## CHAPTER V

## LOCAL CONVEXITY

Definition 5.1 Let X be a TVS($\mathbb{H}$). X is locally convex (lc) if each neighberhood of 0 includes a convex neighberhood of 0.

Example 5.2 (1) Every normed space over $\mathbb{H}$ is a locally convex space over $\mathbb{H}$.

(2) Let $\ell^{\frac{1}{2}}_{\mathbb{H}} = \{ z = (z_n)_{n \in \mathbb{N}} \mid z_n \in \mathbb{H}$ and $\sum_{n=1}^{\infty} |z_n|^{\frac{1}{2}} < \infty \}$

with $\| z \|_{\frac{1}{2}} = \sum_{n=1}^{\infty} |z_n|^{\frac{1}{2}}$. We shall show that $\ell^{\frac{1}{2}}_{\mathbb{H}}$ is not locally convex.

Claim that $\| \cdot \|_{\frac{1}{2}}$ is a paranorm on $\ell^{\frac{1}{2}}_{\mathbb{H}}$. Clearly, $\|0\|_{\frac{1}{2}} = 0$. Let $z \in \ell^{\frac{1}{2}}_{\mathbb{H}}$. Then $\| z \|_{\frac{1}{2}} = \sum_{k=1}^{\infty} |z_k|^{\frac{1}{2}} = \sum_{k=1}^{\infty} |-z_k|^{\frac{1}{2}} = \|-z\|_{\frac{1}{2}}$. Let $w, z \in \ell^{\frac{1}{2}}_{\mathbb{H}}$. Then $\| w + z \|_{\frac{1}{2}} = \sum_{k=1}^{\infty} |w_k + z_k|^{\frac{1}{2}} \leq \sum_{k=1}^{\infty} |w_k|^{\frac{1}{2}} + \sum_{k=1}^{\infty} |z_k|^{\frac{1}{2}} = \| w_k\|_{\frac{1}{2}} + \|z_k\|_{\frac{1}{2}}$.

let $(t_k)$ be a sequence in $\mathbb{H}$ such that $t_k \to t$ for some $t \in \mathbb{H}$ and let $(z_k)_{k \in \mathbb{N}} \subseteq \ell^{\frac{1}{2}}_{\mathbb{H}}$ be a sequence such that $\| z_k - z \|_{\frac{1}{2}} \to 0$ for some $z \in \ell^{\frac{1}{2}}_{\mathbb{H}}$.

Then $\|t_k z_k - t z\|_{\frac{1}{2}} = \sum_{n=1}^{\infty} | t_k z_k^n - t z^n|^{\frac{1}{2}} = \sum_{n=1}^{\infty} | (t_k - t)(z_k^n - z^n) + t(z_k^n - z^n) + (t_k - t) z^n |^{\frac{1}{2}} \leq \sum_{n=1}^{\infty} | t_k - t|^{\frac{1}{2}} |z_k^n - z^n|^{\frac{1}{2}} + |t|^{\frac{1}{2}} \sum_{n=1}^{\infty} |z_k^n - z^n|^{\frac{1}{2}} + \sum_{n=1}^{\infty} |t_k - t|^{\frac{1}{2}} |z^n|^{\frac{1}{2}}$

As $k \to \infty$, we get that $|t_k - t|^{\frac{1}{2}} \to 0$ and $|z_k^n - z^n|^{\frac{1}{2}} \to 0$, since the absolute value $| \cdot |$ is continuous and + is continuous. Hence $\|t_k z_k - t z\|_{\frac{1}{2}} \to 0$ as $k \to \infty$ so we have the claim. It is clear that $\| \cdot \|_{\frac{1}{2}}$

is not a seminorm. Let $U = \{ z \in \ell^{1/2}_{\mathbb{H}} \mid \| z \|_{1/2} \leq 1 \}$. We mustshow that U includes no convex neighberhood of 0. Let $V \in N(\ell^{1/2}_{\mathbb{H}})$ be convex. Then there exists an $\varepsilon > 0$ auch that $V \supseteq \{ z \mid \| z \|_{1/2} \leq \varepsilon \}$. Let $\varepsilon' = \varepsilon^2 \in \mathbb{H}$.

