CHAPTER III

TRANSFORMATION SEMIGROUPS

First, we recall for the following notation of transformation semigroups: For any set X, let

P_X = the partial transformation semigroup on X,

T_X = the full transformation semigroup on X,

I_X = the 1-1 partial transformation semigroup on X (the symmetric inverse semigroup on X),

G_X = the symmetric group on X,

CP_X = the transformation semigroup of all constant partial transformations of X (including 0),

CT_X = the transformation semigroup of all constant transformations of X,

U_X = the transformation semigroup of all almost identical partial transformations of X,

V_X = the transformation semigroup of all almost identical transformations of X and

W_X = the transformation semigroup of all almost identical 1-1 partial transformations of X.

Since for any set X, I_X, G_X and W_X are inverse semigroups, we have by Theorem 1.3 that for any set X, I_X, G_X and W_X are absolutely closed.

It is known that for any set X, P_X and T_X are absolutely closed (see [2],[3],[4],[5] and [6]). The first main purpose of the research is to generalize this result by proving that for any set X, every ideal of P_X and every ideal of T_X is absolutely closed. This implies that for any set X, the transformation semigroups CP_X and CT_X are absolutely closed since CP_X and CT_X are ideals of P_X and T_X, respectively.

Our second main purpose is to prove that for any set X, the transformation semigroups U_X and V_X are absolutely closed.

The following notation will be used:

Let S be a semigroup. If S has an identity, set $S^1 = S$, and if S does not have an identity, let S^1 be the semigroup S with an identity adjoined, usually denoted by 1.

Let X be a set. For any nonempty subset A of X and for $x \in X$, let A_x denote the constant partial transformation of X with domain A and range $\{x\}$. Hence

$$CP_X = \{A_x \mid \emptyset \neq A \subseteq X,\ x \in X\} \cup \{0\},$$

and for $X \neq \emptyset$,

$$CT_X = \{X_x \mid x \in X\}.$$

If $X = \emptyset$, then $CT_X = \{0\}$.

We need the two following lemmas to prove that for any set X, every ideal of T_X is absolutely closed. The first lemma follows directly from the remark given in Chapter I, page 8.

Lemma 3.1. Let U be a subsemigroup of a semigroup S. Assume that

$$u_0 = x_1u_1,$$
$$x_iu_{2i} = x_{i+1}u_{2i+1}\,,\ u_{2i-1}y_i = u_{2i}y_{i+1} \quad (i = 1,\dots,m-1),$$
$$u_{2m-1}y_m = u_{2m}\,,$$

where $u_o, u_1,\dots,u_{2m} \in U$, $x_1,\dots, x_m$, $y_1,\dots, y_m \in S$.
Then $u_0y_1 = x_mu_{2m}$.

Lemma 3.2. Let X be a set and U a subsemigroup of T_X containing CT_X. Let S be a semigroup containing U as a subsemigroup. Let $q \in X$. For each $s \in S^1$, define $\varphi_s \in T_X$ as follows : For $p \in X$,

$$p\varphi_s = \begin{cases} p(X_ps) & \text{if } X_ps \in U\,, \\ q & \text{if } X_ps \notin U\,. \end{cases}$$

Then the following statements hold:

(1) For any $\alpha, \beta \in U$, $s, t \in S^1$, $\alpha s = \beta t$ implies $\alpha\varphi_s = \beta\varphi_t$.

(2) For any $\alpha, \beta \in U$, $t \in S^1$, $\alpha = \beta t$ implies $\alpha = \beta\varphi_t$.

Proof: To prove (1), let $\alpha, \beta \in U$ and $s, t \in S^1$ be such that $\alpha s = \beta t$. To show that $\alpha\varphi_s = \beta\varphi_t$ in T_X, let $p \in X$. Since X_p, $\alpha, \beta \in U \subseteq T_X$, $X_p\alpha = X_{p\alpha}$ and $X_p\beta = X_{p\beta}$. Therefore, we have that $X_{p\alpha}s = X_p\alpha s = X_p\beta t = X_{p\beta}t$ in S^1. It follows that if $X_{p\alpha}s \in U$, then

$$\begin{aligned}(p\alpha)\varphi_s &= (p\alpha)(X_{p\alpha}s)\\ &= ((p\beta)X_{p\alpha})(X_{p\alpha}s)\\ &= (p\beta)(X_{p\alpha}(X_{p\alpha}s))\\ &= (p\beta)(X_{p\alpha}s)\\ &= (p\beta)(X_{p\beta}t)\\ &= (p\beta)\varphi_t\end{aligned}$$

since $(p\beta)X_{p\alpha} = p\alpha$, $X_{p\alpha}X_{p\alpha} = X_{p\alpha}$ and $X_{p\alpha}s = X_{p\beta}t$. But

$$(p\alpha)\varphi_s = \begin{cases} (p\alpha)(X_{p\alpha}s) & \text{if } X_{p\alpha}s \in U, \\ q & \text{if } X_{p\alpha}s \notin U \end{cases}$$

and

$$(p\beta)\varphi_t = \begin{cases} (p\beta)(X_{p\beta}t) & \text{if } X_{p\beta}t \in U, \\ q & \text{if } X_{p\beta}t \notin U, \end{cases}$$

therefore $p(\alpha\varphi_s) = (p\alpha)\varphi_s = (p\beta)\varphi_t = p(\beta\varphi_t)$. Since p is an arbitary point in X, it follows that $\alpha\varphi_s = \beta\varphi_t$.

