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MASS LIMIT OF HOLOGRAPHIC DEGENERATE STAR UNDER EXTERNAL MAGNETIC FIELD

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ด้วยหลักฮอโลกราฟี บัวร์และคณะได้เสนอความสมมูลระหว่างตัวดำเนินการคอนฟอร์มัลแบบซ้อนสถานะในการลอวกาศมินโคว์สกี้สี่มิติและก๊าซเฟอร์มิแบบซ้อนสถานะในกาลอวกาศแอนติเดอร์ซิทเตอร์ห้ามิติและนำไปสู่ความสนใจในการศึกษาดาวที่มีสภาพซ้อนสถานะในกาลอวกาศมินโคว์สกี้สี่มิติ ในงานนี้เราทำการคำนวณสมการสถานะของดาวภายใต้สภาวะอุณหภูมิจำกัดที่ไม่เป็นศูนย์และสนามแม่เหล็กภายนอกที่ไม่เป็นศูนย์ เราจะตรวจสอบดูพฤติกรรมของดาวประเภทนี้ที่อุณหภูมิไม่เป็นศูนย์ภายใต้การมีอยู่ของสนามแม่เหล็กภายนอก ผลที่เราพบคือเมื่ออุณหภูมิเพิ่มขึ้น ขีดจำกัดมวล มวลสะสม ความหนาแน่นพลังงาน และความดันของดาวมีค่าสูงขึ้นตามไปด้วย ในขณะที่การเพิ่มขึ้นของสนามแม่เหล็กกลับมีผลทางตรงกันข้าม พฤติกรรมเหล่านี้เราสามารถวิเคราะห์ได้จากสมการสถานะของดาว

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Motivated by the holographic principle, Boer et al. proposed a correspondence between degenerate conformal operator in $4D$ Minkowski spacetime and degenerate Fermi gas in $5D$ Anti-de Sitter space (AdS_5). Consequently, it is interesting to study a degenerate star in the AdS_5 space to explore the relation to the real degenerate star. In this work, we determine the equations of state of the star in 5 dimensional AdS_5 space at finite temperature in the presence of external magnetic field. We then work out properties of the star. Interestingly, our results demonstrate that the mass limit increases with the temperature. However, the effect of the magnetic field is opposite. Such behavior can be understood from the equations of state of the degenerate star.

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Chapter I

INTRODUCTION

In 1939, Richard Chace Tolman, Julius Robert Oppenheimer and George Michael Volkoff [1, 2] derived the equation of hydrostatic equilibrium for a spherical symmetric star in the framework of general relativity. Although this equation enables us to understand the existence of the maximum mass of spherical symmetric star above which the star would collapse to a black hole, their numerical calculation did not include strong interaction between matters in the star. This is because the perturbative method cannot be used in this strong-coupling regime observed at low temperature. In 1993, Gerardus 't Hooft [3] proposed an important work on dimensional reduction in quantum gravity. In the following year, Leonard Susskind [4] published the paper with title “the world as a hologram”. His follow up to 't Hooft's paper further elaborates the idea of hologram analogy. Their works, which are well-known as the holographic principle, suggest that the information of a volume of space can be thought of as being encoded on a boundary of the region.

Fortunately, the first realization of this principle which is called AdS/CFT correspondence was discovered later by Juan Martn Maldacena [5] in 1997. This correspondence enables us to study the strongly coupled gauge theory on four dimensional Minkowski spacetime (M_4), a cousin of quantum chromodynamics, by avoiding the uncontrollable non-perturbative calculation; due to the weak-strong duality, we can deal with this problem by just doing calculations in the tractable weakly interacting string theory in five dimensional Anti de Sitter space (AdS_5).

In 2009 and 2010, Jan de Boer, Kyriakos Papadodimas and Erik Verlinde [6, 7] found the matching between the degenerate conformal field operator in M_4 and degenerate Fermi gas in AdS_5. With the aim to find interesting behavior of the conformal field operator, they study the Fermi-gas filled star, in particular a neutron star, at zero temperature in Anti de Sitter space. In other words, this study may give rise to an understanding of the phenomena in the gauge theory through the dynamics of the degenerate star in higher-dimensional space.

In this thesis, we will use the holographic principle to compute mass limit of degenerate star at finite temperature in the presence of external magnetic field and study other properties of star in five dimensional Anti de Sitter space in order to understand a degenerate star in four dimension under influence of the strong interaction (if we can find the duality in the future).

Chapter II

THEORETICAL BACKGROUND

In this chapter, we recall briefly concepts and some basic knowledge for this thesis. We divide into five parts. First, the idea of holographic principle which we express its statement and discuss the applications. Second, the relevant thermodynamics and statistical mechanics, we review some concepts and the necessary formulation. Third, Einstein equations for a spherically symmetric star, we consider first in 4 dimensional space-time before extending to 5 dimensional AdS space in the next chapter. Forth, the equations of state of degenerate star which we examine a model of an ideal(non-interacting) degenerate Fermi gas. Finally, we see non relativistic Landau energy level when a particle move in a uniform magnetic field.

2.1 Holographic principle

Holographic principle is one of the most interesting idea which came from studying quantum gravity and string theory. Inspiration of this principle arises from studying the space-time geometry of black holes which is the analogy between properties of black holes and thermodynamics, a black hole's area and entropy [8, 9]. It states that the description of a volume of space can be thought of as encoded on a boundary of the region. It was proposed by Gerard 't Hooft and Leonard Susskind in 1993 and 1994, respectively [3, 4]. Holographic ideas have found several applications in physics and one of the most interesting realizations of this idea is the AdS/CFT correspondece which is a duality between string theory in Anti de Sitter space and a conformal field theory(without gravity) living on the boundary of the AdS space [9]. This application was conjectured by Juan Martn Maldacena in 1997 [5]. Mostly, AdS/CFT is applied in the study of non-perturbative phenomena in the supersymmetry quantum chromodynamics (SUSY QCD).

In conformal field theory(on the boundary), we can find an expectation value of an operator via a generating functional, $\langle e^{\left(\int_{S^d}\phi_0 O\right)}\rangle$ where ϕ_0 is a restriction of the scalar field ϕ at the boundary of AdS_{d+1}, O is an operator and S^d is a d-sphere. For string theory, we have $Z_S\left(\phi_0\right) = e^{\left(-I_S(\phi)\right)}$ is the supergravity partition function on B_{d+1}(open ball in $d+1$ sphere). The relation of the conformal field theory on the boundary to the supergravity in the AdS space is that [10]

$$\langle e^{\int_{S^d}\phi_0 O}\rangle_{CFT} = Z_S\left(\phi_0\right). \tag{2.1}$$

For gauge theory, it is similar to the scalar field. We assume the AdS theory has a gauge group G(global symmetry group of the conformal field theory on the boundary), dimension n, with gauge fields $A^a, a = 1, \ldots, n$. J_a are currents in the boundary and A_0 is an arbitrary source. Thus relation between the supergravity or string theory partition function and generating functional in the conformal field theory is expressed by [10]

$$\langle e^{\int_{S^d} J_a A_0^a}\rangle_{CFT} = Z_S\left(A_0\right). \tag{2.2}$$

Then the partition function of string theory on $AdS_5 \times S^5$ should agree with the partition function of $N = 4$ super Yang-Mills theory on the boundary of this space [11]. So the relation becomes

$$e^{-I_{SUGRA}} \simeq Z_{string} = Z_{gauge} = e^{-W}, \tag{2.3}$$

where $W = \beta F$ is the free energy of the gauge theory divided by the temperature [11]. This idea hopefully enable us to computes the properties of the strongly coupled gauge theory on the boundary via partition function of the string theory in the bulk of the same space(AdS space).

2.2 Thermodynamics and statistical mechanics

2.2.1 First law and the standard thermodynamic relations

The first law of thermodynamics is a principle of conservation of energy. This principle is very important in physics. It tells us that a store of energy in the system, called the internal energy dU, can be changed by causing the system to do work, dW, or by adding heat, dQ, to the system [12]. The change of the internal energy is in the form

$$dU = dQ - dW. \tag{2.4}$$

The first law is always true. It does not depend on a change of state whether it is reversible or irreversible [13], Then

$$dU = dQ_{rev} - dW_{rev} = dQ_{irr} - dW_{irr}. \tag{2.5}$$

Let us consider adding another particle to the thermodynamic system. If the number of particle is allowed to vary, then we put $dW = \mu dN$ by hand as the energy change due to the change of the particle number by dN particles. The quantity, μ, is called the chemical potential, represents the resistance of the system against addition of particles. So equation (2.4) becomes

$$\begin{aligned} dU &= dQ - dW + \mu dN, \\ &= TdS - PdV + \mu dN, \end{aligned} \tag{2.6}$$

where T, S, P and V are temperature, entropy, pressure and volume, respectively. From equation (2.6), we can determine T, P and μ via

$$T = \frac{\partial U}{\partial S} |_{V,N} \qquad -P = \frac{\partial U}{\partial V} |_{S,N} \qquad \mu = \frac{\partial U}{\partial N} |_{S,V} .$$

The chemical potential, is then given by

$$\mu = \frac{\partial U}{\partial N} \qquad or \qquad \mu = \frac{\left(\frac{\partial U}{\partial V}\right)}{\left(\frac{\partial N}{\partial V}\right)} = \frac{d\rho}{dn}, \tag{2.7}$$

where ρ and n are the energy density and the number density, respectively. Moreover, when we consider an adiabatic process ($dQ = TdS = 0$), we have

$$dU = -PdV + \mu dN \qquad \longrightarrow \qquad \frac{dU}{dV} + P = \mu \frac{dN}{dV},$$

$$\rho + P = \mu n. \tag{2.8}$$

So we have the standard thermodynamic relations of energy density, pressure, number density and chemical potential following equations (2.7) and (2.8).

2.2.2 Quantum statistical mechanics of Fermi-Dirac statistics

Statistical mechanics has made the connection between a microscopic system and macroscopic world. The fundamental concept of statistical mechanics is that all macroscopic observable quantities of a state follow from taking mean values of microscopic properties, weighted with probability densities. Then the role of statistical mechanics is to find a way or process of taking mean values, of the microscopic quantities by realization that a system can assume a large number of microstates. This idea is now to be transferred to quantum systems [12, 13].

In classical statistical mechanics, a microstate corresponds to a certain point in phase space $(\vec{r}_i(t), \vec{p}_i(t))$. For quantum mechanics, we do not determine the coordinates and momenta of the particles simultaneously and possibly think state of particles or a system described by a wave function (wave-particle duality). So we replace the classical phase-space coordinate $(\vec{r}_i(t), \vec{p}_i(t))$ to the time evolution of wave function $\Psi(\vec{r}_1, \ldots, \vec{r}_n, t)$ of the system [13]. Furthermore, we promote measurable observables to mathematical operators. When we measure the system, it jumps to an eigenstate of the dynamical variable that is being measured according to eigenvalue equation.

$$\hat{A}\Psi = a\Psi. \tag{2.9}$$

In general, eigenvalues a of the system can assume only certain values. But in the macroscopic world, the eigenvalues (*e.g.* energy) are very close to each other. Then they should have a lot of states having eigenvalues between two-eigenvalues. We define $\Psi^{(i)}(\vec{r}_1, \ldots, \vec{r}_n, t)$ to be the specific microstates corresponding to different wave functions. In quantum statistical mechanics, we can measure the most general expectation values of operator by

$$\langle \hat{A} \rangle = \sum_{i,k} \rho_{ki} \langle \Psi^{(i)} | \hat{A} | \Psi^{(k)} \rangle. \tag{2.10}$$

Notice that expectation values in quantum statistical mechanics are multiplication of the probability ρ_{ki} in statistical mechanics and the expectation value $\langle \Psi^{(i)} | \hat{A} | \Psi^{(k)} \rangle$ in quantum mechanics. If we let $\Psi^{(i)} = \sum_l a_l^{(i)} \psi_l$ and $i = k$ in equation (2.10), then $\rho_{ki} \to \rho_i$ and

$$\begin{aligned} \langle \hat{A} \rangle &= \sum_{k,k'} \left(\sum_i \rho_i a_k^{(i)*} a_{k'}^{(i)} \right) \langle \psi_k | \hat{A} | \psi_{k'} \rangle \\ &= \sum_{k,k'} \rho_{k'k} \langle \psi_k | \hat{A} | \psi_{k'} \rangle = \sum_{k,k'} \langle \psi_{k'} | \hat{\rho} | \psi_k \rangle \langle \psi_k | \hat{A} | \psi_{k'} \rangle, \\ &= \sum_{k'} \langle \psi_{k'} | \hat{\rho} \hat{A} | \psi_{k'} \rangle = Tr\left(\hat{\rho} \hat{A} \right), \end{aligned} \tag{2.11}$$

where $\rho_{k'k} = \sum_i \rho_i a_k^{(i)*} a_{k'}^{(i)}$ and we now let probability $\rho_{k'k}$ be the matrix elements of an operator $\hat{\rho} = \sum_i \rho_i |\Psi_{k'}^{(i)}\rangle\langle\Psi_k^{(i)}|$, it is called the density operator [13].

In statistical mechanics, the grand canonical density operator is given by

$$\rho_n = \frac{e^{-\frac{(E_n - \mu N)}{k_B T}}}{\sum_{n,N} e^{-\frac{(E_n - \mu N)}{k_B T}}}, \tag{2.12}$$

where E_n, μ, N, k_B and T are energy, chemical potential, number of particle, Boltzmann constant and temperature, respectively. The grand canonical partition function becomes

$$Z\left(T,V,\mu\right)=\sum_{n,N}e^{-\frac{(E_n-\mu N)}{k_BT}}. \tag{2.13}$$

Since

$$\begin{aligned}Tr\left(e^{-\frac{(\hat{H}-\mu\hat{N})}{k_BT}}\right) &= \sum_n\langle\psi_n|e^{-\frac{(\hat{H}-\mu\hat{N})}{k_BT}}|\psi_n\rangle=\sum_n\langle\psi_n|\sum_{k=0}^{\infty}\frac{\left(-\frac{(\hat{H}-\mu\hat{N})}{k_BT}\right)^k}{k!}|\psi_n\rangle,\\ &= \sum_{n,N}\langle\psi_n|\sum_{k=0}^{\infty}\frac{\left(-\frac{(E_n-\mu N)}{k_BT}\right)^k}{k!}|\psi_n\rangle=\sum_{n,N}\langle\psi_n|e^{-\frac{(E_n-\mu N)}{k_BT}}|\psi_n\rangle,\\ &= \sum_{n,N}e^{-\frac{(E_n-\mu N)}{k_BT}}=Z\left(T,V,\mu\right), \qquad (2.14)\end{aligned}$$

where $\hat{H}$ and $\hat{N}$ are energy and number operator, respectively. We can change equations (2.12) and (2.13) to an operator equation, in quantum statistical mechanics, which can be written generally in an any basis

$$\hat{\rho}=\frac{e^{-\frac{(\hat{H}-\mu\hat{N})}{k_BT}}}{Tr\left(e^{-\frac{(\hat{H}-\mu\hat{N})}{k_BT}}\right)}. \tag{2.15}$$

In the classical point of view, we can always determine all coordinates and momenta of the particles at each time. Therefore we can keep track of particles even though they may look alike. But from the point of view in quantum mechanics, we cannot specify all coordinates and momenta more accurately in phase space than the size h^3 (3-dimensional space) of a phase-space cell according to uncertainty principle. we can mention only the total probability of finding a particle in a phase-space cell, then we cannot label the particle and follow the trajectory of them, *i.e.*, identical particles are truly indistinguishable [13].

Because of the indistinguishability of identical particles which does not exist in classical mechanics, we see that the quantum Hamiltonian should be invariance under the enumerated changing of the particle coordinates and momenta. We can define the permutation operator $\hat{P}_{ik}$ which commute Halmiltonian operator $\hat{H}$. Then the eigenvalue problem of the $\hat{P}_{ik}$ has a form

$$\begin{aligned}\hat{P}_{ik}\Psi\left(\ldots,\vec{r}_i,\ldots,\vec{r}_k,\ldots\right) &= \lambda\Psi\left(\ldots,\vec{r}_i,\ldots,\vec{r}_k,\ldots\right),\\ \Psi\left(\ldots,\vec{r}_k,\ldots,\vec{r}_i,\ldots\right) &= \lambda\Psi\left(\ldots,\vec{r}_i,\ldots,\vec{r}_k,\ldots\right),\end{aligned}$$

where λ is a eigenvalue of the permutation operator $\hat{P}_{ik}$. If we permute again, it holds that

$$\hat{P}_{ik}^2 \Psi\left(\ldots, \vec{r}_i, \ldots, \vec{r}_k, \ldots\right) = \Psi\left(\ldots, \vec{r}_i, \ldots, \vec{r}_k, \ldots\right) = \lambda^2 \Psi\left(\ldots, \vec{r}_i, \ldots, \vec{r}_k, \ldots\right). \quad (2.16)$$

Thus, the permutation operator $\hat{P}_{ik}$ can have only the eigenvalues $\lambda = \pm 1$. If $\lambda = +1$, the wave function is said that symmetric and if $\lambda = -1$, the wave function is said to be anti-symmetric. In other words, there exist two kinds of particles in the nature. Particles which are described by symmetric wave functions, they are called bosons, and particles that are described by anti-symmetric wave functions, they are called fermions. There are two interesting property of bosons and fermions. The wave function construction in such a way they are either completely symmetric or completely anti-symmetric. The fermions must obey the Pauli exclusion principle, *i.e.*, two equal fermions cannot occupy the same one-particle state. But bosons can violate this principle, *i.e.*, bosons can occupy the same one-particle state. Moreover, the spin-statistics theorem, implies that fermions have half-integer spin and bosons have integer spin [14, 15].

We shall now consider ideal and non-interacting quantum systems. Hamiltonian operator can be split into a sum of one-particle operators

$$\hat{H}\left(\vec{r}_1, \ldots, \vec{r}_n, \vec{p}_1, \ldots, \vec{p}_n\right) = \sum_{i=1}^{n} \hat{h}\left(\vec{r}_i, \vec{p}_i\right).$$

Each operator $\hat{h}\left(\vec{r}_i, \vec{p}_i\right)$ satisfy $\hat{h}\psi_k\left(\vec{r}\right) = \epsilon_k \psi_k\left(\vec{r}\right)$, where $\psi_k\left(\vec{r}\right)$ is a one-particle functions, then the general eigenfunction is

$$\Psi^E_{k_1,\ldots,k_n}\left(\vec{r}_1, \ldots, \vec{r}_n\right) = \prod_{i=1}^{n} \psi_{k_i}\left(\vec{r}_i\right).$$

This product wave function can be written in Dirac's notation, we have ket state vectors are

$$|k_1, \ldots, k_n\rangle = |k_1\rangle|k_2\rangle \ldots |k_n\rangle. \quad (2.17)$$

It means that a particle no.1 is in the quantum state k_1, a particle no.2 is in the state k_2, etc. And the bra state vectors are

$$\langle k_1, \ldots, k_n| = \langle k_n|\langle k_{n-1}| \ldots \langle k_1|. \quad (2.18)$$

When we consider the anti-symmetric wave functions, the fermions, they can be written in Dirac's notation

$$\begin{aligned} |k_1, \ldots, k_n\rangle^A &= \frac{1}{\sqrt{n!}} \sum_P \left(-1\right)^P \hat{P}|k_1, \ldots, k_n\rangle, \\ &= \frac{1}{\sqrt{n!}} \sum_P \left(-1\right)^P \hat{P}|k_{P_1}, \ldots, k_{P_n}\rangle, \end{aligned} \quad (2.19)$$

for the symmetric wave functions, bosons, they are similarly

$$\begin{aligned} |k_1,\ldots,k_n\rangle^S &= \frac{1}{\sqrt{n!n_1!n_2!\ldots}}\sum_P \hat{P}|k_1,\ldots,k_n\rangle, \\ &= \frac{1}{\sqrt{n!n_1!n_2!\ldots}}\sum_P \hat{P}|k_{P_1},\ldots,k_{P_n}\rangle, \end{aligned} \tag{2.20}$$

and if we act the Hamiltonian operator on the wave functions, the eigenvalue problem are

$$\hat{H}|k_1,\ldots,k_n\rangle^{S,A} = E|k_1,\ldots,k_n\rangle^{S,A} \qquad \text{with} \qquad E = \sum_{i=1}^{n}\epsilon_k. \tag{2.21}$$

If we enumerate the one-particle state $|k\rangle$ by the index k, then each state $|k\rangle$ has the occupation number $\{n_1, n_2, \ldots\}$ which determine n-particle state. For fermions, each occupation number have values $n_k = 0, 1$ but occupation number of bosons have values $n_k = 0, 1, \ldots, n$. Moreover, we get the occupation number condition, $\sum_{k=1}^{\infty} n_k = n$, and the energy of equation (2.21) can be expressed in terms of the occupation numbers, $E = \sum_{k=1}^{\infty} n_k\epsilon_k$ [13].

We instead represent the quantum numbers of the occupied states by the occupation numbers.

$$|n_1, n_2, \ldots\rangle^{S,A} \equiv |k_1, k_2, \ldots, k_n\rangle^{S,A}. \tag{2.22}$$

Then, we have

$$\hat{H}|n_1, n_2, \ldots\rangle^{S,A} = E|n_1, n_2, \ldots\rangle^{S,A} \qquad \text{with} \qquad E = \sum_{k=1}^{\infty} n_k\epsilon_k, \tag{2.23a}$$

$$\hat{N}|n_1, n_2, \ldots\rangle^{S,A} = n|n_1, n_2, \ldots\rangle^{S,A} \qquad \text{with} \qquad n = \sum_{k=1}^{\infty} n_k, \tag{2.23b}$$

where $\hat{N}$ is a number operator. From equation (2.14), it changes to

$$Z\left(T,V,\mu\right) = \sum_{n_k} e^{-\frac{\sum_{k=1}^{\infty} n_k(\epsilon_k-\mu)}{k_BT}} \quad \text{where} \;\; n_k = \begin{cases} 0,1 & \text{Fermions} \\ 0,1,2,\ldots & \text{Bosons} \end{cases}. \tag{2.24}$$

We consider only fermions, then

$$\begin{aligned} Z^{FD}\left(T,V,\mu\right) &= \sum_{n_1,n_2,\ldots=0}^{1} \left(e^{-\frac{(\epsilon_1-\mu)}{k_BT}}\right)^{n_1} \left(e^{-\frac{(\epsilon_2-\mu)}{k_BT}}\right)^{n_2} \ldots, \\ &= \prod_{k=1}^{\infty}\sum_{n_k=0}^{1} \left(e^{-\frac{(\epsilon_k-\mu)}{k_BT}}\right)^{n_k}, \\ &= \prod_{k=1}^{\infty}\left(1 + e^{-\frac{(\epsilon_k-\mu)}{k_BT}}\right), \end{aligned} \tag{2.25}$$

and

$$\ln Z = \sum_k \ln\left(1 + e^{-\frac{(\epsilon_k-\mu)}{k_BT}}\right). \tag{2.26}$$

From statistical mechanics, the grand canonical potential is

$$\Phi\left(T,V,\mu\right) = -k_BT\ln Z\left(T,V,\mu\right) = U - TS - \mu N = -PV, \tag{2.27}$$

then

$$P = \frac{k_BT}{V}\ln Z\left(T,V,\mu\right), \tag{2.28}$$

and another thermodynamics quantities

$$N\left(T,V,\mu\right) = k_BT\frac{\partial}{\partial\mu}\ln Z|_{T,V} = \sum_{k=1}^{\infty}\frac{1}{\left(e^{\frac{(\epsilon_k-\mu)}{k_BT}}+1\right)}, \tag{2.29a}$$

$$U\left(T,V,\mu\right) = -\frac{\partial}{\partial\beta}\ln Z|_{z,V} = \sum_{k=1}^{\infty}\frac{\epsilon_k}{\left(e^{\frac{(\epsilon_k-\mu)}{k_BT}}+1\right)}, \tag{2.29b}$$

where $z = e^{\frac{\mu}{k_BT}}$ [13].

2.3 Einstein equations of spherically symmetric star

General relativity is the physical theory of gravity, it was constructed by Albert Einstein in 1915. This theory use the notion of the metric to measure the distance between two points in space-time, denoted by $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. The distance square is invariant under Lorentz transformations and the form of metric tensor, $g_{\mu\nu}$, come from the energy and matter that is present in a given region of space-time. The existences of the energy and matter determine geometry of space-time and the equation which connects energy-matter and geometry together is called the Einstein's equation. So we may say that the fundamental idea of this theory is to deeply relate gravity, energy-momentum tensor, and the curvature of space-time together.