Choose $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \varepsilon^2$. Let $x_0 = \frac{1}{n_0} \sum_{k=1}^{n_0} \varepsilon' \delta^k$ where $\delta^k$ is the sequence z in $\ell^{1/2}_{\mathbb{H}}$ such that $z_k = 1$ and $z_\ell = 0$ if $\ell \neq k$. Since $\| \varepsilon'\delta^k \|_{1/2} = \sum_{k=1}^{\infty} \| \varepsilon' z_k \|^{1/2} = (\varepsilon^2)^{1/2} = \varepsilon \leq \varepsilon, \varepsilon'\delta^k \in \{ z \in \ell^{1/2}_{\mathbb{H}} \mid \| z \|_{1/2} \leq \varepsilon \}$ ;

hence $\varepsilon'\delta^k \in V$ for all $k \in \mathbb{N}$. Let $x_0 = \frac{1}{n_0} \sum_{k=1}^{n_0} \varepsilon'\delta^k = (\frac{\varepsilon'}{n_0}, \frac{\varepsilon'}{n_0}, \ldots, \frac{\varepsilon'}{n_0}, 0,0,0,\ldots)$. Claim that if V is convex then $\sum_{i=1}^{n} \lambda_i x_i \in V$ where $0 \leq \lambda_i \leq 1$ for all i, $\sum_{i=1}^{n} \lambda_i = 1$ and $x_i \in V$ for all $i \in \{1,2,3,\ldots,n\}$. We shall prove this by induction on n. If n = 1 then $1.x_i = x_i \in V$ ; so it is true for n = 1. Suppose that it has been show for $n - 1 \geq 1$ that every element $\sum_{i=1}^{n-1} \lambda_i x_i \in V$, where $\lambda_i \geq 0$ forall i and $\sum_{i=1}^{n-1} \lambda_i = 1$. Consider $\sum_{i=1}^{n} \lambda_i x_i$ where $\lambda_i \geq 0$ for all i and $\sum_{i=1}^{n} \lambda_i = 1$. Let $\alpha = \sum_{i=1}^{n-1} \lambda_i$.

<u>Case 1</u> $\alpha = 0$. Then $\lambda_i = 0$ for all $i \leq n - 1$. Hence $\sum_{i=1}^{n} \lambda_i x_i = \lambda_n x_n = 1.x_n = x_n \in V$.

<u>Case 2</u> $\alpha \neq 0$. By the induction hypothesis, $y \sum_{i=1}^{n-1} \frac{\lambda_i}{\alpha} x_i \in V$.

Since V is convex, $\sum_{i=1}^{n} \lambda_i x_i = \alpha y + (1 - \alpha) x_n \in V$ so we have the claim.

Since $\delta^k \in V$ for all $k = 1,2,\ldots,n_0$ and $\sum_{k=1}^{n_0} \frac{1}{n_0} = 1$ by the previous claim, $X_0 = \frac{1}{n_0} \sum_{k=1}^{n_0} \delta^k = \sum_{k=1}^{n_0} \frac{1}{n_0} ( \delta^k ) \in V$. Thus $\| x_0 \|_{\frac{1}{2}} = \sum_{k=1}^{n_0} (\frac{1}{n_0})^{\frac{1}{2}}$ $= n_0 (\frac{1}{n_0^{\frac{1}{2}}}) = n_0^{\frac{1}{2}} > 1$ so $x_0 \in U$ therefore $V \subseteq U$. But V was arbitary ; so U includes no convex neighberhood of therefore $\ell^{\frac{1}{2}}_{\mathbb{H}}$ is not locally convex. #

<u>Definition 5.3</u> Let X be a TVS($\mathbb{H}$). Then $B \subseteq X$ is called a <u>barrel</u> if and only if B is a balanced convex absorbing closed subset of X.

<u>Theorem 5.4</u> Let X be a locally convex TVS($\mathbb{H}$). Then X has a local base of neighberhoods of 0 which are barrels.