The statement (2) follows from (1) and the fact that $\varphi_1 = 1_X$, the identity map on X. #

<u>Theorem 3.3</u>. For any set X, every ideal of T_X is absolutely closed.

<u>Proof</u>: Let X be a set and I an ideal of T_X. For the case $X = \emptyset$, I is absolutely closed since $|I| = 1$. Assume that $X \neq \emptyset$. To prove that I is absolutely closed, let S be a semigroup containing I as a subsemigroup. Suppose that $\mathrm{Dom}(I,S) \neq I$. Let $d \in \mathrm{Dom}(I,S) \setminus I$. By Theorem 1.1, there exist $\alpha_0, \alpha_1, \ldots, \alpha_{2m} \in I$, $s_1, \ldots, s_m$, $t_1, \ldots, t_m \in S$ such that

$$\text{(I)} \quad \begin{cases} d = \alpha_0 t_1,\ \alpha_0 = s_1\alpha_1, \\ s_i\alpha_{2i} = s_{i+1}\alpha_{2i+1},\ \alpha_{2i-1}t_i = \alpha_{2i}t_{i+1} \quad (i=1,\ldots,m-1), \\ \alpha_{2m-1}t_m = \alpha_{2m}\ . \end{cases}$$

Then $d = s_m\alpha_{2m}$. Since for $\alpha \in I$ and for $x \in X$, $X_x = \alpha X_x \in I$, we have that $CT_X \subseteq I$. Let q be a fixed point in X. For each $s \in S^1$, define

$\varphi_s \in T_X$ as follows: For $p \in X$,

$$p\varphi_s = \begin{cases} p(X_p s) & \text{if } X_p s \in I, \\ q & \text{if } X_p s \notin I. \end{cases}$$

Since $\alpha_{2m} = \alpha_{2m-1}t_m$ and $\alpha_{2i-1}t_i = \alpha_{2i}t_{i+1}$ $(i = 1,\dots,m-1)$, by Lemma 3.2, we get that

$$\text{(II)} \begin{cases} \alpha_{2m} = \alpha_{2m-1}\varphi_{t_m} , \\ \alpha_{2i-1}\varphi_{t_i} = \alpha_{2i}\varphi_{t_{i+1}} \qquad (i = 1,\dots,m-1). \end{cases}$$

Since T_X is regular and $\alpha_{2m} \in I \subseteq T_X$, there exists $\beta \in T_X$ such that $\alpha_{2m} = \alpha_{2m}\beta\alpha_{2m}$. From (II), we have

$$\text{(III)} \begin{cases} \alpha_{2m} = \alpha_{2m-1}(\varphi_{t_m}\beta\alpha_{2m}) , \\ \alpha_{2i-1}(\varphi_{t_i}\beta\alpha_{2m}) = \alpha_{2i}(\varphi_{t_{i+1}}\beta\alpha_{2m}) \qquad (i = 1,\dots,m-1). \end{cases}$$

Since I is an ideal of T_X and $\alpha_{2m} \in I$, so $\varphi_{t_i}\beta\alpha_{2m} \in I \subseteq S$ for all $i \in \{1,\dots,m\}$. By (I) and (III), we have the following system of equalities:

$$\alpha_0 = s_1\alpha_1 ,$$

$$s_i\alpha_{2i} = s_{i+1}\alpha_{2i+1},$$

$$\alpha_{2i-1}(\varphi_{t_i}\beta\alpha_{2m}) = \alpha_{2i}(\varphi_{t_{i+1}}\beta\alpha_{2m}) \quad (i=1,\dots,m-1),$$

$$\alpha_{2m-1}(\varphi_{t_m}\beta\alpha_{2m}) = \alpha_{2m} .$$

By Lemma 3.1, $\alpha_0(\varphi_{t_1}\beta\alpha_{2m}) = s_m\alpha_{2m}$. Since $d = s_m\alpha_{2m}$, $d = \alpha_0(\varphi_{t_1}\beta\alpha_{2m})$.

Therefore $d \in I$ because $\alpha_0, \alpha_{2m} \in I$ and $\varphi_{t_1}, \beta \in T_X$. This is a contradiction. Hence $\mathrm{Dom}(I,S) = I$. This proves that I is absolutely closed as required. #

Let X be a set. Then T_X is an ideal of itself. The transformation semigroup CT_X is an ideal of T_X since $\alpha X_x = X_x$ and $X_x\alpha = X_{x\alpha}$ for all $\alpha \in T_X$, $x \in X$. Hence the two following corollaries are obtained directly from Theorem 3.3.