2.3.1 Einstein equations and Tolman-Oppenheimer-Volkoff (TOV) equation in 4 dimensions

We find solutions of Einstein's equation of hydrostatic equilibrium for a spherical symmetric star. The general form of metric in 4 dimensional space-time, spherical coordinates, is defined by [16]

$$ds^2 = A(r)c^2dt^2 - B(r)dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{2.30}$$

where the two arbitrary functions $A(r)$ and $B(r)$ are determined by the energy and matter that we consider and require that $A(r)c^2 \to c^2$, $B(r) \to 1$ as $r \to \infty$ [17]. We assume the energy and matter to be described by a perfect fluid, so that [18]

$$T^{\mu}_{\ \nu} = \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & -P_r & 0 & 0 \\ 0 & 0 & -P_{\theta_1} & 0 \\ 0 & 0 & 0 & -P_{\theta_2} \end{pmatrix}. \tag{2.31}$$

The Lagrangian of equation (2.30) is [17]

$$L = A(r)c^2\dot{t}^2 - B(r)\dot{r}^2 - r^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right), \tag{2.32}$$

and we can find Christoffle symbols

$$\Gamma^{t}_{\ tr} = \Gamma^{t}_{\ rt} = \frac{A'}{2A}, \tag{2.33a}$$

$$\Gamma^{r}_{\ tt} = \frac{A'c^2}{2B}, \quad \Gamma^{r}_{\ rr} = \frac{B'}{2B}, \quad \Gamma^{r}_{\ \theta\theta} = -\frac{r}{B}, \quad \Gamma^{r}_{\ \phi\phi} = -\frac{r\sin^2\theta}{B}, \tag{2.33b}$$

$$\Gamma^{\theta}_{\ r\theta} = \Gamma^{\theta}_{\ \theta r} = \frac{1}{r}, \quad \Gamma^{\theta}_{\ \phi\phi} = -\sin\theta\cos\theta, \tag{2.33c}$$

$$\Gamma^{\phi}_{\ r\phi} = \Gamma^{\phi}_{\ r\phi} = \frac{1}{r}, \quad \Gamma^{\phi}_{\ \theta\phi} = \Gamma^{\phi}_{\ \phi\theta} = \cot\theta. \tag{2.33d}$$

The others of Christoffle symbol are all zero. The Riemann tensor

$$\begin{aligned} R^{tr}_{\ \ tr} &= \frac{A''}{2AB} - \frac{A'B'}{4AB^2} - \frac{A'^2}{4A^2B}, & R^{r\theta}_{\ \ r\theta} &= -\frac{B'}{2rB^2}, \\ R^{t\theta}_{\ \ t\theta} &= \frac{A'}{2rAB}, & R^{r\phi}_{\ \ r\phi} &= -\frac{B'}{2rB^2}, \\ R^{t\phi}_{\ \ t\phi} &= \frac{A'}{2rAB}, & R^{\theta\phi}_{\ \ \theta\phi} &= -\frac{B-1}{r^2B}. \end{aligned}$$

There are only four non zero components of the Ricci tensor. They are

$$R^t{}_t = R^{rt}{}_{rt} + R^{\theta t}{}_{\theta t} + R^{\phi t}{}_{\phi t},$$
$$= \frac{A''}{2AB} - \frac{A'B'}{4AB^2} - \frac{A'^2}{4A^2B} + \frac{A'}{rAB}, \tag{2.34a}$$
$$R^r{}_r = R^{tr}{}_{tr} + R^{\theta r}{}_{\theta r} + R^{\phi r}{}_{\phi r},$$
$$= \frac{A''}{2AB} - \frac{A'B'}{4AB^2} - \frac{A'^2}{4A^2B} - \frac{B'}{rB^2}, \tag{2.34b}$$
$$R^\theta{}_\theta = R^{t\theta}{}_{t\theta} + R^{r\theta}{}_{r\theta} + R^{\phi\theta}{}_{\phi\theta},$$
$$= \frac{A'}{2rAB} - \frac{B'}{2rB^2} - \frac{B-1}{r^2B}, \tag{2.34c}$$
$$R^\phi{}_\phi = R^{t\phi}{}_{t\phi} + R^{r\phi}{}_{r\phi} + R^{\theta\phi}{}_{\theta\phi},$$
$$= \frac{A'}{2rAB} - \frac{B'}{2rB^2} - \frac{B-1}{r^2B}. \tag{2.34d}$$

From Einstein equation

$$G^\mu{}_\nu = R^\mu{}_\nu - g^\mu{}_\nu \frac{R}{2} = \frac{8\pi G}{c^4} T^\mu{}_\nu, \tag{2.35}$$

where G and c are Newton's constant and speed of light, respectively. We can calculate $G^\mu{}_\nu$

$$G^t{}_t = -R^{r\theta}{}_{r\theta} - R^{r\phi}{}_{r\phi} - R^{\theta\phi}{}_{\theta\phi} = \frac{B'}{rB^2} + \frac{B-1}{r^2B}, \tag{2.36a}$$
$$G^r{}_r = -R^{t\theta}{}_{t\theta} - R^{t\phi}{}_{t\phi} - R^{\theta\phi}{}_{\theta\phi} = -\frac{A'}{rAB} + \frac{B-1}{r^2B}, \tag{2.36b}$$
$$G^\theta{}_\theta = -R^{tr}{}_{tr} - R^{t\theta}{}_{t\theta} - R^{r\theta}{}_{r\theta}$$
$$= -\frac{A''}{2AB} + \frac{A'B'}{4AB^2} + \frac{A'^2}{4A^2B} - \frac{A'}{2rAB} + \frac{B'}{2rB^2}, \tag{2.36c}$$
$$G^\phi{}_\phi = -R^{tr}{}_{tr} - R^{t\phi}{}_{t\phi} - R^{r\phi}{}_{r\phi} = G^\theta{}_\theta. \tag{2.36d}$$

Consider $G^t{}_t$ component in perfect fluid case, we can solve to find the function $B(r)$

$$G^t{}_t = \frac{B'}{rB^2} + \frac{B-1}{r^2B} = \frac{8\pi G}{c^4}\rho c^2,$$
$$\frac{rB'}{B^2} + \frac{B-1}{B} = \frac{8\pi G}{c^2}\rho r^2 \quad \longrightarrow \quad r\frac{B-1}{B} = \frac{8\pi G}{c^2}\int_0^r \rho r^2 dr,$$
$$\therefore \qquad B(r) = \frac{1}{1 - \frac{2G\int_0^r \rho 4\pi r^2 dr}{c^2 r}} = \frac{1}{1 - \frac{2GM}{c^2 r}} \quad \textit{where} \quad M = \int_0^r \rho 4\pi r^2 dr. \tag{2.37}$$

Consider $G^r{}_r$ component in perfect fluid case, we have

$$G^r{}_r = \frac{B-1}{r^2 B} - \frac{A'}{rAB} = -\frac{8\pi G}{c^4} P_r,$$

$$\frac{2GM}{c^2 r^3} - \left(1 - \frac{2GM}{c^2 r}\right)\frac{A'}{rA} = -\frac{8\pi G}{c^4} P_r \quad \longrightarrow \quad \frac{A'}{A} = \frac{\frac{2GM}{c^2} + \frac{8\pi G}{c^4} P_r r^3}{r^2\left(1 - \frac{2GM}{c^2 r}\right)},$$

$$\therefore \quad \frac{A'}{A} = \frac{2GM}{c^2 r^2}\left(1 + \frac{4\pi r^3}{M}\frac{P_r}{c^2}\right)\left(1 - \frac{2GM}{c^2 r}\right)^{-1}. \tag{2.38}$$

Moreover, we consider the energy momentum conserved $\nabla_\mu T^\mu{}_\nu = 0$ by let $\nu = r$, $c = 1$ and $P_r = P_{\theta_1} = P_{\theta_2} = P$, it leads to the important equation, called the Tolman-Oppenheimer-Volkoff equation (TOV equation)

$$\begin{aligned}
\nabla_\mu T^\mu{}_\nu &= \partial_\mu T^\mu{}_r + \Gamma^\mu{}_{\mu\alpha} T^\alpha{}_r - \Gamma^\alpha{}_{r\mu} T^\mu{}_\alpha, \\
&= \partial_\mu T^\mu{}_r + \Gamma^\mu{}_{\mu r} T^r{}_r - \left(\Gamma^t{}_{rt} T^t{}_t + \Gamma^r{}_{rr} T^r{}_r + \Gamma^\theta{}_{r\theta} T^\theta{}_\theta + \Gamma^\phi{}_{r\phi} T^\phi{}_\phi\right), \\
&= -\frac{dP_r}{dr} + \left(\Gamma^t{}_{tr} + \Gamma^r{}_{rr} + \Gamma^\theta{}_{\theta r} + \Gamma^\phi{}_{\phi r}\right)(-P_r) \\
&\quad - \left(\Gamma^t{}_{rt}\rho + \Gamma^r{}_{rr}(-P_r) + \Gamma^\theta{}_{r\theta}(-P_\theta) + \Gamma^\phi{}_{r\phi}(-P_\phi)\right), \\
&= -\frac{dP}{dr} - (\rho + P)\Gamma^t{}_{tr} = 0,
\end{aligned}$$

$$\therefore \quad \frac{dP}{dr} = -\frac{(\rho + P)}{2}\frac{A'}{A}, \tag{2.39a}$$

$$= -\frac{GM\rho}{r^2}\left(1 + \frac{P}{\rho}\right)\left(1 + \frac{4\pi r^3 P}{M}\right)\left(1 - \frac{2GM}{r}\right)^{-1}. \tag{2.39b}$$

The TOV equation completely determines the structure of a spherically symmetric body of isotropic material which is in static gravitational equilibrium. Furthermore, equations (2.37) and (2.39b) use to describe the interior of a spherical, static, relativistic star.

2.3.2 Relation between thermodynamic property and TOV equation

Notice that the TOV equation (2.39a) concerns energy density and pressure of matter. From thermodynamics, we know that both quantities are related by equation (2.8). Then we can connect the thermodynamic relation and energy conserved condition with the TOV equation (2.39a). Taking derivative of equation (2.8), we get

$$\frac{d\rho}{dr} + \frac{dP}{dr} = \frac{d\mu}{dr} n + \mu\frac{dn}{dr},$$

use equation (2.7) ($\therefore \mu\frac{dn}{dr} = \frac{d\rho}{dn}\cdot\frac{dn}{dr} = \frac{d\rho}{dr}$)

$$\frac{dP}{dr} = -\frac{d\rho}{dr} + \frac{d\mu}{dr}n + \frac{d\rho}{dr} = n\frac{d\mu}{dr}, \tag{2.40}$$

substitute into TOV equation (2.39a)

$$\frac{dP}{dr} + \frac{1}{2A}\frac{dA}{dr}(\rho + P) = 0 \quad \longrightarrow \quad n\frac{d\mu}{dr} + \frac{1}{2A}\frac{dA}{dr}\mu n = 0,$$
$$\frac{\mu'}{\mu} + \frac{A'}{2A} = 0 \quad \longrightarrow \quad \ln\left(\mu\sqrt{A}\right) = C,$$
$$\therefore \quad \mu(r) = \frac{e^C}{\sqrt{A(r)}}, \tag{2.41}$$

where C is a constant. If we consider at zero temperature and $r = 0$, then we have $\mu(T = 0)$ equal to Fermi energy, denoted by ϵ_F.

2.4 Degenerate star

The term compact star (sometimes called compact object) is used to refer collectively to white dwarfs, neutron stars, other exotic dense stars, and black holes. They form the endpoint of stellar evolution. A star radiate all the time so it loses nuclear energy reservoir in a finite time. When the nuclear fuel of the star has been consumed, the gas pressure of the hot interior can no longer support the gravity of matter in the star and the star collapses to a denser state. We call the compact star which is built by degenerate matter that a degenerate star. There are two major differences between degenerate stars and normal stars. First, they do not use nuclear fuel to generate thermal pressure against the gravitational collapse, they are supported by the pressure of degenerate matter. Second, the size of degenerate stars is smaller than normal stars when both of them are having the same mass [19, 20].

White dwarf was the first object of compact star which was studied. Initially, Frederick William Herschel [21] investigated double stars. Then many astronomers, *e.g.* Friedrich Georg Wilhelm von Struve, Friedrich Wilhelm Bessel, Otto Wilhelm von Struve, Walter Adams etc., had studied continuously via observation. Willem Jacob Luyten [22, 23, 24, 25] appeared to had been the first to used the term white dwarf in 1922 and the term was later popularized by Arthur Stanley Eddington [26]. In December 1926, Ralph Howard Fowler [27] applied non-relativistic Fermi-Dirac statistics to explain electron degeneracy pressure holding up the star from gravitational collapse. Then Subrahmanyan Chandrasekhar

[28, 29] improved this idea to describe the structure of white dwarf star in 1930. He uses the relativistic form of the Fermi-Dirac statistics for the degenerate case. He showed that the white dwarf should have a maximum mass of 1.4 times that of the sun (now known as the Chandrasekhar limit, is an upper bound on the mass of bodies made from electron-degenerate matter). In 1932, James Chadwick [30] discovered the neutron. Immediately, the ideas formulated by Fowler for the electron was generalized to neutron. The existence of a new class of compact star was predicted with a large core of degenerate neutron, the neutron star. In 1934, Walter Baade and Fritz Zwicky [31, 19] proposed the idea of neutron star, pointing out that they would be at very high density and small radius, and would be much more gravitationally bound than ordinary star. The first neutron star model calculations were performed by Richard Chace Tolman, Julius Robert Oppenheimer and George Michael Volkoff in 1939 [1, 2], describing the matter in such a star as an ideal degenerate neutron gas. Their calculations also showed the existence of a maximum mass, like in the case of white dwarf, above which the star is not stable and collapses into a black hole. They found a maximum stable mass of 0.75 times that of the sun [1, 2, 20]. An upper bound of the mass of stars which compose degenerate neutron, the neutron star, is called the TolmanOppenheimerVolkoff limit (or TOV limit). The TOV limit is analogous to the Chandrasekhar limit for white dwarf star.

The coupled equations of relativistic stellar structure is derived first by Oppenheimer and Volkoff (eqautions (2.37), (2.39a)). By rewriting them we can arrive at the physical interpretation

$$dM(r) = 4\pi r^2 \rho(r) dr, \tag{2.42a}$$

$$4\pi r^2 dP(r) = -\frac{GM(r)dM(r)}{r^2}\left(1+\frac{P(r)}{\rho(r)}\right)\left(1+\frac{4\pi r^3 P(r)}{M(r)}\right)\left(1-\frac{2GM(r)}{r}\right)^{-1}. \tag{2.42b}$$

Let us think of a shell of matter in the star by radius r and thickness dr, equation (2.42a) gives the mass-energy in this shell and equation (2.42b) expresses the balance between the force acting on a shell of matter due to material pressure from within and the weight of matter weighing down on it from without. The first factor of equation (2.42b) is the attractive Newtonian force of gravity acting on the shell by the mass interior to it and the other three factors are the exact corrections for general relativity. So these equations express the balance at each r between the internal pressure as it supports the overlaying material against the gravitational collapse of the mass-energy interior to r. Since the derivative of pressure is negative, it is clear that the pressure decreases monotonically in a star. Moreover, the equation of state $P = P(\rho)$ is the manner in which the properties of dense matter enter the equations of stellar structure [32].

There are two boundary conditions for the TOV equation. At $r = 0$, mass of the star becomes zero, $M(r = 0) = 0$, and the pressure in the center of the star can be an arbitrary value, $P(r = 0) = P_0 > 0$. Consider the pressure drop to zero, it cannot support overlaying material against the gravitational collapse exerted on it from the mass within and so marks the edge of the star. The point R where the pressure vanishes defines the radius of the star. Thus at the edge of the star, $r = R$, the pressure go to zero, $P(r = R) = 0$, and the mass of the star is then read off at this point, $M(r = R) = M_{total}$ [32, 33].

Electron and neutron are fermions, particles of half odd-integer spin. They obey the Pauli exclusion principle, not more than one fermion can occupy a given quantum state. We neglect all interactions so the simplest model for the equation of state of white dwarf and neutron star can be thought of as an ideal degenerate Fermi gas. Ideal in this context means that all interactions are ignored and degenerate means that all quantum states up to a given energy, called the Fermi energy, are occupied. Thus, in summing over the occupied states over the energies, we want to sum or integrate over momentum states. From quantum mechanics we recall normalization of momentum states in a box of dimension L [32], so that

$$\frac{1}{L^3}\sum_k \cdots \longrightarrow \int \frac{d^3k}{(2\pi)^3}\cdots = \frac{1}{2\pi^2}\int_0^{k_F} k^2 dk \cdots .$$

For degenerate systems, all energy states are filled in order up to the Fermi energy or in the case that momentum eigenstates are used, up to the Fermi momentum, and we let subscript F to denote the Fermi energy and momentum and set $c = \hbar = 1$. Then the energy density, pressure and number density at zero temperature are given by [32, 33]

$$\rho = \frac{g_s}{2\pi^2}\int_0^{k_F} dk k^2\sqrt{k^2+m^2}, \tag{2.43a}$$

$$P = \frac{1}{3}\frac{g_s}{2\pi^2}\int_0^{k_F} dk \frac{k^4}{\sqrt{k^2+m^2}}, \tag{2.43b}$$

$$n = \frac{g_s}{2\pi^2}\int_0^{k_F} dk k^2, \tag{2.43c}$$

where $g_s = (2s+1)$ denotes the degenerate spin state, for electron and neutron $s = \frac{1}{2}$ and $g_s = 2$, and the Fermi momentum k_F is related to the chemical potential μ via [32, 33]

$$\mu = \sqrt{k_F^2 + m^2}. \tag{2.44}$$

We can integrate analytically to find energy density, pressure and number density [32, 33], we have

$$\rho = \frac{1}{4\pi^2}\left(\mu k_F\left(\mu^2 - \frac{1}{2}m^2\right) - \frac{1}{2}m^4 \ln\left(\frac{\mu + k_F}{m}\right)\right), \tag{2.45a}$$

$$P = \frac{1}{12\pi^2}\left(\mu k_F\left(\mu^2 - \frac{5}{2}m^2\right) + \frac{3}{2}m^4 \ln\left(\frac{\mu + k_F}{m}\right)\right), \tag{2.45b}$$

$$n = \frac{k_F^3}{3\pi^2}. \tag{2.45c}$$

These equations of state are roughly approximation to easy solving. Indeed, at very high density, the nuclear force, especially the repulsive components, will become important. Electron would react with proton to form neutron via inverse beta decay or electron capture and the nuclear force therefore provides additional resistance to gravitational attraction beyond that provided by the neutron Fermi pressure. Nevertheless, the Fermi gas model for the equations of state use the fundamental idea, *i.e.* the role of the Pauli exclusion principle, to deal with dense matter and this model can be improved by (1) include the Fermi distribution at finite temperature, (2) consider many species of matter, (3) rotation of the star, (4) effect of the Coulomb force and magnetic field, (5) effect of nuclear interactions and finally (6) phase transitions, such as quark deconfinement or kaon condensation [19, 32]. So we can say that the main uncertainty in degenerate star model is the equation of state. After the work of Tolman, Oppenheimer and Volkoff, there are now many models of degenerate star giving different mass limits and other properties.

Unfortunately, when we consider the effects of strong interaction, the underlying theory for this interaction is the quantum chromodynamics (QCD). The perturbative method cannot be used in the strong-coupling regime observed at low temperature. However, we can use the holographic principle and the weak-strong duality to deal with this problem by performing calculations of the partition function in five dimensional Anti de Sitter space. Although we do not consider strong interaction in AdS_5 partition function, the effect of interaction has been included in four dimensional spacetime via duality. We use an ideal degenerate Fermi gas model in five dimensional Anti de Sitter space to study a degenerate star in the 5 dimensional AdS space. At present, the physical relevance of the 5 dimensional degenerate star in the AdS space to the realistic degenerate star in 4 dimension is still unclear [7]. In the future, we hope that we can use duality to describe behavior of a degenerate star in four dimensional spacetime.

2.5 Landau energy level

A particle that has a spin also has a certain intrinsic magnetic moment μ. The intrinsic magnetic moment of the electron is $\mu_B = -\frac{e\hbar}{2m}$, where e, $\hbar$, and m are electric charge of the electron, reduced Planck constant and mass of the electron, respectively. This quantity is called Bohr magneton. The magnetic moment of heavy particles is measured in nuclear magnetons, $\mu_N = \frac{e\hbar}{2m_p}$ where m_p is a mass of the proton. The intrinsic magnetic moment of the proton, μ_p, and neutron, μ_n, are found by experiment. The magnetic moment of the proton is parallel to the spin but the magnetic moment of the neutron is opposite to the spin [34].

In classical mechanics, Hamiltonian of a charged particle in an electromagnetic field is

$$H = \frac{1}{2m}\left(p - \frac{q}{c}A\right)^2 + q\phi, \tag{2.46}$$

where p, ϕ and A are generalized momentum of the particles, the electric scalar potential, and the components of the magnetic vector potential, respectively. If the particle has a spin, then the intrinsic magnetic moment of the particle interacts directly with the magnetic field. Since the spin is a purely quantum effect, it vanishes in the classical limit. In quantum mechanics, we include an extra term, $-\mu\cdot B$ corresponding to the energy of the magnetic moment μ in the magnetic field B, and promote dynamical variable to the operator [34]. Thus the Hamiltonian of a particle with a spin is

$$\hat{H} = \frac{1}{2m}\left(\hat{\mathbf{p}} - \frac{q}{c}\hat{\mathbf{A}}\right)^2 + q\phi - \hat{\mu}\cdot\mathbf{B}. \tag{2.47}$$

In general, we must keep in mind that the momentum operator $\hat{p}$ does not commute with the magnetic vector potential A to expanding the square. Let us consider a particle move in a uniform magnetic field [35]. Let $A_x = -By$, $A_y = A_z = 0 = \phi$ and the wave function in the form $\psi = e^{(\frac{i}{\hbar})(p_x x + p_z z)}\chi(y)$. Then we can obtain the expression for the energy levels of a particle in a uniform magnetic field [34]

$$E = (n + \frac{1}{2} + \sigma)\hbar\omega_H + \frac{p_z^2}{2m}, \tag{2.48}$$

where $\omega_H = \frac{|e|B}{mc}$ and $n = 0, 1, 2, \ldots$. Notice that the energy in the x-y plane gives the discrete energy values corresponding to motion in a plane perpendicular to the field, it is called Landau energy level [34].

Chapter III

HOLOGRAPHIC DEGENERATE STAR UNDER EXTERNAL MAGNETIC FIELD

Analytic calculations are performed rigorously in this chapter. We use the notion of holographic description to study a degenerate star. In de Boer et al's work [6, 7], they propose that composite operators in the CFT correspond to a degenerate Fermi gas in AdS space. Thus, we must extend Einstein equations from 4D to 5D in AdS space and find the coupled equations of motion to study properties of strongly coupled degenerate star. Moreover, we take into account the external magnetic field and finite temperature in our system. Energy's system is separated to be energy level, called Landau energy level, and equations of state of the system obey Fermi-Dirac statistics.

3.1 The equation of hydrostatic equilibrium for a spherical symmetric star in d dimensions

We solve Einstein's equation in d-dimensional. From the Einstein's equation, we have

$$G^{\mu}{}_{\nu} = R^{\mu}{}_{\nu} - g^{\mu}{}_{\nu}\frac{R}{2} = V_{d-2}C_{d-1}T^{\mu}{}_{\nu}, \tag{3.1}$$

where $R^{\mu}{}_{\nu}$, $g^{\mu}{}_{\nu}$, R, $T^{\mu}{}_{\nu}$, V_{d-2}, C_{d-1} are Ricci tensor, metric tensor, Ricci scalar, energy momentum tensor, the area of S^{d-2} and constant$\left(\frac{16\pi G}{(d-2)V_{d-2}c^4}\right)$, respectively. Momentum tensor's form is perfect fluid

$$T^{\mu}{}_{\nu} = \begin{pmatrix} \rho c^2 & & & & \\ & -P_r & & & \\ & & -P_{\theta_1} & & \\ & & & \ddots & \\ & & & & -P_{\theta_{d-2}} \end{pmatrix}, \tag{3.2}$$

and we use a spherically symmetric metric in d dimension [36]

$$\begin{aligned} ds^2 &= A(r)c^2dt^2 - B(r)dr^2 - r^2 d\Omega_{d-2}^2 \\ &= A(r)c^2dt^2 - B(r)dr^2 - r^2d\theta_1^2 - r^2\sin^2\theta_1\left(d\theta_2^2 + \cdots + \prod_{i=2}^{d-3}\sin^2\theta_i d\theta_{d-2}^2\right) \\ &= A(r)c^2dt^2 - B(r)dr^2 - r^2d\theta_1^2 - r^2\sin^2\theta_1\left(d\theta_2^2 + \sum_{j=3}^{d-2}\prod_{i=2}^{j-1}\sin^2\theta_i d\theta_j^2\right) \end{aligned} \quad (3.3)$$

Thus, the Lagrangian of this metric is given by

$$L = A(r)c^2\dot{t}^2 - B(r)\dot{r}^2 - r^2\dot{\theta}_1^2 - r^2\sin^2\theta_1\left(\dot{\theta}_2^2 + \sum_{j=3}^{d-2}\prod_{i=2}^{j-1}\sin^2\theta_i\dot{\theta}_j^2\right). \quad (3.4)$$

In this thesis, we use the functions $A(r)$ and $B(r)$ to find formulation but our calculation adjust them to $A(r)^2$ and $B(r)^2$, respectively. Use Lagrange's equation to find the equations of motion and affine connection

$$\partial_\tau\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q}. \quad (3.5)$$

In each component, we show first step, the equations of motion and affine connection, respectively. Consider time component, we have $\partial_\tau\left(\frac{\partial L}{\partial \dot{t}}\right) = \frac{\partial L}{\partial t}$,

$$\partial_\tau\left(2Ac^2\dot{t}\right) = 0,$$

we get the equation of motion in t component is inform

$$\ddot{t} + \frac{A'}{A}\dot{r}\dot{t} = 0, \quad (3.6)$$

and the affine connection in t component is

$$\Gamma^t{}_{rt} = \Gamma^t{}_{tr} = \frac{A'}{2A}. \quad (3.7)$$

Consider r component, we have $\partial_\tau\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r}$,

$$-2B'\dot{r}^2 - 2B\ddot{r} = A'c^2\dot{t}^2 - B'\dot{r}^2 - 2r\dot{\theta}_1^2 - 2r\sin^2\theta_1\left(\dot{\theta}_2^2 + \sum_{j=3}^{d-2}\prod_{i=2}^{j-1}\sin^2\theta_i\dot{\theta}_j^2\right),$$

we obtain the equation of motion in r component,

$$\ddot{r} + \frac{A'c^2}{2B}\dot{t}^2 + \frac{B'}{2B}\dot{r}^2 - \frac{r}{B}\dot{\theta}_1^2 - \frac{r\sin^2\theta_1}{B}\left(\dot{\theta}_2^2 + \sum_{j=3}^{d-2}\prod_{i=2}^{j-1}\sin^2\theta_i\dot{\theta}_j^2\right) = 0, \quad (3.8)$$

and the affine connection in r component is

$$\Gamma^r{}_{tt} = \frac{A'c^2}{2B}, \Gamma^r{}_{rr} = \frac{B'}{2B}, \Gamma^r{}_{\theta_1\theta_1} = \frac{-r}{B}, \Gamma^r{}_{\theta_2\theta_2} = \frac{-r\sin^2\theta_1}{B},$$

$$\ldots, \Gamma^r{}_{\theta_j\theta_j} = \frac{-r\sin^2\theta_1}{B}\prod_{i=2}^{j-1}\sin^2\theta_i. \quad (3.9)$$

