CHAPTER II

INVERTIBLE MATRICES

The purpose of this chapter is to characterize any invertible matrix over a commutative semiring with 0,1 which has some special properties.

It is known that a square matrix A over a field is invertible if and only if $\det A \neq 0$. The theory of matrices over a ring gives a generalization of this result as follows: If R is a commutative ring with identity, then a square matrix A over R is invertible if and only if detA is an invertible element in R (see [2]).

If R is a commutative ring with identity, $\det A = \det^{+}A - \det^{-}A$ for any square matrix A over R, so the above known results may be restated as follow:

(1) A square matrix A over a field is invertible if and only if $\det^{+}A \neq \det^{-}A$.

(2) If R is a commutative ring with identity 1, then a square matrix A over R is invertible if and only if $(\det^{+}A)x = 1+(\det^{-}A)x$ for some $x \in R$.

Note that (1) is a special case of (2), and the condition that $(\det^{+}A)x = 1+(\det^{-}A)x$ for some $x \in R$ in (2) implies that $\det^{+}A \neq \det^{-}A$ if R contains more than one element, but the converse is not true.

Unfortunately, each implication in (2) is not true if R is a commutative semiring with 0,1, as shown by the following example.

<u>Example</u>. Let S be the Boolean algebra {0,1} with 0+0 = 0, 0+1 = 1+0 = 1+1 = 1 and $0\cdot0 = 0\cdot1 = 1\cdot0 = 0$, $1\cdot1 = 1$, and let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \varepsilon M_2(S)$, $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \varepsilon M_2(S)$.

Since $AA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, A is invertible over S. Because $(\det^+A)\cdot x = 0\cdot x = 0$ and $1+(\det^-A)\cdot x = 1+1\cdot x = 1$ for any $x \varepsilon S$, we have that $(\det^+A)\cdot x \neq 1+(\det^-A)\cdot x$ for any $x \varepsilon S$.

Since $\det^+B = 1$ and $\det^-B = 0$, $(\det^+B)\cdot1 = 1+(\det^-B)\cdot1$. If $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \varepsilon M_2(S)$ is such that $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then $\begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, so b+d = 0, d = 1 which is impossible since 0+1 = 1+1 = 1. Hence B is not invertible over S. #

The first theorem gives a necessary condition for a square matrix over any commutative semiring with 0,1 to be invertible in terms of the positive determinant and negative determinant.

<u>Theorem 2.1</u>. Let A be a square matrix over a commutative semiring S with 0,1. If A is invertible over S, then $\det^+A \neq \det^-A$.

<u>Proof</u>. Assume $A \varepsilon M_n(S)$ is invertible, and let $B \varepsilon M_n(S)$ be the inverse of A. By Lemma 1 of [8], there exists an element $r \varepsilon S$ such that

$$\det^+(AB) = (\det^+A)(\det^+B)+(\det^-A)(\det^-B)+r,$$

$$\det^-(AB) = (\det^+A)(\det^-B)+(\det^-A)(\det^+B)+r.$$

Since $AB = I_n$, $\det^+ I_n = 1$ and $\det^- I_n = 0$. Then

$$1 = (\det^+ A)(\det^+ B) + (\det^- A)(\det^- B) + r,$$

$$0 = (\det^+ A)(\det^- B) + (\det^- A)(\det^+ B) + r.$$

Because $1 \neq 0$, it follows from these two equalities that $\det^+ A \neq \det^- A$. #

The above example shows that the converse of Theorem 2.1 is not true.

Consider the following semirings of numbers: $(\mathbb{N} \cup \{0\},+,\cdot)$, $(\mathbb{Q}^+ \cup \{0\},+,\cdot)$, $(\mathbb{R}^+ \cup \{0\},+,\cdot)$ where $+$ and $\cdot$ are the usual addition and the usual multiplication, respectively. These semirings have the following two properties: They have no zero divisors and 0 is the only element which has an additive inverse. To characterize an invertible matrix over each of these semirings, we generally characterize an invertible matrix over a commutative semiring with 0,1 which has the above two properties as follows.

