CHAPTER III

FEYNMAN PATH INTEGRAL IN QUANTUM MECHANICS

In proceeding to evaluate the density of states, the basic ideas of constructing the Feynman path integral will be presented in this chapter. We present in this chapter the mathematical formulation of the quantum-mechanical transformation or the propagator, in the form of a path integral. As an illustration, we apply this method to the harmonic oscillator.

Feynman Propagator [36]

If a particle moves from one point to another point there are many possible paths which the particle can take. In terms of classical mechanics, when we consider the particle as a point, there is a principle of least action which expresses the condition that determines a particular path from among all the possible paths. For simplicity, we will restrict ourselves to the case of a particle moving in one dimension. Thus the position at any time can be specified by a coordinate x, a function of time. By the path, we therefore mean a function x(t).

If a particle starts from the point x_a at an initial time t_a and goes to the final point x_b at time t_b, there will be many possible paths in which the particle can travel. For each path there exists the action S,

$$S = \int_{t_a}^{t_b} L(\dot{x}, x, t)\, dt, \qquad (3.1)$$

where L is the lagrangian for the system.

The principle of least action states that the particular path x(t) in which the particle can travel is that for which S is a minimum. That is to say, the value of S is unchanged in the first order if the path x(t) is slightly modified. This particular path x(t) is called the classical path.

In the next chapter, we shall determine the classical action of random systems, which is quite difficult. It is as well first to become familiar with a simple system of classical action, which can then serve as a guide to the more difficult determination of the classical action of random systems. The determination of the classical action of a harmonic oscillator will therefore now be studied.

For a harmonic oscillator, the Lagrangian is

$$L = \frac{m}{2}\left(\dot{x}^2 - \omega^2 x^2\right),$$

therefore

$$S = \int_{t_a}^{t_b} \frac{m}{2}\left(\dot{x}^2 - \omega^2 x^2\right) dt,$$

Suppose the path $x(t)$ is varied from $\bar{x}$ by an amount $\delta x(t)$. The end points x_a and x_b are fixed, that is

$$\delta x(t_a) = \delta x(t_b) = 0 \tag{3.2}$$

From the principle of least action, the condition that $\bar{x}$ must yield a minimum value of S is

$$\delta S = 0. \tag{3.3}$$

Hence

$$\delta S = \frac{m}{2}\left\{\int_{t_a}^{t_b} 2\dot{\bar{x}}\frac{d}{dt}\delta x\,dt - \omega^2\int_{t_a}^{t_b} 2\bar{x}\,\delta x\,dt\right\} = 0$$

Thus

$$\frac{m}{2}\left\{2\dot{\bar{x}}\delta x\Big|_{t_b}^{t_a} - \int 2\ddot{\bar{x}}\,\delta x\,dt - \omega^2\int 2\bar{x}\,\delta x\,dt\right\} = 0 \tag{3.4}$$

By applying the condition in eq. (3.2) to the first term in the left hand side of eq. (3.4), we obtain

$$\frac{m}{2}\left\{-\int 2\ddot{\bar{x}}\,\delta x\,dt - \omega^2\int 2\bar{x}\,\delta\dot{x}\,dt\right\} = 0.$$

Thus

$$\ddot{\bar{x}} + \omega^2 \bar{x} = 0 . \qquad (3.5)$$

The solution of eq. (3.5) is

$$\bar{x}(t) = A \sin \omega t + B \cos \omega t, \qquad (3.6)$$

where A and B are constants. By applying the boundary condition $x(t_a) = x_a$ and $x(t_b) = x_b$ to eq. (3.6) and making the period T of the harmonic oscillator equal to $t_b - t_a$, we can obtain the constants A and B,

$$A = \frac{x_b - x_a \cos \omega T}{\sin \omega T},$$

$$B = x_a$$

Now $$S_{cl} = \int_0^T \frac{m}{2}\left(\dot{\bar{x}}^2 - \omega \bar{x}^2\right) dt$$

$$= \frac{m}{2}\left\{ \dot{\bar{x}} x \Big|_0^T - \int_0^T \bar{x}\, \ddot{\bar{x}}\, dt - \omega^2 \int_0^T \bar{x}^2 dt \right\}$$

$$= \frac{m}{2}\left[\dot{\bar{x}}(T)\,\bar{x}(T) - \dot{\bar{x}}(0)\,\bar{x}(0) - \left\{ \int_0^T \bar{x}\left(\ddot{\bar{x}} + \omega^2 \bar{x}\right) dt \right\} \right] \qquad (3.7)$$

and since $\ddot{\bar{x}} + \omega^2 \bar{x} = 0$,

$$S_{cl} = \frac{m}{2}\left[\dot{\bar{x}}(T)\bar{x}(T) - \dot{\bar{x}}(0)\bar{x}(0) \right].$$