Consider θ_1 component, we have $\partial_\tau\left(\frac{\partial L}{\partial\dot{\theta}_1}\right)=\frac{\partial L}{\partial\theta_1}$,

$$-\left(2r\dot{r}\dot{\theta}_1+r^2\ddot{\theta}_1\right)=-r^2\sin\theta_1\cos\theta_1\left(\dot{\theta}_2^2+\sum_{j=3}^{d-2}\prod_{i=2}^{j-1}\sin^2\theta_i\dot{\theta}_j^2\right),$$

we get the equation of motion in θ_1 component is inform

$$\ddot{\theta}_1+\frac{2}{r}\dot{r}\dot{\theta}_1-\sin\theta_1\cos\theta_1\left(\dot{\theta}_2^2+\sum_{j=3}^{d-2}\prod_{i=2}^{j-1}\sin^2\theta_i\dot{\theta}_j^2\right)=0, \tag{3.10}$$

and the affine connection in θ_1 component is

$$\begin{aligned}\Gamma^{\theta_1}{}_{r\theta_1}=\Gamma^{\theta_1}{}_{\theta_1 r}=\frac{1}{r},\Gamma^{\theta_1}{}_{\theta_2\theta_2}&=-\sin\theta_1\cos\theta_1,\\ \ldots,\Gamma^{\theta_1}{}_{\theta_j\theta_j}&=-\sin\theta_1\cos\theta_1\prod_{i=2}^{j-1}\sin^2\theta_i,\end{aligned} \tag{3.11}$$

which $3\leqslant j\leqslant d-2$. Consider θ_2 component, we have $\partial_\tau\left(\frac{\partial L}{\partial\dot{\theta}_2}\right)=\frac{\partial L}{\partial\theta_2}$,

$$\begin{aligned}&-\left(2r\dot{r}\sin^2\theta_1\dot{\theta}_2+2r^2\sin\theta_1\cos\theta_1\dot{\theta}_1\dot{\theta}_2+r^2\sin^2\theta_1\ddot{\theta}_2\right)\\ &\qquad=-r^2\sin^2\theta_1\sin\theta_2\cos\theta_2\sum_{j=3}^{d-2}\prod_{i=2}^{j-1}\sin^2\theta_i\dot{\theta}_j^2,\end{aligned}$$

we get the equation of motion in θ_2 component is inform

$$\ddot{\theta}_2+\frac{2}{r}\dot{r}\dot{\theta}_2+2\cot\theta_1\dot{\theta}_1\dot{\theta}_2-\sin\theta_2\cos\theta_2\sum_{j=4}^{d-2}\prod_{i=3}^{j-1}\sin^2\theta_i\dot{\theta}_j^2\ =0, \tag{3.12}$$

and the affine connection in θ_2 component is

$$\begin{aligned}\Gamma^{\theta_2}{}_{r\theta_2}=\Gamma^{\theta_2}{}_{\theta_2 r}=\frac{1}{r},\Gamma^{\theta_2}{}_{\theta_1\theta_2}&=\Gamma^{\theta_2}{}_{\theta_2\theta_1}=\cot\theta_1,\\ \ldots,\Gamma^{\theta_2}{}_{\theta_j\theta_j}&=-\sin\theta_2\cos\theta_2\prod_{i=3}^{j-1}\sin^2\theta_i.\end{aligned} \tag{3.13}$$

which $4\leqslant j\leqslant d-2$. Consider θ_j component, we have $\partial_\tau\left(\frac{\partial L}{\partial\dot{\theta}_j}\right)=\frac{\partial L}{\partial\theta_j}$ which $j\geqslant 3$

$$\begin{aligned}&2r\dot{r}\sin^2\theta_1\prod_{i=2}^{j-1}\sin^2\theta_i\dot{\theta}_j+2r^2\sin\theta_1\cos\theta_1\prod_{i=2}^{j-1}\sin^2\theta_i\dot{\theta}_1\dot{\theta}_j\\ &+2r^2\sin^2\theta_1\sum_{l=2}^{j-1}\prod_{\substack{i=2\\ i\neq l}}^{j-1}\sin^2\theta_i\sin\theta_l\cos\theta_l\dot{\theta}_l\dot{\theta}_j+r^2\sin^2\theta_1\prod_{i=2}^{j-1}\sin^2\theta_i\ddot{\theta}_j\\ &=r^2\sin^2\theta_1\sum_{k=j+1}^{d-2}\prod_{\substack{i=2\\ i\neq j}}^{k-1}\sin^2\theta_i\sin\theta_j\cos\theta_j\dot{\theta}_k^2,\end{aligned}$$

we get the equation of motion in θ_j component is inform

$$\ddot{\theta_j} + \frac{2}{r}\dot{r}\dot{\theta}_j + 2\cot\theta_1\dot{\theta}_1\dot{\theta}_j + \frac{2\sum_{l=2}^{j-1}\prod_{\substack{i=2\\i\neq l}}^{j-1}\sin\theta_l\cos\theta_l\sin^2\theta_i}{\prod_{i=2}^{j-1}\sin^2\theta_i}\dot{\theta}_l\dot{\theta}_j$$
$$-\sum_{k=j+1}^{d-2}\prod_{i=j+1}^{k-1}\sin\theta_j\cos\theta_j\sin^2\theta_i\dot{\theta}_k^2 = 0, \tag{3.14}$$

and the affine connection in θ_j component is

$$\Gamma^{\theta_j}{}_{r\theta_j} = \Gamma^{\theta_j}{}_{\theta_j r} = \frac{1}{r}, \Gamma^{\theta_j}{}_{\theta_1\theta_j} = \Gamma^{\theta_j}{}_{\theta_j\theta_1} = \cot\theta_1, \Gamma^{\theta_j}{}_{\theta_l\theta_j} = \Gamma^{\theta_j}{}_{\theta_j\theta_l}$$
$$= \frac{\prod_{\substack{i=2\\i\neq l}}^{j-1}\sin\theta_l\cos\theta_l\sin^2\theta_i}{\prod_{i=2}^{j-1}\sin^2\theta_i} = \cot\theta_l, \Gamma^{\theta_j}{}_{\theta_k\theta_k} = -\sin\theta_j\cos\theta_j\prod_{i=j+1}^{k-1}\sin^2\theta_i, \tag{3.15}$$

which $2 \leqslant l \leqslant j-1$ and $j+1 \leqslant k \leqslant d-2$. Find Ricci scalar from

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu\Gamma^{\rho}{}_{\nu\sigma} - \partial_\nu\Gamma^{\rho}{}_{\mu\sigma} + \Gamma^{\rho}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\nu\sigma} - \Gamma^{\rho}{}_{\nu\lambda}\Gamma^{\lambda}{}_{\mu\sigma}, \tag{3.16a}$$
$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}, \tag{3.16b}$$
$$R = R^{\mu}{}_{\mu} = g^{\mu\nu}R_{\mu\nu}. \tag{3.16c}$$

For the component t

$$R_{tt} = R^{\lambda}{}_{t\lambda t} = R^{t}{}_{ttt} + R^{r}{}_{trt} + R^{\theta_1}{}_{t\theta_1 t} + R^{\theta_2}{}_{t\theta_2 t} + \ldots + R^{\theta_i}{}_{t\theta_i t} + \ldots + R^{\theta_{d-2}}{}_{t\theta_{d-2}t},$$

$$R^{t}{}_{ttt} = \partial_t\Gamma^{t}{}_{tt} - \partial_t\Gamma^{t}{}_{tt} + \Gamma^{t}{}_{t\lambda}\Gamma^{\lambda}{}_{tt} - \Gamma^{t}{}_{t\lambda}\Gamma^{\lambda}{}_{tt} = 0,$$

$$\begin{aligned} R^{r}{}_{trt} &= \partial_r\Gamma^{r}{}_{tt} - \partial_t\Gamma^{r}{}_{rt} + \Gamma^{r}{}_{r\lambda}\Gamma^{\lambda}{}_{tt} - \Gamma^{r}{}_{t\lambda}\Gamma^{\lambda}{}_{rt}, \\ &= \left(\frac{A''c^2}{2B} - \frac{A'B'c^2}{2B^2}\right) + \left(\frac{A'B'c^2}{4B^2}\right) - \left(\frac{(A')^2c^2}{4AB}\right), \end{aligned}$$

$$\begin{aligned} R^{\theta_1}{}_{t\theta_1 t} &= \partial_{\theta_1}\Gamma^{\theta_1}{}_{tt} - \partial_t\Gamma^{\theta_1}{}_{\theta_1 t} + \Gamma^{\theta_1}{}_{\theta_1\lambda}\Gamma^{\lambda}{}_{tt} - \Gamma^{\theta_1}{}_{t\lambda}\Gamma^{\lambda}{}_{\theta_1 t}, \\ &= \left(\frac{A'c^2}{2rB}\right), \end{aligned}$$

$$\begin{aligned} R^{\theta_2}{}_{t\theta_2 t} &= \partial_{\theta_2}\Gamma^{\theta_2}{}_{tt} - \partial_t\Gamma^{\theta_2}{}_{\theta_2 t} + \Gamma^{\theta_2}{}_{\theta_2\lambda}\Gamma^{\lambda}{}_{tt} - \Gamma^{\theta_2}{}_{t\lambda}\Gamma^{\lambda}{}_{\theta_2 t}, \\ &= \left(\frac{A'c^2}{2rB}\right), \end{aligned}$$

$$\begin{aligned} R^{\theta_i}{}_{t\theta_i t} &= \partial_{\theta_i}\Gamma^{\theta_i}{}_{tt} - \partial_t\Gamma^{\theta_i}{}_{\theta_i t} + \Gamma^{\theta_i}{}_{\theta_i\lambda}\Gamma^{\lambda}{}_{tt} - \Gamma^{\theta_i}{}_{t\lambda}\Gamma^{\lambda}{}_{\theta_i t}, \\ &= \left(\frac{A'c^2}{2rB}\right), \end{aligned}$$

then

$$R_{tt} = \frac{A''c^2}{2B} - \frac{A'B'c^2}{4B^2} - \frac{(A')^2 c^2}{4AB} + (d-2)\frac{A'c^2}{2rB}, \tag{3.17a}$$

$$g^{tt}R_{tt} = \frac{A''}{2AB} - \frac{A'B'}{4AB^2} - \frac{(A')^2}{4A^2B} + (d-2)\frac{A'}{2rAB}. \tag{3.17b}$$

For the component r

$$R_{rr} = R^{\lambda}{}_{r\lambda r} = R^{t}{}_{rtr} + R^{r}{}_{rrr} + R^{\theta_1}{}_{r\theta_1 r} + R^{\theta_2}{}_{r\theta_2 r} + \ldots + R^{\theta_i}{}_{r\theta_i r} + \ldots + R^{\theta_{d-2}}{}_{r\theta_{d-2} r}$$

$$\begin{aligned} R^{t}{}_{rtr} &= \partial_t \Gamma^{t}{}_{rr} - \partial_r \Gamma^{t}{}_{tr} + \Gamma^{t}{}_{t\lambda}\Gamma^{\lambda}{}_{rr} - \Gamma^{t}{}_{r\lambda}\Gamma^{\lambda}{}_{tr}, \\ &= -\left(\frac{A''}{2A} - \frac{(A')^2}{2A^2}\right) + \left(\frac{A'B'}{4AB}\right) - \left(\frac{(A')^2}{4A^2}\right) = \frac{(A')^2}{4A^2} - \frac{A''}{2A} + \frac{A'B'}{4AB}, \end{aligned}$$

$$R^{r}{}_{rrr} = \partial_r \Gamma^{r}{}_{rr} - \partial_r \Gamma^{r}{}_{rr} + \Gamma^{r}{}_{r\lambda}\Gamma^{\lambda}{}_{rr} - \Gamma^{r}{}_{r\lambda}\Gamma^{\lambda}{}_{rr} = 0,$$

$$\begin{aligned} R^{\theta_1}{}_{r\theta_1 r} &= \partial_{\theta_1} \Gamma^{\theta_1}{}_{rr} - \partial_r \Gamma^{\theta_1}{}_{\theta_1 r} + \Gamma^{\theta_1}{}_{\theta_1\lambda}\Gamma^{\lambda}{}_{rr} - \Gamma^{\theta_1}{}_{r\lambda}\Gamma^{\lambda}{}_{\theta_1 r}, \\ &= -\left(-\frac{1}{r^2}\right) + \left(\frac{B'}{2rB}\right) - \left(\frac{1}{r^2}\right) = \frac{B'}{2rB}, \end{aligned}$$

$$\begin{aligned} R^{\theta_2}{}_{r\theta_2 r} &= \partial_{\theta_2} \Gamma^{\theta_2}{}_{rr} - \partial_r \Gamma^{\theta_2}{}_{\theta_2 r} + \Gamma^{\theta_2}{}_{\theta_2\lambda}\Gamma^{\lambda}{}_{rr} - \Gamma^{\theta_2}{}_{r\lambda}\Gamma^{\lambda}{}_{\theta_2 r}, \\ &= -\left(-\frac{1}{r^2}\right) + \left(\frac{B'}{2rB}\right) - \left(\frac{1}{r^2}\right) = \frac{B'}{2rB}, \end{aligned}$$

$$\begin{aligned} R^{\theta_i}{}_{r\theta_i r} &= \partial_{\theta_i} \Gamma^{\theta_i}{}_{rr} - \partial_r \Gamma^{\theta_i}{}_{\theta_i r} + \Gamma^{\theta_i}{}_{\theta_i\lambda}\Gamma^{\lambda}{}_{rr} - \Gamma^{\theta_i}{}_{r\lambda}\Gamma^{\lambda}{}_{\theta_i r}, \\ &= -\left(-\frac{1}{r^2}\right) + \left(\frac{B'}{2rB}\right) - \left(\frac{1}{r^2}\right) = \frac{B'}{2rB}, \end{aligned}$$

then

$$R_{rr} = \frac{(A')^2}{4A^2} - \frac{A''}{2A} + \frac{A'B'}{4AB} + (d-2)\frac{B'}{2rB}, \tag{3.18a}$$

$$g^{rr}R_{rr} = \frac{A''}{2AB} - \frac{(A')^2}{4A^2B} - \frac{A'B'}{4AB^2} - (d-2)\frac{B'}{2rB^2}. \tag{3.18b}$$

For the component θ_1

$$\begin{aligned} R_{\theta_1\theta_1} &= R^{\lambda}{}_{\theta_1\lambda\theta_1} \\ &= R^{t}{}_{\theta_1 t\theta_1} + R^{r}{}_{\theta_1 r\theta_1} + R^{\theta_1}{}_{\theta_1\theta_1\theta_1} + R^{\theta_2}{}_{\theta_1\theta_2\theta_1} + \ldots + R^{\theta_i}{}_{\theta_1\theta_i\theta_1} + \ldots + R^{\theta_{d-2}}{}_{\theta_1\theta_{d-2}\theta_1}, \end{aligned}$$

$$\begin{aligned} R^{t}{}_{\theta_1 t\theta_1} &= \partial_t \Gamma^{t}{}_{\theta_1\theta_1} - \partial_{\theta_1} \Gamma^{t}{}_{t\theta_1} + \Gamma^{t}{}_{t\lambda}\Gamma^{\lambda}{}_{\theta_1\theta_1} - \Gamma^{t}{}_{\theta_1\lambda}\Gamma^{\lambda}{}_{t\theta_1}, \\ &= -\left(-\frac{rA'}{2AB}\right), \end{aligned}$$

$$\begin{aligned} R^{r}{}_{\theta_1 r\theta_1} &= \partial_r\Gamma^{r}{}_{\theta_1\theta_1} - \partial_{\theta_1}\Gamma^{r}{}_{r\theta_1} + \Gamma^{r}{}_{r\lambda}\Gamma^{\lambda}{}_{\theta_1\theta_1} - \Gamma^{r}{}_{\theta_1\lambda}\Gamma^{\lambda}{}_{r\theta_1}, \\ &= \left(-\frac{1}{B} + \frac{rB'}{B^2}\right) + \left(-\frac{rB'}{2B^2}\right) - \left(-\frac{1}{B}\right) = \frac{rB'}{2B^2}, \end{aligned}$$

$$R^{\theta_1}{}_{\theta_1\theta_1\theta_1} = \partial_{\theta_1}\Gamma^{\theta_1}{}_{\theta_1\theta_1} - \partial_{\theta_1}\Gamma^{\theta_1}{}_{\theta_1\theta_1} + \Gamma^{\theta_1}{}_{\theta_1\lambda}\Gamma^{\lambda}{}_{\theta_1\theta_1} - \Gamma^{\theta_1}{}_{\theta_1\lambda}\Gamma^{\lambda}{}_{\theta_1\theta_1} = 0,$$

$$\begin{aligned} R^{\theta_2}{}_{\theta_1\theta_2\theta_1} &= \partial_{\theta_2}\Gamma^{\theta_2}{}_{\theta_1\theta_1} - \partial_{\theta_1}\Gamma^{\theta_2}{}_{\theta_2\theta_1} + \Gamma^{\theta_2}{}_{\theta_2\lambda}\Gamma^{\lambda}{}_{\theta_1\theta_1} - \Gamma^{\theta_2}{}_{\theta_1\lambda}\Gamma^{\lambda}{}_{\theta_2\theta_1}, \\ &= -\left(-\csc^2\theta_1\right) + \left(-\frac{1}{B}\right) - \left(\cot^2\theta_1\right) = 1 - \frac{1}{B}, \end{aligned}$$

$$\begin{aligned} R^{\theta_i}{}_{\theta_1\theta_i\theta_1} &= \partial_{\theta_i}\Gamma^{\theta_i}{}_{\theta_1\theta_1} - \partial_{\theta_1}\Gamma^{\theta_i}{}_{\theta_i\theta_1} + \Gamma^{\theta_i}{}_{\theta_i\lambda}\Gamma^{\lambda}{}_{\theta_1\theta_1} - \Gamma^{\theta_i}{}_{\theta_1\lambda}\Gamma^{\lambda}{}_{\theta_i\theta_1}, \\ &= -\left(-\csc^2\theta_1\right) + \left(-\frac{1}{B}\right) - \left(\cot^2\theta_1\right) = 1 - \frac{1}{B}, \end{aligned}$$

then

$$R_{\theta_1\theta_1} = -\frac{rA'}{2AB} + \frac{rB'}{2B^2} + (d-3)\left(1 - \frac{1}{B}\right), \tag{3.19a}$$

$$g^{\theta_1\theta_1}R_{\theta_1\theta_1} = \frac{A'}{2rAB} - \frac{B'}{2rB^2} - \frac{(d-3)}{r^2}\left(1 - \frac{1}{B}\right). \tag{3.19b}$$

For the component θ_2

$$\begin{aligned} R_{\theta_2\theta_2} &= R^{\lambda}{}_{\theta_2\lambda\theta_2} \\ &= R^{t}{}_{\theta_2 t\theta_2} + R^{r}{}_{\theta_2 r\theta_2} + R^{\theta_1}{}_{\theta_2\theta_1\theta_2} + R^{\theta_2}{}_{\theta_2\theta_2\theta_2} + \ldots + R^{\theta_i}{}_{\theta_2\theta_i\theta_2} + \ldots + R^{\theta_{d-2}}{}_{\theta_2\theta_{d-2}\theta_2}, \end{aligned}$$

$$\begin{aligned} R^{t}{}_{\theta_2 t\theta_2} &= \partial_t\Gamma^{t}{}_{\theta_2\theta_2} - \partial_{\theta_2}\Gamma^{t}{}_{t\theta_2} + \Gamma^{t}{}_{t\lambda}\Gamma^{\lambda}{}_{\theta_2\theta_2} - \Gamma^{t}{}_{\theta_2\lambda}\Gamma^{\lambda}{}_{t\theta_2}, \\ &= \left(-\frac{rA'\sin^2\theta_1}{2AB}\right), \end{aligned}$$

$$\begin{aligned} R^{r}{}_{\theta_2 r\theta_2} &= \partial_r\Gamma^{r}{}_{\theta_2\theta_2} - \partial_{\theta_2}\Gamma^{r}{}_{r\theta_2} + \Gamma^{r}{}_{r\lambda}\Gamma^{\lambda}{}_{\theta_2\theta_2} - \Gamma^{r}{}_{\theta_2\lambda}\Gamma^{\lambda}{}_{r\theta_2}, \\ &= \left(-\frac{\sin^2\theta_1}{B} + \frac{rB'\sin^2\theta_1}{B^2}\right) + \left(-\frac{rB'\sin^2\theta_1}{2B^2}\right) - \left(-\frac{\sin^2\theta_1}{B}\right) \\ &= \frac{rB'\sin^2\theta_1}{2B^2}, \end{aligned}$$

$$\begin{aligned} R^{\theta_1}{}_{\theta_2\theta_1\theta_2} &= \partial_{\theta_1}\Gamma^{\theta_1}{}_{\theta_2\theta_2} - \partial_{\theta_2}\Gamma^{\theta_1}{}_{\theta_1\theta_2} + \Gamma^{\theta_1}{}_{\theta_1\lambda}\Gamma^{\lambda}{}_{\theta_2\theta_2} - \Gamma^{\theta_1}{}_{\theta_2\lambda}\Gamma^{\lambda}{}_{\theta_1\theta_2}, \\ &= \left(-\cos^2\theta_1 + \sin^2\theta_1\right) + \left(-\frac{\sin^2\theta_1}{B}\right) - \left(-\cos^2\theta_1\right) = \sin^2\theta_1\left(1 - \frac{1}{B}\right), \end{aligned}$$

$$R^{\theta_2}{}_{\theta_2\theta_2\theta_2} = \partial_{\theta_2}\Gamma^{\theta_2}{}_{\theta_2\theta_2} - \partial_{\theta_2}\Gamma^{\theta_2}{}_{\theta_2\theta_2} + \Gamma^{\theta_2}{}_{\theta_2\lambda}\Gamma^{\lambda}{}_{\theta_2\theta_2} - \Gamma^{\theta_2}{}_{\theta_2\lambda}\Gamma^{\lambda}{}_{\theta_2\theta_2} = 0,$$

$$\begin{aligned} R^{\theta_i}{}_{\theta_2\theta_i\theta_2} &= \partial_{\theta_i}\Gamma^{\theta_i}{}_{\theta_2\theta_2} - \partial_{\theta_2}\Gamma^{\theta_i}{}_{\theta_i\theta_2} + \Gamma^{\theta_i}{}_{\theta_i\lambda}\Gamma^{\lambda}{}_{\theta_2\theta_2} - \Gamma^{\theta_i}{}_{\theta_2\lambda}\Gamma^{\lambda}{}_{\theta_i\theta_2}, \\ &= -\left(-\csc^2\theta_2\right) + \left(-\frac{\sin^2\theta_1}{B} - \cos^2\theta_1\right) - \left(\cot^2\theta_2\right) = \sin^2\theta_1\left(1-\frac{1}{B}\right), \end{aligned}$$

then

$$R_{\theta_2\theta_2} = -\frac{rA'\sin^2\theta_1}{2AB} + \frac{rB'\sin^2\theta_1}{2B^2} + (d-3)\sin^2\theta_1\left(1-\frac{1}{B}\right), \tag{3.20a}$$

$$g^{\theta_2\theta_2}R_{\theta_2\theta_2} = \frac{A'}{2rAB} - \frac{B'}{2rB^2} - \frac{(d-3)}{r^2}\left(1-\frac{1}{B}\right). \tag{3.20b}$$

For the component θ_i, $i \geqslant 3$

$$\begin{aligned} R_{\theta_i\theta_i} &= R^{\lambda}{}_{\theta_i\lambda\theta_i} \\ &= R^{t}{}_{\theta_i t\theta_i} + R^{r}{}_{\theta_i r\theta_i} + R^{\theta_1}{}_{\theta_i\theta_1\theta_i} + R^{\theta_2}{}_{\theta_i\theta_2\theta_i} + \ldots + R^{\theta_j}{}_{\theta_i\theta_j\theta_i} + \ldots + R^{\theta_{d-2}}{}_{\theta_i\theta_{d-2}\theta_i} \end{aligned}$$

$$\begin{aligned} R^{t}{}_{\theta_i t\theta_i} &= \partial_t\Gamma^{t}{}_{\theta_i\theta_i} - \partial_{\theta_i}\Gamma^{t}{}_{t\theta_i} + \Gamma^{t}{}_{t\lambda}\Gamma^{\lambda}{}_{\theta_i\theta_i} - \Gamma^{t}{}_{\theta_i\lambda}\Gamma^{\lambda}{}_{t\theta_i}, \\ &= \left(-\frac{rA'\sin^2\theta_1}{2AB}\right)\prod_{k=2}^{i-1}\sin^2\theta_k, \end{aligned}$$

$$\begin{aligned} R^{r}{}_{\theta_i r\theta_i} &= \partial_r\Gamma^{r}{}_{\theta_i\theta_i} - \partial_{\theta_i}\Gamma^{r}{}_{r\theta_i} + \Gamma^{r}{}_{r\lambda}\Gamma^{\lambda}{}_{\theta_i\theta_i} - \Gamma^{r}{}_{\theta_i\lambda}\Gamma^{\lambda}{}_{r\theta_i}, \\ &= \left(-\frac{\sin^2\theta_1}{B}\prod_{k=2}^{i-1}\sin^2\theta_k + \frac{rB'\sin^2\theta_1}{B^2}\prod_{k=2}^{i-1}\sin^2\theta_k\right) \\ &\quad + \left(-\frac{rB'\sin^2\theta_1}{2B^2}\prod_{k=2}^{i-1}\sin^2\theta_k\right) - \left(-\frac{\sin^2\theta_1}{B}\prod_{k=2}^{i-1}\sin^2\theta_k\right), \\ &= \frac{rB'\sin^2\theta_1}{2B^2}\prod_{k=2}^{i-1}\sin^2\theta_k, \end{aligned}$$

$$\begin{aligned} R^{\theta_1}{}_{\theta_i\theta_1\theta_i} &= \partial_{\theta_1}\Gamma^{\theta_1}{}_{\theta_i\theta_i} - \partial_{\theta_i}\Gamma^{\theta_1}{}_{\theta_1\theta_i} + \Gamma^{\theta_1}{}_{\theta_1\lambda}\Gamma^{\lambda}{}_{\theta_i\theta_i} - \Gamma^{\theta_1}{}_{\theta_i\lambda}\Gamma^{\lambda}{}_{\theta_1\theta_i}, \\ &= \left(-\cos^2\theta_1\prod_{k=2}^{i-1}\sin^2\theta_k + \sin^2\theta_1\prod_{k=2}^{i-1}\sin^2\theta_k\right) \\ &\quad + \left(-\frac{\sin^2\theta_1}{B}\prod_{k=2}^{i-1}\sin^2\theta_k\right) - \left(-\cos^2\theta_1\prod_{k=2}^{i-1}\sin^2\theta_k\right), \\ &= \sin^2\theta_1\prod_{k=2}^{i-1}\sin^2\theta_k\left(1-\frac{1}{B}\right), \end{aligned}$$

$$
\begin{aligned}
R^{\theta_2}{}_{\theta_i\theta_2\theta_i} &= \partial_{\theta_2}\Gamma^{\theta_2}{}_{\theta_i\theta_i} - \partial_{\theta_i}\Gamma^{\theta_2}{}_{\theta_2\theta_i} + \Gamma^{\theta_2}{}_{\theta_2\lambda}\Gamma^{\lambda}{}_{\theta_i\theta_i} - \Gamma^{\theta_2}{}_{\theta_i\lambda}\Gamma^{\lambda}{}_{\theta_2\theta_i}, \\
&= \left(-\cos^2\theta_2\prod_{k=3}^{i-1}\sin^2\theta_k + \sin^2\theta_2\prod_{k=3}^{i-1}\sin^2\theta_k\right) \\
&\quad + \left(-\frac{\sin^2\theta_1}{B}\prod_{k=2}^{i-1}\sin^2\theta_k - \cos^2\theta_1\prod_{k=2}^{i-1}\sin^2\theta_k\right) - \left(-\cos^2\theta_2\prod_{k=3}^{i-1}\sin^2\theta_k\right), \\
&= \sin^2\theta_1\prod_{k=2}^{i-1}\sin^2\theta_k\left(1-\frac{1}{B}\right),
\end{aligned}
$$

we divide $R^{\theta_j}{}_{\theta_i\theta_j\theta_i}$ in three case. Case I, $\theta_j = \theta_i$, we have

$$
R^{\theta_i}{}_{\theta_i\theta_i\theta_i} = \partial_{\theta_i}\Gamma^{\theta_i}{}_{\theta_i\theta_i} - \partial_{\theta_i}\Gamma^{\theta_i}{}_{\theta_i\theta_i} + \Gamma^{\theta_i}{}_{\theta_i\lambda}\Gamma^{\lambda}{}_{\theta_i\theta_i} - \Gamma^{\theta_i}{}_{\theta_i\lambda}\Gamma^{\lambda}{}_{\theta_i\theta_i} = 0,
$$