<u>Theorem 2.2</u>. Let S be a commutative semiring with 0,1 . Assume that S has no zero divisors and 0 is the only element of S which has an additive inverse. Then a square matrix A over S is invertible if and only if every row and every column of A has exactly one nonzero element and every nonzero element of A is an invertible element of S.

<u>Proof</u>. By the assumption, we have that for $x, y \in S$, $xy = 0$ implies $x = 0$ or $y = 0$, and $x+y = 0$ implies $x = y = 0$.

Let A be an $n\times n$ matrix over S. First we assume A is invertible. Since A is invertible, every row and every column of A has at least one nonzero element. Let B be the inverse of A. Then $AB = BA = I_n$, so

$$\sum_{h=1}^{n} A_{ih}B_{hj} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \sum_{h=1}^{n} B_{ih}A_{hj} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

To show that every row of A has exactly one nonzero element, suppose not. Then there exists $k, \ell, \ell' \in \{1,2,\ldots,n\}$ such that $\ell \neq \ell'$ and $A_{k\ell} \neq 0$, $A_{k\ell'} \neq 0$. For $j \in \{1,2,\ldots,n\}$, $j \neq k$, we have $\sum_{t=1}^{n} A_{kt}B_{tj} = 0$ which implies $A_{kt}B_{tj} = 0$ for every $t \in \{1,2,\ldots,n\}$. In particular, $A_{k\ell}B_{\ell j} = 0$ for every $j \neq k$. It follows that $B_{\ell j} = 0$ for every $j \neq k$ since $A_{k\ell} \neq 0$. Because $\sum_{t=1}^{n} B_{\ell t}A_{t\ell} = 1$ and $B_{\ell t} = 0$ for every $t \neq k$, we have that $B_{\ell k}A_{k\ell} = 1$. Hence $B_{\ell k} \neq 0$. Since $\ell \neq \ell'$, $\sum_{t=1}^{n} B_{\ell t}A_{t\ell'} = 0$, so $B_{\ell t}A_{t\ell'} = 0$ for every $t \in \{1,2,\ldots,n\}$. Therefore $B_{\ell k}A_{k\ell'} = 0$ which is a contradiction because $B_{\ell k} \neq 0$ and $A_{k\ell'} \neq 0$. This proves that every row of A has exactly one nonzero element. It can be shown similarly that every column of A has exactly one nonzero element.

Next, we shall show that every nonzero element of A is an invertible element in S. Let $i \in \{1,2,\ldots,n\}$ and assume that A_{ik} is a unique nonzero element of the ith row of A. Then $1 = \sum_{h=1}^{n} A_{ih}B_{hi} = A_{ik}B_{ki}$ since $A_{ih} = 0$ for all $h \neq k$. Hence A_{ik} is an invertible element of S.

Conversely, assume that A has the properties that every row and every column of A has exactly one nonzero element and every nonzero element of A is an invertible element of S. Since every row of A has exactly one nonzero element, it follows that for every $i \in \{1,2,\ldots,n\}$, there exists a unique $k_i \in \{1,2,\ldots,n\}$ such that $A_{ik_i} \neq 0$. Then each A_{ik_i} is an invertible element of S because every nonzero element of A is invertible. Since every column of A has exactly one nonzero element, we have that $A_{jk_i} = 0$ for all i,j in

$\{1,2,\ldots,n\}, j \neq i$. Let B be a square matrix over S defined by

$$B_{ij} = \begin{cases} A_{ji}^{-1} & \text{if } A_{ji} \neq 0, \\ 0 & \text{otherwise}. \end{cases}$$

