(I) $\quad (y\alpha^{-1})\varphi = a_y\theta^{-1}$ if $y\alpha^{-1} \cap A \neq \varnothing$ and

(II) $\quad (y\alpha^{-1})\varphi \in \{x\theta^{-1} \mid x \in \nabla\theta \setminus A\}$ if $y\alpha^{-1} \cap A = \varnothing$.

We have from the construction of φ that $\bigcup_{y\in\nabla\alpha} (y\alpha^{-1})\varphi \subseteq \Delta\theta$ and $f_{a_y} \in a_y\theta^{-1} = (y\alpha^{-1})\varphi$ for all $y \in \nabla\alpha$ such that $y\alpha^{-1} \cap A \neq \varnothing$. Define a mapping $\lambda\colon \Delta\alpha \to \Delta\theta$ by

(III) $\quad (y\alpha^{-1})\lambda \subseteq (y\alpha^{-1})\varphi$ for all $y \in \nabla\alpha$ and

(IV) $\quad (y\alpha^{-1})\lambda = \{f_{a_y}\}$ if $y\alpha^{-1} \cap A \neq \varnothing$.

Since $|\nabla\lambda| \leq \min\{|X|, |Y|\} = |\nabla\theta| = |\nabla\theta \setminus A|$, there exists $\eta_1 \in \mathcal{I}(X,Y)$ such that $\Delta\eta_1 \subseteq \nabla\theta \setminus A$ and $\nabla\eta_1 = \nabla\lambda$. Extend the mapping η_1 to a mapping η_2 such that $\eta_2\colon X \to X$ and $|A\eta_2| < |A|$. Define a mapping η as follows: $\eta = \eta_1$ if $\mathcal{S}(X,Y) = \mathcal{PT}(X,Y)$ or $\mathcal{I}(X,Y)$ and $\eta = \eta_2$ if $\mathcal{S}(X,Y) = \mathcal{T}(X,Y)$. Then $\eta \in \mathcal{S}(X,Y)$.

Next, we claim that the following statements hold:

(1) $\lambda, \eta \in U$.

(2) $\Delta\lambda\theta = \Delta\alpha$ and $(x(\lambda\theta))(\lambda\theta)^{-1} = (x\alpha)\alpha^{-1}$ for all $x \in \Delta\alpha$.

(3) There exists $\gamma \in \mathcal{S}(X,Y)$ such that $\alpha = \lambda\theta\gamma$.

(4) $\eta\theta\gamma \in U$.

(5) There exists $\beta \in \mathcal{S}(X,Y)$ such that $\lambda = \beta\theta\eta$.

If these statements hold, then we have the following zigzag in $\mathcal{S}(X,Y)$ over U with value α:

$$\begin{aligned}
\alpha &= \lambda\theta\gamma \quad , \quad \lambda \in U \quad , \gamma \in \mathcal{S}(X,Y) && (by(1),(3)),\\
&= (\beta\theta\eta)\theta\gamma \quad , \quad \eta \in U \quad , \beta \in \mathcal{S}(X,Y) \quad , \lambda = \beta\theta\eta && (by(1),(5)),\\
&= \beta\theta(\eta\theta\gamma) \quad , \eta\theta\gamma \in U && (by(4)),
\end{aligned}$$

which implies by the Zigzag Theorem that $\alpha \in \mathrm{Dom}(U, \mathcal{S}(X,Y))$, as required. Now, it remains to show that these statements hold.

By (IV), we have that $(y\alpha^{-1} \cap A)\lambda = \{f_{a_y}\}$ for all $y \in \nabla\alpha$ such that $y\alpha^{-1} \cap A \neq \varnothing$. It follows that

$$\begin{aligned}
A\lambda &= (A \cap \Delta\lambda)\lambda \\
&= (A \cap \Delta\alpha)\lambda \qquad (\text{since } \Delta\alpha = \Delta\lambda) \\
&= \left(A \cap \left(\bigcup_{y \in \nabla\alpha} y\alpha^{-1}\right)\right)\lambda \\
&= \left(\bigcup_{y \in \nabla\alpha} (A \cap y\alpha^{-1})\right)\lambda \\
&= \{f_{a_y} \mid y \in \nabla\alpha \text{ and } y\alpha^{-1} \cap A \neq \varnothing\} \\
&\subseteq \{f_x \mid x \in A\}.
\end{aligned}$$

Hence $A\lambda \cap (Y \setminus \{f_x \mid x \in A\}) = \varnothing$, so $\lambda \in U$.By definition of η, $|A\eta| = |A\eta_2| < |A|$ if $\mathcal{S}(X,Y) = \mathcal{T}(X,Y)$ and $|A\eta| = |A\eta_1| = 0 < |A|$ if $\mathcal{S}(X,Y) = \mathcal{PT}(X,Y)$ or $\mathcal{I}(X,Y)$. These imply that $\eta \in U$. Hence (1) holds.