Case II, $j < i$, we have

$$
\begin{aligned}
R^{\theta_j}{}_{\theta_i\theta_j\theta_i} &= \partial_{\theta_j}\Gamma^{\theta_j}{}_{\theta_i\theta_i} - \partial_{\theta_i}\Gamma^{\theta_j}{}_{\theta_j\theta_i} + \Gamma^{\theta_j}{}_{\theta_j\lambda}\Gamma^{\lambda}{}_{\theta_i\theta_i} - \Gamma^{\theta_j}{}_{\theta_i\lambda}\Gamma^{\lambda}{}_{\theta_j\theta_i}, \\
&= \left(-\cos^2\theta_j\prod_{k=j+1}^{i-1}\sin^2\theta_k + \sin^2\theta_j\prod_{k=j+1}^{i-1}\sin^2\theta_k\right) \\
&\quad + \left(-\frac{\sin^2\theta_1}{B}\prod_{k=2}^{i-1}\sin^2\theta_k - \cos^2\theta_1\prod_{k=2}^{i-1}\sin^2\theta_k - \cos^2\theta_2\prod_{k=3}^{i-1}\sin^2\theta_k\right. \\
&\quad \left. -\cos^2\theta_3\prod_{k=4}^{i-1}\sin^2\theta_k - \ldots - \cos^2\theta_{j-1}\prod_{k=j}^{i-1}\sin^2\theta_k\right) \\
&\quad - \left(-\cos^2\theta_j\prod_{k=j+1}^{i-1}\sin^2\theta_k\right), \\
&= \sin^2\theta_1\prod_{k=2}^{i-1}\sin^2\theta_k\left(1-\frac{1}{B}\right),
\end{aligned}
$$

Case III, $j > i$, we have

$$
\begin{aligned}
R^{\theta_j}{}_{\theta_i\theta_j\theta_i} &= \partial_{\theta_j}\Gamma^{\theta_j}{}_{\theta_i\theta_i} - \partial_{\theta_i}\Gamma^{\theta_j}{}_{\theta_j\theta_i} + \Gamma^{\theta_j}{}_{\theta_j\lambda}\Gamma^{\lambda}{}_{\theta_i\theta_i} - \Gamma^{\theta_j}{}_{\theta_i\lambda}\Gamma^{\lambda}{}_{\theta_j\theta_i} \\
&= -\left(-\csc^2\theta_i\right) + \left(-\frac{\sin^2\theta_1}{B}\prod_{k=2}^{i-1}\sin^2\theta_k - \cos^2\theta_1\prod_{k=2}^{i-1}\sin^2\theta_k - \right. \\
&\quad \left. \cos^2\theta_2\prod_{k=3}^{i-1}\sin^2\theta_k - \cos^2\theta_3\prod_{k=4}^{i-1}\sin^2\theta_k - \ldots - \cos^2\theta_{i-1}\right) - \left(\cot^2\theta_i\right), \\
&= \sin^2\theta_1\prod_{k=2}^{i-1}\sin^2\theta_k\left(1-\frac{1}{B}\right),
\end{aligned}
$$

then we have

$$R_{\theta_i\theta_i} = -\frac{rA'\sin^2\theta_1}{2AB}\prod_{k=2}^{i-1}\sin^2\theta_k + \frac{rB'\sin^2\theta_1}{2B^2}\prod_{k=2}^{i-1}\sin^2\theta_k + (d-3)\sin^2\theta_1\prod_{k=2}^{i-1}\sin^2\theta_k\left(1-\frac{1}{B}\right), \tag{3.21a}$$

$$g^{\theta_i\theta_i}R_{\theta_i\theta_i} = \frac{A'}{2rAB} - \frac{B'}{2rB^2} - \frac{(d-3)}{r^2}\left(1-\frac{1}{B}\right). \tag{3.21b}$$

Thus

$$\begin{aligned}
R^t{}_t &= \frac{A''}{2AB} - \frac{A'B'}{4AB^2} - \frac{(A')^2}{4A^2B} + (d-2)\frac{A'}{2rAB},\\
R^r{}_r &= \frac{A''}{2AB} - \frac{A'B'}{4AB^2} - \frac{(A')^2}{4A^2B} + (d-2)\frac{B'}{2rB^2},\\
R^{\theta_1}{}_{\theta_1} &= \frac{A'}{2rAB} - \frac{B'}{2rB^2} - \frac{(d-3)}{r^2}\left(1-\frac{1}{B}\right),\\
R^{\theta_2}{}_{\theta_2} &= \frac{A'}{2rAB} - \frac{B'}{2rB^2} - \frac{(d-3)}{r^2}\left(1-\frac{1}{B}\right),\\
R^{\theta_i}{}_{\theta_i} &= \frac{A'}{2rAB} - \frac{B'}{2rB^2} - \frac{(d-3)}{r^2}\left(1-\frac{1}{B}\right).
\end{aligned}$$

Consider $G^t{}_t = R^t{}_t - \frac{g^t{}_t}{2}\left(R^t{}_t + R^r{}_r + R^{\theta_1}{}_{\theta_1} + R^{\theta_2}{}_{\theta_2} + \ldots + R^{\theta_i}{}_{\theta_i} + \ldots + R^{\theta_{d-2}}{}_{\theta_{d-2}}\right) = V_{d-2}C_{d-1}T^t{}_t \to R^t{}_t - \left(R^r{}_r + \ldots + R^{\theta_{d-2}}{}_{\theta_{d-2}}\right) = 2V_{d-2}C_{d-1}\rho c^2$, then

$$(d-2)\frac{B'}{rB^2} + \frac{(d-2)(d-3)}{r^2}\left(1-\frac{1}{B}\right) = 2V_{d-2}C_{d-1}\rho c^2, \tag{3.22}$$

$$B' - \frac{(d-3)}{r}B = B^2\left(\frac{2rV_{d-2}C_{d-1}\rho c^2}{(d-2)} - \frac{(d-3)}{r}\right). \tag{3.23}$$

Change $B \to B^2$, so

$$B' - \frac{(d-3)}{2r}B = B^3\left(\frac{rV_{d-2}C_{d-1}\rho c^2}{(d-2)} - \frac{(d-3)}{2r}\right). \tag{3.24}$$

Let $b = B^{1-n}$ and $n = 3$, we have $b = B^{1-3} = B^{-2}$ moreover, $\frac{db}{dr} = \frac{(-2)}{B^3}\frac{dB}{dr} \to \frac{dB}{dr} = \frac{B^3}{(-2)}\frac{db}{dr}$. So

$$\frac{B^3}{(-2)}\frac{db}{dr} - \frac{(d-3)}{2r}B = B^3\left(\frac{rV_{d-2}C_{d-1}\rho c^2}{(d-2)} - \frac{(d-3)}{2r}\right),$$

$$b' + (d-3)\frac{b}{r} = \left(\frac{(d-3)}{r} - \frac{2rV_{d-2}C_{d-1}\rho c^2}{(d-2)}\right). \tag{3.25}$$

Put equation (3.25)$\times r^{d-3}$, we have

$$(d-2)\,r^{d-3}b' + (d-3)\,(d-2)\,br^{d-4} = \left((d-3)\,(d-2)\,r^{(d-4)} - 2r^{d-2}V_{d-2}C_{d-1}\rho c^2\right),$$

or

$$\left(r^{(d-3)}b\right)' = \left((d-3)\,r^{(d-4)} - \frac{2r^{d-2}V_{d-2}C_{d-1}\rho c^2}{(d-2)}\right).$$

We get b in form

$$b = 1 - \frac{2c^2V_{d-2}C_{d-1}}{(d-2)\,r^{d-3}}\int \rho r^{d-2}dr,$$

or

$$B^2 = \frac{1}{1 - \frac{2c^2V_{d-2}C_{d-1}}{(d-2)r^{d-3}}\int \rho r^{d-2}dr}. \tag{3.26}$$

If we consider an AdS Space(with a negative cosmological constant, Λ), then the Einstein's equation becomes

$$G^{\mu}{}_{\nu} + \Lambda g^{\mu}{}_{\nu} = V_{d-2}C_{d-1}T^{\mu}{}_{\nu}, \tag{3.27}$$

and equation (3.23) becomes

$$B' - \frac{(d-3)}{r}B = B^2\left(\frac{2rV_{d-2}C_{d-1}\rho c^2}{(d-2)} - \frac{(d-3)}{r} - \frac{2\Lambda r}{(d-2)}\right). \tag{3.28}$$

Change $B \to B^2$, so that

$$B' - \frac{(d-3)}{2r}B = B^3\left(\frac{rV_{d-2}C_{d-1}\rho c^2}{(d-2)} - \frac{(d-3)}{2r} - \frac{\Lambda r}{(d-2)}\right). \tag{3.29}$$

Let $b = B^{1-n}$ and $n = 3$, we have $b = B^{1-3} = B^{-2}$ moreover, $\frac{db}{dr} = \frac{(-2)}{B^3}\frac{dB}{dr} \to \frac{dB}{dr} = \frac{B^3}{(-2)}\frac{db}{dr}$. So

$$\frac{B^3}{(-2)}\frac{db}{dr} - \frac{(d-3)}{2r}B = B^3\left(\frac{rV_{d-2}C_{d-1}\rho c^2}{(d-2)} - \frac{(d-3)}{2r} - \frac{\Lambda r}{(d-2)}\right),$$

$$b' + (d-3)\,\frac{b}{r} = \left(\frac{(d-3)}{r} - \frac{2rV_{d-2}C_{d-1}\rho c^2}{(d-2)} + \frac{2\Lambda r}{(d-2)}\right). \tag{3.30}$$

Put equation (3.30)$\times r^{d-3}$, we have

$$(d-2)\,r^{d-3}b' + (d-3)\,(d-2)\,br^{d-4} = \left((d-3)\,(d-2)\,r^{(d-4)} - 2r^{d-2}V_{d-2}C_{d-1}\rho c^2\right),$$

or

$$\left(r^{(d-3)}b\right)' = \left((d-3)\,r^{(d-4)} - \frac{2r^{d-2}V_{d-2}C_{d-1}\rho c^2}{(d-2)}\right).$$

Then

$$b = 1 - \frac{2c^2 V_{d-2} C_{d-1}}{(d-2)\, r^{d-3}} \int \rho r^{d-2} dr + \frac{2\Lambda r^2}{(d-2)\,(d-1)},$$

or

$$B^2 = \frac{1}{1 - \frac{2c^2 V_{d-2} C_{d-1}}{(d-2) r^{d-3}} \int \rho r^{d-2} dr + \frac{2\Lambda r^2}{(d-2)(d-1)}}. \tag{3.31}$$

Let $\frac{2\Lambda}{(d-2)(d-1)} = \frac{1}{l^2}$, then

$$B^2 = \frac{1}{1 - \frac{2c^2 V_{d-2} C_{d-1}}{(d-2) r^{d-3}} \int \rho r^{d-2} dr + \frac{r^2}{l^2}} = \frac{1}{1 - \frac{M C_{d-1}}{r^{d-3}} + \frac{r^2}{l^2}}. \tag{3.32}$$

Moreover, we have

$$M\,(r) = \frac{2V_{d-2}}{(d-2)} \int \rho r^{d-2} dr. \tag{3.33}$$

Consider $G^r{}_r = R^r{}_r - \frac{g^r{}_r}{2}\left(R^t{}_t + R^r{}_r + R^{\theta_1}{}_{\theta_1} + R^{\theta_2}{}_{\theta_2} + \ldots + R^{\theta_i}{}_{\theta_i} + \ldots + R^{\theta_{d-2}}{}_{\theta_{d-2}}\right) = V_{d-2} C_{d-1} T^r{}_r \to R^r{}_r - \left(R^t{}_t + \ldots + R^{\theta_{d-2}}{}_{\theta_{d-2}}\right) = 2V_{d-2} C_{d-1} P_r$, then

$$\frac{(d-2)\, A'}{rAB} - \frac{(d-2)\,(d-3)}{r^2}\left(1 - \frac{1}{B}\right) = 2V_{d-2} C_{d-1} P_r.$$

Use equation (3.22) from $G^t{}_t$, then we have

$$\frac{(d-2)\, A'}{rAB} + \frac{(d-2)\, B'}{rB^2} = 2V_{d-2} C_{d-1} \left(\rho c^2 + P_r\right). \tag{3.34}$$

Let equation (3.34) $\times \frac{rB}{(d-2)}$, so

$$\frac{A'}{A} + \frac{B'}{B} = \frac{2V_{d-2} C_{d-1}}{(d-2)} rB \left(\rho c^2 + P_r\right). \tag{3.35}$$

For $A \to A^2$, $B \to B^2 \therefore \frac{A'}{A} \to \frac{\left(A^2\right)'}{A^2} = \frac{2A'}{A}$, $\frac{B'}{B} \to \frac{2B'}{B}$, then equation (3.35) becomes

$$\frac{A'}{A} + \frac{B'}{B} = \frac{V_{d-2} C_{d-1} r B^2}{(d-2)} \left(\rho c^2 + P_r\right),$$

Solve this equation to find relations between A and B, we have

$$A^2\,(r) = \frac{e^{2\chi(r)}}{B^2\,(r)}, \tag{3.36}$$

where

$$\chi\,(r) = \frac{V_{d-2} C_{d-1}}{(d-2)} \int \left(\rho\,(r)\, c^2 + P_r\,(r)\right) r B^2\,(r)\, dr. \tag{3.37}$$

So we have coupled equations of motion from equation (3.33) and (3.37)

$$M'(r) = \frac{2V_{d-2}}{(d-2)}\rho(r)\, r^{d-2}, \tag{3.38a}$$

$$\chi'(r) = \frac{V_{d-2}C_{d-1}}{(d-2)}\left(\rho(r)\, c^2 + P_r(r)\right) rB^2(r). \tag{3.38b}$$

Moreover, when we consider the energy momentum conserved $\nabla_\mu T^\mu_{\ \nu} = 0$ by let $\nu = r, c = 1$ and $P_r = P_{\theta_1} = \ldots = P_{\theta_i} = \ldots = P_{\theta_{d-2}} = P$, $A \to A^2$, it leads to the TOV equation in d dimension that is given by

$$\frac{dP}{dr} = -(\rho + P)\frac{A'}{A},$$

and use thermodynamic relation, we have

$$\mu(r) = \frac{\epsilon_F}{A(r)}. \tag{3.39}$$

Then $A(r) = \frac{\epsilon_F}{\mu(r)} \to \frac{e^{\chi(r)}}{B(r)} = \frac{\epsilon_F}{\mu(r)} \to \chi(r) - \ln B(r) = \ln \epsilon_F - \ln \mu(r) \therefore \chi'(r) = \frac{B'(r)}{B(r)} - \frac{\mu'(r)}{\mu(r)}$ and the coupled equations of motion can be written new form of relation between mass and chemical potential

$$M'(r) = \frac{2V_{d-2}}{(d-2)}\rho(r)\, r^{d-2}, \tag{3.40a}$$

$$\mu'(r) = \mu(r)\left(\frac{B'(r)}{B(r)} - \frac{V_{d-2}C_{d-1}}{(d-2)}\left(\rho(r)\, c^2 + P_r(r)\right) rB^2(r)\right). \tag{3.40b}$$

3.2 Relativistic Landau energy level in 5 dimension

We solve Dirac equation to find relativistic energy level under electromagnetic field. Our calculation considers only positive energy solution and uses Dirac gamma matrices. Dirac equation is given by

$$i\hbar\gamma^\mu\partial_\mu\psi - mc\psi = 0, \tag{3.41}$$

where $\hbar$, c and m are Planck constant, speed of light and mass of particle, respectively and Dirac gamma matrices are

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}, \tag{3.42}$$

where 1 and $\vec{\sigma}$ are 2×2 identity matrix and Pauli matrices[Appendix A], respectively. We consider solution in two cases. First, positive energy solution, we have solution $\psi(x) = u(p)e^{-ipx} = u(p)e^{-iEt+i\vec{p}\cdot\vec{x}}$ and Dirac equation is $(\gamma^\mu p_\mu - m)u(p) = 0$. Second, negative energy solution, we have solution $\psi(x) = v(p)e^{ipx} = v(p)e^{iEt-i\vec{p}\cdot\vec{x}}$ and Dirac equation is $(\gamma^\mu p_\mu + m)v(p) = 0$. We concern only positive energy solution because we consider particle not the anti-particle. Let $\hbar = c = 1$, then

$$\begin{aligned}(\gamma^\mu p_\mu - m) &= \gamma^0 p_0 + \gamma^1 p_1 + \gamma^2 p_2 + \gamma^3 p_3 - m1_{4\times4} = \gamma^0 E - \vec{\gamma}\cdot\vec{p} - m1_{4\times4},\\ &= \begin{pmatrix}(E-m)\,1_{2\times2} & -\vec{\sigma}\cdot\vec{p}\\ \vec{\sigma}\cdot\vec{p} & -(E+m)\,1_{2\times2}\end{pmatrix}. \end{aligned}\tag{3.43}$$

Consider particle in an external magnetic field $\therefore p_\mu \to p_\mu - qA_\mu$

$$(\gamma^\mu(p_\mu - qA_\mu) - m) = \begin{pmatrix}(E-qA_0-m)\,1_{2\times2} & -\vec{\sigma}\cdot\left(\vec{p}-q\vec{A}\right)\\ \vec{\sigma}\cdot\left(\vec{p}-q\vec{A}\right) & -(E-qA_0+m)\,1_{2\times2}\end{pmatrix} = 0.\tag{3.44}$$

Let magnetic field be uniform in z-direction $\therefore A_0 = A_x = A_z = 0$, $A_y = Bx$. So

$$(\gamma^\mu(p_\mu - qA_\mu) - m) = \begin{pmatrix}(E-m)\,1_{2\times2} & -\vec{\sigma}\cdot\left(\vec{p}-q\vec{A}\right)\\ \vec{\sigma}\cdot\left(\vec{p}-q\vec{A}\right) & -(E+m)\,1_{2\times2}\end{pmatrix}.\tag{3.45}$$

Let $u(p) = \binom{\phi}{\chi}$, where ϕ and χ are two-component spinor. From $(\gamma^\mu(p_\mu - qA) - m)u(p) = 0$, we have $(E-m)\phi = \vec{\sigma}\cdot\left(\vec{p}-q\vec{A}\right)\chi$ and $(E+m)\chi = \vec{\sigma}\cdot\left(\vec{p}-q\vec{A}\right)\phi$, therefore

$$\left(E^2 - m^2\right)\phi = \left(\vec{\sigma}\cdot\left(\vec{p}-q\vec{A}\right)\right)^2\phi.\tag{3.46}$$

Use Appendix A.1, equation (A.5), we have

$$\begin{aligned}\left(\vec{\sigma}\cdot\left(\vec{p}-q\vec{A}\right)\right)^2 &= \left(\vec{\sigma}\cdot\left(\vec{p}-q\vec{A}\right)\right)\left(\vec{\sigma}\cdot\left(\vec{p}-q\vec{A}\right)\right),\\ &= \left(\vec{p}-q\vec{A}\right)\cdot\left(\vec{p}-q\vec{A}\right) + i\vec{\sigma}\cdot\left(\left(\vec{p}-q\vec{A}\right)\times\left(\vec{p}-q\vec{A}\right)\right),\\ &= \left(\vec{p}-q\vec{A}\right)^2 - iq\vec{\sigma}\cdot\left(\vec{p}\times\vec{A}+\vec{A}\times\vec{p}\right).\end{aligned}$$

Consider term $\left(\vec{p}\times\vec{A}+\vec{A}\times\vec{p}\right)$

$$\begin{aligned}\left(\vec{p}\times\vec{A}+\vec{A}\times\vec{p}\right)_i &= \epsilon_{ijk}\left(p_jA_k + A_jp_k\right) = \epsilon_{ijk}\left(p_jA_k - A_kp_j\right),\\ &= \epsilon_{ijk}[p_j, A_k] = -i\epsilon_{ijk}[\nabla_j, A_k],\\ &= -i\left(\vec{\nabla}\times\vec{A}\right)_i = -i\left(\vec{B}\right)_i.\end{aligned}$$

Then

$$\begin{aligned}\left(\vec{\sigma}\cdot\left(\vec{p}-q\vec{A}\right)\right)^2 &= \left(\vec{p}-q\vec{A}\right)^2 - iq\vec{\sigma}\cdot\left(-i\vec{B}\right)\\ &= \left(\vec{p}-q\vec{A}\right)^2 - q\vec{\sigma}\cdot\vec{B}\\ &= p^2 + q^2B^2x^2 - qB\left(\sigma_z + 2xp_y\right).\end{aligned}\tag{3.47}$$

We get equation [37]

$$\{p_x^2 + p_y^2 + p_z^2 - 2qBxp_y + q^2B^2x^2 - qB\sigma_z\}\phi = \left(E^2 - m^2\right)\phi. \tag{3.48}$$

But we want to study the Fermi gas in 5-dimensional AdS space, then we put new momentum p_w and coordinate w and consider solution in the form $\phi = e^{i(p_y y + p_z z + p_w w)} f(x)$. So equation (3.48) becomes

$$\{p_x^2 + p_y^2 + p_z^2 + p_w^2 - 2qBxp_y + q^2B^2x^2 - qB\sigma_z\}\phi = \left(E^2 - m^2\right)\phi. \tag{3.49}$$

Change variables to operators and substitute ϕ by our solution. we get

$$\{-\frac{d^2}{dx^2} - \frac{d^2}{dy^2} - \frac{d^2}{dz^2} - \frac{d^2}{dw^2} - 2qBxp_y + q^2B^2x^2 - qB\sigma_z\}e^{i(p_y y + p_z z + p_w w)} f(x)$$
$$= \left(E^2 - m^2\right) e^{i(p_y y + p_z z + p_w w)} f(x).$$

So

$$\{-\frac{d^2}{dx^2} + (qBx - p_y)^2 - qB\sigma_z\}f = \left(E^2 - m^2 - p_z^2 - p_w^2\right) f.$$

Let $\xi = \sqrt{qB}\left(x - \frac{p_y}{qB}\right)$, $a = \frac{E^2 - m^2 - p_z^2 - p_w^2}{qB}$ $\therefore$ $\frac{d^2}{dx^2} = \frac{d}{dx}\left(\frac{d}{dx}\right) = \frac{d}{d\xi}\left(\frac{d}{d\xi}\cdot\frac{d\xi}{dx}\right)\frac{d\xi}{dx}$, $\frac{d^2}{dx^2} = \frac{d}{d\xi}\left(\frac{d}{d\xi}\cdot\sqrt{qB}\right)\sqrt{qB} = qB\frac{d^2}{d\xi^2}$ and $\xi^2 = qB\left(x - \frac{p_y}{qB}\right)^2$, we get

$$\left(-\frac{d^2}{d\xi^2} + \xi^2 - \sigma_z\right) f = \left(\frac{E^2 - m^2 - p_z^2 - p_w^2}{qB}\right) f = af. \tag{3.50}$$

The spinor function f may be chosen so that it is an eigenfunction of the operator σ_z; $\sigma_z f = \mu f$, where $\mu = \pm 1$ $\left(\because \sigma_z = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\right)$ [37]. See appendix A.2, we have each function component f_μ in function $f = \left(\begin{smallmatrix} f_{\mu=1} \\ f_{\mu=-1} \end{smallmatrix}\right)$ is a Hermite function. Moreover, we have energy condition following from ordinary differential equation form which is given by

$$E_n^2 = m^2c^4 + p_z^2c^2 + p_w^2c^2 + (2n - \mu + 1)\, 2mc^2\mu_B B. \;\; (n = 0, 1, 2, \dots \;\; , \mu = \pm 1)$$

If we let $j = n - \frac{\mu}{2}$ (we consider the case, $\mu = 0$, throwing away the Zeeman energy term), then we have

$$E_j^2 = m^2c^4 + p_z^2c^2 + p_w^2c^2 + \left(j + \frac{1}{2}\right) 4mc^2\mu_B B, \tag{3.51a}$$

$$= m^2c^4 + p_n^2c^2 + \left(j + \frac{1}{2}\right) 4mc^2\mu_B B. \qquad \left(p_n^2 = p_z^2 + p_w^2\right) \tag{3.51b}$$

From equations (3.51a) and (3.51b), energy is quantized in $x - y$ plane and can be degenerate, *i.e.*, there are several states with the same one-particle energy. We find the number of states g_j of a discrete energy level j

$$\begin{aligned} g_j &= \frac{g_s}{h^2}\int dp_x dp_y dx dy = \frac{g_s L_x L_y}{h^2}2\pi\int_{p_j}^{p_{j+1}} p dp = \frac{g_s \pi L_x L_y}{h^2}\left(p_{j+1}^2 - p_j^2\right), \\ &= \frac{g_s \pi L_x L_y}{h^2}\left(4m\mu_B B\right). \qquad \left(\because p_j^2 c^2 = \left(p_x^2 + p_y^2\right)c^2 = 4jmc^2\mu_B B\right) \end{aligned} \tag{3.52}$$

where $g_s(= 2s+1$ in the case of fermions) is a spin degeneracy and the degeneracy factor g_j is independent of j and vanishes for $B \to 0$.