Differentiating eq. (3.6) with respect to t, we obtain

$$\dot{\bar{x}}(t) = A\omega \cos \omega t - B\omega \sin \omega t \qquad (3.8)$$

Substituting eq. (3.6) when t = 0 and t = T and eq. (3.7) when t = 0 and t = T into eq. (3.7), and also making use of the values of A and B, we obtain

$$S_{cl} = \frac{m\omega}{2 \sin \omega T}\left\{ \cos \omega T\left(x_a^2 + x_b^2 \right) - 2x_a x_b \right\} \qquad (3.9)$$

Thus we can obtain the classical action of the harmonic oscillator.

After discussing the path in which a particle can move from a to b from the classical point of view and showing how the classical path and classical action can be found, we will next look at this from the quantum mechanical point of view.

Quantum mechanics deals with probabilities, that is, it states that we cannot specify the position of a particle, we can only know the probability of its being found in a given place. The probability that a particle will be found to have a path x(t) lying somewhere within the space time continuum is the absolute square of a probability amplitude. The probability amplitude is associated with the entire motion of a particle as a function of time, rather than simply with the position of the particle at a particular time. Thus when we consider the path by which the particles goes from a to b, we must specify how much each trajectory contributes to the probability amplitude K(b, a). It is not just the particular path of extreme action which contributes; rather, it is the case that

all the paths contribute. The contribution $\emptyset\,[x(t)]$ from a single path depends on the classical action for that path in units of $\hbar$.

$$\emptyset[x(t)] = \text{const} \exp\left[\frac{i}{\hbar} S\{x(t)\}\right]$$

The amplitude K(b,a), is thus the sum over all trajectories between the end points a and b of contributions $\emptyset\,[x(t)]$,

$$K(b,a) = \sum_{\substack{\text{over all} \\ \text{paths from a to b}}} \emptyset[x(t)]$$

$$K(b,a) = \sum_{\substack{\text{over all} \\ \text{paths from a to b}}} \text{const} \exp\left[\frac{i}{\hbar} S\{x(t)\}\right]. \quad (3.10)$$

We have thus described the physical ideas concerned in the construction of the amplitude for a particle to reach a particular point in space and time by closely following its motion in getting there. So if we want to find the probability amplitude of the particle going from a to b, we have to carry out the sum in eq. (3.10). But the number of paths from a to b is infinite, so eq. (3.10) is very difficult to work with. Another method and more efficient method of computing the sum over all paths will now be described.

We choose a subset of all paths by first separating the independent time into small interval, ε. This gives us a set of successive times $t_1, t_2, t_3, \ldots$ between the values t_a and t_b, where $t_{i+1} = t_i + \varepsilon$, At each time t_i we select some special point x_i and constructing a path by connecting all the points so selected to form a line. This process is shown in Fig. 8. It is possible to define a sum over all paths constructed in

this manner by taking a multiple integral over all values of x_i for i between 1 and n-1, where

$$n\varepsilon = t_b - t_a, \quad t_0 = t_a, \quad t_n = t_b,$$

$$x_0 = x_a, \quad x_n = x_b$$

By using this method, eq. (3.10) then becomes

$$K(b,a) \approx \int\int\cdots\int \text{const} \exp\left[\frac{i}{\hbar} S\{x(t)\}\right] dx_1\, dx_2 \cdots dx_{n-1}. \tag{3.11}$$

We do not integrate x_0 or x_n because these are the fixed end points x_a and x_b. In order to achieve the correct measure, eq. (3.11) must be taken in the limit $\varepsilon \to 0$ and some normalizing factor A^{-n} which depends on ε must be provided in order that the limit of eq. (3.11) becomes

$$K(b,a) \approx \lim_{\varepsilon \to 0} \frac{1}{A} \int\int\cdots\int \text{const} \exp\left[\frac{i}{\hbar} S\{x(t)\}\right] \frac{dx_1}{A}\frac{dx_2}{A}\cdots\frac{dx_{n-1}}{A}. \tag{3.12}$$

Eq. (3.12) can also be written in a less restrictive notation as

$$K(b,a) \approx N \int\int\cdots\int \text{const} \exp\left[\frac{i}{\hbar} S\{x(t)\}\right] \mathcal{D}(\text{path}).$$

This is called a path integral and the amplitude K(b,a) is known as the Feynman propagator.