<u>Corollary 3.4</u>. For any set X, T_X is absolutely closed.

<u>Corollary 3.5</u>. For any set X, CT_X is absolutely closed.

Next, we shall prove that for any set X, every ideal of P_X is absolutely closed. To prove this, Lemma 3.1 and the following lemma are required:

<u>Lemma 3.6</u>. Let X be a set and U a subsemigroup of P_X containing CT_X. Let S be a semigroup containing U as a subsemigroup. For each $s \in S^1$, define $\psi_s \in P_X$ as follows: For $p \in X$,

$$p\psi_s = \begin{cases} p(X_p s) & \text{if } X_p s \in U \setminus \{0\}, \\ \text{undefined} & \text{if } X_p s \notin U \setminus \{0\}. \end{cases}$$

Then

(1) For each $\alpha, \beta \in U$, $s, t \in S^1$, $\alpha s = \beta t$ implies $\alpha\psi_s = \beta\psi_t$.

(2) For each $\alpha, \beta \in U$, $t \in S^1$, $\alpha = \beta t$ implies $\alpha = \beta\psi_t$.

Proof: To prove (1), let $\alpha, \beta \in U$ and $s, t \in S^1$ be such that $\alpha s = \beta t$. To show that $\Delta\alpha\psi_s \subseteq \Delta\beta\psi_t$, let $p \in \Delta\alpha\psi_s$. Then $p \in \Delta\alpha$ and $X_{p\alpha}s \in U\setminus\{0\}$. If $p \notin \Delta\beta$, then $X_p\beta = 0$, so $0t = X_p\beta t = X_p\alpha s = X_{p\alpha}s \in U$, and hence $0 = 0(0t) = 0t = X_{p\alpha}s$, a contradiction. Hence $p \in \Delta\beta$. It then follows that $X_{p\beta}t = X_p\beta t = X_p\alpha s = X_{p\alpha}s \in U\setminus\{0\}$. Therefore $p\beta \in \Delta\psi_t$, so $p \in \Delta\beta\psi_t$.

The proof that $\Delta\beta\psi_t \subseteq \Delta\alpha\psi_s$ can be given similarly. Hence $\Delta\alpha\psi_s = \Delta\beta\psi_t$.

Next, to show that $\alpha\psi_s = \beta\psi_t$, let $p \in \Delta\alpha\psi_s (= \Delta\beta\psi_t)$. Then $X_{p\alpha}s, X_{p\beta}t \in U\setminus\{0\}$. Since $\alpha s = \beta t$, it follows that $p(\alpha\psi_s) = (p\alpha)\psi_s = (p\alpha)(X_{p\alpha}s) = (p\alpha)(X_p\alpha s) = (p\alpha)(X_p\beta t) = (p\alpha)(X_{p\beta}t) = (p\alpha)(X_{p\beta}(X_{p\beta}t)) = ((p\alpha)X_{p\beta})(X_{p\beta}t) = (p\beta)(X_{p\beta}t) = (p\beta)\psi_t = p(\beta\psi_t)$. Therefore $\alpha\psi_s = \beta\psi_t$.

The statement (2) follows from (1) and the fact that $\psi_1 = 1_X$, the identity map on X. #

Theorem 3.7. For any set X, every ideal of P_X is absolutely closed.

Proof: Let X be a set and I an ideal of P_X. If $I = \{0\}$, then I is absolutely closed by Theorem 1.3.

Assume that $I \neq \{0\}$. Since for each $\alpha \in P_X\setminus\{0\}$, $X_x = X_p\alpha X_x$ for all $x \in X$ and $p \in \Delta\alpha$, it follows that $CT_X \subseteq I$. To show that I is absolutely closed, let S be a semigroup containing I as a subsemigroup. Suppose that $Dom(I,S) \neq I$. Let $d \in Dom(I,S)\setminus I$. By Theorem 1.1, there exist $\alpha_0, \alpha_1, \ldots, \alpha_{2m} \in I$, $s_1, \ldots, s_m$, $t_1, \ldots, t_m \in S$ such that

$$(\mathrm{I}) \quad \begin{cases} d = \alpha_0 t_1, \ \alpha_0 = s_1\alpha_1, \\ s_i\alpha_{2i} = s_{i+1}\alpha_{2i+1}, \ \alpha_{2i-1}t_i = \alpha_{2i}t_{i+1} \quad (i=1,\ldots,m-1), \\ \alpha_{2m-1}t_m = \alpha_{2m} . \end{cases}$$