<u>Proof</u> : Let $U \in N(X)$ be closed and convex. Since $U \in N(X)$, there exists a balanced neighberhood W of 0 such that $W \subseteq U$. Let V = the convex hull of W. Then $V \in N(X)$, V is balanced, V is convex and $V \subseteq U$. Since U is closed, $\bar{V} \subseteq U$ so $\bar{V}$ is balanced, convex and closed. Since $\bar{V} \in N(X)$, by Theorem 3.18, $\bar{V}$ is absorbing therefore $\bar{V}$ is barrel ; hence X has a local base of neighberhoods of 0 which are barrels. #

<u>Lamma 5.5</u> Let X be a vector space over $\mathbb{H}$ and B a filter base of covex sets such that for each $U \in B$, $\frac{1}{2}$ U includes a member of B. Then B is additive.

<u>Proof</u> : Let $U \in B$. By assumption, $\frac{1}{2} U \subseteq V$ for some $V \in B$. Since U is covex, $U \subseteq \frac{1}{2}U + \frac{1}{2}U \subseteq V + V$. So B is additive. #

<u>Theorem 5.6</u> Let X be a locally convex Space over $\mathbb{H}$ and B a totally bounded subset of X. Then the balanced convex closure of B is also totally bounded.

Proof : Claim that if B is totally bounded then $\bar{B}$ is totally bounded. Let $U \in N(X)$. Choose a closed set $V \in N(X)$ such that $V \subseteq U$. Since B is totally bounded, $B = \bigcup_{j=1}^{n} B_j$ where $B_j$ is small of order V for all $j = 1,2,\ldots,n$. Since $B_j$ is small of order V, $B_j - B_j \subseteq V$ for all $j = 1,2,\ldots,n$. Hence, by Theorem 3.24, $\bar{B}_j - \bar{B}_j \subseteq \overline{B_j - B_j} \subseteq \bar{V} = V$ for all $j = 1,2,\ldots,n$ so $\bar{B}_j$ is small of order V for each $j = 1,2,\ldots,n$ Since $V \subseteq U$, $\bar{B}_j$ is small of order U for each $j = 1,2,\ldots,n$. But $\bar{B} = \overline{\bigcup_{j=1}^{n} B_j} = \bigcup_{j=1}^{n} \bar{B}_j$ where $\bar{B}_j$ is small of order U for each $j = 1,2,\ldots,n$ thus $\bar{B}$ is totally bounded and we have the claim.

Let $B_1$ be the balanced convex hull of B, that is, for all $x \in B_1$, $x = \sum_{i=1}^{s} r_i b_i$ where $r_i \in \mathbb{H}$, $b_i \in B$ and $\sum_{i=1}^{s} |r_i| \leq 1$, $i = 1,2,\ldots,s$. We must show that $B_1$ is totally bounded. If we can show that $B_1$ is totally bounded then $\bar{B}_1$ is also totally bounded by the claim. Hence the balanced covex closure of B is also totally bounded. Let $U \in N(X)$. We must show that there exists a finite set $G \subseteq X$ such that $B_1 \subseteq G + U$. Since X is a lc space over $\mathbb{H}$, there exists a balanced convex set $V \in N(X)$ such that $V + V \subseteq U$. Since B is totally bounded, there exists a finite set $F \subseteq X$, say $F = \{f_1, f_2, \ldots, f_n\}$ such that $B \subseteq F + V$. Since F is bounded, there exists an $m \in \mathbb{N}$ such that $F \subseteq mV$. Let $t \in \mathbb{H}^n$. Define $\|t\|_1 = \sum_{j=1}^{n} |t_j|$ . Let D be the unit disc in $\mathbb{H}^n$ ; that is, $D = \{z \in \mathbb{H}^n \mid \|z\|_1 \leq 1\}$. Then D is compact in $\mathbb{H}^n$ so D is totally bounded. Hence there exists a finite set $A \subseteq \mathbb{H}^n$ such that $D \subseteq A + \{z \in \mathbb{H}^n \mid \|z\|_1 \leq \frac{1}{m}\}$. Then for each $t \in D$, $\|t - a\| \leq \frac{1}{m}$ for some $a \in A$.