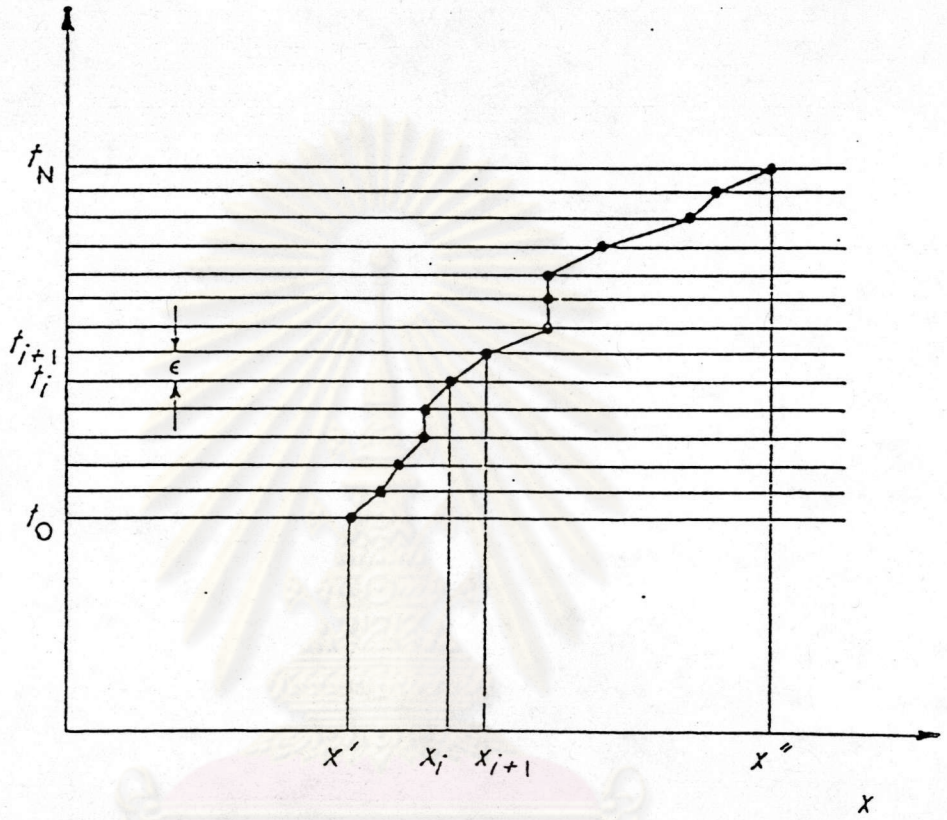

Fig. 8 Diagram showing how the path integrals can be constructed.

Propagator from Schrödinger's Equation

So far, we have followed Feynman's argument in writing down the propagator in the form of path integral. It will now be shown that the propagator in this form can also be derived directly from the Schrödinger's equation. The time-dependent Schrödinger's equation is

$$\left[i\hbar \frac{\partial}{\partial t} - H \right] \psi(\vec{r}, t) = 0.$$

We can define the one-electron Green function of this equation as the solution of

$$\left[i\hbar \frac{\partial}{\partial t} - H \right] G(\vec{r}, \vec{r}\,'; t, t') = \delta(\vec{r} - \vec{r}\,') \delta(t - t').$$

Thus the Green function can be written as

$$G(\vec{r}, \vec{r}\,'; t, t') = \left\langle \vec{r} \left| \exp\left\{ -\frac{i}{\hbar} H(t - t') \right\} \right| \vec{r}\,' \right\rangle. \quad (3.13)$$