Then $d = s_m\alpha_{2m}$. For each $s \in S^1$, define $\phi_s \in P_X$ as follows: For $p \in X$,

$$p\phi_s = \begin{cases} p(X_p s) & \text{if } X_p s \in I\setminus\{0\}, \\ \text{undefined} & \text{if } X_p s \notin I\setminus\{0\}. \end{cases}$$

Since $\alpha_{2m} = \alpha_{2m-1}t_m$, $\alpha_{2i-1}t_i = \alpha_{2i}t_{i+1}$ $(i = 1,\ldots,m-1)$ and $CT_X \subseteq I$, by Lemma 3.6, we get that

$$\text{(II)} \begin{cases} \alpha_{2m} = \alpha_{2m-1}\phi_{t_m}, \\ \alpha_{2i-1}\phi_{t_i} = \alpha_{2i}\phi_{t_{i+1}} \quad (i = 1,\ldots,m-1). \end{cases}$$

Since P_X is regular and $\alpha_{2m} \in I \subseteq P_X$, there exists $\beta \in P_X$ such that $\alpha_{2m} = \alpha_{2m}\beta\alpha_{2m}$. From (II), we have that

$$\text{(III)} \begin{cases} \alpha_{2m} = \alpha_{2m-1}(\phi_{t_m}\beta\alpha_{2m}), \\ \alpha_{2i-1}(\phi_{t_i}\beta\alpha_{2m}) = \alpha_{2i}(\phi_{t_{i+1}}\beta\alpha_{2m}) \quad (i = 1,\ldots,m-1). \end{cases}$$

Since I is an ideal of P_X and $\alpha_{2m} \in I$, so $\phi_{t_i}\beta\alpha_{2m} \in I \subseteq S$ for all $i \in \{1,\ldots,m\}$. By (I) and (III), we have the following system of equalities:

$$\alpha_0 = s_1\alpha_1,$$
$$s_i\alpha_{2i} = s_{i+1}\alpha_{2i+1},\ \alpha_{2i-1}(\phi_{t_i}\beta\alpha_{2m}) = \alpha_{2i}(\phi_{t_{i+1}}\beta\alpha_{2m})(i=1,\ldots,m-1),$$
$$\alpha_{2m-1}(\phi_{t_m}\beta\alpha_{2m}) = \alpha_{2m}.$$

By Lemma 3.1, $\alpha_0(\phi_{t_1}\beta\alpha_{2m}) = s_m\alpha_{2m}$. Since $d = s_m\alpha_{2m}$, $d = \alpha_0(\phi_{t_1}\beta\alpha_{2m})$.

Therefore $d \in I$ because $\alpha_0, \alpha_{2m} \in I$ and $\phi_{t_1}, \beta \in P_X$. It contradicts the choice of d. Hence Dom(I,S) = I. This proves that I is absolutely closed, as required. #

Let X be a set. Then P_X is an ideal of itself. It is easy to see that for $\alpha \in P_X$, $\emptyset \neq A \subseteq X$ and $x \in X$,

$$A_x\alpha = \begin{cases} 0 & \text{if } x \notin \Delta\alpha , \\ A_{x\alpha} & \text{if } x \in \Delta\alpha \end{cases}$$

and

$$\alpha A_x = \begin{cases} 0 & \text{if } \nabla\alpha \cap A = \emptyset , \\ (\Delta\alpha A_x)_x & \text{if } \nabla\alpha \cap A \neq \emptyset . \end{cases}$$

It then follows that CP_X is an ideal of P_X. Hence the two following corollaries are consequences of Theorem 3.7.

<u>Corollary 3.8</u>. For any set X, P_X is absolutely closed.

<u>Corollary 3.9</u>. For any set X, CP_X is absolutely closed.

Our next step of this chapter is to show that for any set X, the transformation semigroup of all almost identical partial transformations of X, U_X, and the transformation semigroup of all almost identical transformations of X, V_X, are absolutely closed.

Let X be a set.

Recall that for $\alpha \in P_X$, $S(\alpha) = \{x \in \Delta\alpha \mid x\alpha \neq x\}$

(the shift of α) and α is almost identical if and only if $S(\alpha)$ is finite,

$$U_X = \{\alpha \in P_X \mid S(\alpha) \text{ is finite}\}$$

and

$$V_X = \{\alpha \in T_X \mid S(\alpha) \text{ is finite}\}.$$

Observe that $V_X \subseteq U_X$.

It is known that for $\alpha \in P_X$, $\alpha^2 = \alpha$ if and only if $\nabla\alpha \subseteq \Delta\alpha$ and $x\alpha = x$ for all $x \in \nabla\alpha$.

The following fact which is required to use is easily seen: For $\alpha \in P_X$, if $S(\alpha)$ is finite, then $S(\alpha) \cup S(\alpha)\alpha$ is a finite subset of X. In general, if F is a finite subset of U_X or V_X, then $\bigcup_{\alpha\in F} (S(\alpha) \cup S(\alpha)\alpha)$ is a finite subset of X.