Let $G = \{ \sum_{i=1}^{n} a_i f_i \mid a = (a_1, a_2, \ldots, a_n) \in A \}$. Then G is a finite set. Claim that $B_1 \subseteq G + U$. Let $x \in B_1$. Since $B_1$ is the balanced convex hull of B, $x = \sum_{i=1}^{s} s_i b_i$ where $r_i \in \mathbb{H}$, $b_i \in B$ and $\sum_{i=1}^{s} |r_i| \leq 1$, $i = 1,2,\ldots,s$. Since $B \subseteq F + V$, $b_i = f_{j(i)} + \upsilon_i$ for some $f_{j(i)} \in F$ and $\upsilon_i \in V$, $i = 1,2,\ldots,s$. Let $w = \sum_{i=1}^{s} r_i f_{j(i)}$ and $\upsilon = \sum_{i=1}^{s} r_i \upsilon_i$. Then $x = \sum_{i=1}^{s} r_i b_i = \sum_{i=1}^{s} r_i (f_{j(i)} + \upsilon'_i) = \sum_{i=1}^{s} r_i f_{j(i)} + \sum_{i=1}^{s} r_i \upsilon_i = w + \upsilon$. Since V is balanced and convex, $\upsilon_i \in V$ and $\sum_{i=1}^{s} |r_i| \leq 1$, $\upsilon = \sum_{i=1}^{s} r_i \upsilon_i \in V$. We must show that $w \in G + V$. Write $w = \sum_{i=1}^{n} q_i f_i$ where each $q_i = 0$ or $q_i$ is the sum of the $r_j$ and each $r_j$ appears exacty once. Thus $\sum_{i=1}^{n} |q_i| \leq \sum_{i=1}^{s} |r_i| \leq 1$, so $q = (q_1, q_2, \ldots, q_n) \in D$. Choose $a = (a_1, a_2, \ldots, a_n) \in A$ such that $\|q - a\| \leq \frac{1}{m}$. Then $w = \sum_{i=1}^{n} q_i f_i = \sum_{i=1}^{n} a_i f_i + \sum_{i=1}^{n} (q_i - a_i) f_i$. Let $g = \sum_{i=1}^{n} a_i f_i$ and $\upsilon_1 = \sum_{i=1}^{n} (q_i - a_i) f_i$ so $w = g + \upsilon_1$. Clearly $g \in G$. We must show that $\upsilon_1 \in V$. Write $\upsilon_1 = \sum_{i=1}^{n} (q_i - a_i) f_i = \sum_{i=1}^{n} m(q_i - a_i)(\frac{f_i}{m})$. Since $F \subseteq mV$, $f_i/_m \in V$ for each i. $\sum_{i=1}^{n} |m(q_i - a_i)| = m \sum_{i=1}^{n} |q_i - a_i| = m\|q - a\|_1 \leq m(^1/m) = 1$. Since V is balanced and covex, $f_i/_m \in V$ for each i and $\sum_{i=1}^{n} |m(q_i - a_i)| \leq 1$, $\upsilon_1 = \sum_{i=1}^{n} m(q_i - a_i)(^{f_i}/_m) \in V$. Hence $x = w + \upsilon \in G + V + V \subseteq G + U$. Since $x \in B_1$ was arbitrary, $B_1 \subseteq G + U$ so we have the claim. #

Lemma 5.7 Let $(X, \|\cdot\|)$ be a seminormed space over $\mathbb{H}$. Then $\|\cdot\|$ is continuous if and only if $\{x \mid \|x\| \le 1\} \in N(X)$.

Proof : $(\Longrightarrow)$ The statement is clearly true.

$(\Longleftarrow)$ For $r > 0$, let $U_r = \{x \mid \|x\| \le r\}$. Let $(x_\delta)_{\delta \in D}$ be a net in X such that $x_\delta \to 0$. Since $U_\varepsilon = \varepsilon U_1 \in N(X)$ for all $\varepsilon > 0$, there exists a $\delta \in D$ such that $\delta' \ge \delta$ implies that $x_{\delta'} \in U_\varepsilon$ so $\|x_{\delta'}\| \le \varepsilon$ for all $\delta' \ge \delta$. Now $\|x_{\delta'}\| = \|x_{\delta'}\| - \|0\| = \|x_{\delta'}\| - 0 \le$ so $\|x_\delta\| \to 0$ ; hence $\|\cdot\|$ is continuous at 0. Let $a \in X$. Let $(x_\delta)_{\delta \in D}$ be a net in X such that $X_\delta \to a$. We must show that $\|x_\delta\| \to \|a\|$. Since $x_\delta \to a$, there exists a $\delta \in D$ such that for all $\delta' \in D$, $\delta' \ge \delta$ implies that $x_{\delta'} - a \; U_{\varepsilon/2}$. Hence $\big| \|x_\delta\| - \|a\| \big| \le \|x_\delta - a\| \le \varepsilon_{/2} < \varepsilon$ so $\|x_\delta\| \to \|a\|$ therefore $\|\cdot\|$ is continuous at a. Since a was arbitrary, $\|\cdot\|$ is continuous on X. #

Theorem 5.8 Let $(X, T)$ be a locally convex TVS($\mathbb{H}$). Then there exists a set P of seminorms such that $T = \sigma p$.