Let us divide the time interval $t - t'$ into n equal small subintervals, so that $t - t' = n\varepsilon$. By making use of the identity

$$\exp\left(-\frac{i\varepsilon H n}{\hbar} \right) = \lim_{\varepsilon \to 0} \left(1 - \frac{i\varepsilon H}{\hbar} \right)^n,$$

eq. (3.12) becomes

$$G(\vec{r}, \vec{r}\,'; t, t') = \lim_{\varepsilon \to 0} \left\langle \vec{r} \left| \frac{\left(1 - \frac{i\varepsilon H}{\hbar}\right) \cdots \left(1 - \frac{i\varepsilon H}{\hbar}\right)}{\text{n factor}} \right| \vec{r}\,' \right\rangle. \quad (3.14)$$

According to the rules of quantum mechanics we can insert a complete set of states between each pair of factors in eq. (3.14):

$$G(\vec{r}, \vec{r}\,'; t, t') = \lim_{\varepsilon \to 0} \int \int \cdots \int \left\langle \vec{r} \left| \left(1 - \frac{i\varepsilon H}{\hbar}\right) \right| \vec{r}_{n-1} \right\rangle d\vec{r}_{n-1}$$

$$\cdot \left\langle \vec{r}_{n-1} \left| \left(1 - \frac{i\varepsilon H}{\hbar}\right) \right| \vec{r}_{n-2} \right\rangle d\vec{r}_{n-2} \cdots$$

$$\cdot d\vec{r}_2 \left\langle \vec{r}_2 \left| \left(1 - \frac{i\varepsilon H}{\hbar}\right) \right| \vec{r}_1 \right\rangle d\vec{r}_1 \left\langle \vec{r}_1 \left| \left(1 - \frac{i\varepsilon H}{\hbar}\right) \right| \vec{r}\,' \right\rangle \quad (3.15)$$

We now consider the Hamiltonian of the system in the position representation

$$H = \frac{p^2}{2m} + V(\vec{r}),$$

where p is the momentum operator.

Then

$$\left\langle \vec{r}_{i+1} \left| \left(1 - \frac{i\varepsilon H}{\hbar} \right) \right| \vec{r}_i \right\rangle = \int \langle \vec{r}_{i+1} | \vec{p} \rangle d\vec{p} \left\langle \vec{p} \left| 1 - \frac{i\varepsilon}{\hbar} \left\{ \frac{p^2}{2m} + V(\vec{r}) \right\} \right| \vec{r}_i \right\rangle$$

$$= \int \langle \vec{r}_{i+1} | \vec{p} \rangle d\vec{p} \langle \vec{p} | \vec{r}_i \rangle \cdot \left[1 - \frac{i\varepsilon}{\hbar} \left\{ \frac{p^2}{2m} + V(\vec{r}) \right\} \right] \tag{3.16}$$

From quantum mechanics, the momentum eigenfunctions $\langle \vec{r} | \vec{p} \rangle$ are

$$\langle \vec{r} | \vec{p} \rangle = \frac{1}{(2\pi\hbar)^{3/2}} \exp\left(\frac{i}{\hbar} \vec{p} \cdot \vec{r} \right)$$

Therefore

$$\left\langle \vec{r}_{i+1} \left| \left(1 - \frac{i\varepsilon H}{\hbar} \right) \right| \vec{r}_i \right\rangle = \frac{1}{(2\pi\hbar)^3} \int \exp\left\{ \frac{i}{\hbar} (\vec{r}_{i+1} - \vec{r}_i) \cdot \vec{p} \right\}$$

$$\cdot \left[1 - \frac{i\varepsilon}{\hbar} \left\{ \frac{p^2}{2m} + V(\vec{r}) \right\} \right] d\vec{p} \tag{3.17}$$