For convenience, we introduce the following notation: For $A \subseteq X$ and $x \in X$, let $\overline{A}_x \in T_X$ be defined by

$$y\overline{A}_x = \begin{cases} x & \text{if } y \in A, \\ y & \text{if } y \notin A. \end{cases}$$

For $A \subseteq X$ and $x \in X$, the following statements clearly hold:

(1) $\nabla\overline{A}_x = \{x\} \cup (X \setminus A)$.

(2) $$S(\overline{A}_x) = \begin{cases} A & \text{if } x \notin A, \\ A \setminus \{x\} & \text{if } x \in A. \end{cases}$$

(3) $\overline{A}_x \in V_X$ if and only if A is finite.

(4) $(\overline{A}_x)^2 = \overline{A}_x$.

(5) For $\alpha \in T_X$, if $S(\alpha) \subseteq A$, then $\overline{A}_x\alpha = \overline{A}_{x\alpha}$.

The following lemma is required to show that for any set X, V_X is absolutely closed.

Lemma 3.10. Let X be a set and S a semigroup containing V_X as a subsemigroup. Let F be a finite subset of V_X and $A = \bigcup_{\alpha\in F} (S(\alpha)\cup S(\alpha)\alpha)$ Let $q \in X$. For each $s \in S^1$, define $\varphi_s \in V_X$ as follows: For $p \in X$,

$$p\varphi_s = \begin{cases} p(\bar{A}_p s) & \text{if } p \in A \text{ and } \bar{A}_p s \in V_X , \\ q & \text{if } p \in A \text{ and } \bar{A}_p s \notin V_X , \\ p & \text{if } p \notin A. \end{cases}$$

Then:

(1) For each $\alpha, \beta \in F$, $s, t \in S^1$, $\alpha s = \beta t$ implies $\alpha\varphi_s = \beta\varphi_t$.

(2) For each $\alpha, \beta \in F$, $t \in S^1$, $\alpha = \beta t$ implies $\alpha = \beta\varphi_t$.

Proof: To prove (1), let $\alpha, \beta \in F$ and $s, t \in S^1$ be such that $\alpha s = \beta t$. Then $S(\alpha)\cup S(\alpha)\alpha \subseteq A$ and $S(\beta)\cup S(\beta)\beta \subseteq A$ which imply that $A\alpha \subseteq A$ and $A\beta \subseteq A$. Let $p \in X$. Then $(\bar{A}_{p\alpha})^2 = \bar{A}_{p\alpha}$.

Case 1: $p \in A$. Then $p\alpha, p\beta \in A$. Therefore we have that $(p\beta)\bar{A}_{p\alpha} = p\alpha$. Since $S(\alpha) \subseteq A$ and $S(\beta) \subseteq A$, $\bar{A}_p\alpha = \bar{A}_{p\alpha}$ and $\bar{A}_p\beta = \bar{A}_{p\beta}$. Thus $\bar{A}_{p\alpha}s = (\bar{A}_p\alpha)s = \bar{A}_p(\alpha s) = \bar{A}_p(\beta t) = (\bar{A}_p\beta)t = \bar{A}_{p\beta}t$. Therefore, if $\bar{A}_{p\alpha}s \in V_X$, then

$$\begin{aligned} (p\alpha)(\bar{A}_{p\alpha}s) &= ((p\beta)\bar{A}_{p\alpha})(\bar{A}_{p\alpha}s) \\ &= (p\beta)(\bar{A}_{p\alpha}(\bar{A}_{p\alpha}s)) \\ &= (p\beta)(\bar{A}_{p\alpha}s) \\ &= (p\beta)(\bar{A}_{p\beta}t) . \end{aligned}$$

Since $p\alpha, p\beta \in A$, we have that

$$(p\alpha)\varphi_s = \begin{cases} (p\alpha)(\bar{A}_{p\alpha}s) & \text{if } \bar{A}_{p\alpha}s \in V_X , \\ q & \text{if } \bar{A}_{p\alpha}s \notin V_X \end{cases}$$

and

$$(p\beta)\phi_t = \begin{cases} (p\beta)(\bar{A}_{p\beta}t) & \text{if } \bar{A}_{p\beta}t \in V_X , \\ q & \text{if } \bar{A}_{p\beta}t \notin V_X , \end{cases}$$

so $(p\alpha)\varphi_s = (p\beta)\varphi_t$. This implies that $p(\alpha\varphi_s) = p(\beta\varphi_t)$.

<u>Case 2: $p \notin A$</u>. Then $p\alpha = p = p\beta$. Since $p \notin A$, $p\varphi_s = p = p\varphi_t$. Hence $p(\alpha\varphi_s) = (p\alpha)\varphi_s = p\varphi_s = p = p\varphi_t = (p\beta)\varphi_t = p(\beta\varphi_t)$.

This proves that $\alpha\varphi_s = \beta\varphi_t$.