3.3 Pressure and energy density under magnetic field at finite temperature

We consider equation (2.26). For a large volume, the sum over all one-particle states can be rewritten in terms of an integral, $\sum_k \to \int \frac{d^{d-1}\vec{r}d^{d-1}\vec{p}}{h^{d-1}}$ where $d =$ space-time dimension. We consider 5 dimensional AdS space $\therefore d = 5$

$$\begin{aligned} \ln Z &= \int \frac{d^4 r d^4 p}{h^4}\ln\left(1 + e^{-\frac{(\epsilon-\mu)}{k_B T}}\right), \\ &= \frac{1}{h^4}\int dx dy dz dx_{AdS} dp_x dp_y dp_z dp_{AdS}\ln\left(1 + e^{-\frac{(\epsilon-\mu)}{k_B T}}\right), \\ &= \frac{L_x L_y L_z L_{AdS}}{h^4}\int dp_x dp_y dp_z dp_{AdS}\ln\left(1 + e^{-\frac{(\epsilon-\mu)}{k_B T}}\right), \end{aligned} \tag{3.53}$$

but the energy levels are degenerate when the magnetic field is applied

$$\begin{aligned} \ln Z &= \frac{g_s}{h^2}\int_{-\infty}^{\infty} dp_z dp_{AdS} dz dx_{AdS}\sum_{j=0}^{\infty} g_j \ln\left(1 + e^{-\frac{\left(E_j-\mu\right)}{k_B T}}\right), \\ &= \frac{g_s L_x L_y L_z L_{AdS}}{h^4}\left(4\pi m\mu_B B\right)\int_{-\infty}^{\infty} dp_z dp_{AdS}\sum_{j=0}^{\infty}\ln\left(1 + e^{-\frac{\left(E_j-\mu\right)}{k_B T}}\right), \\ &= \left(\frac{4g_s\pi m\mu_B BV}{h^4}\right)(2\pi)\int_0^{\infty} p_n dp_n \sum_{j=0}^{\infty}\ln\left(1 + e^{-\frac{\left(E_j-\mu\right)}{k_B T}}\right), \end{aligned} \tag{3.54}$$

where $(V = L_x L_y L_z L_{AdS}, p_n^2 = p_z^2 + p_{AdS}^2)$ and use Appendix A, equation (A.8), then we have

$$f(x) = \ln\left(1 + ze^{-\frac{\sqrt{m^2c^4 + p_n^2c^2 + 4xmc^2\mu_B B}}{k_B T}}\right), \tag{3.55a}$$

$$f'(x)|_{x=0} = \frac{ze^{-\frac{\sqrt{m^2c^4 + p_n^2c^2 + 4xmc^2\mu_B B}}{k_B T}}}{\left(1 + ze^{-\frac{\sqrt{m^2c^4 + p_n^2c^2 + 4xmc^2\mu_B B}}{k_B T}}\right)} \cdot \frac{(-4mc^2\mu_B B)}{2k_B T\left(\sqrt{m^2c^4 + p_n^2c^2 + 4xmc^2\mu_B B}\right)}|_{x=0},$$

$$= \frac{(-2mc^2\mu_B B)}{k_B T\left(\sqrt{m^2c^4 + p_n^2c^2}\right)} \cdot \frac{ze^{-\frac{\sqrt{m^2c^4 + p_n^2c^2}}{k_B T}}}{1 + ze^{-\frac{\sqrt{m^2c^4 + p_n^2c^2}}{k_B T}}}, \tag{3.55b}$$

where $z = e^{\frac{\mu}{k_B T}}$. Therefore

$$\ln Z \approx \left(\frac{8g_s\pi^2 m\mu_B BV}{h^4}\right)\left\{\int_0^\infty dx \int_0^\infty dp_n p_n \ln\left(1 + ze^{-\frac{\sqrt{m^2c^4 + p_n^2c^2 + 4xmc^2\mu_B B}}{k_B T}}\right)\right.$$

$$\left. - \left(\frac{mc^2\mu_B B}{12k_B T}\right)\int_0^\infty \frac{p_n ze^{-\frac{\sqrt{m^2c^4 + p_n^2c^2}}{k_B T}}}{\left(\sqrt{m^2c^4 + p_n^2c^2}\right)\left(1 + ze^{-\frac{\sqrt{m^2c^4 + p_n^2c^2}}{k_B T}}\right)} dp_n\right\},$$

$$= \ln Z_0 + \ln Z_B. \tag{3.56}$$

Consider 1^{st} term in $\ln Z$ $(\ln Z_0)$ and let $\epsilon = \sqrt{m^2c^4 + p_n^2c^2 + 4xmc^2\mu_B B} = \sqrt{m^2c^4 + (p')^2 c^2}; p'^2 = p_n^2 + 4xm\mu_B B \therefore d\epsilon = \frac{2mc^2\mu_B B}{\epsilon}dx \rightarrow dx = \frac{\epsilon}{2mc^2\mu_B B}d\epsilon$

$$\int_0^\infty \int_0^\infty dx dp_n p_n \ln\left(1 + ze^{-\frac{\epsilon}{k_B T}}\right)$$

$$= \frac{1}{2mc^2\mu_B B}\int_{mc^2}^\infty d\epsilon\epsilon \int_0^{\sqrt{\frac{\epsilon^2}{c^2} - m^2c^2}} dp_n p_n \ln\left(1 + ze^{-\frac{\epsilon}{k_B T}}\right),$$

$$= \frac{1}{4mc^2\mu_B B}\int_{mc^2}^\infty d\epsilon\epsilon\left(\frac{\epsilon^2}{c^2} - m^2c^2\right)\ln\left(1 + ze^{-\frac{\epsilon}{k_B T}}\right). \tag{3.57}$$

From $\epsilon^2 = m^2c^4 + (p')^2 c^2$, $\therefore 2\epsilon d\epsilon = 2p'c^2 dp'$ and $p'^2 = \frac{\epsilon^2}{c^2} - m^2c^2$, when $\epsilon \rightarrow mc^2 \quad p' \rightarrow 0$ and $\epsilon \rightarrow \infty \quad p' \rightarrow \infty$. So

$$\frac{1}{4mc^2\mu_B B}\int_{mc^2}^\infty d\epsilon\epsilon\left(\frac{\epsilon^2}{c^2} - m^2c^2\right)\ln\left(1 + ze^{-\frac{\epsilon}{k_B T}}\right) = \frac{c^2}{4mc^2\mu_B B}\int_0^\infty dp' p'^3 \ln\left(1 + ze^{-\frac{\epsilon}{k_B T}}\right).$$

The logarithm can be removed by integrating by parts $\left(\int udv = uv - \int vdu\right)$. Let $u = \ln\left(1 + ze^{-\frac{\epsilon}{k_BT}}\right)$, $dv = p'^3 dp'$ $\therefore du = \frac{ze^{-\frac{\epsilon}{k_BT}}}{k_BT\left(1+ze^{-\frac{\epsilon}{k_BT}}\right)} \cdot \frac{d\epsilon}{dp'}dp'$ and $v = \frac{p'^4}{4}$

$$\begin{aligned}\frac{1}{4m\mu_B B}\int_0^\infty dp' p'^3 \ln\left(1 + ze^{-\frac{\epsilon}{k_BT}}\right) =& \frac{1}{4m\mu_B B}\left(\frac{p'^4}{4}\ln\left(1 + ze^{-\frac{\epsilon}{k_BT}}\right)\Big|_0^\infty\right. \\ &\left. + \frac{1}{4k_BT}\int_0^\infty \frac{p'^4}{z^{-1}e^{\frac{\epsilon}{k_BT}}+1}\cdot\frac{d\epsilon}{dp'}dp'\right).\end{aligned}$$

Consider limit of the first term $p' \to 0$ $\therefore$ the first term $\to 0$ and $\epsilon = \sqrt{m^2c^4 + (p')^2c^2}$ $\therefore p' \to \infty$ $\epsilon \to 0$, we have $\ln(1) = 0$, then the first term vanishes and $\frac{d\epsilon}{dp'} = \frac{p'c^2}{\epsilon}$,

$$\frac{1}{4m\mu_B B}\int_0^\infty dp' p'^3 \ln\left(1 + ze^{-\frac{\epsilon}{k_BT}}\right) = \frac{1}{4m\mu_B B}\left(\frac{c^2}{4k_BT}\right)\int_0^\infty \frac{p'^5}{\epsilon\left(z^{-1}e^{\frac{\epsilon}{k_BT}}+1\right)}dp'$$

$\therefore 1^{st}$ term in $\ln Z$ becomes

$$\begin{aligned}\ln Z_0 &= \left(\frac{8g_s\pi^2 m\mu_B BV}{h^4}\right)\left(\frac{c^2}{16k_BTm\mu_B B}\right)\int_0^\infty \frac{p'^5}{\epsilon\left(z^{-1}e^{\frac{\epsilon}{k_BT}}+1\right)}, \\ &= \left(\frac{g_s\pi^2c^2V}{2h^4k_BT}\right)\int_0^\infty \frac{p'^5}{\epsilon\left(z^{-1}e^{\frac{\epsilon}{k_BT}}+1\right)}dp'. \qquad (3.58)\end{aligned}$$

This is just the partition function of a free relativistic Fermi gas without a magnetic field in $5D$ at finite temperature (the first term in $\ln Z$ reproduces the limiting case $B \to 0$). The higher order terms thus represent corrections to the free case. Consider $\int_0^\infty \frac{p'^5}{\epsilon\left(z^{-1}e^{\frac{\epsilon}{k_BT}}+1\right)}dp'$. From $\epsilon^2 = m^2c^2 + p'^2c^2$, then

$$\int_0^\infty \frac{p'^5}{\epsilon\left(z^{-1}e^{\frac{\epsilon}{k_BT}}+1\right)}dp' = \int_{mc^2}^\infty \frac{\left(\frac{\epsilon^2}{c^2} - m^2c^2\right)^2}{c^2\left(e^{\frac{\epsilon-\mu}{k_BT}}+1\right)} = \int_{mc^2}^\infty \frac{f(\epsilon)\,d\epsilon}{c^2\left(e^{\frac{\epsilon-\mu}{k_BT}}+1\right)}, \qquad (3.59)$$

where $f(\epsilon) = \left(\frac{\epsilon^2}{c^2} - m^2c^2\right)^2$. Now we use a trick to integrate by change of variable, we call integrate trick [38, 39]. Let $\frac{(\epsilon-\mu)}{k_BT} = y$ $\therefore d\epsilon = k_BTdy$ and $\epsilon \to mc^2$ $\;y \to \frac{mc^2-\mu}{k_BT}$, $\epsilon \to \infty$ $\;y \to \infty$. Consider

$$\begin{aligned}\int_{mc^2}^\infty \frac{f(\epsilon)\,d\epsilon}{\left(e^{\frac{\epsilon-\mu}{k_BT}}+1\right)} &= \int_{\frac{mc^2-\mu}{k_BT}}^\infty \frac{f(\mu + k_BTdy)}{e^y+1}k_BTdy, \\ &= \int_{\frac{mc^2-\mu}{k_BT}}^0 \frac{f(\mu + k_BTdy)}{e^y+1}k_BTdy + \int_0^\infty \frac{f(\mu + k_BTdy)}{e^y+1}k_BTdy. \qquad (3.60)\end{aligned}$$

In 1^{st} term, change $T \to -T$

$$\begin{aligned}\int_{\frac{mc^2-\mu}{k_BT}}^\infty \frac{f(\mu + k_BTdy)}{e^y+1}k_BTdy =& \int_0^{\frac{\mu-mc^2}{k_BT}} \frac{f(\mu - k_BTdy)}{e^{-y}+1}k_BTdy \\ &+ \int_0^\infty \frac{f(\mu + k_BTdy)}{e^y+1}k_BTdy, \qquad (3.61)\end{aligned}$$

and $1-\frac{1}{ey+1}=\frac{e^y+1-1}{e^y+1}=\frac{e^y}{e^y+1}=\frac{1}{1+e^{-y}}$. If we let $\mu-k_BT=\epsilon \therefore dy=\frac{d\epsilon}{-k_BT}$ $y\to 0$, $\epsilon\to\mu$ and $y\to\frac{\mu-mc^2}{k_BT}$, $\epsilon\to mc^2$. So

$$\int_{mc^2}^{\infty}\frac{f(\epsilon)\,d\epsilon}{\left(e^{\frac{(\epsilon-\mu)}{k_BT}}+1\right)}=\int_{mc^2}^{\mu}f(\epsilon)\,d\epsilon-k_BT\int_0^{\frac{\mu-mc^2}{k_BT}}\frac{f(\mu-k_BT)}{(e^y+1)}dy$$
$$+\int_0^{\infty}\frac{f(\mu+k_BTy)}{(e^y+1)}dy, \qquad (3.62)$$

therefore

$$\ln Z_0=\left(\frac{g_s\pi^2V}{2h^4k_BT}\right)\left\{\int_{mc^2}^{\mu}\left(\frac{\epsilon^2}{c^2}-m^2c^2\right)^2d\epsilon-k_BT\int_0^{\frac{\mu-mc^2}{k_BT}}\frac{\left(\frac{(\mu-k_BTy)^2}{c^2}-m^2c^2\right)^2}{e^y+1}dy\right.$$
$$\left.+k_BT\int_0^{\infty}\frac{\left(\frac{(\mu+k_BTy)^2}{c^2}-m^2c^2\right)^2}{e^y+1}dy\right\}. \qquad (3.63)$$

Consider 2^{nd} term in $\ln Z$. Let $\epsilon|_{x=0}=\sqrt{m^2c^4+p_n^2c^2}\therefore 2\epsilon|_{x=0}d\epsilon|_{x=0}=2p_nc^2dp_n$, $p\to 0$ $\quad\epsilon|_{x=0}\to mc^2$, $p\to\infty$ $\quad\epsilon|_{x=0}\to\infty$

$$\int_0^{\infty}\frac{p_n}{\epsilon|_{x=0}}\cdot\frac{ze^{-\frac{\epsilon|_{x=0}}{k_BT}}}{\left(1+ze^{-\frac{\epsilon|_{x=0}}{k_BT}}\right)}dp_n = \int_{mc^2}^{\infty}\frac{p_n}{\epsilon|_{x=0}}\cdot\frac{ze^{-\frac{\epsilon|_{x=0}}{k_BT}}}{\left(1+ze^{-\frac{\epsilon|_{x=0}}{k_BT}}\right)}\cdot\frac{\epsilon|_{x=0}}{p_nc^2}d\epsilon|_{x=0},$$
$$= \int_{mc^2}^{\infty}\frac{ze^{-\frac{\epsilon|_{x=0}}{k_BT}}}{c^2\left(1+ze^{-\frac{\epsilon|_{x=0}}{k_BT}}\right)}d\epsilon|_{x=0},$$
$$= \frac{1}{c^2}\int_{mc^2}^{\infty}\frac{d\epsilon|_{x=0}}{\left(z^{-1}e^{\frac{\epsilon|_{x=0}}{k_BT}}+1\right)}. \qquad (3.64)$$

Therefore

$$\ln Z_B = \left(\frac{8g_s\pi^2m\mu_BBV}{h^4}\right)\left(-\frac{mc^2\mu_BB}{12k_BT}\right)\left(\frac{1}{c^2}\right)\int_{mc^2}^{\infty}\frac{d\epsilon|_{x=0}}{\left(z^{-1}e^{\frac{\epsilon|_{x=0}}{k_BT}}+1\right)},$$
$$= \left(-\frac{2g_s\pi^2m^2\mu_B^2B^2V}{3h^4k_BT}\right)\int_{mc^2}^{\infty}\frac{d\epsilon|_{x=0}}{\left(z^{-1}e^{\frac{\epsilon|_{x=0}}{k_BT}}+1\right)}. \qquad (3.65)$$

Use integrate trick following equation (3.62), then

$$\ln Z_B=\left(-\frac{2g_s\pi^2m^2\mu_B^2B^2V}{3h^4k_BT}\right)\left\{\left(\mu-mc^2\right)-k_BT\int_0^{\frac{\mu-mc^2}{k_BT}}\frac{dy}{e^y+1}\right.$$
$$\left.+k_BT\int_0^{\infty}\frac{dy}{e^y+1}\right\}. \qquad (3.66)$$

From thermodynamics (equation(2.28))

$$P = \frac{k_BT}{V}\ln Z = \frac{k_BT}{V}\left(\ln Z_0 + \ln Z_B\right),$$
$$= \left(\frac{g_s\pi^2}{2h^4}\right)\left\{\int_{mc^2}^{\mu}\left(\frac{\epsilon^2}{c^2} - m^2c^2\right)^2 d\epsilon - k_BT\int_0^{\frac{\mu - mc^2}{k_BT}} \frac{\left(\frac{(\mu - k_BTy)^2}{c^2} - m^2c^2\right)^2}{e^y+1}dy\right.$$
$$\left. + k_BT\int_0^{\infty}\frac{\left(\frac{(\mu + k_BTy)^2}{c^2} - m^2c^2\right)^2}{e^y+1}dy\right\} - \left(\frac{2\pi^2m^2\mu_B^2B^2}{3h^4}\right)\left\{\left(\mu - mc^2\right)\right.$$
$$\left. - k_BT\int_0^{\frac{\mu - mc^2}{k_BT}}\frac{dy}{e^y+1} + k_BT\int_0^{\infty}\frac{dy}{e^y+1}\right\}. \tag{3.67}$$

Find energy from equation (2.29b) and $\sum_k \to \int \frac{d^{d-1}\vec{r}d^{d-1}\vec{p}}{h^{d-1}}$, $d=5$

$$U = \int \frac{d^4rd^4p}{h^4}\frac{\epsilon}{z^{-1}e^{\frac{\epsilon}{k_BT}}+1} = \frac{1}{h^4}\int dxdydzdx_{AdS}dp_xdp_ydp_zdp_{AdS}\frac{\epsilon}{z^{-1}e^{\frac{\epsilon}{k_BT}}+1}.$$

We apply a magnetic field

$$\begin{aligned} U &= \frac{g_s}{h^2}\int_{-\infty}^{\infty}dp_zdp_{AdS}dzdx_{AdS}\sum_{j=0}^{\infty}g_j\frac{E_j}{z^{-1}e^{\frac{E_j}{k_BT}}+1}, \\ &= \frac{g_sL_xL_yL_zL_{AdS}}{h^4}\left(4\pi m\mu_BB\right)\int_{-\infty}^{\infty}dp_zdp_{AdS}\sum_{j=0}^{\infty}\frac{E_j}{z^{-1}e^{\frac{E_j}{k_BT}}}, \\ &= \left(\frac{8g_s\pi^2m\mu_BBV}{h^4}\right)\int_0^{\infty}p_ndp_n\sum_{j=0}^{\infty}\frac{E_j}{z^{-1}e^{\frac{E_j}{k_BT}}}, \end{aligned} \tag{3.68}$$

use Appendix A, equation (A.8)

$$f(x) = \frac{\sqrt{m^2c^4 + p_n^2c^2 + 4xmc^2\mu_BB}}{z^{-1}e^{\frac{\sqrt{m^2c^4+p_n^2c^2+4xmc^2\mu_BB}}{k_BT}}+1}, \tag{3.69a}$$

$$f'(x)|_{x=0} = \frac{2mc^2\mu_BB}{\left(\sqrt{m^2c^4+p_n^2c^2+4xmc^2\mu_BB}\right)\left(z^{-1}e^{\frac{\sqrt{m^2c^4+p_n^2c^2+4xmc^2\mu_BB}}{k_BT}}+1\right)}|_{x=0}$$
$$- \frac{\left(2mc^2\mu_BB\right)z^{-1}e^{\frac{\sqrt{m^2c^4+p_n^2c^2+4xmc^2\mu_BB}}{k_BT}}}{\left(z^{-1}e^{\frac{\sqrt{m^2c^4+p_n^2c^2+4xmc^2\mu_BB}}{k_BT}}+1\right)^2}|_{x=0}. \tag{3.69b}$$

So

$$U \approx \left(\frac{8g_s\pi^2 m\mu_B BV}{h^4}\right)\left\{\int_0^\infty dx\int_0^\infty dp_n p_n \frac{\sqrt{m^2c^4+p_n^2c^2+4xmc^2\mu_B B}}{\left(z^{-1}e^{\frac{\sqrt{m^2c^4+p_n^2c^2+4xmc^2\mu_B B}}{k_BT}}+1\right)}\right.$$

$$+\left(\frac{mc^2\mu_B B}{12}\right)\int_0^\infty dp_n p_n\left[\frac{1}{\left(\sqrt{m^2c^4+p_n^2c^2}\right)\left(z^{-1}e^{\frac{\sqrt{m^2c^4+p_n^2c^2}}{k_BT}}+1\right)}\right.$$

$$\left.\left.-\frac{z^{-1}e^{\sqrt{m^2c^4+p_n^2c^2}}}{k_BT\left(z^{-1}e^{\frac{\sqrt{m^2c^4+p_n^2c^2}}{k_BT}}+1\right)^2}\right]\right\}. \tag{3.70}$$

Consider 1^{st} term in U. Let $\epsilon=\sqrt{m^2c^4+p_n^2c^2+4xmc^2\mu_B B}=\sqrt{m^2c^4+(p')^2c^2}$; $p'^2=p_n^2+4xm\mu_B B$. $\therefore d\epsilon=\frac{2mc^2\mu_B B}{\epsilon}dx\rightarrow dx=\frac{\epsilon}{2mc^2\mu_B B}d\epsilon$

$$\begin{aligned}\int_0^\infty dx\int_0^\infty dp_n p_n\frac{\epsilon}{z^{-1}e^{\frac{\epsilon}{k_BT}}+1} &= \frac{1}{2mc^2\mu_B B}\int_{mc^2}^\infty d\epsilon\frac{\epsilon^2}{z^{-1}e^{\frac{\epsilon}{k_BT}}+1}\int_0^{\sqrt{\frac{\epsilon^2}{c^2}-m^2c^2}}dp_n p_n,\\ &= \frac{1}{4mc^2\mu_B B}\int_{mc^2}^\infty d\epsilon\frac{\epsilon^2}{z^{-1}e^{\frac{\epsilon}{k_BT}}+1}\left(\frac{\epsilon^2}{c^2}-m^2c^2\right)\end{aligned} \tag{3.71}$$

Use integrate trick, equation (3.62)

$$\begin{aligned}U_0=\left(\frac{2g_s\pi^2V}{h^4c^2}\right)&\left\{\int_{mc^2}^\mu \epsilon^2\left(\frac{\epsilon^2}{c^2}-m^2c^2\right)d\epsilon\right.\\ &-k_BT\int_0^{\frac{\mu-mc^2}{k_BT}}\frac{(\mu-k_BTy)^2\left(\frac{(\mu-k_BTy)^2}{c^2}-m^2c^2\right)}{(e^y+1)}dy\\ &\left.+k_BT\int_0^\infty\frac{(\mu+k_BTy)^2\left(\frac{(\mu+k_BTy)^2}{c^2}-m^2c^2\right)}{(e^y+1)}dy\right\}.\end{aligned} \tag{3.72}$$

Consider 2^{nd} term in U. Let $\epsilon|_{x=0}=\sqrt{m^2c^4+p_n^2c^2}$ $\therefore 2\epsilon|_{x=0}d\epsilon|_{x=0}=2p_nc^2dp_n$, $p\rightarrow 0\quad \epsilon|_{x=0}\rightarrow mc^2$, $p\rightarrow\infty\quad \epsilon|_{x=0}\rightarrow\infty$

$$\int_0^\infty dp_np_n\left[\frac{1}{(\epsilon|_{x=0})\left(z^{-1}e^{\frac{\epsilon|_{x=0}}{k_BT}}+1\right)}-\frac{z^{-1}e^{\epsilon|_{x=0}}}{k_BT\left(z^{-1}e^{\frac{\epsilon|_{x=0}}{k_BT}}+1\right)^2}\right]$$

$$=\int_{mc^2}^\infty\frac{d\epsilon|_{x=0}}{c^2}\left[\frac{1}{\left(z^{-1}e^{\frac{\epsilon|_{x=0}}{k_BT}}+1\right)}-\frac{(\epsilon|_{x=0})\,z^{-1}e^{\frac{\epsilon|_{x=0}}{k_BT}}}{k_BT\left(z^{-1}e^{\frac{\epsilon|_{x=0}}{k_BT}}\right)^2}\right]. \tag{3.73}$$

Again, we use integrate trick, equation (3.62), and use $\frac{1}{(e^{-y}+1)^2} = \frac{1}{(e^{-y}+1)} \cdot \frac{1}{(e^{-y}+1)} = \left(1 - \frac{1}{e^y+1}\right)\left(1 - \frac{1}{e^y+1}\right) = \left(1 - \frac{2}{e^y+1} + \frac{1}{(e^y+1)^2}\right)$. We have

$$U_B = \left(\frac{8g_s\pi^2 m\mu_B BV}{h^4}\right)\left(\frac{mc^2\mu_B B}{12}\right)\left(\frac{1}{c^2}\right)\left\{\left[\left(\mu - mc^2\right) - k_BT\int_0^{\frac{\mu-mc^2}{k_BT}} \frac{dy}{(e^y+1)} + k_BT\int_0^{\infty}\frac{dy}{(e^y+1)}\right] - \left[\int_{mc^2}^{\mu}\frac{\epsilon e^{\frac{\epsilon-\mu}{k_BT}}}{k_BT}d\epsilon - 2k_BT\int_0^{\frac{\mu-mc^2}{k_BT}}\frac{(\mu-k_BTy)\,e^{-y}}{k_BT\,(e^y+1)}dy + k_BT\int_0^{\frac{\mu-mc^2}{k_BT}}\frac{(\mu-k_BTy)\,e^{-y}}{k_BT\,(e^y+1)^2}dy + k_BT\int_0^{\infty}\frac{(\mu+k_BTy)\,e^{y}}{k_BT\,(e^y+1)^2}dy\right]\right\}. \quad (3.74)$$

Find energy density, inherited from energy, $\rho c^2 = \frac{U}{V} = \frac{U_0+U_B}{V}$, then

$$\begin{aligned}\rho c^2 = &\left(\frac{2g_s\pi^2}{h^4c^2}\right)\left\{\int_{mc^2}^{\mu}\epsilon^2\left(\frac{\epsilon^2}{c^2} - m^2c^2\right)d\epsilon\right.\\ &- k_BT\int_0^{\frac{\mu-mc^2}{k_BT}}\frac{(\mu-k_BTy)^2\left(\frac{(\mu-k_BTy)^2}{c^2}-m^2c^2\right)}{(e^y+1)}dy\\ &\left.+ k_BT\int_0^{\infty}\frac{(\mu+k_BTy)^2\left(\frac{(\mu+k_BTy)^2}{c^2}-m^2c^2\right)}{(e^y+1)}dy\right\}\\ &+\left(\frac{2g_s\pi^2m^2\mu_B^2B^2}{3h^4}\right)\left\{\left[(\mu-mc^2)\right.\right.\\ &\left.- k_BT\int_0^{\frac{\mu-mc^2}{k_BT}}\frac{dy}{(e^y+1)} + k_BT\int_0^{\infty}\frac{dy}{(e^y+1)}\right] - \left[\int_{mc^2}^{\mu}\frac{\epsilon e^{\frac{\epsilon-\mu}{k_BT}}}{k_BT}d\epsilon\right.\\ &- 2\int_0^{\frac{\mu-mc^2}{k_BT}}\frac{(\mu-k_BTy)\,e^{-y}}{(e^y+1)}dy + \int_0^{\frac{\mu-mc^2}{k_BT}}\frac{(\mu-k_BTy)\,e^{-y}}{(e^y+1)^2}dy\\ &\left.\left.+\int_0^{\infty}\frac{(\mu+k_BTy)\,e^{y}}{(e^y+1)^2}dy\right]\right\}. \end{aligned} \quad (3.75)$$

Chapter IV

NUMERICAL RESULTS AND DISCUSSION

In this chapter, we present our results from the analytic approximation and numerical analyses. Since we consider the system under an external magnetic field at finite temperature, we divide the results to five cases, *i.e.* zero temperature and zero magnetic field, finite temperature and zero magnetic field, zero temperature and finite magnetic field, finite temperature and finite magnetic field, and variation of the radius of curvature of AdS space, respectively. We study and discuss the effect of temperature, magnetic field and the radius of curvature of AdS space to the mass limit and other properties of a degenerate star in AdS space in each case.