We now replace $1 - \frac{i\varepsilon}{\hbar}\left\{ \frac{p^2}{2m} + V(\vec{r}) \right\}$ by the corresponding exponential; the error introduced here is $O(\varepsilon^2)$, so that the total error from all the n factors can be neglected. Eq. (3.17) thus becomes

$$\left\langle \vec{r}_{i+1} \left| \left(1 - \frac{i\varepsilon H}{\hbar} \right) \right| \vec{r}_i \right\rangle = \frac{1}{(2\pi\hbar)^3} \int \exp\left[-\frac{i\varepsilon}{\hbar} \left\{ \frac{p^2}{2m} - \frac{(\vec{r}_{i+1} - \vec{r}_i)\cdot\vec{p}}{\varepsilon} \right\} \right] d\vec{p}$$

$$\cdot \exp\left\{ -\frac{i\varepsilon}{\hbar} V(\vec{r}) \right\}$$

$$= \frac{1}{(2\pi\hbar)^3} \int \exp\left[-\frac{i\varepsilon}{2m\hbar} \left\{ p^2 - \frac{2m(\vec{r}_{i+1} - \vec{r}_i)\cdot\vec{p}}{\varepsilon} \right.\right.$$

$$\left.\left. + \left(\frac{m}{\varepsilon}(\vec{r}_{i+1} - \vec{r}_i) \right)^2 - \left(\frac{m}{\varepsilon}(\vec{r}_{i+1} - \vec{r}_i) \right)^2 \right\} \right] d\vec{p}$$

$$\cdot \exp\left\{ -\frac{i\varepsilon}{\hbar} V(\vec{r}) \right\}$$

$$= \frac{1}{(2\pi\hbar)^3} \int \exp\left[-\frac{i\varepsilon}{2m\hbar} \left\{ \vec{p} - \frac{m}{\varepsilon}(\vec{r}_{i+1} - \vec{r}_i) \right\}^2 \right] d\vec{p}$$

$$\cdot \exp\left\{ \frac{i\varepsilon}{2m\hbar} \left(\frac{m}{\varepsilon}(\vec{r}_{i+1} - \vec{r}_i) \right)^2 \right\} \cdot \exp\left\{ -\frac{i\varepsilon}{\hbar} V(\vec{r}) \right\}$$

so

$$\left\langle \vec{r}_{i+1} \left| \left(1 - \frac{i\varepsilon H}{\hbar} \right) \right| \vec{r}_i \right\rangle = \frac{1}{(2\pi\hbar)^3} \left(\frac{2m\pi\hbar}{i\varepsilon} \right)^{3/2} \cdot \exp\left\{ \frac{im}{2\hbar\varepsilon} (\vec{r}_{i+1} - \vec{r}_i)^2 \right.$$

$$\left. - \frac{i\varepsilon}{\hbar} V(\vec{r}) \right\}$$

$$= \left(\frac{m}{2\pi\hbar i\varepsilon} \right)^{3/2} \exp\left[\frac{i\varepsilon}{\hbar} \left\{ \frac{m}{2} \left(\frac{\vec{r}_{i+1} - \vec{r}_i}{\varepsilon} \right)^2 - V(\vec{r}) \right\} \right]$$

(3.18)

Substituting for eq. (3.18) in eq. (3.15), we obtain

$$G(\vec{r},\vec{r}\,';t,t') = \lim_{\varepsilon\to 0}\frac{1}{A}\int\int\cdots\int \exp\left[\frac{i\varepsilon}{\hbar}\sum_{i=0}^{n-1}\left\{\frac{m}{2}\left(\frac{\vec{r}_{i+1}-\vec{r}_i}{\varepsilon}\right)^2 - V(\vec{r})\right\}\right] \cdot \frac{d\vec{r}_1}{A}\frac{d\vec{r}_2}{A}\cdots\frac{d\vec{r}_{n-1}}{A} \tag{3.19}$$

where $\frac{1}{A} = \left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{3/2}$, $\vec{r}_n = \vec{r}$, $\vec{r}_0 = \vec{r}\,'$.

In an obvious notation

$$G(\vec{r},\vec{r}\,';t,t') = \lim_{\varepsilon\to 0}\frac{1}{A}\int\int\cdots\int \exp\left[\frac{i}{\hbar}\int_{t'}^{t}\left\{\frac{m}{2}\dot{\vec{r}}^2 - V(\vec{r})\right\}dt\right] \cdot \frac{d\vec{r}_1}{A}\frac{d\vec{r}_2}{A}\cdots\frac{d\vec{r}_{n-1}}{A}\,. \tag{3.20}$$

It can be shown that the time-dependent Green function (3.19) of Schrödinger's equation has exactly the same form as the Feynman propagator (3.12). The latter can be written in this form by using the argument discussed at the beginning of this chapter. As a simple example of how to obtain $G(\vec{r},\vec{r}\,';t,t')$ written in the form of eq. (3.20), let us consider the case of a free electron.