Claim that $AB = I_n (= BA)$. Since $A_{it} = 0$ for all $i,t \in \{1,2,\ldots,n\}$, $t \neq k_i$, it follows that for $i,j \in \{1,2,\ldots,n\}$, $(AB)_{ij} = \sum_{t=1}^{n} A_{it}B_{tj} = A_{ik_i}B_{k_ij}$. Then $(AB)_{ii} = A_{ik_i}B_{k_ii} = A_{ik_i}A_{ik_i}^{-1} = 1$ for every $i \in \{1,2,\ldots,n\}$. If $i,j \in \{1,2,\ldots,n\}$ with $i \neq j$, then $A_{jk_i} = 0$, so $B_{k_ij} = 0$ which implies $(AB)_{ij} = 0$. #

Corollary 2.3. A square matrix A over the semiring $(\mathbb{N} \cup \{0\},+,\cdot)$ is invertible if and only if A is a permutation matrix.

Proof. It follows directly from Theorem 2.2 and the fact that 1 is the only invertible element of the semiring $(\mathbb{N} \cup \{0\},+,\cdot)$. #

Corollary 2.4. Let S be a subsemiring of the semiring $(\mathbb{R}^+ \cup \{0\},+,\cdot)$ containing 0,1. Assume that every element of S is (multiplicatively) invertible. Then a square matrix A over S is invertible if and only if every row and every column of A has exactly one nonzero element

Proof. Since every nonzero element of the semiring $(S,+,\cdot)$ is invertible, this corollary is obtained by Theorem 2.2. #

Corollary 2.5. Let S be the semiring $(\mathbb{Q}^+ \cup \{0\},+,\cdot)$ or the semiring $(\mathbb{R}^+ \cup \{0\},+,\cdot)$. Then a square matrix A over S is invertible if and only if every row and every column of A has exactly one nonzero ele element.

Proof. It follows directly from Corollary 2.4. #

Let S be a set of real numbers with a minimum element m and a maximum element M. Then (S,max,min) is a commutative semiring with zero m and identity M. For $x,y \in S$, if $\min\{x,y\} = m$, then $x = m$ or $y = m$, and if $\max\{x,y\} = m$ then $x = y = m$. Also, for $x,y \in S$, $\min\{x,y\} = M$ implies $x = y = M$. Hence by Theorem 2.2, we have the following corollary.

Corollary 2.6. Let S be a set of real numbers with a minimum element and a maximum element. Then a square matrix A over the semiring (S,max,min) is invertible if and only if A is a permutation matrix.

In particular, a square matrix A over the semiring ([0,1],max,min) is invertible if and only if A is a permutation matrix.

Let S be a semilattice semiring with 0,1. Since $x+x = x$ and $x^2 = x$ for every x in S, it can be easily seen that S has the following properties:

(1) For $a,b \in S$, $a+b = 0$ implies $a = b = 0$.

(2) For $a,b \in S$, $ab = 1$ implies $a = b = 1$.

In any semilattice semiring S with 0,1, 0 is the only element of S which has an additive inverse (since $x+y = 0$ implies $x = y = 0$) but S may have a zero divisors. A Boolean algebra with more than two elements is an example. Then Theorem 2.2 cannot be used to characterize an invertible matrix over a similattice semiring with 0,1.

The next theorem gives a characterization of an invertible matrix over a semilattice semiring with 0,1 in terms of the positive determinant, negative determinant and the product of any two elements in the same column.

<u>Theorem 2.7</u> . Let S be a semilattice semiring with 0,1 and A a square matrix over S. Then A is invertible if and only if $\det^+A + \det^-A = 1$ and the product of any two elements of A in the same column is 0.

<u>Proof</u>. Let A be an nxn matrix over S.