Before going into the detail of each case, we integrate equations (3.67) and (3.75) to obtain

$$\begin{aligned} P = &\left(\frac{g_s\pi^2}{30c^4h^4}\right)\left(3\mu(r)^5 - 10m^2c^4\mu(r)^3 + 15m^4c^8\mu(r) - 8m^5c^{10} - 10k_B^2T^2m^2c^4\pi^2\mu(r)\right. \\ &+ 7k_B^4T^4\pi^4\mu(r) + 10k_B^2T^2\pi^2\mu(r)^3 - 120k_B^3T^3m^2c^4Li_3\left(-e^{\frac{mc^2-\mu(r)}{k_BT}}\right) \\ &+ 360k_B^4T^4mc^2Li_4\left(-e^{\frac{mc^2-\mu(r)}{k_BT}}\right) - 360k_B^5T^5Li_5\left(-e^{\frac{mc^2-\mu(r)}{k_BT}}\right) \\ &\left.- 20k_BTm^2c^4\mu_B^2B^2\ln\left(1+e^{\frac{\mu(r)-mc^2}{k_BT}}\right)\right), \end{aligned} \tag{4.1}$$

$$\begin{aligned} \rho c^2 = &\left(\frac{g_s2\pi^2}{15c^4h^4}\right)\left(3\mu(r)^5 - 5m^2c^4\mu(r)^3 + 2m^5c^{10} - 5k_B^2T^2m^2c^4\pi^2\mu(r) + 7k_B^4T^4\pi^4\mu(r)\right. \\ &+ 10k_B^2T^2\pi^2\mu(r)^3 + 30k_B^2T^2m^3c^6Li_2\left(-e^{\frac{mc^2-\mu(r)}{k_BT}}\right) - 150k_B^3T^3m^2c^4Li_3\left(-e^{\frac{mc^2-\mu(r)}{k_BT}}\right) \\ &\left.+ 360k_B^4T^4mc^2Li_4\left(-e^{\frac{mc^2-\mu(r)}{k_BT}}\right) - 360k_B^5T^5Li_5\left(-e^{\frac{mc^2-\mu(r)}{k_BT}}\right)\right) + \left(\frac{4m^2\pi^2\mu_B^2B^2}{3h^4}\right) \\ &\left(\frac{mc^2}{\left(1+e^{\frac{\mu(r)-mc^2}{k_BT}}\right)} - \mu(r) - k_BT\ln\left(1+e^{\frac{mc^2-\mu(r)}{k_BT}}\right) + k_BT\ln\left(1+e^{\frac{\mu(r)-mc^2}{k_BT}}\right)\right), \end{aligned} \tag{4.2}$$

where $Li_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}$ is a polylogarithm function. For numerical analysis, we set $G_5 = Gl$, $G = c = \hbar = k_B = \mu_B = l = 1$, $m = 0.1$ where G, G_5, l are the Newton constant in 4 dimensional space-time, the Newton constant in 5 dimensional space-time and radius of the AdS space, respectively. We can transform the numerical results to the SI unit by using the table of dimensional translation in Appendix B. We use the coupled equations of motion between mass and chemical potential (Eqs. (3.40a), (3.40b)) to find the mass limit and study other physical properties. We set the initial conditions at the center of star to be $M(r = 0) = 0$ and $\mu(r = 0) = e = 2.718281828$.

4.1 Case I, zero temperature and zero magnetic field

Case I, the pressure and the energy density (Eqs. (4.1), (4.2)) reduce to

$$P = \left(\frac{g_s \pi^2}{30c^4h^4}\right)\left(3\mu(r)^5 - 10m^2c^4\mu(r)^3 + 15m^4c^8\mu(r) - 8m^5c^{10}\right), \tag{4.3a}$$

$$\rho c^2 = \left(\frac{g_s 2\pi^2}{15c^4h^4}\right)\left(3\mu(r)^5 - 5m^2c^4\mu(r)^3 + 2m^5c^{10}\right). \tag{4.3b}$$

We present here the accumulated mass, the chemical potential, the energy density and the pressure distribution in the degenerate star versus the radius of the star in Figs. 4.1 and 4.2. Relation between the total mass and the central chemical potential of the degenerate star is shown in Fig 4.3. Relation between the total mass and the energy density of the degenerate star is shown in Fig. 4.4.

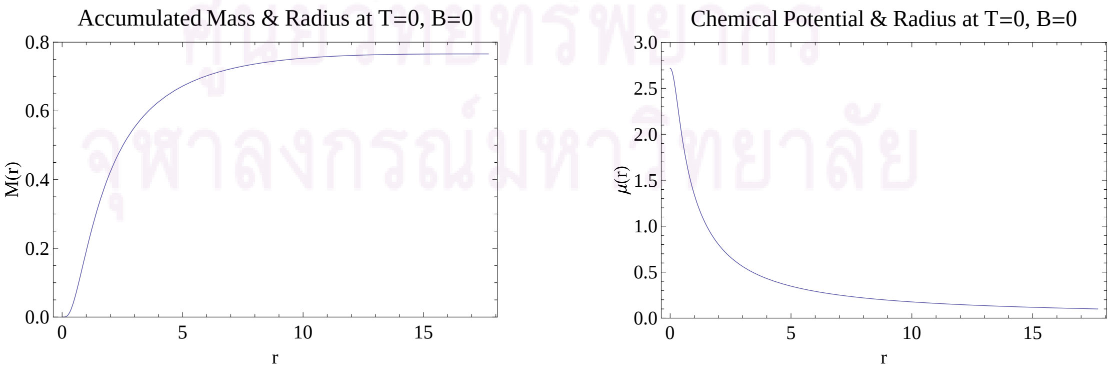

Figure 4.1: The accumulated mass(left) and the chemical potential(right) distribution in the degenerate star at $T = 0$, $B = 0$

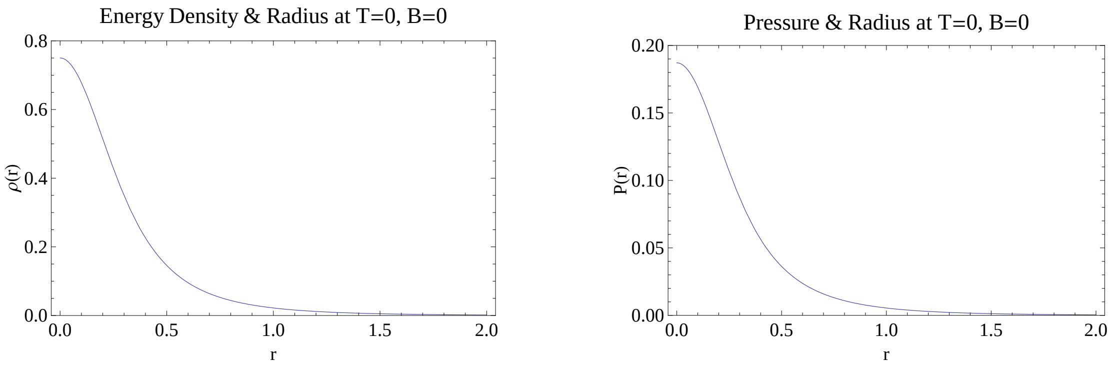

Figure 4.2: The energy density(left) and the pressure(right) distribution in the degenerate star at $T = 0$, $B = 0$

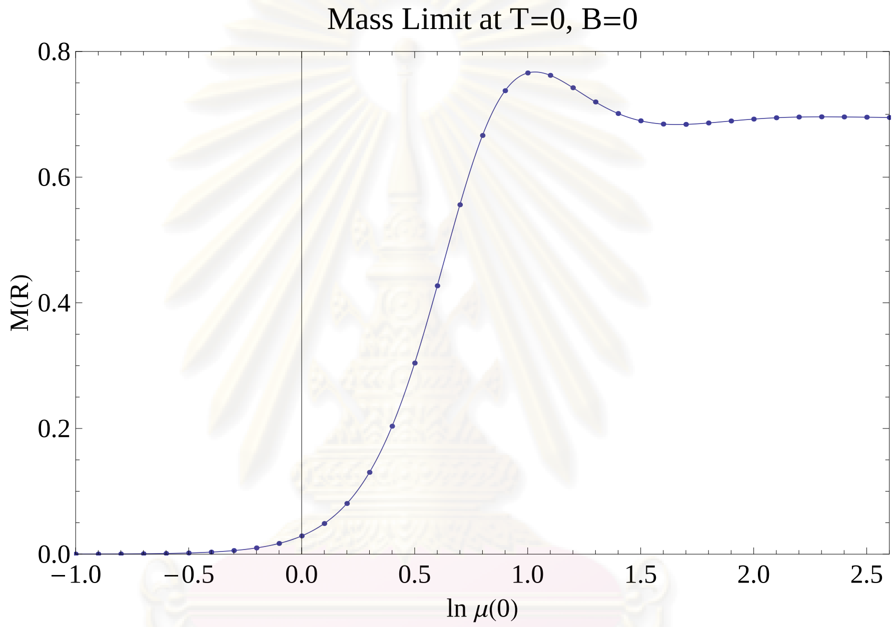

Figure 4.3: The relation between mass and central chemical potential (in logarithmic scale) of the degenerate star at $T = 0$, $B = 0$

In the simulation, the edge of the degenerate star is at $r = 17.6922$ where pressure drops to zero. In Fig. 4.1(left), the accumulated mass grows rapidly, in particular for the interval between $r = 0$ and $r = 5$. Beyond the central region, the accumulated mass increases less rapidly and becomes gradual. The behavior of the accumulated mass is determined by the energy density and the pressure distribution within the star. Initially, both of them, the energy density and pressure in Fig. 4.2, decrease rapidly then they drop to zero more gradually at larger distance. The chemical potential also behaves similarly(Fig. 4.1 (right)). It is clear that the matter in the star becomes extremely dense in the region near the core of the star. Figs. 4.3 and 4.4 show the maximum mass of the degenerate star. From numerical analysis, the maximum mass is found to be 0.767302 for the

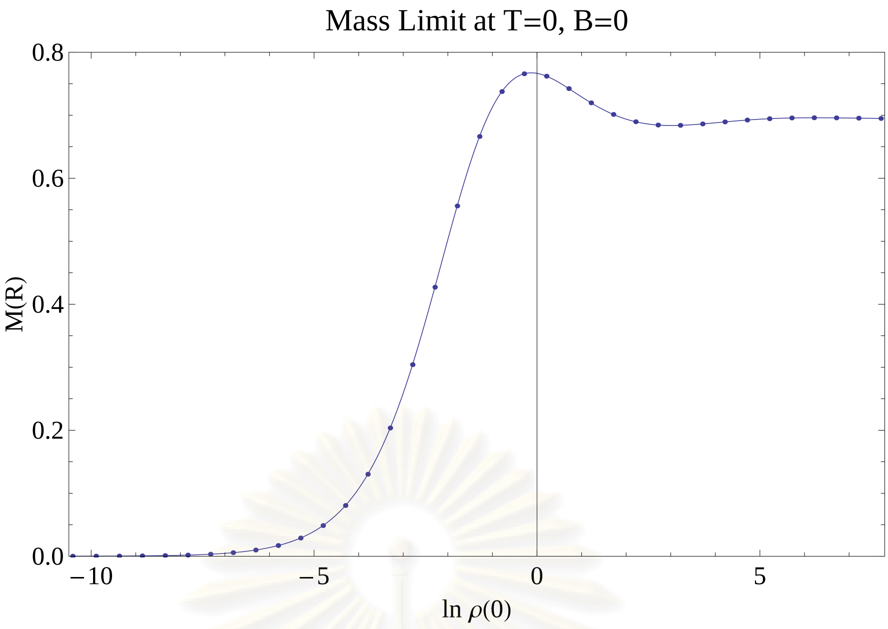

Figure 4.4: The relation between mass and central energy density (in logarithmic scale) of the degenerate star at $T = 0,\ B = 0$

central chemical potential equal to $e^{1.033}$ or at the central energy density equal to $e^{-0.122306}$.

4.2 Case II, finite temperature and zero magnetic field

Case II, we include the effects of the finite temperature and study changing in the mass limit and other properties of the star by comparing the results to the zero temperature case. The pressure and energy density in this case reduce to

$$\begin{aligned} P = \left(\frac{g_s\pi^2}{30c^4h^4}\right)\Big(&3\mu(r)^5 - 10m^2c^4\mu(r)^3 + 15m^4c^8\mu(r) - 8m^5c^{10} - 10k_B^2T^2m^2c^4\pi^2\mu(r) \\ &+ 7k_B^4T^4\pi^4\mu(r) + 10k_B^2T^2\pi^2\mu(r)^3 - 120k_B^3T^3m^2c^4Li_3\left(-e^{\frac{mc^2-\mu(r)}{k_BT}}\right) \\ &+ 360k_B^4T^4mc^2Li_4\left(-e^{\frac{mc^2-\mu(r)}{k_BT}}\right) - 360k_B^5T^5Li_5\left(-e^{\frac{mc^2-\mu(r)}{k_BT}}\right)\Big), \end{aligned} \tag{4.4a}$$

$$\begin{aligned} \rho c^2 = \left(\frac{g_s 2\pi^2}{15c^4h^4}\right)\Big(&3\mu(r)^5 - 5m^2c^4\mu(r)^3 + 2m^5c^{10} - 5k_B^2T^2m^2c^4\pi^2\mu(r) + 7k_B^4T^4\pi^4\mu(r) \\ &+ 10k_B^2T^2\pi^2\mu(r)^3 + 30k_B^2T^2m^3c^6Li_2\left(-e^{\frac{mc^2-\mu(r)}{k_BT}}\right) - 150k_B^3T^3m^2c^4Li_3\left(-e^{\frac{mc^2-\mu(r)}{k_BT}}\right) \\ &+ 360k_B^4T^4mc^2Li_4\left(-e^{\frac{mc^2-\mu(r)}{k_BT}}\right) - 360k_B^5T^5Li_5\left(-e^{\frac{mc^2-\mu(r)}{k_BT}}\right)\Big). \end{aligned} \tag{4.4b}$$

We set temperature values in the simulation unit to be 0.001 and 0.002. For $T \gtrsim 0.003$, the energy density and pressure do not decrease to zero at finite radius and therefore the surface of the star cannot be defined properly. We will further investigate this case in the future work.

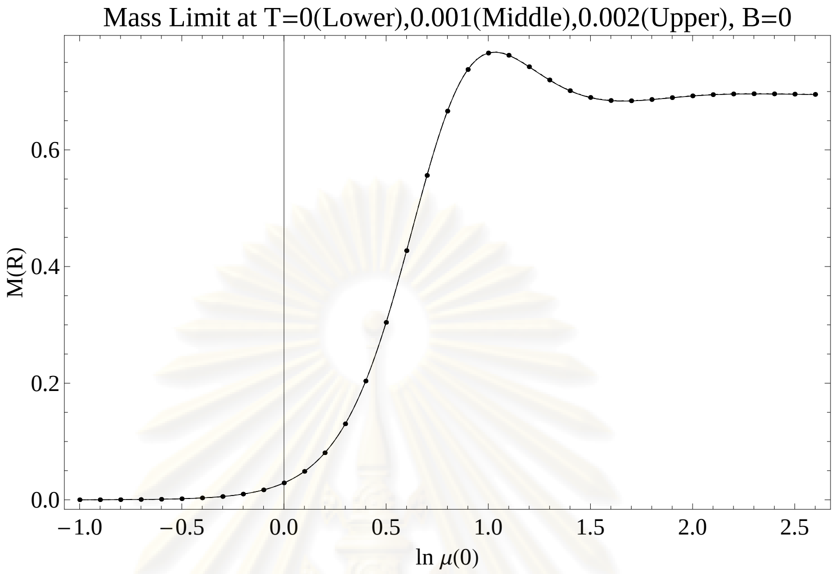

Figure 4.5: The relation between mass and central chemical potential (in logarithmic scale) of the degenerate star at $B = 0$

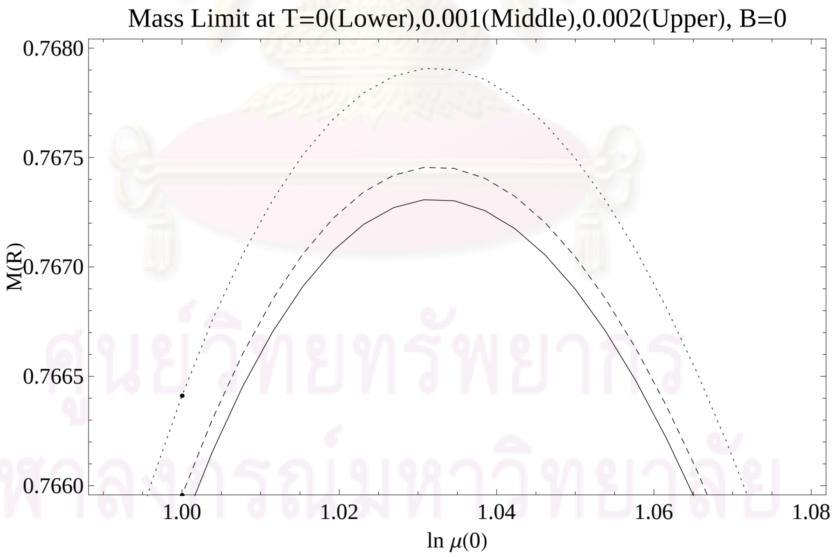

Figure 4.6: Enlargement of the relation between mass and chemical potential density (in logarithmic scale) of the degenerate star at $B = 0$

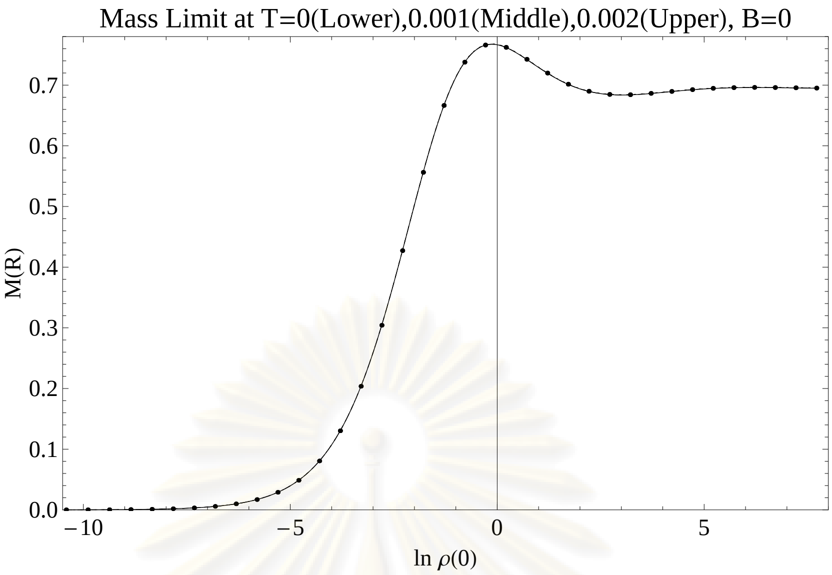

Figure 4.7: The relation between mass and central energy density (in logarithmic scale) of the degenerate star at $B = 0$

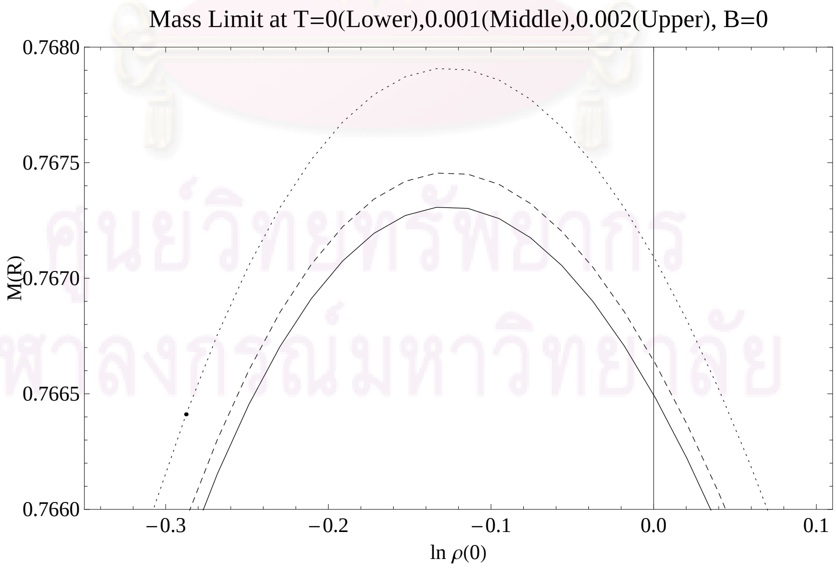

Figure 4.8: Enlargement of the relation between mass and central energy density (in logarithmic scale) of the degenerate star at $B = 0$

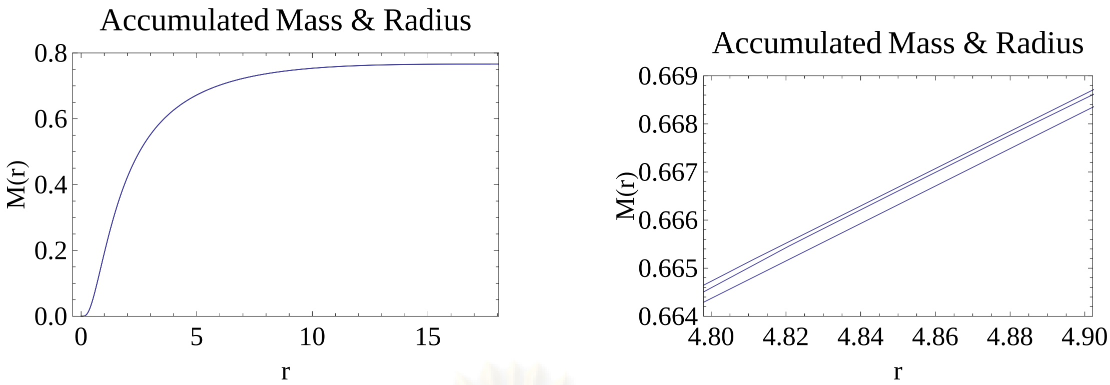

Figure 4.9: The accumulated mass distribution and its enlargement in the degenerate star at $B = 0$, $T = 0$(lower), $= 0.001$ (middle), $= 0.002$ (upper)

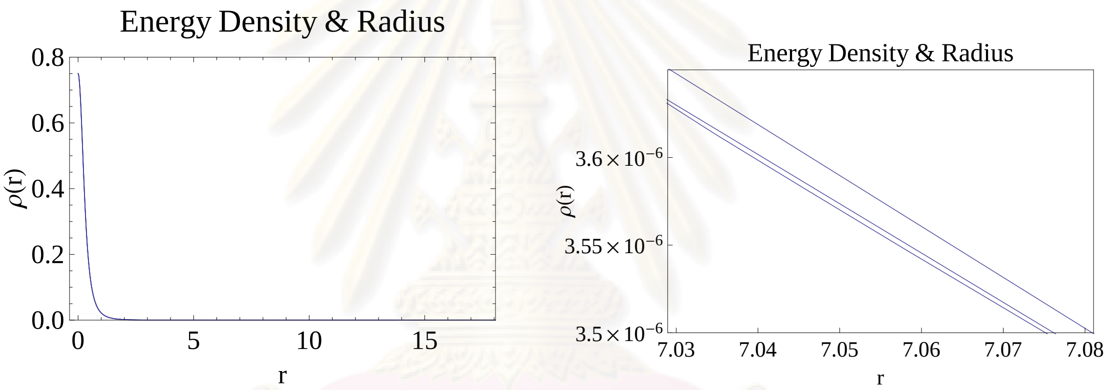

Figure 4.10: The energy density distribution and its enlargement in the degenerate star at $B = 0$, $T = 0$(lower), $= 0.001$ (middle), $= 0.002$ (upper)

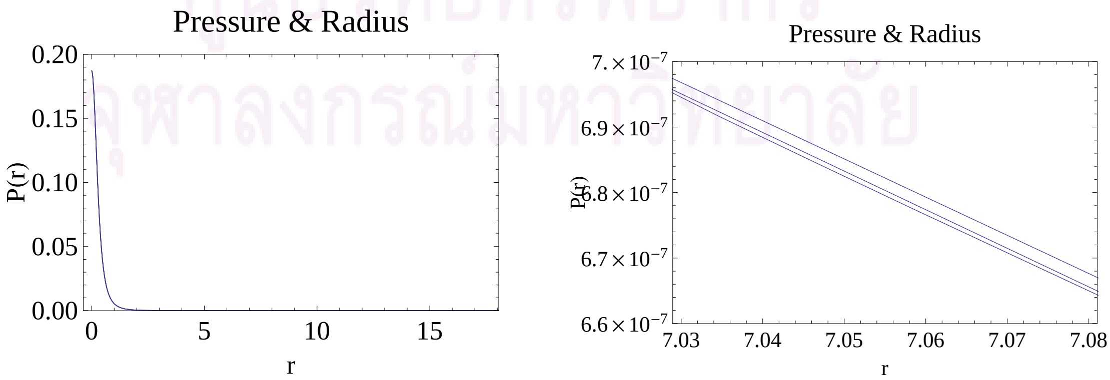

Figure 4.11: The pressure distribution and its enlargement in the degenerate star at $B = 0$, $T = 0$(lower), $= 0.001$ (middle), $= 0.002$ (upper)

In Figs. 4.5 and 4.7 show that temperature increasing rarely affect the mass limit. In addition, the maximum mass are shown in Figs. 4.6 and 4.8 which take the numerical value as 0.767451($r = 23.8829$) at $T = 0.001$ and 0.767903($r = 38.5232$) at $T = 0.002$. This is because the small increase in the temperature affects the Fermi-Dirac distribution very slightly. It enables almost the particles are still in the quantum states as before, degenerate state, and very small part of the particles obtain influence by gaining more pressure. Consequently, when temperature increases, the maximum mass also grows. Temperature increasing results in the increase of pressure and energy density as shown in Figs. 4.10(right) and 4.11(right). However the changes are quite small as are shown in Figs. 4.10(left) and 4.11(left). Certainly, the accumulated mass also increases in figure 4.9.