Next, we shall prove (2). Since $\Delta\lambda = \Delta\alpha$ and $\nabla\lambda \subseteq \Delta\theta$, we have that $\Delta\lambda\theta = \Delta\alpha$. Since for each $y \in \nabla\alpha, (y\alpha^{-1})\varphi = x\theta^{-1}$ for some $x \in \nabla\theta$ (from (I) and (II)), it follows that

(*) $$|(y\alpha^{-1})\varphi\theta| = 1 \quad \text{for all } y \in \nabla\alpha$$

and hence for $y, z \in \nabla\alpha, (y\alpha^{-1})\varphi\theta = (z\alpha^{-1})\varphi\theta$ implies $(y\alpha^{-1})\varphi = (z\alpha^{-1})\varphi$. But φ is a 1–1 mapping, so we have

(**) $$\text{for } y, z \in \nabla\alpha, (y\alpha^{-1})\varphi\theta = (z\alpha^{-1})\varphi\theta \qquad \text{implies } y\alpha^{-1} = z\alpha^{-1}.$$

Let $x \in \Delta\alpha$. Then $x\alpha \in \nabla\alpha$ and $((x\alpha)\alpha^{-1})(\lambda\theta) \subseteq (((x\alpha)\alpha^{-1})\varphi)\theta$ (from(III)), so by (*), we have $|((x\alpha)\alpha^{-1})(\lambda\theta)| = 1$. Since $x \in (x\alpha)\alpha^{-1}$, we have $x(\lambda\theta) \in ((x\alpha)\alpha^{-1})(\lambda\theta)$. Then $\{x(\lambda\theta)\} = ((x\alpha)\alpha^{-1})(\lambda\theta)$ and hence $(x\alpha)\alpha^{-1} \subseteq (x(\lambda\theta))(\lambda\theta)^{-1}$. Conversely, if $z \in (x(\lambda\theta))(\lambda\theta)^{-1}$, then $z(\lambda\theta) = x(\lambda\theta)$ and thus $z(\lambda\theta) \in ((z\alpha)\alpha^{-1})(\lambda\theta) \subseteq ((z\alpha)\alpha^{-1})\varphi\theta$ (from(III)) and $z(\lambda\theta) = x(\lambda\theta) \in ((x\alpha)\alpha^{-1})\varphi\theta$. It follows from (*) that $((z\alpha)\alpha^{-1})(\varphi\theta) = ((x\alpha)\alpha^{-1})\varphi\theta$. By (**), $(z\alpha)\alpha^{-1} = (x\alpha)\alpha^{-1}$, so $z \in (x\alpha)\alpha^{-1}$. Hence we have that $(x(\lambda\theta))(\lambda\theta)^{-1} = (x\alpha)\alpha^{-1}$. Hence (2) holds.

By (2) and Proposition 2.4(i), there exists $\gamma \in \mathcal{S}(X,Y)$ such that $((\lambda\theta)\gamma)|_{\Delta\alpha} = \alpha$. But $\Delta\alpha = \Delta\lambda\theta$, so $\lambda\theta\gamma = \alpha$. Hence (3) holds.

We obtain (4) from the fact that $|A(\eta\theta\gamma)| \leq |A\eta| < |A|$.

By the definition of η_1 and η, we have that $\nabla\lambda = \nabla\eta_1 = (\Delta\eta_1)\eta_1 \subseteq (\nabla\theta)\eta_1 \subseteq (\nabla\theta)\eta = \nabla\theta\eta$. By Proposition 2.4(ii), there exists $\beta \in \mathcal{S}(X,Y)$ such that $\beta(\theta\eta) = \lambda$. Thus (5) is proved.

Hence, the lemma is completely proved. #

Theorem 3.6. *Let X and Y be sets, $\mathcal{S}(X,Y)$ denote any one of $\mathcal{T}(X,Y)$, $\mathcal{PT}(X,Y)$ or $\mathcal{I}(X,Y)$ and $\theta \in \mathcal{S}(Y,X)$. Then $(\mathcal{S}(X,Y),\theta)$ has a proper dense subsemigroup if and only if X and Y are both infinite and $|\nabla\theta| = \min\{|X|,|Y|\}$.*

If X and Y are both infinite and $|\nabla\theta| = \min\{|X|,|Y|\}$, then the following statements hold:

(1) Suppose A is an infinite subset of $\nabla\theta$ such that $|\nabla\theta \setminus A| = |\nabla\theta|$. For each $a \in A$, choose $y_a \in a\theta^{-1}$. Then the set U defined by

$$U = \left\{\alpha \in \mathcal{S}(X,Y) \;\middle|\; |A\alpha \cap (Y \setminus \{y_a \mid a \in A\})| < |A|\right\}.$$

is a proper dense subsemigroup of $(\mathcal{S}(X,Y),\theta)$.