For a free electron $L = \frac{m}{2}\dot{\vec{r}}^2$, therefore by using eq. (3.19), we obtain

$$G(\vec{r},\vec{r}\,';t,t') = \lim_{\varepsilon\to 0}\frac{1}{A}\int\int\cdots\int \exp\left[\frac{i\varepsilon}{\hbar}\sum_{i=0}^{n-1}\frac{m}{2}\left(\frac{\vec{r}_{i+1}-\vec{r}_i}{\varepsilon}\right)^2\right] \cdot \frac{d\vec{r}_1}{A}\frac{d\vec{r}_2}{A}\cdots\frac{d\vec{r}_{n-1}}{A},$$

where $\frac{1}{A} = \left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{3/2}$, $\vec{r}_n = \vec{r}, \vec{r}_0 = \vec{r}\,'$.

The calculation is carried out by direct integrations as follows:

Since $\int_{-\infty}^{\infty} \exp\left\{ a(\vec{r}_0 - \vec{r}_1)^2 + b(\vec{r}_2 - \vec{r}_1)^2 \right\} d\vec{r}_1$

$$= \left(-\frac{\pi}{a+b}\right)^{3/2} \exp\left\{ \frac{ab}{a+b}(\vec{r}_2 - \vec{r}_0)^2 \right\},$$

we have

$$\left(\frac{m}{2\pi i\hbar\varepsilon}\right)^3 \int_{-\infty}^{\infty} \exp\left\{ \frac{im}{2\hbar\varepsilon}(\vec{r}_1 - \vec{r}_0)^2 + (\vec{r}_2 - \vec{r}_1)^2 \right\} d\vec{r}_1$$

$$= \left(\frac{m}{2\pi i\hbar\varepsilon}\right)^3 \left(\frac{\pi i\hbar\varepsilon}{m}\right)^3 \exp\left\{ \frac{im}{2\hbar(2\varepsilon)}(\vec{r}_2 - \vec{r}_0)^2 \right\}$$

$$= \left(\frac{m}{2\pi\hbar(2\varepsilon)}\right)^{3/2} \exp\left\{ \frac{im}{2\hbar(2\varepsilon)}(\vec{r}_2 - \vec{r}_0)^2 \right\}$$

Multiplying the result by $\left(\frac{2\pi i\hbar\varepsilon}{m}\right)^{-1/2} \exp\left\{ \frac{im}{2\hbar\varepsilon}(x_3 - x_2)^2 \right\}$ and integrating over $\vec{r}_2$, we obtain

$$\frac{1}{A}\int\int \exp\left[\frac{im}{2\hbar\varepsilon}\left\{ (\vec{r}_1 - \vec{r}_0)^2 + (\vec{r}_2 - \vec{r}_1)^2 + (\vec{r}_3 - \vec{r}_2)^2 \right\} \right] \frac{d\vec{r}_1}{A} \frac{d\vec{r}_2}{A}$$

$$= \int \left(\frac{m}{2\pi i\hbar(2\varepsilon)}\right)^{3/2} \exp\left[\frac{im}{2\hbar(2\varepsilon)}(\vec{r}_2 - \vec{r}_0)^2 \right]$$

$$\left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{3/2} \exp\left[\frac{im}{2\hbar\varepsilon}(\vec{r}_3 - \vec{r}_2)^2 \right] d\vec{r}_2$$

$$= \left(\frac{m}{2\pi i\hbar(2\varepsilon)}\right)^{3/2}\left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{3/2}\left(\frac{-2\pi\hbar(2\varepsilon)}{3im}\right)^{3/2}\exp\left[\frac{im}{2\hbar(3\varepsilon)}(\vec{r}_3-\vec{r}_0)^2\right]$$

$$= \left(\frac{m}{2\pi i\hbar(3\varepsilon)}\right)^{3/2}\exp\left[\frac{im}{2\hbar(3\varepsilon)}(\vec{r}_3-\vec{r}_0)^2\right].$$