First assume that A is an invertible matrix over S with the inverse matrix B. By Lemma 1 of [8], there exists an element r of S such that

$$\det^+(AB) = (\det^+A)(\det^+B)+(\det^-A)(\det^-B)+r,$$

$$\det^-(AB) = (\det^+A)(\det^-B)+(\det^-A)(\det^+B)+r.$$

But $\det^+(AB) = \det^+I_n = 1$ and $\det^-(AB) = \det^-I_n = 0$, so we have that

$$1 = (\det^+A)(\det^+B)+(\det^-A)(\det^-B)+r,$$

$$0 = (\det^+A)(\det^-B)+(\det^-A)(\det^+B)+r.$$

From the last equality, we have that $(\det^+A)(\det^-B) = (\det^-A)(\det^+B) = r = 0$. Then

$$\begin{aligned} 1 &= (\det^+A)(\det^+B)+(\det^-A)(\det^-B) \\ &= (\det^+A)(\det^+B)+(\det^+A)(\det^-B)+(\det^-A)(\det^+B) \\ &\quad +(\det^-A)(\det^-B) \\ &= \det^+A(\det^+B+\det^-B)+\det^-A(\det^+B+\det^-B) \\ &= (\det^+A+\det^-A)(\det^+B+\det^-B) \end{aligned}$$

which implies that $\det^+A+\det^-A = 1$.

For $i,j \in \{1,2,\ldots,n\}$, if $i \neq j$, then $0 = (AB)_{ij} = \sum_{t=1}^{n} A_{it}B_{tj}$ which implies that $A_{it}B_{tj} = 0$. This proves that $A_{it}B_{tj} = 0$ for all $i,j,t \in \{1,2,\ldots,n\}$, $i \neq j$. If $i,i',j \in \{1,2,\ldots,n\}$, $i \neq i'$, then $A_{ij}B_{ji'} = 0$ and $A_{i'j}B_{jt} = 0$ for every $t \in \{1,2,\ldots,n\}$, $t \neq i'$, so

$$\begin{aligned} A_{ij}A_{i'j} &= (A_{ij}A_{i'j})1 \\ &= (A_{ij}A_{i'j})(BA)_{jj} \end{aligned}$$

$$= (A_{ij}A_{i'j})\left(\sum_{t=1}^{n} B_{jt}A_{tj}\right)$$

$$= \sum_{t=1}^{n} A_{ij}A_{i'j}B_{jt}A_{tj}$$

$$= A_{ij}A_{i'j}B_{ji'}A_{i'j} + \sum_{\substack{t=1 \\ t\neq i'}}^{n} A_{ij}A_{i'j}B_{jt}A_{tj}$$

$$= (A_{ij}B_{ji'})A_{i'j} + \sum_{\substack{t=1 \\ t\neq i'}}^{n} A_{ij}(A_{i'j}B_{jt})A_{tj}$$

$$= 0$$

This proves that the product of any two elements of A in the some column is zero.

Conversely, assume that $\det^{+}A + \det^{-}A = 1$ and $A_{ij}A_{i'j} = 0$ for all $i, i', j \in \{1,2,...,n\}$, $i \neq i'$. Claim that $AA^{T} = I_n$. Since $x^2 = x$ for every x in S, we have that

$$(AA^{T})_{11}(AA^{T})_{22}\cdots(AA^{T})_{nn} = \left(\sum_{t=1}^{n} A_{1t}A_{1t}\right)\left(\sum_{t=1}^{n} A_{2t}A_{2t}\right)\cdots\left(\sum_{t=1}^{n} A_{nt}A_{nt}\right)$$

$$= \left(\sum_{t=1}^{n} A_{1t}\right)\left(\sum_{t=1}^{n} A_{2t}\right)\cdots\left(\sum_{t=1}^{n} A_{nt}\right)$$

$$= \sum_{t_1,...,t_n \in \{1,2,...,n\}} A_{1t_1}A_{2t_2}\cdots A_{nt_n}.$$

Since $A_{ij}A_{i'j} = 0$ for all $i, i', j \in \{1,2,...,n\}$, $i \neq i'$, it follows that $A_{1t_1}A_{2t_2}\cdots A_{nt_n} = 0$ if $t_1, t_2, ..., t_n \in \{1,2,...,n\}$ and they are not all distinct. Hence

$$\sum_{t_1,...,t_n \in \{1,2,...,n\}} A_{1t_1}A_{2t_2}\cdots A_{nt_n} = \sum_{\sigma\in S_n} A_{1\sigma(1)}A_{2\sigma(2)}\cdots A_{n\sigma(n)},$$

so $(AA^{T})_{11}(AA^{T})_{22}\cdots(AA^{T})_{nn} = \sum_{\sigma\in S_n} A_{1\sigma(1)}A_{2\sigma(2)}\cdots A_{n\sigma(n)} = \det^{+}A + \det^{-}A = 1.$