(2) $(\mathcal{S}(X,Y),\theta)$ *has infinitely many proper dense subsemigroups and the cardinality of the collection of such proper dense subsemigroups is not less than* $\min\{|X|,|Y|\}$.

Proof. We shall prove that if $(\mathcal{S}(X,Y),\theta)$ has a proper dense subsemigroup, then X and Y are both infinite and $|\nabla\theta| = \min\{|X|,|Y|\}$ by contrapositive, that is, if X is finite or Y is finite or $|\nabla\theta| < \min\{|X|,|Y|\}$, then $(\mathcal{S}(X,Y),\theta)$ has no proper dense subsemigroup. To prove this, it can be easily seen that it suffices to show that each of the following three cases implies that $(\mathcal{S}(X,Y),\theta)$ has no proper dense subsemigroup:

(i) $|\nabla\theta| < \min\{|X|,|Y|\}$,

(ii) $|X| < \infty$ and $|\nabla\theta| = |X| \leq |Y|$ and

(iii) $|Y| < \infty$ and $|\nabla\theta| = |Y| \leq |X|$.

By Lemma 3.2, (i) implies that $(\mathcal{S}(X,Y),\theta)$ has no proper dense subsemigroup.

Assume that (ii) holds. Then $|\nabla\theta| = |X| < \infty$, so $\nabla\theta = X$ since $\nabla\theta \subseteq X$. It follows by Lemma 3.3 that $(\mathcal{S}(X,Y),\theta)$ has no proper dense subsemigroup.

Next, assume that (iii) holds. Then $|Y| = |\nabla\theta| \leq |\Delta\theta| \leq |Y| < \infty$, and therefore $|\Delta\theta| = |Y|$. Since $\Delta\theta \subseteq Y$, we have that $\Delta\theta = Y$. By Lemma 3.4, $(\mathcal{S}(X,Y),\theta)$ has no proper dense subsemigroup.

The converse and (1) of the theorem follow directly from Lemma 3.5.

Next,to prove (2) assume that X and Y are both infinite and $|\nabla\theta| = min\{|X|,|Y|\}$. Consider the sets $\nabla\theta \times \nabla\theta$ and $\nabla\theta$. Since $|\nabla\theta \times \nabla\theta| = |\nabla\theta|$, there exists a partition $\{A_x \mid x \in \nabla\theta\}$ of $\nabla\theta$ such that $|A_x| = |\nabla\theta|$ for all $x \in \nabla\theta$ (see page 20). Then $|\nabla\theta \setminus A_x| = |\nabla\theta|$ for all $x \in \nabla\theta$. For $x \in \nabla\theta$, choose $y_x \in x\theta^{-1}$. For each $x \in \nabla\theta$, let

$$U_x = \{\alpha \in \mathcal{S}(X,Y) \mid |A_x\alpha \cap (Y \setminus \{y_a \mid a \in A_x\})| < |A_x|\}$$

and $\mathcal{C} = \{U_x \mid x \in \nabla\theta\}$. By Lemma 3.5, U_x is a proper dense subsemigroup of $(\mathcal{S}(X,Y),\theta)$ for all $x \in \nabla\theta$. Let $x, x' \in \nabla\theta$ be distinct. Then $|A_x \cup A_{x'}| = |A_x| = |\{y_a \mid a \in A_x\}|$, so there exists $\alpha \in \mathcal{I}(X,Y)$ such that $\Delta\alpha = A_x \cup A_{x'}$ and $\nabla\alpha = \{y_a \mid a \in A_x\}$. Let $\beta \in \mathcal{S}(X,Y)$ be such that $\beta|_{\Delta\alpha} = \alpha$. Then $A_x\beta \subseteq \{y_a \mid a \in A_x\}$ and $A_{x'}\beta \subseteq \{y_a \mid a \in A_x\} \subseteq Y \setminus \{y_a \mid a \in A_{x'}\}$ and hence $|A_x\beta \cap (Y \setminus \{y_a \mid a \in A_x\})| = 0 < |A_x|$ and $|A_{x'}\beta \cap (Y \setminus \{y_a \mid a \in A_{x'}\})| = |A_{x'}\beta| = |A_{x'}|$. Therefore $\beta \in U_x$ but $\beta \notin U_{x'}$. This shows that the sets U_x's $(x \in \nabla\theta)$ are all distinct. Hence $|\mathcal{C}| = |\nabla\theta| = \min\{|X|, |Y|\}$. #