Proof : Let P be the set of continuous seminorms. Since $0 \in P$, $P \neq \emptyset$. To show $\sigma P \subseteq T$. Let $(x_\delta)_{\delta \in D}$ be a net in $(X, T)$ such that $x_\delta \to 0$. Then $p(x_\delta) \to 0$ for each $p \in P$ so $x_\delta \to 0$ in $\sigma P$ and therefore by Corollary 1.17, $\sigma P \subseteq T$. We must show that $T \subseteq \sigma P$. Let $(x_\delta)_{\delta \in D}$ be a net in $(X, \sigma P)$ such that $x_\delta \to 0$. Let $U \in N(X)$ be a balanced convex set. Let p be the gauge of U. Since $\{x \mid p(x) \le 1\} \in N(X)$, by Lemma 5.7, p is continuous. Then there exists a $\delta \in D$ such that $\delta' \ge \delta$ implies that $p(x_{\delta'}) < 1$ so $x_{\delta'} \in U$ for all $\delta' \ge \delta$ therefore $x_{\delta'} \to 0$ in $(X, T)$. By Corallary 1.17, $T \subseteq \sigma p$ hence $T = \sigma p$. #

Notation : We write (X,P) for any locally convex TVS(ℍ) where topology is $\sigma P$, where P is the family of continuous seminorms.

Theorem 5.9 Let P be a family of seminorms on a vector space X over ℍ. Then for any $U \in N(X, \sigma P)$ there exists an $\varepsilon > 0$ and $p_1, p_2, \ldots p_n \in P$ such that $U \supseteq \bigcap_{i=1}^{n} \{x | p_i(x) < \varepsilon\}$.

Proof : For $\varepsilon > 0$, let $B(\varepsilon) = \{x | p(x) < \varepsilon\}, p \in P$. Then $\{B(\varepsilon)\}_{\varepsilon \in \mathbb{R}^+}$ forms a subbase of neighberhoods of 0 for $(X, \sigma P)$. Let B be the the set of finite intersections of such sets in $\{B(\varepsilon)\}_{\varepsilon \in \mathbb{R}^+}$. Then B is a base for $N(\sigma P)$. So $U \supseteq \bigcap_{i=1}^{n} \{x | p_i(x) < \varepsilon_i\}$. for some $p_i \in P$, $\varepsilon_i > 0$, $i = 1,2,\ldots,n$. Let $\varepsilon = \min\{\varepsilon_i\}$. Then

$$U \supseteq \bigcap_{i=1}^{n} \{x | p_i(x) < \varepsilon_i\} \supseteq \bigcap_{i=1}^{n} \{x | p_i(x) < \varepsilon\}. \quad \#$$

Theorem 5.10 Let (X,P) be a lc space over ℍ and $f \in X^{\#}$. Then $f \in X'$ if ans only if there exists an $M > 0$ and $p_1, p_2, \ldots p_n \in P$ such that $|f(x)| \leq M \sum_{i=1}^{n} p_i(x)$ for all x. If P is the set of all continuous seminorm, $|f| \leq p$ for some $p \in P$.