In this way a recurring process is established by which, after (n-1) steps, we obtain

$$G_0(\vec{r},\vec{r}\,';t) = \left(\frac{m}{2\pi i\hbar n\varepsilon}\right)^{3/2}\exp\left[\frac{im}{2\hbar(n\varepsilon)}(\vec{r}_n-\vec{r}_0)^2\right]$$

since $n\varepsilon = t$, $\vec{r}_n = \vec{r}$ and $\vec{r}_0 = \vec{r}\,'$, therefore

$$G_0(\vec{r},\vec{r}\,';t) = \left(\frac{m}{2\pi i\hbar t}\right)^{3/2}\exp\left[\frac{im}{2\hbar t}(\vec{r}-\vec{r}\,')^2\right].$$

The method of direct integration can be carried out only for this simple case of a free electron. For other cases, the path integral is more difficult to work out. Therefore we will introduce another method to compute other path integrals.

Since the classical path is that for which S is a minimum, this path is therefore completely fixed. Consequently any path $\vec{r}(\tau)$ can be written in terms of the classical path $\vec{r}_c(\tau)$ and of a new variable $\vec{y}(\tau)$, the deviation from the classical path:

$$\vec{r}(\tau) = \vec{r}_c(\tau)+\vec{y}(\tau),$$

and the path differential $\mathcal{D}[\vec{r}(\tau)]$ can be replaced by $\mathcal{D}[\vec{y}(\tau)]$. This is, instead of defining a point on the path by its distance $\vec{r}(\tau)$ from an arbitrary coordinate axis as formerly, we now define it by its deviation $\vec{y}(\tau)$ from the classical path, as shown in

Fig. 9. Since any path $\vec{r}(\tau)$ and the classical path $\vec{r}_c(\tau)$ must reach the same end points,

$$\vec{y}(0) = \vec{y}(t) = 0.$$

In between these end points $\vec{y}(\tau)$ can take any form.

For a particle whose lagrangian has the quadratic form

$$L = a(\tau)\dot{\vec{r}}^2(\tau) + b(\tau)\dot{\vec{r}}(\tau)\vec{r}(\tau) + c(\tau)\vec{r}^2(\tau) + d(\tau)\dot{\vec{r}}(\tau) + e(\tau)\vec{r}(\tau) + f(\tau),$$

the integral for the action can be written

$$S[\vec{r}(\tau)] = S[\vec{r}_c(\tau) + \vec{y}(\tau)]$$

$$= \int_0^t a(\tau)\left\{\dot{\vec{r}}^2{}_c(\tau) + 2\dot{\vec{r}}_c(\tau)\dot{\vec{y}}(\tau) + \dot{\vec{y}}^2(\tau) + \cdots\right\} d\tau$$

If all the terms which do not involve $\vec{y}(\tau)$ are collected, the resulting integral is just the classical action S_{cl}. If all the terms which contain $\vec{y}(\tau)$ as a linear factor are collected, the resulting integral vanishes. All that remain are the second-order terms in $\vec{y}$. Thus

$$S[\vec{r}(\tau)] = S_{cl} + \int_0^t \left[a(\tau)\dot{\vec{y}}^2 + b(\tau)\dot{\vec{y}}\,\vec{y} + c(\tau)\vec{y}^2\right] d\tau .$$

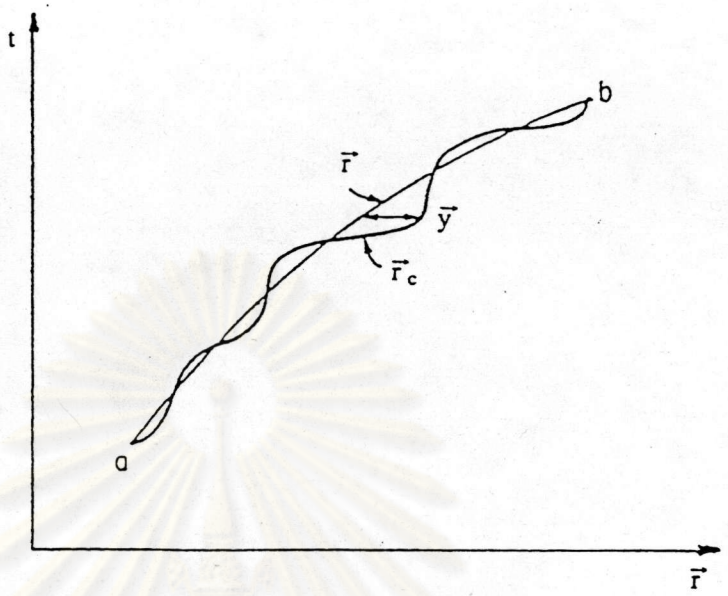

Fig. 9 Diagram showing a path deviating from the classical path.