The statement (2) follows from (1) and the fact that $\varphi_1 = 1_X$, the identity map on X. #

<u>Theorem 3.11</u>. For any set X, V_X is absolutely closed.

<u>Proof</u>: Let X be a set. If $X = \emptyset$, then V_X is absolutely closed since $|V_X| = 1$. Assume that $X \neq \emptyset$. To prove that V_X is absolutely closed, let S be a semigroup containing V_X as a subsemigroup. Let $d \in \mathrm{Dom}(V_X,S)$. Since V_X has an identity, by Corollary 1.2, there exist $\alpha_0,\alpha_1,\ldots,\alpha_{2m} \in V_X$, $s_1,\ldots,s_m,t_1,\ldots,t_m \in S$ such that

$$(\mathrm{I}) \begin{cases} d = \alpha_0 t_1,\ \alpha_0 = s_1\alpha_1, \\ \qquad s_i\alpha_{2i} = s_i\alpha_{2i+1},\ \alpha_{2i-1}t_i = \alpha_{2i}t_{i+1} (i=1,\ldots,m-1), \\ \qquad\qquad \alpha_{2m-1}t_m = \alpha_{2m} . \end{cases}$$

Then $d = s_m \alpha_{2m}$. Let q be a fixed point in X. Let $F = \{\alpha_0, \alpha_1, \ldots, \alpha_{2m}\}$ and $A = \bigcup_{\alpha \varepsilon F} (S(\alpha) \cup S(\alpha)\alpha)$. For each $s \varepsilon S^1$, define $\varphi_s \varepsilon V_X$ as follows: For $p \varepsilon X$,

$$p\varphi_s = \begin{cases} p(\bar{A}_p s) & \text{if } p \varepsilon A \text{ and } \bar{A}_p s \varepsilon V_X, \\ q & \text{if } p \varepsilon A \text{ and } \bar{A}_p s \not\varepsilon V_X, \\ p & \text{if } p \not\varepsilon A. \end{cases}$$

Since $\alpha_{2i-1} t_i = \alpha_{2i} t_{i+1}$ $(i=1,\ldots,m-1)$ and $\alpha_{2m-1} t_m = \alpha_{2m}$, by Lemma 3.10, we get that

$$\text{(II)} \quad \begin{cases} \alpha_{2i-1}\varphi_{t_i} = \alpha_{2i}\varphi_{t_{i+1}} & (i = 1,\ldots,m-1), \\ \alpha_{2m-1}\varphi_{t_m} = \alpha_{2m} . \end{cases}$$

By (I) and (II), we obtain the following system of equalities:

$$\alpha_0 = s_1\alpha_1,$$

$$s_i \alpha_{2i} = s_{i+1}\alpha_{2i+1}, \quad \alpha_{2i-1}\varphi_{t_i} = \alpha_{2i}\varphi_{t_{i+1}} \qquad (i = 1,\ldots,m-1),$$

$$\alpha_{2m-1}\varphi_{t_m} = \alpha_{2m} .$$

Hence by Lemma 3.1, we get that $\alpha_0 \varphi_{t_1} = s_m \alpha_{2m}$. Since $d = s_m \alpha_{2m}$ and $\alpha_0, \varphi_{t_1} \varepsilon V_X$, $d = \alpha_0 \varphi_{t_1} \varepsilon V_X$, so $Dom(V_X, S) \subseteq V_X$. Hence $Dom(V_X,S) = V_X$. This proves that V_X is absolutely closed. #

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Next, we shall show that for any set X, U_X is absolutely closed. The following lemma is required:

<u>Lemma 3.12</u>. Let X be a set and S a semigroup containing U_X as a subsemigroup. Let F be a finite subset of U_X and $A = \bigcup_{\alpha \varepsilon F} (S(\alpha) \cup S(\alpha)\alpha)$.

For each $s \in S^1$, define $\psi_s \in U_X$ as follows: For $p \in X$,

$$p\psi_s = \begin{cases} p(A_p s) & \text{if } p \in A \text{ and } A_p s \in U_X \smallsetminus \{0\}, \\ p & \text{if } p \notin A \text{ and } \{p\}_p s = \{p\}_p , \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Then:

(1) For each $\alpha, \beta \in F$, $s, t \in S^1$, $\alpha s = \beta t$ implies $\alpha\psi_s = \beta\psi_t$.

(2) For each $\alpha, \beta \in F$, $t \in S^1$, $\alpha = \beta t$ implies $\alpha = \beta\psi_t$.