Proof : ($\Rightarrow$) Suppose that $f \in X'$. Then $\{x | |f(x)| \leq 1\} \in N(X)$ so by Theorem 5.9, there exists an $\varepsilon > 0$ and $p_1, p_2, \ldots p_n \in P$ such that $\{x | |f(x)| \leq 1\} \supseteq \bigcap_{i=1}^{n} \{x | p_i(x) < \varepsilon\}$. Let $p = \sum_{i=1}^{n} p_i$. Let $x \in X$ be such that $p(x) < \varepsilon$. Then $x \in \bigcap \{x | p_i(x) < \varepsilon\}$ so $|f(x)| \leq 1$. We must show that $|f(x)| \leq \frac{1}{\varepsilon} \sum_{i=1}^{n} p_i(x)$ for all $x \in X$. Suppose that there exists an $x \in X$ such that $|f(x)| > \frac{1}{\varepsilon} \sum_{i=1}^{n} p_i(x)$. Then there exists a $t > 0$ such that $|f(x)| > t > \frac{1}{\varepsilon} \sum_{i=1}^{n} p_i(x)$. Let $y = (\frac{1}{t})\, x$. Then $p_i(\frac{x}{t}) = \frac{1}{t}\, p_i(x) \leq \frac{1}{t} \sum_{i=1}^{n} p_i(x) < \varepsilon$. But $|f(\frac{x}{t})| = \frac{1}{t}\, |f(x)| > 1$, a contradiction, hence $|f(x)| \leq \frac{1}{\varepsilon} \sum_{i=1}^{n} p_i(x)$ for all $x \in X$.

($\Longleftarrow$) Let $\varepsilon > 0$ be given. By assumption, there exists an M and $p_1, p_2, \ldots, p_n \in p$ ruch that $|f(x)| \leq M \sum_{i=1}^{n} p_1(x)$ for all $x \in X$. Let $p = \sum_{i=1}^{n} p_i$. Let $x \in X$ be such that $p(x) < \frac{\varepsilon}{1+|M|}$. Then $|f(x)| \leq M \frac{\varepsilon}{1+|M|} < \varepsilon$ so f is continuous at 0. Since f is linear and continuous at 0, f is continuous everywhere. #

<u>Theorem 5.11</u> Let (X, P) be a locally convex space over $\mathbb{H}$. Then $S \subseteq X$ is bounded if and only if p(S) is bounded for each $p \in P$.

<u>Proof</u> : ($\Longrightarrow$) Suppose that S is bounded. Let $p \in P$. We must show that p(S) is bounded : Since p is a seminorm, $U = \{ x | p(x) < 1\} \in N(X)$. Since S is bounded, there exists an $m \in \mathbb{N}$ such that $S \subseteq mU$. Then $p(x) < m$ for all $x \in S$ so p(S) is bounded.

($\Longleftarrow$) Let $(x_n)$ be a sequence in S. Then, for each $p \in P$, $p({}^{x_n}/_n) = p^{(x_n)}/_n \to 0$, since S is bounded ; hence ${}^{x_n}/_n \to 0$, by Theorem 3.29. #

<u>Definition 5.12</u> Let X, Y be vector spaces over $\mathbb{H}$ and A a set of maps from X into Y. A is called <u>total over X</u> if and only if $f(x) = 0$ for all $f \in A$ implies that $x = 0$.

Theorem 5.13  Let (X, P) be a lc space over $\mathbb{H}$. Then X is separated if and only if P is total.

Proof : ($\Leftarrow$) Let y = 0. Since P is total, there exists a $p \in P$ such that $p(y) > 0$. Let $U = \{x | p(x) < (\frac{1}{2})p(y)\}$. Then $U \in N(X)$ and $y \in U$. So X is separated.

($\Rightarrow$) Suppose that P is not total. Then there exists an $x \in X$ such that $x = 0$ and $p(x) = 0$ for all $p \in P$. By Theorem 5.9, $x \in U$ for all $U \in N(X)$. By Theorem 3.22, X is not separated. #

Corallary 5.14  Let (X,P) be a locally convex separated space over $\mathbb{H}$. Then $X'$ is total over X.

Proof : Let $x \in X \setminus \{0\}$. By Theorem 5.13, since X is separated P is total. Hence there exists a $p \in P$ such that $p(x) = 0$. Let $y = (X, p)$. Then Y is seminormed space over $\mathbb{H}$. By Theorem 3.6, since $\{0\}$ is closed and $x \notin \{0\}$, there exists an $f \in Y'$ such that $f(x) = 1 \neq 0$. Since $\sigma p \supseteq \sigma p$, $f \in X'$ so $X'$ is total . #

Theorem 5.15  Let (X, P) be a locally convex space over $\mathbb{H}$ and $f \in S'$, where S is a vector subspace of X. Then there exists an $F \in X'$ such that $F = f$ on S.