Thus the Green function can be written

$$G(\vec{r},\vec{r}';t) = N\int \exp\left[\frac{i}{\hbar}\left\{S_{cl}+\int_0^t\left[a(\tau)\dot{\vec{y}}^2+b(\tau)\dot{\vec{y}}\,\vec{y}+c(\tau)\vec{y}^2\right]d\tau\right\}\right]\mathcal{D}\left[\vec{y}(\tau)\right].$$

Since the integral over paths $\vec{y}(\tau)$ does not depend upon the classical path and all paths $\vec{y}(\tau)$ start from and return to the point $\vec{y}=0$, therefore $G(\vec{r},\vec{r}';t)$ can be written as

$$G(\vec{r},\vec{r}';t) = \exp\left(\frac{i}{\hbar}S_{cl}\right)N\int_0^0 \exp\left[\frac{i}{\hbar}\left\{\int_0^t\left[a(\tau)\dot{\vec{y}}^2+b(\tau)\dot{\vec{y}}\,\vec{y}+c(\tau)\vec{y}^2\right]d\tau\right\}\right]\mathcal{D}\left[\vec{y}(\tau)\right]. \quad (3.22)$$

Consequently it can be seen that the path integral (3.20) which was specified by the distance $\vec{r}(\tau)$ from an arbitrary coordinate axis, and which depends upon the end-point positions, can be reduced to a product of two functions one of which does not depend upon the end-point positions. The new product depends upon the classical action which can be obtained without too much difficulty, and a path integral depending only on the times at the end points.

Consider the path integral for a one-dimensional harmonic oscillator. This is

$$G(\vec{x}, \vec{x}'; t) = \int_{x'}^{x} \exp\left[\frac{i}{\hbar}\int_0^t \left\{\frac{m}{2}\left(\dot{x}^2 - \omega^2 x^2\right)\right\} d\tau\right] \mathfrak{D}[x(\tau)] .$$

Using the method which has just been described, this path integral can be reduced to a product of two function,

$$G(\vec{x}, \vec{x}'; t) = \exp\left[\frac{i}{\hbar} S_{cl}\right] \int_0^0 \exp\left[\frac{i}{\hbar}\int_0^t \left\{\frac{m}{2}\left(\dot{y}^2 - \omega^2 y^2\right)\right\} d\tau\right] \mathfrak{D}[y(\tau)] .$$

Since the classical action S_{cl} of the harmonic oscillator has already been calculated at the beginning of this chapter, we can write

$$S_{cl} = \frac{m\omega}{2 \sin \omega t}\left\{\cos \omega t\left(x'^2 + x^2\right) - 2xx'\right\} .$$

We have therefore to evaluate the remaining path integral

$$\int_0^0 \exp\left[\frac{i}{\hbar}\int_0^t \frac{m}{2}\left(\dot{y}^2 - \omega^2 y^2\right)d\tau\right]\mathcal{D}\left[y(\tau)\right],$$

which we will later call F(t). There are two methods of evaluating F(t). One is to expand y(τ) as a Fourier since series,

$$y(\tau) = \sum_n a_n \sin\frac{n\pi\tau}{t},$$

and then consider the paths as functions of the coefficients a_n instead of functions of y at any particular time τ . The details of this method for evaluating F(t) is given by Feynman and Hibbs. The result is

$$F(t) = \left[\frac{m\omega}{2\pi i\hbar \sin\omega t}\right]^{1/2}.$$

Thus the time-dependent Green function of the harmonic oscillator is

$$G(\vec{r},\vec{r}';t) = \left[\frac{m\omega}{2\pi i\hbar \sin\omega t}\right]^{1/2}\cdot\exp\left[\frac{i}{\hbar}\frac{m\omega}{2\sin\omega t}\left\{\cos\omega t\,(x'^2+x^2)+2x'x\right\}\right].$$

In the next chapter, the Feynman averaged propagator of two-dimensional electron gas with random potential in transverse magnetic field will be investigated.