Thus $(AA^{T})_{ii} = 1$ for every $i \in \{1,2,...,n\}$. If $i, j \in \{1,2,...,n\}$, $i \neq j$, then $A_{it}A_{jt} = 0$ for every $t \in \{1,2,...,n\}$ and hence $(AA^{T})_{ij} = \sum_{t=1}^{n} A_{it}A_{jt}$

$= 0.$ #

If A is a square matrix over a commutative semiring with 0,1, then $\det^+A = \det^+A^T$ and $\det^-A = \det^-A^T$, and A is invertible if and only if A^T is invertible. From these facts, the following corollary is clearly obtained.

Corollary 2.8. Let S be a semilattice semiring with 0,1 and A a square matrix over S. Then A is invertible if and only if $\det^+A + \det^-A = 1$ and the product of any two elements of A in the same row is 0.

Next, we characterize any invertible matrix over a semilattice semiring with 0,1 in term of the product of any two elements in the same column and the sum of all elements in each row.

Theorem 2.9. Let S be a semilattice semiring with 0,1 and A a square matrix over S. Then A is invertible if and only if the product of any two elements of A in the same column is 0 and the sum of all elements of A in each row is 1.

Proof. Let A be an $n\times n$ matrix over S.

First assume A is invertible. Then by Theorem 2.7, $A_{ij}A_{i'j} = 0$ for all $i,i',j \in \{1,2,\ldots,n\}$, $i \neq i'$ and $\det^+A + \det^-A = 1$ which imply that

$$\left(\sum_{t=1}^{n} A_{1t}\right)\left(\sum_{t=1}^{n} A_{2t}\right)\cdots\left(\sum_{t=1}^{n} A_{nt}\right) = \sum_{t_1,\ldots,t_n\in\{1,2,\ldots,n\}} A_{1t_1}A_{2t_2}\cdots A_{nt_n}$$

$$= \sum_{\sigma\in\mathcal{S}_n} A_{1\sigma(1)}A_{2\sigma(2)}\cdots A_{n\sigma(n)}$$

$$= \det^+A + \det^-A$$

$$= 1 .$$

Hence $\sum_{t=1}^{n} A_{it} = 1$ for all $i \in \{1,2,\ldots,n\}$.

Conversely, assume $A_{ij}A_{i'j} = 0$ for all $i,i',j \in \{1,2,\ldots,n\}$, $i \neq i'$ and $\sum_{t=1}^{n} A_{it} = 1$ for every $i \in \{1,2,\ldots,n\}$. Then

$$1 = \left(\sum_{t=1}^{n} A_{1t}\right)\left(\sum_{t=1}^{n} A_{2t}\right)\cdots\left(\sum_{t=1}^{n} A_{nt}\right)$$

$$= \sum_{t_1,\ldots,t_n \in \{1,2,\ldots,n\}} A_{1t_1}A_{2t_2}\cdots A_{nt_n}$$

$$= \sum_{\sigma \in S_n} A_{1\sigma(1)}A_{2\sigma(2)}\cdots A_{n\sigma(n)}$$

$$= \det{}^{+}A + \det{}^{-}A.$$

By Theorem 2.7, A is invertible. #

Corollary 2.10 Let S be a semilattice semiring with 0,1 and A a square matrix over S. Then A is invertible if and only if the product of any two elements of A in the same row is 0 and the sum of all elements of A in each column is 1.

Proof It follows directly from Theorem 2.9 and the fact that A is invertible if and only if A^T is invertible. #