Proof : Let $Q = \{ p|_S | p \in P \}$. By Theorem 5.10, since $f \in S'$, there exist an $M > 0$ and $q_1, q_2, \ldots, q_n \in Q$ such that $|f(x)| \leq M \sum_{i=1}^{n} q_i(x)$ for all $x \in S$ where $q_i = p_i|_S$, $p_i \in P$. Let $p(x) = M \sum_{i=1}^{n} p_i(x)$ for all $x \in X$. By Theorem 2.7, f can be extended to $F \in X^{\#}$ with $|F(x)| \leq p(x)$ for all $x \in X$ ; hence, by Theorem 5.10, F is continuous. #

<u>Corollary 5.16</u> Let $X$ be a locally convex space over $\mathbb{H}$, $S$ a subspace and $x_0 \in X \setminus \bar{S}$. Then there exists an $F \in X'$ such that $F(x_0) = 1$ and $F = 0$ on $S$.

<u>Proof</u> : Defince $f : S + \langle x_0 \rangle \to \mathbb{H}$ as follows : let $x \in S + \langle x_0 \rangle$ Then $x = s + tx_0$ for some $s \in S$ and $t \in \mathbb{H}$. Define $f(x) = t$. Since $x_0 \notin \bar{S}$ a subspace, $x_0 = 0$ ; hence $f$ is well-defined.
$\ker f = \{x \in s + \langle x_0 \rangle \mid f(x) = 0\} = S$. Since $x_0 \notin \bar{S}$, $\ker f = S$ is not dense in $S + \langle x_0 \rangle$. We want to show that $\ker f$ is closed. Suppose that $\ker f = S$ is not closed. Then there exists a $p \in \bar{S}$ such that $p \notin S$. Hence $f(p) \neq 0$ and $f(\frac{p}{f(p)}) = \frac{f(p)}{f(p)} = 1$ so $f(x_0 - \frac{p}{f(p)}) = f(x_0) - f(\frac{p}{f(p)}) = 1 - 1 = 0$ Hence $x_o - \frac{p}{f(p)} \in S$ so $x_0 - \frac{p}{f(p)} \in \bar{S}$. Since $p \in \bar{S}$ and $\bar{S}$ is a subspace of $X$, $\frac{p}{f(p)} \in \bar{S}$ therefore $x_0 \in \bar{S}$, a contradction. Thus $\ker f$ is closed so by Theorem 3.47, $f$ is continuous. By Theorem 5.15, there exists an $F \in X'$ such that $F = f$ on $S + \langle x_0 \rangle$ so $F(x_0) = 1$ and $F = 0$ on $S$. #

<u>Corollary 5.17</u> Let $X$ be a locally convex space over $\mathbb{H}$. Then $S$ is fundamental if and only if for all $f \in X'$, $f = 0$ on $S$ implies that $f = 0$.

<u>Proof</u> : ($\Leftarrow$) Suppose that $S$ is not fundamental. Let $x \in X \setminus \overline{\langle S \rangle}$. By Corollary 5.16, there exists an $F \in X'$ such that $F(x) = 1$ and $F = 0$ on $\overline{\langle S \rangle}$ so $F = 0$ on $S$ but $F \neq 0$.

($\Rightarrow$) Suppose that $S$ is fundamental. Let $f \in X'$ be such that $f = 0$ on $S$. We must show that $f = 0$. Let $x \in X$. Then $x \in \overline{\langle S \rangle}$ hence either $x \in \langle s \rangle$ or $x$ is a cluster point of $\langle s \rangle$. If $x \in \langle S \rangle$ then $x = \sum_{i=1}^{n} \lambda_i s_i$ for some $\lambda_i \in \mathbb{H}$ and $s_i \in S$, $i = 1,2,\ldots,n$.