4.3 Case III, zero temperature and finite magnetic field

Case III, we turn on the magnetic field and study the mass limit and other properties at zero temperature by comparing to the results of Case I. The pressure and energy density in this case become to

$$P = \left(\frac{g_s \pi^2}{30c^4h^4}\right)\left(3\mu(r)^5 - 10m^2c^4\mu(r)^3 + 15m^4c^8\mu(r) - 8m^5c^{10}\right), \tag{4.5a}$$

$$\rho c^2 = \left(\frac{g_s 2\pi^2}{15c^4h^4}\right)\left(3\mu(r)^5 - 5m^2c^4\mu(r)^3 + 2m^5c^{10}\right) - \mu(r)\left(\frac{4m^2\pi^2\mu_B^2B^2}{3h^4}\right). \tag{4.5b}$$

Notice that the pressure of the star has the same form as the pressure in Case I since the correction term of the magnetic field contains the temperature. But the energy density becomes smaller due to the contribution from the term $-\mu(r)\left(\frac{4m^2\pi^2\mu_B^2B^2}{3h^4}\right)$. We let the numerical values of the magnetic field to be 0.01 and 0.1.

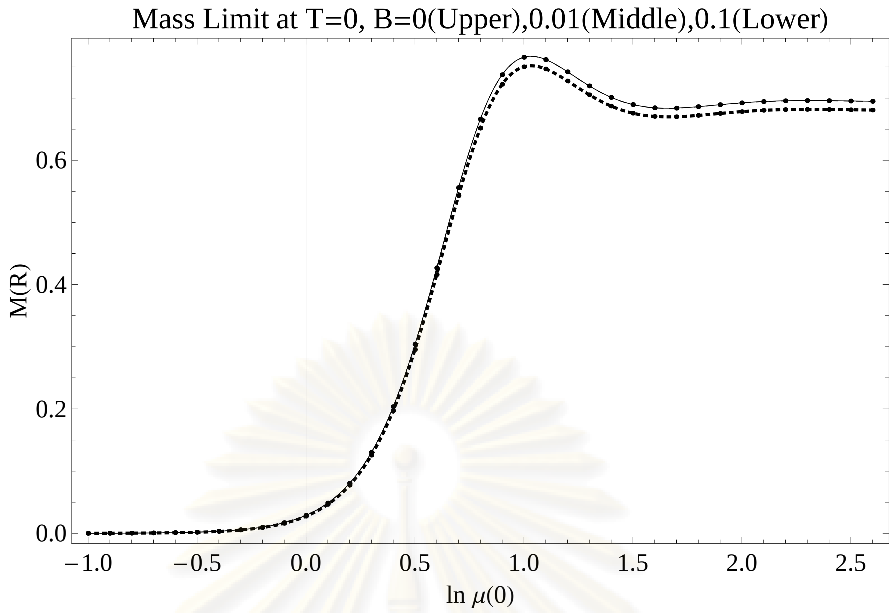

Figure 4.12: The relation between mass and central chemical potential (in logarithmic scale) of the degenerate star at $T = 0$

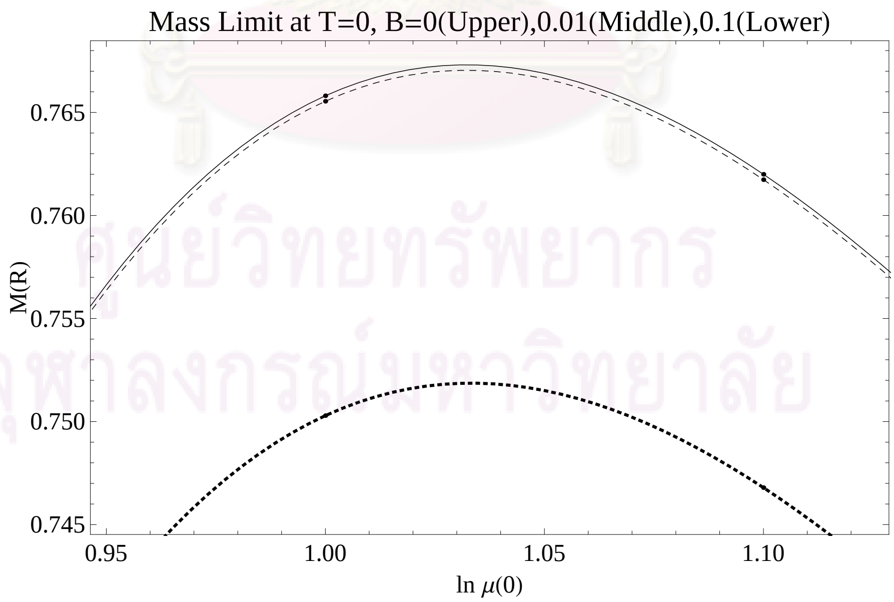

Figure 4.13: Enlargement of the relation between mass and central chemical potential (in logarithmic scale) of the degenerate star at $T = 0$

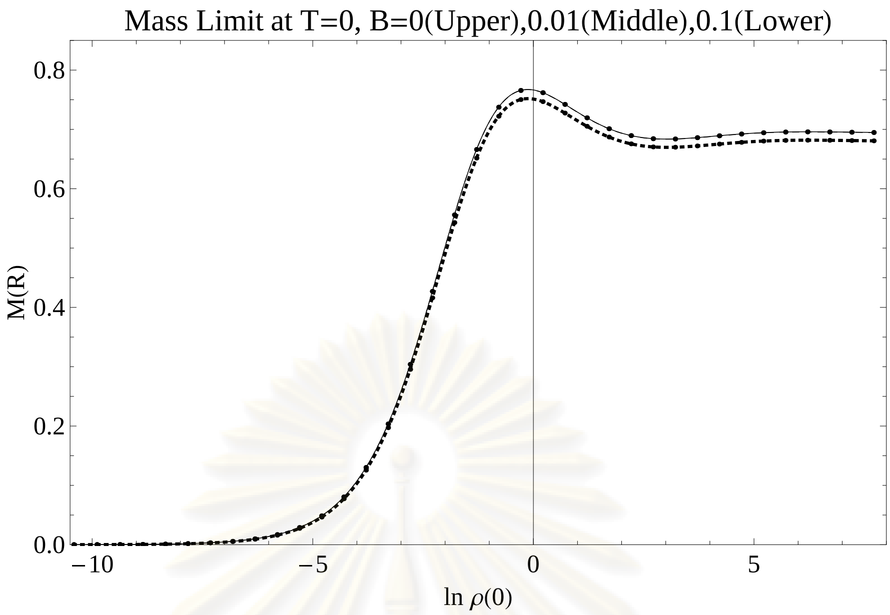

Figure 4.14: The relation between mass and central energy density (in logarithmic scale) of the degenerate star at $T = 0$

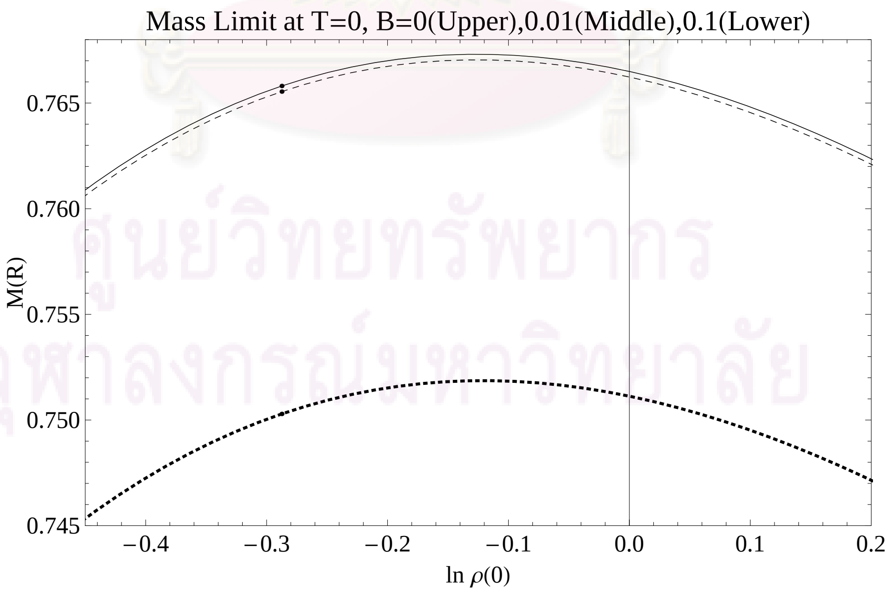

Figure 4.15: Enlargement of the relation between mass and central energy density (in logarithmic scale) of the degenerate star at $T = 0$

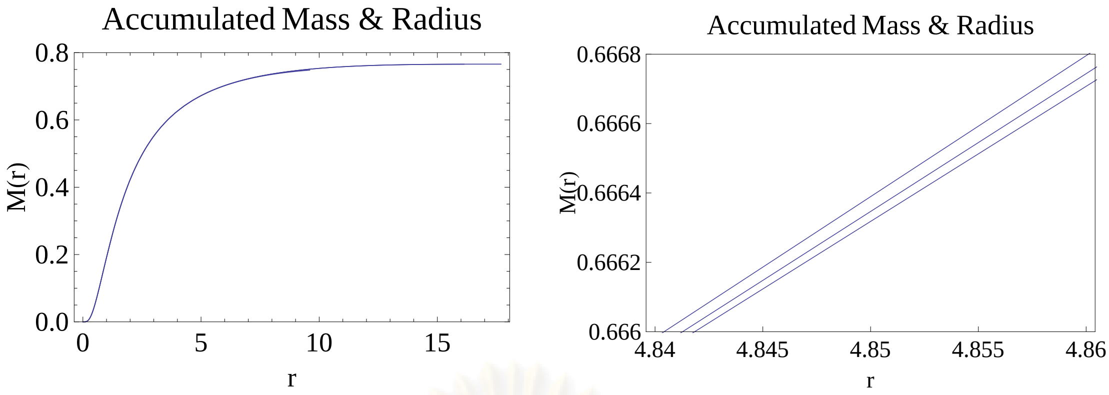

Figure 4.16: The accumulated mass distribution and its enlargement in the degenerate star at $T = 0$, $B = 0$(upper), $= 0.01$ (middle), $= 0.11$ (lower)

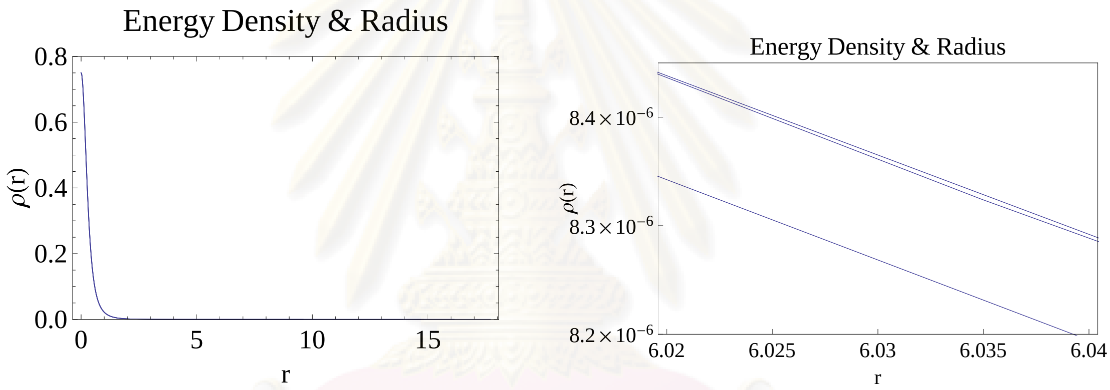

Figure 4.17: The energy density distribution and its enlargement in the degenerate star at $T = 0$, $B = 0$(upper), $= 0.01$ (middle), $= 0.11$ (lower)

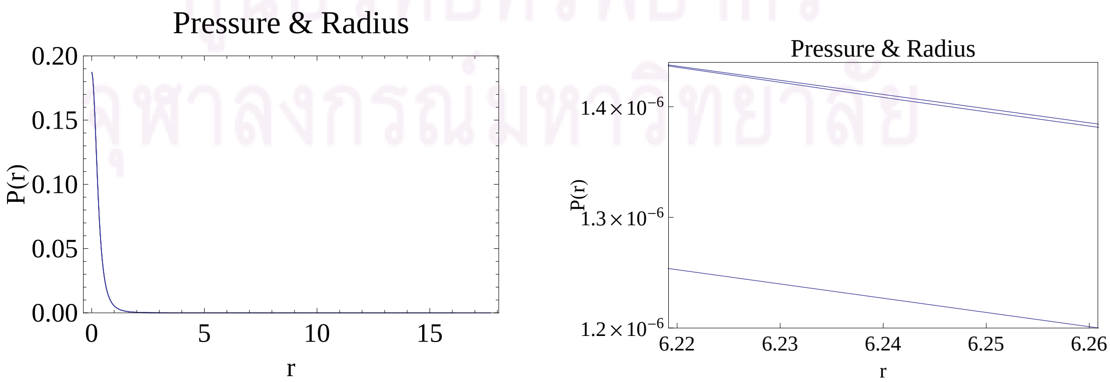

Figure 4.18: The pressure distribution and its enlargement in the degenerate star at $T = 0$, $B = 0$(upper), $= 0.01$ (middle), $= 0.11$ (lower)

Figs. 4.12 - 4.15 show that the mass limit decreases when the magnetic field increases with the maximum mass equal to 0.767036($r = 16.1099$) at $B = 0.01$ and 0.751856($r = 9.92797$) at $B = 0.1$. We can see the cause of this effect from the equation of state in the energy density part (Eq. 4.5b), since the coupled equations of motion between mass and the chemical potential of the star (Eqs. (3.40a) and (3.40b)) involve the energy density. Decreasing the energy density leads to the decrease of mass and the chemical potential of the star. The decrease of the chemical potential leads to the decrease in the pressure of the star subsequently. Numerical analysis confirms these behaviour as are shown in Figs. 4.16, 4.17 and 4.18.

4.4 Case IV, finite temperature and finite magnetic field

Case IV, we consider finite both temperature and magnetic field to study the mass limit of the star. Then the equations of state have the full form according to equations (4.1) and (4.2).

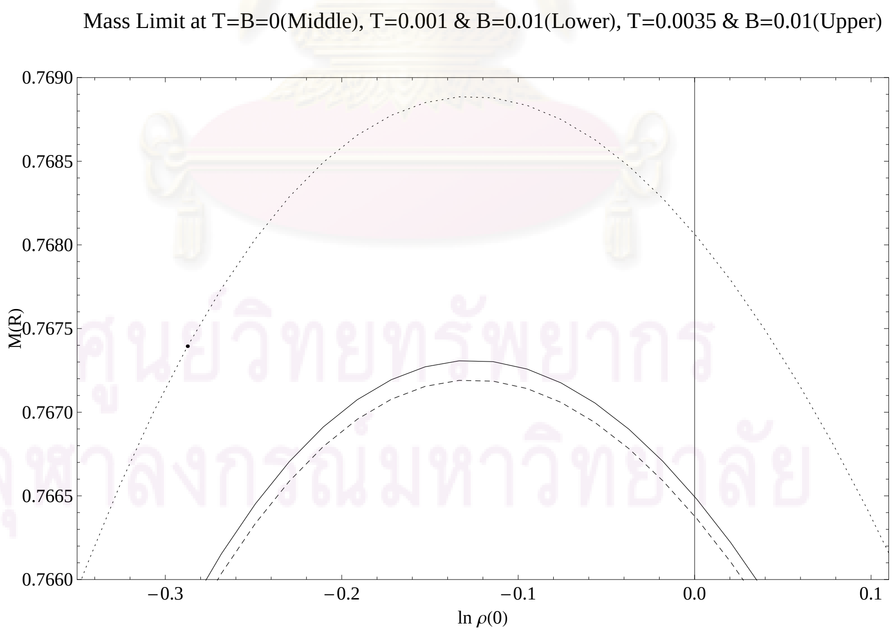

Figure 4.19: The relation between mass and central energy density (in logarithmic scale) of the degenerate star

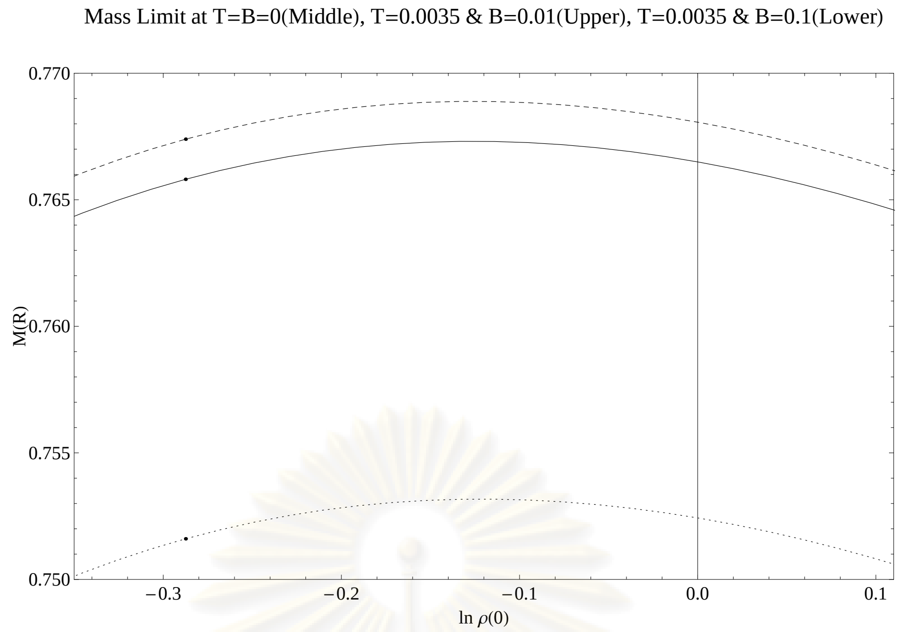

Figure 4.20: The relation between mass and central energy density (in logarithmic scale) of the degenerate star

In this case, we see the similar behaviour as in the second and the third case, temperature increase leads to the increase of the mass limit whereas the effect of the magnetic field is the opposite. In Fig. 4.19, although we set the temperature $T = 0.001$, the magnetic field $B = 0.01$ has more the effect on the profile of the star. The mass limit is smaller than the mass limit in the case of the zero temperature and magnetic field. Surely, when we raise the temperature, the mass limit also grows up (the upper line in the Fig 4.19). In Fig. 4.20, we set the temperature $T = 0.0035$ and the magnetic field $B = 0.01$. In this case, it turns out that the mass limit has increased with respect to the case when $T = 0, B = 0$. When we change the magnetic field to $B = 0.10$, the mass limit becomes smaller than the zero-field zero-temperature mass limit. Namely, the influence of the magnetic has overcome those of the temperature.

4.5 Case V, variation of the curvature radius at zero temperature and magnetic field

We vary the curvature radius of AdS space, l, and study the changes in the profile of the star in this section. For simplicity, we will set the temperature and the external magnetic field to be zero. We let the curvature radius to be $1, 3, 5$ and 7, and observe considerable changes in the mass limit of the star as are shown

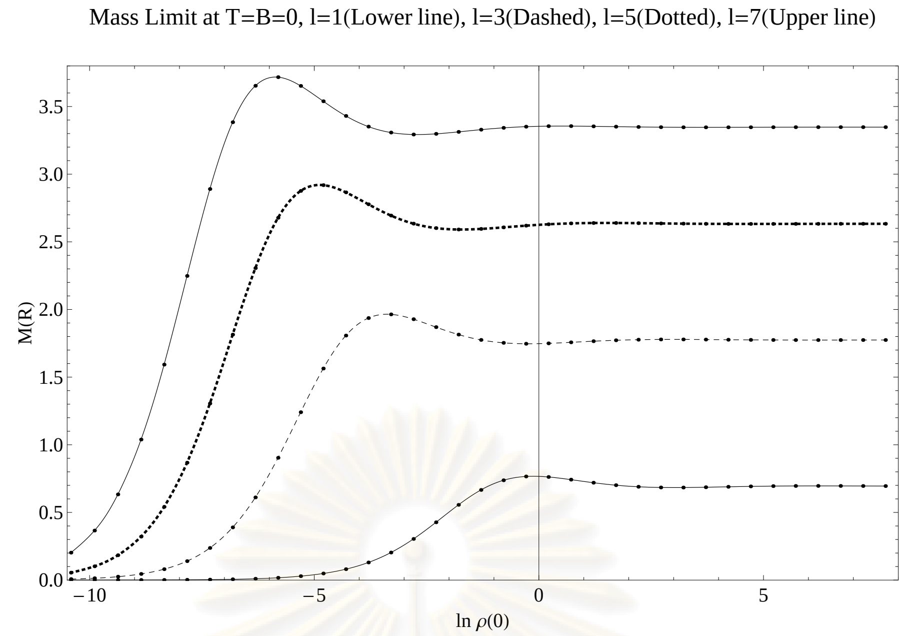

Figure 4.21: The relation between mass and central energy density (in logarithmic scale) of the degenerate star

in Fig. 4.21. The mass limit of the degenerate star increases evidently when we raise the curvature radius of the AdS space. Moreover, the peak of the mass limit curve shifts to the lower central density side. For $l = 3$, the maximum mass is 1.96473($r = 27.4029$) for the central chemical potential $\mu = e^{0.3825}$ or the central energy density $\rho = e^{-3.38048}$. For $l = 5$, the maximum mass is 2.92023($r = 33.5921$) for the central chemical potential $\mu = e^{0.083}$ or the central energy density $\rho = e^{-4.88441}$. For $l = 7$, the maximum mass is 3.71782($r = 38.4035$) for the central chemical potential $\mu = e^{-0.1115}$ or the central energy density $\rho = e^{-5.86373}$.

Chapter V

CONCLUSIONS

Both temperature and magnetic field affect the mass limit and other properties of the degenerate star. The increase of temperature enables the pressure and the energy density of the star to increase, *i.e.* the degenerate pressure also increases. Then the mass limit becomes slightly greater due to the larger pressure. This is the typical behavior of the Fermi gas at finite temperature.

In the presence of external magnetic field, the mass limit decreases when the magnetic field increases. As we can see from Eqs. (4.1) and (4.2), an increase in the magnetic field will result in a smaller energy and pressure density as well as a smaller chemical potential.

The radius of curvature of the AdS space also affects the mass limit evidently. When the radius of curvature increases, the mass limit increases appreciably as are shown in Fig. 4.21. Interestingly, the peak of the mass limit curve shifts to the lower central density side.

REFERENCES

[1] Tolman, R. C. Static solutions of Einstein's field equtions for spheres of fluid. *Phys. Rev.* **55(4)**, (1939): 364-373.

[2] Oppenheimer, J. R., and Volkoff, G. M. On massive neutron cores. *Phys. Rev.* **55(4)**, (1939): 374-381.

[3] 't Hooft, G. Dimensional reduction in quantum gravity. Utrecht Preprint THU-93/26.

[4] Susskind, L. The world as a hologram. *J. Math. Phys.* **36**, (1995): 6377-6396.

[5] Maldacena, J. M. The large N limit of superconformal field theories and supergravity. *Adv. Theor. Math. Phys.* **2**, (1998) 231[Int. J. Theor. Phys. **38**, 1113 (1999)].

[6] de Boer, J., Papadodimas, K., and Verlinde, E. Holographic neutron stars. *JHEP* **1010**, 020 (October 2010).

[7] Arsiwalla, X., de Boer, J., Papadodimas, K., and Verlinde, E. Degenerate stars and gravitational collapse in AdS/CFT. *JHEP* **1101**, 144 (January 2011).

[8] Bousso, R. The holographic principle. *Rev. Mod. Phys.* **74(3)**, (August 2002): 825874.

[9] Jones, J. L. A brief history of the holographic principle. 16 June 2006. (Unpublished Manuscript)

[10] Witten, E. Anti-de Sitter space and holography. *Adv. Theor. Math. Phys.* **2**, (1998): 253-291.

[11] Aharony, O., Gubser, S. S., Maldacena, J. M., Ooguri, H., and Oz, Y. Large N field theories, string theory and gravity. *Phys. Rept.* **323**, (2000): 183-386.

[12] Reichl, L. E. *A modern course in statistical physics.* Canada: John Wiley & Sons, 1998.

[13] Greiner, W., Neise, L., and Stöcker, H. *Thermodynamics and statistical mechanics.* New York: Springer-Verlag, 1995.

[14] Fierz, M. Über die relativistische theorie kräftefreier teilchen mit beliebigem spin. *Helv. Phys. Acta* **12**, (1939): 3-37.

[15] Pauli, W. The connection between spin and statistics. *Phys. Rev.* **58**, (1940): 716-722.

[16] Hobson, M. P., Efstathiou, G. P., and Lasenby, A. N. *General relativity: An introduction for physicists.* New York: Cambridge University Press, 2006.

[17] Martin, J. L. *General realtivity: A guide to its consequences for gravity and cosmology.* Chichester: Ellis Horwood Limited, 1988.

[18] Ryder, L. *Introduction to general relativity.* New York: Cambridge University Press, 2009.

[19] Shapiro, S. L. and Teukolsky, S. A. *Black holes, white dwarfs, and neutron stars: The physics of compact objects.* Weinheim: WILEY-VCH Verlag GmbH & Co. KGaA, 2004.

[20] Camenzind, M. *Compact objects in astrophysics: white dwarfs, neutron stars and black holes.* Springer, 2007.

[21] Herschel, F. W. Catalogue of double stars. *Philosophical Transactions of the Royal Society of London* **75**, (1785): 40-126.

[22] Luyten, W. J. Note on some faint early type stars with large proper motions. *Publications of the Astronomical Society of the Pacific* **34**, (February 1922): 54-55.

[23] Luyten, W. J. Additional note on faint early-type stars with large proper-motions. *Publications of the Astronomical Society of the Pacific* **34**, (April 1922): 132.

[24] Luyten, W. J. The mean parallax of early-type stars of determined proper motion and apparent magnitude. *Publications of the Astronomical Society of the Pacific* **34**, (June 1922): 156-160.

[25] Luyten, W. J. Third note on faint early type stars with large proper motion. *Publications of the Astronomical Society of the Pacific* **34**, (December 1922): 356-357.

[26] Eddington, A. S. On the relation between the masses and luminosities of the stars. *Monthly Notices of the Royal Astronomical Society* **84**, (March 1924): 308-332.

[27] Fowler, R. H. On dense matter. *Monthly Notices of the Royal Astronomical Society* **87**, (1926): 114-122.

[28] Chandrasekhar, S. The density of white dwarf stars. *Phil. Mag.* **11**, (1931): 592-596.

[29] Chandrasekhar, S. The maximum mass of ideal white dwarfs. *Astrophys. J.* **74**, (1931): 81-82.

[30] Chadwick, J. The existence of a neutron. *Proc. Roy. Soc., A,* **136**, (1932): 692-708.