<u>Proof</u>: To prove (1), let $\alpha, \beta \in F$ and $s, t \in S^1$ be such that $\alpha s = \beta t$. Then $A\alpha \subseteq A$ and $A\beta \subseteq A$. To show that $\Delta\alpha\psi_s = \Delta\beta\psi_t$, it suffices to show that

$$\{p\varepsilon X | p\varepsilon\Delta\alpha \cap A \text{ and } p\alpha\varepsilon\Delta\psi_s\} = \{p\varepsilon X | p\varepsilon\Delta\beta \cap A \text{ and } p\beta\varepsilon\Delta\psi_t\}$$

and

$$\{p\varepsilon X | p\varepsilon\Delta\alpha \smallsetminus A \text{ and } p\alpha\varepsilon\Delta\psi_s\} = \{p\varepsilon X | p\varepsilon\Delta\beta \smallsetminus A \text{ and } p\beta\varepsilon\Delta\psi_t\} .$$

To show that $\{p\varepsilon X | p\varepsilon\Delta\alpha \cap A \text{ and } p\alpha\varepsilon\Delta\psi_s\} \subseteq \{p\varepsilon X | p\varepsilon\Delta\beta \cap A \text{ and } p\beta\varepsilon\Delta\psi_t\}$, let $p \in \Delta\alpha \cap A$ and $p\alpha \in \Delta\psi_s$. Then $p\alpha \in A$. Since $p\alpha \in \Delta\psi_s$, $A_{p\alpha} s \varepsilon U_X \smallsetminus \{0\}$. If $p \notin \Delta\beta$, then $A_p\beta = 0$, so $0t = A_p\beta t = A_p\alpha s = A_{p\alpha} s \in U_X$ and hence $0 = 0(0t) = 0t = A_{p\alpha} s$, a contradiction. Hence $p \in \Delta\beta$. Therefore $p \in \Delta\beta \cap A$. From $A_{p\beta} t = A_p\beta t = A_p\alpha s = A_{p\alpha} s \in U_X \smallsetminus \{0\}$, we have that $p\beta \in \Delta\psi_t$.

The proof that $\{p\varepsilon X | p\varepsilon\Delta\beta \cap A \text{ and } p\beta\varepsilon\Delta\psi_t\} \subseteq \{p\varepsilon X | p\varepsilon\Delta\alpha \cap A \text{ and } p\alpha\varepsilon\Delta\psi_s\}$ can be given similarly. Hence $\{p\varepsilon X | p\varepsilon\Delta\alpha \cap A \text{ and } p\alpha\varepsilon\Delta\psi_s\} = \{p\varepsilon X | p\varepsilon\Delta\beta \cap A \text{ and } p\beta\varepsilon\Delta\psi_t\}$.

Next, to show that $\{p\varepsilon X | p\varepsilon\Delta\alpha \smallsetminus A \text{ and } p\alpha\varepsilon\Delta\psi_s\} \subseteq \{p\varepsilon X | p\varepsilon\Delta\beta \smallsetminus A \text{ and } p\beta\varepsilon\Delta\psi_t\}$, let $p \in \Delta\alpha \smallsetminus A$ and $p\alpha \in \Delta\psi_s$. Since $S(\alpha) \subseteq A$,

$p\alpha = p$. Since $p\alpha \in \Delta\phi_s$, $\{p\alpha\}_{p\alpha}s = \{p\alpha\}_{p\alpha}$. Then $\{p\}_p s = \{p\}_p$. If $p \notin \Delta\beta$, then $\{p\}_p\beta = 0$ and hence $\{p\}_p = \{p\}_p s = \{p\}_{p\alpha}s = \{p\}_p\alpha s = \{p\}_p\beta t = 0t$, which implies that $0 = 0(0t) = 0t = \{p\}_p$, a contradiction. Hence $p \in \Delta\beta$. Therefore $p \in \Delta\beta \setminus A$. Since $S(\beta) \subseteq A$, $p\beta = p$. It follows that $\{p\beta\}_{p\beta}t = \{p\}_{p\beta}t = \{p\}_p\beta t = \{p\}_p\alpha s = \{p\}_{p\alpha}s = \{p\}_p s = \{p\}_p = \{p\beta\}_{p\beta}$. Therefore $p\beta \in \Delta\phi_t$.

The proof that $\{p\in X \mid p\in\Delta\beta \setminus A \text{ and } p\beta\in\Delta\phi_t\} \subseteq \{p\in X \mid p\in\Delta\alpha \setminus A \text{ and } p\alpha\in\Delta\phi_s\}$ can be given similarly. Hence $\{p\in X \mid p\in\Delta\alpha \setminus A \text{ and } p\alpha\in\Delta\phi_s\} = \{p\in X \mid p\in\Delta\beta \setminus A \text{ and } p\beta\in\Delta\phi_t\}$.

Therefore $\Delta\alpha\phi_s = \Delta\beta\phi_t$.

Now, let $p \in \Delta\alpha\phi_s$ $(= \Delta\beta\phi_t)$. Then $p\alpha \in \Delta\phi_s$ and $p\beta \in \Delta\phi_t$.