Then $f(x) = f\left(\sum_{i=1}^{n} \lambda_i s_i\right) = \sum_{i=1}^{n} \lambda_i f(s_i) = 0$. If $x$ is a cluster point of $<S>$ then there exists a net $(x_\delta)_{\delta \in D}$ in $<S>$ such that $x_\delta \to x$. Since $f$ is continuous, $f(x_\delta) \to f(x)$. Since $x_\delta \in <S>$ for all $\delta \in D$, $x_\delta = \sum_{i=1}^{n} t_\delta^i s_\delta^i$ for some $t_\delta^i \in \mathbb{H}$ and $s_\delta^i \in S$. Then $f(x_\delta) = f\left(\sum_{i=1}^{n} t_\delta^i s_\delta^i\right) = \sum_{i=1}^{n} t_\delta^i f(s_\delta^i) = 0$. Hence $f(x) = 0$. So $f = 0$. #

Theorem 5.18 Let $(Y, P)$ be a locally convex TVS($\mathbb{H}$) and $X$ a TVS($\mathbb{H}$). A linear map $f : X \to Y$ is continuous if and only if $p \circ f$ is continuous for each $p \in P$.

Proof : ($\Longrightarrow$) $p \circ f$ is the composition of continuous maps for each $p \in P$.

($\Longleftarrow$) Let $(x_\delta)_{\delta \in D}$ be a net in $X$ such that $x \longrightarrow 0$. Since $p \in P$ is continuous, $p(f(x_\delta)) \to 0$ ; hence, by Theorem 3.11, $f(x_\delta) \to 0$ in $P$. Thus $f$ is continuous at $x = 0$ ; hence everywhere. #

Lemma 5.19 Let $X$ be a vector space over $\mathbb{H}$. Let $p,q$ be seminorms on $X$. Let $f \in X^{\#}$ satisfy $|f(x)| \leq p(x) + q(x)$ for all $x$. Then there exist $q, h \in X^{\#}$ with $|g(x)| \leq p(x)$, $|h(x)| \leq g(x)$ and $f = g + h$.



Proof : Let $Y = X \times X$. Define $r : Y \to \mathbb{H}$ by $r(x,y) = p(x) + q(y)$. Then $r$ is a seminorm on $Y$. Let $D = \{(x, x) \mid x \in X\}$. Define $U : D \to \mathbb{R}$ by $u(x, x) = f(x)$. Since $|f(x)| \leq p(x) + q(x)$ for all $x$, $|u(x, x)| \leq r(x, x)$ for all $x$. By Theorem 2.7, we can extend $u$ to $Y$ with $|u(x, y)| \leq r(x, y)$. Then $f(x) = u(x, 0) + u(0, x)$, $|u(x, 0)| \leq r(x, 0) = p(x)$ and $|u(0, x)| \leq r(0, x) = q(x)$. Let $g(x) = u(x, 0)$ and $h(x) = u(0,x)$. Then $g, h \in X^{\#}$ with $|g(x)| \leq p(x)$, $|h(x)| \leq q(x)$

and $f = g + h$. #

<u>Theorem 5.20</u> Let $\Phi$ be a collection of locally convex topologies on a vector space X over $\mathbb{H}$. Then $f \in (x, v\Phi)'$ if and only if there exists $T_1, T_2, \ldots, T_n \in \Phi$ ; $g_1, g_2, \ldots g_n \in X^{\#}$ such that each $g_i \in (X, T_i)'$ and $f = \sum_{i=1}^{n} g_i$.

<u>Proof</u> : ($\Longleftarrow$) Since $v\Phi \supseteq T_i$ for all $i = 1,2,\ldots,n$ and $g_i \in (x, v\Phi)'$, $f = \sum_{i=1}^{n} g_i \in (X, v\Phi)'$.

($\Longrightarrow$) Let $P(T)$ be the set of all continuous seminorms on $(X, T)$ for each $T \in \Phi$. By theorem 5.8, $(X, T) = (X, P(T))$ for all $T \in \Phi$. By Theorem 5.10, there exist $p_1, p_2, \ldots p_n \in P(T)$ and an $M > 0$ such that $|f(x)| \leq M \sum_{i=1}^{n} p_i(x)$ for all x. By Lemma 5.19, there exist $g_1, g_2, \ldots, g_n \in X^{\#}$ and $T_1, T_2, \ldots T_n \in \Phi$ such that $g_i \in X^{\#}$ for all i such that $f = \sum_{i=1}^{n} g_i$ and $|g_i(x)| \leq p_i(x)$ for all x. Since $p_i$ is continuous for all i and $|g_i(x)| \leq p_i(x)$ for all x, $g_i$ is continuous for all i. Hence $g_i \in (X, T_i)'$ is such that $f = \sum_{i=1}^{n} g_i$. #