[31] Baade, W. and Zwicky, F. Supernovae and cosmic rays. *Phys. Rev.* **45**, (1934): 138.

[32] Glendenning, N. K. *Special and general relativity: With applications to white dwarfs, neutron stars and black holes.* Springer, 2007.

[33] Schmitt, A. *Dense matter in compact stars: A pedagogical introduction.* Springer, 2010.

[34] Landau, L. D. and Lifshitz, E. M. *Course of theoretical physics Vol 3, Quantum mechanics: Non-relativistic theory.* Oxford: Pergamon Press, 1965.

[35] Landau, L. D. Diamagnetismus der metalle. *Z. Phys.* **64**, (1930): 629-637.

[36] Blumenson, L. E. A derivation of n-dimensional spherical coordinates. *Mathematical Association of America* **67**, (1960): 63-66.

[37] Akhiezer, A. I. and Berestetskii, V. B. *Quantum electrodynamics. Interscience Publishers.* 1965.

[38] Cetina, E., Magaña, F. and Valladares, A. A. The free-electron gas in n dimensions. *American Journal of Physics* **45**, (1977): 960-963.

[39] Landau, L. D. and Lifshitz, E. M. *Course of theoretical physics Vol 5, Statistical physics.* Oxford: Pergamon Press, 1980.

APPENDICES

Appendix A

Useful Calculation

A.1 Pauli matrices

Pauli matrices are a set of 2×2-matrix, denoted by σ. They are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{A.1}$$

where subscripts 1, 2 and 3 refer to x, y and z, respectively. Properties of Pauli matrices are firstly,

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = -\,i\sigma_1\sigma_2\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I, \tag{A.2a}$$

$$\sigma_i^\dagger = \sigma_i, \tag{A.2b}$$

$$\det(\sigma_i) = -1, \tag{A.2c}$$

$$Tr(\sigma_i) = 0 \qquad for \quad i = 1, 2, 3. \tag{A.2d}$$

Secondly, Pauli matrices satisfy the following anticommutation and commutation relations

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij} \cdot I_{2\times 2}, \tag{A.3a}$$

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k. \tag{A.3b}$$

Thirdly, Pauli vector is defined by $\vec{\sigma} = \sigma_1\hat{x} + \sigma_2\hat{y} + \sigma_3\hat{z}$. Consider $\vec{a} = a_1\hat{x} + a_2\hat{y} + a_3\hat{z}$, Thus

$$\vec{\sigma} \cdot \vec{a} = \sum_k a_k\sigma_k = \begin{pmatrix} +a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 \end{pmatrix}, \tag{A.4}$$

and we have a very important identity

$$\begin{aligned}
(\vec{\sigma} \cdot \vec{a})\left(\vec{\sigma} \cdot \vec{b}\right) &= \left(\sum_j \sigma_j a_j\right)\left(\sum_k \sigma_k b_k\right) = \sum_j \sum_k \left(\frac{1}{2}\{\sigma_j, \sigma_k\} + \frac{1}{2}[\sigma_j, \sigma_k]\right) a_j b_k, \\
&= \sum_j \sum_k \left(\delta_{jk} + i\epsilon_{jkl}\sigma_l\right) a_j b_k, \\
&= \left(\vec{a} \cdot \vec{b}\right) + i\vec{\sigma} \cdot \left(\vec{a} \times \vec{b}\right).
\end{aligned} \tag{A.5}$$

A.2 Hermite function

When we have the differential equation in the form

$$\psi_n''(x) + \left(2n + 1 - x^2\right)\psi_n(x) = 0. \tag{A.6}$$

Solution of this equation is a Hermite function

$$\psi_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} e^{\frac{x^2}{2}} \frac{d^n}{dx^n}\left(e^{-x^2}\right). \tag{A.7}$$

A.3 Euler-Maclaurin formula

The Euler-Maclaurin formula provides a powerful approximate connection between integral and summation.

$$\sum_{j=0}^{\infty} f\left(\overbrace{j + \frac{1}{2}}^{\text{x}}\right) \approx \int_0^{\infty} f(x)\,dx + \frac{1}{24} f'(0) + \ldots \quad . \tag{A.8}$$

Appendix B

Table and Codes

B.1 Dimensional translation table

Table B. 1: Dimensional translation table of physical quantities

quantity	dimensionless variable	physical variable
density	ρ	$\rho_0\rho$
pressure	P	$\rho_0 P$
mass	M	$\left(\frac{c^{10}}{G^4\rho_0}\right)^{\frac{1}{3}} M$
radius	r	$\left(\frac{c^4}{G\rho_0}\right)^{\frac{1}{3}} r$
temperature	T	$\frac{\left(\rho_0 c^4 \hbar^4\right)^{\frac{1}{5}}}{k_B} T$
magnetic field	B	$\frac{\left(\rho_0 c^4 \hbar^4\right)^{\frac{1}{5}}}{\mu_B} B$

$\rho_0 = \frac{\left(\frac{m_p}{m_s}c^2\right)^5}{c^4\hbar^4}$ where m_p and m_s are rest mass of particles (*i.e.* electron, neutron etc.) and mass that use in simulation, respectively.

B.2 Mathematica Codes

We present the principal mathematica codes for simulation. Our codes presented here consider the case for zero temperature and magnetic field and the curvature radius of AdS space $l = 1$. For other cases, we can adjust the parameters to match the case of our interest. There are three main codes.

1. Mathematica code for finding the mass limit of the star.

```
For[j = 0, j ≤ 36, j++, {Clear;, G = 1(*6.67428*10^-11 and G5=Gl(e^(2R/l)-1)≈Gl,
   the distance R is the radion, l is a curvature radius of AdS5*);,
  c = 1(*299792458*);, h = 2 * Pi(*6.626068*10^-34*);, k = 1(*1.3806503*10^-23*);,
  μB = 1(*for electron 9.27400915*10^-24 and for neutron -9.662364*10^-24*);,
  V3 = 2 * Pi^2;, l = 1.;, C4 = (16 * Pi * G * l)/(3 * V3);, x = 1/2;,
  m = .1(*for electron 9.10938188*10^-31 and for neutron 1.67492729*10^-27*);,
  ρ0 = (((1.67492729*10^-27)/m) (299 792 458)^2)^5 / ((299 792 458)^4 ((6.626068*10^-34)/(2 Pi))^4);,
  r0 = ((299 792 458)^4 / ((6.67428 * 10^-11) (ρ0)))^(1/3);, T = 10^-6(*10^8*);, B = 0.;,
  P[r_] := (2 π^2)/(30 c^4 h^4)
    (3 (μ[r])^5 - 10 m^2 c^4 (μ[r])^3 + 15 m^4 c^8 μ[r] - 8 m^5 c^10 - 10 k^2 T^2 m^2 c^4 π^2 μ[r] + 7 k^4 T^4 π^4 μ[r] +
     10 k^2 T^2 π^2 (μ[r])^3 - 120 k^3 T^3 m^2 c^4 PolyLog[3, -e^((m c^2-μ[r])/(k T))] + 360 k^4 T^4 m c^2 PolyLog[4, -e^((m c^2-μ[r])/(k T))] -
     360 k^5 T^5 PolyLog[5, -e^((m c^2-μ[r])/(k T))] - 20 k T m^2 c^4 (μB)^2 B^2 Log[1 + e^((μ[r]-m c^2)/(k T))]);,
  ρ[r_] := 2/(15 c^4 h^4) 2 π^2 (3 (μ[r])^5 - 5 m^2 c^4 (μ[r])^3 + 2 m^5 c^10 - 5 k^2 T^2 m^2 c^4 π^2 μ[r] + 7 k^4 T^4 π^4 μ[r] +
     10 k^2 T^2 π^2 (μ[r])^3 + 30 k^2 T^2 m^3 c^6 PolyLog[2, -e^((m c^2-μ[r])/(k T))] - 150 k^3 T^3 m^2 c^4 PolyLog[3, -e^((m c^2-μ[r])/(k T))] +
     360 k^4 T^4 m c^2 PolyLog[4, -e^((m c^2-μ[r])/(k T))] - 360 k^5 T^5 PolyLog[5, -e^((m c^2-μ[r])/(k T))]) + (2 * 2 m^2 π^2 (μB)^2 B^2)/(3 h^4)
    ((m c^2)/((1 + e^((μ[r]-m c^2)/(k T)))) - μ[r] - k T Log[1 + e^((m c^2-μ[r])/(k T))] + k T Log[1 + e^((μ[r]-m c^2)/(k T))]);, $RecursionLimit = Infinity;,
```

Mathematica code for simulation of the mass limit of the star Part 1

```
solution =
  NDSolve[{
    ∂r M[r] == 2/3 * V3 * r^3 * ρ[r],
    ∂r μ[r] == μ[r] (((C4 M'[r])/(2*r^2) - (C4 M[r])/r^3 - r/l^2)/(1 - (C4 M[r])/r^2 + r^2/l^2) - ((V3 * C4 * r * (1 - (C4 M[r])/r^2 + r^2/l^2)^-1)/3 (ρ[r] + P[r]))),
    M[10^-12] == 0, μ[10^-12] == E^(-1.+(.1 j))}, {M, μ}, {r, 10^-12, 100}, MaxSteps → 1 000 000 000];

  R0 = Last[Last[FindRoot[{μ[r] /. solution} - Sqrt[(m c^2)^2 + 2 m c^2 μB B] == 0, {r, 1.}]]];,
  R1 = Last[Last[FindRoot[{ρ[r] /. solution} == 0, {r, 1.}]]];,
  R2 = Last[Last[FindRoot[{(ρ[r] + P[r]) /. solution} == 0, {r, 1.}]]];,
  R = Last[Last[FindRoot[{P[r] /. solution} == 0, {r, 1.}]]];,
  Print[R0, "    ", R1, "    ", R2, "    ", R, "    ", First[μ[R] /. solution], "    ",
    First[Log[μ[10^-12]] /. solution], "    ", First[Log[ρ[10^-12]] /. solution], "    ",
    First[M[R] /. solution], "    ", r0 * R, "    ", ((ρ0) (r0)^4)/(299 792 458)^2 First[M[R] /. solution],
    "    ", (((ρ0) (r0)^4)/(299 792 458)^2 First[M[R] /. solution])/(1.98892 * 10^30)];, μ[j] = First[Log[μ[10^-12]] /. solution];,
  ρ[j] = First[Log[ρ[10^-12]] /. solution];, M[j] = First[M[R] /. solution];}]
Table[{μ[j], M[j]}, {j, 0, 36}]
ListPlot[Table[{μ[j], M[j]}, {j, 0, 36}], PlotRange → {{-1.0, 2.6}, {0, .8}},
 Frame → True, FrameLabel → TraditionalForm /@ {"ln μ(0)", "M(R)"},
 BaseStyle → {FontFamily → "Times", FontSize → 16}, PlotLabel → "Mass Limit at T=0, B=0",
 Mesh → Full, InterpolationOrder → 4, Joined → True] (*G=c=kB=μB=l=1,h=2π,T=10^-6→0,B=0,m=.1*)
Table[{ρ[j], M[j]}, {j, 0, 36}]
ListPlot[Table[{ρ[j], M[j]}, {j, 0, 36}], PlotRange → {{-10.5, 7.8}, {0, .8}},
 Frame → True, FrameLabel → TraditionalForm /@ {"ln ρ(0)", "M(R)"},
 BaseStyle → {FontFamily → "Times", FontSize → 16}, PlotLabel → "Mass Limit at T=0, B=0",
 Mesh → Full, InterpolationOrder → 4, Joined → True] (*G=c=kB=μB=l=1,h=2π,T=10^-6→0,B=0,m=.1*)
```

Mathematica code for simulation of the mass limit of the star Part 2

2. Mathematica code for finding the relation between the mass, chemical potential and energy density and radius of the star.

Clear;

G = 1(*6.67428*10^{-11} and G$_5$=Gl$\left(e^{\frac{2R}{l}}-1\right)$≈Gl, the distance R is the radion, l is a curvature radius of AdS$_5$*); c = 1(*299792458*); h = 2 * Pi(*6.626068*10^{-34}*); k = 1 (*1.3806503*10^{-23}*); μB = 1(*for electron 9.27400915*10^{-24} and for neutron -9.662364*10^{-24}*);

V$_3$ = 2 * Pi^2; l = 1; C$_4$ = $\frac{16 * Pi * G * l}{3 * V_3}$; m = .1 (*for electron 9.10938188*10^{-31} and for neutron 1.67492729*10^{-27}*);

$$\rho 0 = \frac{\left(\left(\frac{1.67492729*10^{-27}}{m}\right)(299\,792\,458)^2\right)^5}{(299\,792\,458)^4\left(\frac{6.626068*10^{-34}}{2\,Pi}\right)^4};$$

$$r0 = \left(\frac{(299\,792\,458)^4}{\left(6.67428 * 10^{-11}\right)(\rho 0)}\right)^{\frac{1}{3}}; \; T = 10^{-6}(*10^8*); \; B = 0.;$$

P[r_] :=

$$\frac{2\pi^2}{30 c^4 h^4}\Big(3(\mu[r])^5 - 10 m^2 c^4 (\mu[r])^3 + 15 m^4 c^8 \mu[r] - 8 m^5 c^{10} - 10 k^2 T^2 m^2 c^4 \pi^2 \mu[r] + 7 k^4 T^4 \pi^4 \mu[r] + 10 k^2 T^2 \pi^2 (\mu[r])^3 - 120 k^3 T^3 m^2 c^4 \, PolyLog\left[3, -e^{\frac{m c^2-\mu[r]}{k T}}\right] + 360 k^4 T^4 m c^2 \, PolyLog\left[4, -e^{\frac{m c^2-\mu[r]}{k T}}\right] - 360 k^5 T^5 \, PolyLog\left[5, -e^{\frac{m c^2-\mu[r]}{k T}}\right] - 20 k T m^2 c^4 (\mu B)^2 B^2 \, Log\left[1 + e^{\frac{\mu[r]-m c^2}{k T}}\right]\Big);$$

$$\rho[r\_] := \frac{2}{15 c^4 h^4} 2\pi^2 \Big(3(\mu[r])^5 - 5 m^2 c^4 (\mu[r])^3 + 2 m^5 c^{10} - 5 k^2 T^2 m^2 c^4 \pi^2 \mu[r] + 7 k^4 T^4 \pi^4 \mu[r] + 10 k^2 T^2 \pi^2 (\mu[r])^3 + 30 k^2 T^2 m^3 c^6 \, PolyLog\left[2, -e^{\frac{m c^2-\mu[r]}{k T}}\right] - 150 k^3 T^3 m^2 c^4 \, PolyLog\left[3, -e^{\frac{m c^2-\mu[r]}{k T}}\right] + 360 k^4 T^4 m c^2 \, PolyLog\left[4, -e^{\frac{m c^2-\mu[r]}{k T}}\right] - 360 k^5 T^5 \, PolyLog\left[5, -e^{\frac{m c^2-\mu[r]}{k T}}\right]\Big) + \frac{2 * 2 m^2 \pi^2 (\mu B)^2 B^2}{3 h^4}\left(\frac{m c^2}{\left(1 + e^{\frac{\mu[r]-m c^2}{k T}}\right)} - \mu[r] - k T \, Log\left[1 + e^{\frac{m c^2-\mu[r]}{k T}}\right] + k T \, Log\left[1 + e^{\frac{\mu[r]-m c^2}{k T}}\right]\right);$$

Mathematica code for simulation of the mass, chemical potential, pressure and energy density distributions Part 1

```
$RecursionLimit = Infinity;
solution =
 NDSolve[{
   ∂_r M[r] == 2/3 * V_3 * r^3 * ρ[r],
   ∂_r μ[r] == μ[r] (((C_4 M'[r])/(2*r^2) - (C_4 M[r])/r^3 - r/l^2)/(1 - (C_4 M[r])/r^2 + r^2/l^2) - ((V_3 * C_4 * r * (1 - (C_4 M[r])/r^2 + r^2/l^2)^-1)/3 (ρ[r] + P[r]))),
   M[10^-12] == 0, μ[10^-12] == E^1.}, {M, μ}, {r, 10^-12, 1000}, MaxSteps → 1000000000];
R0 = Last[Last[FindRoot[{μ[r] /. solution} - Sqrt[(m c^2)^2 + 2 m c^2 μB B] == 0, {r, 1.}]]];
R1 = Last[Last[FindRoot[{ρ[r] /. solution} == 0, {r, 1.}]]];
R2 = Last[Last[FindRoot[{(ρ[r] + P[r]) /. solution} == 0, {r, 1.}]]];
R = Last[Last[FindRoot[{P[r] /. solution} == 0, {r, 1.}]]];
Print[R0, "    ", R1, "    ", R2, "    ", R, "    ",
 First[μ[R] /. solution], "    ", First[Log[μ[10^-12]] /. solution], "    ",
 First[Log[ρ[10^-12]] /. solution], "    ", First[M[R] /. solution], "    ", r0 * R, "    ",
 ((ρ0) (r0)^4)/(299792458)^2 First[M[R] /. solution], "    ", (((ρ0) (r0)^4)/(299792458)^2 First[M[R] /. solution])/(1.98892 * 10^30)]
Plot[{M[r] /. solution}, {r, 0, R}, PlotRange → {0, 0.8},
 PlotLabel → "Accumulated Mass & Radius at T=0, B=0",
 Frame → True, FrameLabel → TraditionalForm /@ {"r", "M(r)"},
 BaseStyle → {FontFamily → "Times", FontSize → 16}]
Plot[{μ[r] /. solution}, {r, 0, R}, PlotRange → {0, 3},
 PlotLabel → "Chemical Potential & Radius at T=0, B=0",
 Frame → True, FrameLabel → TraditionalForm /@ {"r", "μ(r)"},
 BaseStyle → {FontFamily → "Times", FontSize → 16}]
Plot[{ρ[r] /. solution}, {r, 0, 2.}, PlotRange → {0, .8},
 PlotLabel → "Energy Density & Radius at T=0, B=0",
 Frame → True, FrameLabel → TraditionalForm /@ {"r", "ρ(r)"},
 BaseStyle → {FontFamily → "Times", FontSize → 16}]
Plot[{P[r] /. solution}, {r, 0, 2.}, PlotRange → {0, 0.2},
 PlotLabel → "Pressure & Radius at T=0, B=0", Frame → True,
 FrameLabel → TraditionalForm /@ {"r", "P(r)"}, BaseStyle → {FontFamily → "Times", FontSize → 16}]
```

Mathematica code for simulation of the mass, chemical potential, pressure and energy density distributions Part 2

3. Mathematica code for comparing the mass, chemical potential, pressure and energy density distributions when some parameter values are changed.

```
For[j = 0, j ≤ 2, j++, {Clear;, G = 1(*6.67428*10^-11 and G5=Gl(e^(2R/l)-1)≈Gl,
  the distance R is the radion, l is a curvature radius of AdS5*);,
  c = 1(*299792458*);, h = 2 * Pi(*6.626068*10^-34*);, k = 1(*1.3806503*10^-23*);,
  μB = 1(*for electron 9.27400915*10^-24 and for neutron -9.662364*10^-24*);,
  V3 = 2 * Pi^2;, l = 1.;, C4 = (16 * Pi * G * l)/(3 * V3);, x = 1/2;,
  m = .1(*for electron 9.10938188*10^-31 and for neutron 1.67492729*10^-27*);,
  ρ0 = (((1.67492729*10^-27)/m) (299 792 458)^2)^5/((299 792 458)^4 ((6.626068*10^-34)/(2 Pi))^4);,
  r0 = ((299 792 458)^4/((6.67428 * 10^-11) (ρ0)))^(1/3);, T = 10^-6 + (0.001 j) (*10^8*);,
  B = 0.(*0.-(0.035j)+0.045(j)^2*);,
  P[r_] := (2 π^2)/(30 c^4 h^4) (3 (μ[r])^5 - 10 m^2 c^4 (μ[r])^3 + 15 m^4 c^8 μ[r] - 8 m^5 c^10 - 10 k^2 T^2 m^2 c^4 π^2 μ[r] +
      7 k^4 T^4 π^4 μ[r] + 10 k^2 T^2 π^2 (μ[r])^3 - 120 k^3 T^3 m^2 c^4 PolyLog[3, -e^((m c^2-μ[r])/(k T))] + 360 k^4 T^4 m c^2
      PolyLog[4, -e^((m c^2-μ[r])/(k T))] - 360 k^5 T^5 PolyLog[5, -e^((m c^2-μ[r])/(k T))] - 20 k T m^2 c^4 (μB)^2 B^2 Log[1 + e^((μ[r]-m c^2)/(k T))]);,
  ρ[r_] := 2/(15 c^4 h^4) 2 π^2 (3 (μ[r])^5 - 5 m^2 c^4 (μ[r])^3 + 2 m^5 c^10 - 5 k^2 T^2 m^2 c^4 π^2 μ[r] + 7 k^4 T^4 π^4 μ[r] +
      10 k^2 T^2 π^2 (μ[r])^3 + 30 k^2 T^2 m^3 c^6 PolyLog[2, -e^((m c^2-μ[r])/(k T))] - 150 k^3 T^3 m^2 c^4 PolyLog[3, -e^((m c^2-μ[r])/(k T))] +
      360 k^4 T^4 m c^2 PolyLog[4, -e^((m c^2-μ[r])/(k T))] - 360 k^5 T^5 PolyLog[5, -e^((m c^2-μ[r])/(k T))]) +
    (2 * 2 m^2 π^2 (μB)^2 B^2)/(3 h^4) ((m c^2)/((1 + e^((μ[r]-m c^2)/(k T)))) - μ[r] - k T Log[1 + e^((m c^2-μ[r])/(k T))] + k T Log[1 + e^((μ[r]-m c^2)/(k T))]);,
  $RecursionLimit = Infinity;,
  solution =
   NDSolve[{
     ∂r M[r] == 2/3 * V3 * r^3 * ρ[r],
     ∂r μ[r] == μ[r] ((((C4 M'[r])/(2*r^2) - (C4 M[r])/r^3 - r/l^2))/((1 - (C4 M[r])/r^2 + r^2/l^2)) - ((V3 * C4 * r * (1 - (C4 M[r])/r^2 + r^2/l^2)^-1)/3 (ρ[r] + P[r]))),
     M[10^-12] == 0, μ[10^-12] == E}, {M, μ}, {r, 10^-12, 1000}, MaxSteps → 1 000 000 000];,
```

Mathematica code for simulation of comparing the mass, chemical potential, pressure and energy density distributions Part 1

```
R0 = Last[Last[FindRoot[{μ[r] /. solution} - Sqrt[(m c^2)^2 + 2 m c^2 μB B] == 0, {r, 1.}]]];,
  R1 = Last[Last[FindRoot[{ρ[r] /. solution} == 0, {r, 1.}]]];,
  R2 = Last[Last[FindRoot[{(ρ[r] + P[r]) /. solution} == 0, {r, 1.}]]];,
  R = Last[Last[FindRoot[{P[r] /. solution} == 0, {r, 1.}]]];,
  Print[Plot[{M[r] /. solution}, {r, 0, R}, PlotRange → {0, 0.8},
     PlotLabel → "Accumulated Mass & Radius", Frame → True, FrameLabel →
      TraditionalForm /@ {"r", "M(r)"}, BaseStyle → {FontFamily → "Times", FontSize → 16}]];,
  Print[Plot[{μ[r] /. solution}, {r, 0, R}, PlotRange → {0, 3.},
     PlotLabel → "Chemical Potential & Radius", Frame → True, FrameLabel →
      TraditionalForm /@ {"r", "μ(r)"}, BaseStyle → {FontFamily → "Times", FontSize → 16}]];,
  Print[Plot[{ρ[r] /. solution}, {r, 0, 2.}, PlotRange → {0, 0.8},
     PlotLabel → "Energy Density & Radius", Frame → True, FrameLabel →
      TraditionalForm /@ {"r", "ρ(r)"}, BaseStyle → {FontFamily → "Times", FontSize → 16}]];,
  Print[Plot[{P[r] /. solution}, {r, 0, 2.}, PlotRange → {0, 0.2}, PlotLabel →
      "Pressure & Radius", Frame → True, FrameLabel → TraditionalForm /@ {"r", "P(r)"},
     BaseStyle → {FontFamily → "Times", FontSize → 16}]];,
  M[j] = Plot[{M[r] /. solution}, {r, 0, R}, PlotRange → {0, 0.8}, AxesLabel → {"r", "M"}],
  μ[j] = Plot[{μ[r] /. solution}, {r, 0, R}, PlotRange → {0, 3.}, AxesLabel → {"r", "μ"}],
  ρ[j] = Plot[{ρ[r] /. solution}, {r, 0, R}, PlotRange → {0, 0.8},
    AxesLabel → {"r", "ρ"}, PlotRange → Automatic],
  P[j] = Plot[{P[r] /. solution}, {r, 0, R}, PlotRange → {0, 0.2},
    AxesLabel → {"r", "P"}, PlotRange → Automatic]}]
Show[M[0], M[1], M[2], PlotLabel → "Accumulated Mass & Radius", Frame → True,
 FrameLabel → TraditionalForm /@ {"r", "M(r)"},
 BaseStyle → {FontFamily → "Times", FontSize → 16}]
Show[μ[0], μ[1], μ[2], PlotRange → {{0, R}, {.0, 3.}}, PlotLabel → "Chemical Potential & Radius",
 Frame → True, FrameLabel → TraditionalForm /@ {"r", "μ(r)"},
 BaseStyle → {FontFamily → "Times", FontSize → 16}]
Show[ρ[0], ρ[1], ρ[2], PlotLabel → "Energy Density & Radius",
 Frame → True, FrameLabel → TraditionalForm /@ {"r", "ρ(r)"},
 BaseStyle → {FontFamily → "Times", FontSize → 16}]
Show[P[0], P[1], P[2], PlotLabel → "Pressure & Radius", Frame → True,
 FrameLabel → TraditionalForm /@ {"r", "P(r)"}, BaseStyle → {FontFamily → "Times", FontSize → 16}]
```

Mathematica code for simulation of comparing the mass, chemical potential, pressure and energy density distributions Part 2

VITAE

Mr. Tossaporn Chullaphan was born in 3 June 1986 and received his Bachelor's degree in physics from Prince of Songkla University in 2008. While he was studying at Prince of Songkla University, he received scholarship from Human Resource Development in Science Project (HRDSP). His research interests are in theoretical physics and mathematical physics.

Presentations

1. Mass limit of holographic degenerate star at finite temperature in the presence of external magnetic field: The 19^{th} National Graduate Research Conference, Rajabhat Rajanagarindra University, Chachoengsao, Thailand, 23 - 24 December 2010.

International Schools

1. 1^{st} CERN School Thailand, Chulalongkorn University, Bangkok, 4 - 13 October 2010.

2. The Siam GR $\bigoplus$ HEP $\bigoplus$ COSMO Symposium IV, Naresuan University, Thailand, 26 -28 July 2009.