Case 1: $p \in A$. Then $p\alpha, p\beta \in A$. Since $p\alpha \in \Delta\phi_s$ and $p\beta \in \Delta\phi_t$, we have $A_{p\alpha}s, A_{p\beta}t \in U_X \setminus \{0\}$, so $(p\alpha)\phi_s = (p\alpha)(A_{p\alpha}s)$ and $(p\beta)\phi_t = (p\beta)(A_{p\beta}t)$. From $p\alpha, p\beta \in A$, we get that $(p\beta)A_{p\alpha} = p\alpha$ and $A_{p\alpha} = A_{p\alpha}A_{p\alpha}$. Since $\alpha s = \beta t$, $A_{p\alpha}s = A_p\alpha s = A_p\beta t = A_{p\beta}t$. Therefore

$$\begin{aligned} p(\alpha\phi_s) &= (p\alpha)\phi_s \\ &= (p\alpha)(A_{p\alpha}s) \\ &= ((p\beta)A_{p\alpha})(A_{p\alpha}s) \\ &= (p\beta)(A_{p\alpha}(A_{p\alpha}s)) \\ &= (p\beta)(A_{p\alpha}s) \\ &= (p\beta)(A_{p\beta}t) \\ &= (p\beta)\phi_t \\ &= p(\beta\phi_t)\ . \end{aligned}$$

Case 2: $p \notin A$. Then $p\alpha = p = p\beta \notin A$. But $p\alpha \in \Delta\phi_s$ and $p\beta \in \Delta\phi_t$, so $(p\alpha)\phi_s = p\alpha$ and $(p\beta)\phi_t = p\beta$ which imply that $p(\alpha\phi_s) = p(\beta\phi_t)$.

This proves that $\alpha\phi_s = \beta\phi_t$, as required.

The statement (2) follows from (1) and the fact that $\phi_1 = 1_X$, the identity map on X. #

<u>Theorem 3.13</u>. For any set X, U_X is absolutely closed.

<u>Proof</u>: Let X be a set. For the case $X = \emptyset$, U_X is absolutely closed since $|U_X| = 1$. Assume that $X \neq \emptyset$. To prove that U_X is absolutely closed, let S be a semigroup containing U_X as a subsemigroup. Let $d \in Dom(U_X, S)$. Since $1_X \in U_X$, by Corollary 1.2, there exist $\alpha_0, \alpha_1, \ldots, \alpha_{2m} \in U_X$, $s_1, \ldots, s_m, t_1, \ldots, t_m \in S$ such that

$$
(I) \quad \begin{cases} d = \alpha_0 t_1, \ \alpha_0 = s_1\alpha_1, \\ s_i\alpha_{2i} = s_{i+1}\alpha_{2i+1}, \alpha_{2i-1}t_i = \alpha_{2i}t_{i+1} \quad (i=1,\ldots,m-1), \\ \alpha_{2m-1}t_m = \alpha_{2m} . \end{cases}
$$

Then $d = s_m\alpha_{2m}$. Let $F = \{\alpha_0, \alpha_1, \ldots, \alpha_{2m}\}$ and $A = \bigcup_{\alpha \in F} (S(\alpha) \cup S(\alpha)\alpha)$.

For each $s \in S^1$, define $\phi_s \in U_X$ as follows: For $p \in X$,

$$
p\phi_s = \begin{cases} p(A_p s) & \text{if } p \in A \text{ and } A_p s \in U_X \setminus \{0\}, \\ p & \text{if } p \notin A \text{ and } \{p\}_p s = \{p\}_p , \\ \text{undefined} & \text{otherwise.} \end{cases}
$$

Since $\alpha_{2i-1}t_i = \alpha_{2i}t_{i+1}$ $(i=1,\ldots,m-1)$ and $\alpha_{2m}t_m = \alpha_{2m}$, by Lemma 3.12, we get that

$$
(II) \quad \begin{cases} \alpha_{2i-1}\phi_{t_i} = \alpha_{2i}\phi_{t_{i+1}} \quad (i = 1,\ldots,m-1), \\ \alpha_{2m-1}\phi_{t_m} = \alpha_{2m} . \end{cases}
$$

By (I) and (II), we obtain the following system of equalities:

$$\alpha_0 = s_1\alpha_1,$$

$$s_i\alpha_{2i} = s_{i+1}\alpha_{2i+1},\ \alpha_{2i-1}\psi_{t_i} = \alpha_{2i}\psi_{t_{i+1}}\quad (i = 1,\ldots,m-1),$$

$$\alpha_{2m-1}\psi_{t_m} = \alpha_{2m}\ ,$$

so by Lemma 3.1, we get that $\alpha_0\psi_{t_1} = s_m\alpha_{2m}$. Since $d = s_m\alpha_{2m}$ and α_0, $\psi_{t_1} \in U_X$, $d = \alpha_0\psi_{t_1} \in U_X$, so $\mathrm{Dom}(U_X,S) \subseteq U_X$. Hence $\mathrm{Dom}(U_X,S) = U_X$. This proves that U_X is absolutely closed. #

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