ลำดับใหญ่สุดของการส่งเสถียรเสมือนแบบเอกรูป

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MAXIMAL SEQUENCES OF UNIFORMLY VIRTUALLY STABLE MAPS

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เรานิยามความใหญ่สุดให้ลำดับเกี่ยวเนื่องของการส่งเสถียรเสมือนแบบเอกรูป พร้อมทั้งศึกษาโครงสร้างและเงื่อนไขที่รับประกันการมีอยู่ของลำดับใหญ่สุด นอกจากนี้เราสร้างตัวอย่างของการส่งเสถียรเสมือนแบบเอกรูปที่มีลำดับใหญ่สุดตามกำหนด และการส่งเสถียรเสมือนแบบเอกรูปที่ไม่มีลำดับใหญ่สุด

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The notion of maximality of sequences associated with uniformly virtually stable maps is defined. Their structures and conditions that guarantee their existence are investigated. Also, some examples of uniformly virtually stable maps with respect to prescribed sequences and uniformly virtually stable maps having no maximal sequences are constructed.

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LIST OF NOTATIONS

Let (X, τ) be a Hausdorff space, $f : X \to X$ and (Y, d) a metric space.

$\mathbb{N}_0$	the set of natural numbers with 0
S^1	the set $\{z \in \mathbb{C} : \lvert z \rvert = 1\}$
$\mathfrak{N}_X(x)$, $\mathfrak{N}(x)$	the set of neighborhoods of x in X
f^n	the map $\underbrace{f \circ f \circ \cdots \circ f}_{n \text{ copies}}$
$\mathrm{F}(f)$	the set $\{x \in X : f(x) = x\}$
$\mathrm{C}(f)$	the set $\{x \in X : \lim_{n \to \infty} f^n(x) \text{ exists}\}$
$\mathrm{B}_d(x; r)$	the set $\{y \in Y : d(x, y) < r\}$
$i, j, \ldots, n$	nonnegative integers
r, s, t	nonnegative (real) numbers

CHAPTER I
INTRODUCTION

This dissertation is in the field of fixed point theory which has the motivation from the researches of P. Chaoha, W. Atiponrat and S. Iampiboonvatana. In 2007, P. Chaoha introduced the virtual nonexpansiveness of continuous selfmaps on metric spaces and showed that the fixed point sets of maps of this type are retracts of convergence sets (see [1]). In 2009, the generalization of virtual nonexpansiveness of maps on the Hausdorff spaces was shown by P. Chaoha and W. Atiponrat, and called it the virtual stability of (continuous) maps. Also, in some situations, every fixed point set of a virtually stable map is still a retract of convergence set of that map (see [2]). Recently, S. Iampiboonvatana generalized the virtual stability of maps by dropping the continuity. Again the fixed point sets of this re-defined maps are retracts of convergence sets. Moreover the notion of uniform virtual stability of maps was defined, also. The uniform virtual stability of maps are indeed dependent on some sequences. These sequences seem to have an implicit structure and, in some sense, they are allowed to extend to bigger sequences. This leads us to the concept of maximality of such sequences.

In this thesis, we give the definition of such maximality and also investigate sequences which are maximal in this sense. We call these sequences maximal sequences. In chapter II (preliminaries), all necessary basic knowledge for this thesis will be reviewed (without proofs). In chapter III (maximal sequences), the precise definition of the maximality of sequences will be given and also basic and non-basic properties will be shown. The last chapter (examples of uniformly virtually stable maps with respect to prescribed sequences) gives some interesting examples of uniformly virtually stable maps whose its associated maximal sequence is prescribed, and uniformly virtually stable maps without maximal sequences.

CHAPTER II
PRELIMINARIES

Throughout this dissertation,

(1) the word *sequence* is referred to *a strictly increasing sequence of natural numbers* (and also, for the notation of the sequence: (a_n));

(2) every infinite set A of natural numbers is determined by the sequence whose image is A, similarly, every sequence (a_n) is determined by its image, $\{a_n : n \in \mathbb{N}\}$. Then there is no confusion when the same notion is used for both set and sequence;

(3) all spaces are assumed to be Hausdorff spaces.

In this chapter, the basic knowledge for our invertigation is stated. The first section gives the notion of virtual stability of maps, and also, some of its properties. The second section states (without proofs) some results in number and set theories used in this thesis.

2.1 Virtual Stability

Definition 2.1.1. [3] A selfmap f having a fixed point on a Hausdorff space X is said to be:

(1) *virtually stable* if for each fixed point x of f and each neighborhood U of x, there are a neighborhood V and a strictly increasing sequence (a_n) of natural numbers such that $f^{a_n}(V) \subseteq U$ for all $n \in \mathbb{N}$, and denoted by $\mathrm{VS}(X)$ (briefly, VS) the set of virtually stable maps on X. That is,

$$f \in \mathrm{VS} \iff \forall x \in \mathrm{F}(f) \forall U \in \mathfrak{N}(x) \exists V \in \mathfrak{N}(x) \exists (a_n) \forall n \in \mathbb{N} \left[f^{a_n}(V) \subseteq U \right].$$

(2) *uniformly virtually stable* if there is a strictly increasing sequence (a_n) of natural numbers such that for each fixed point x and each neighborhood U of x, there exists a neighborhood V of x satisfying a condition: $f^{a_n}(V) \subseteq U$ for all $n \in \mathbb{N}$; in this case, f is called a *uniformly virtually stable map with respect to* (a_n) and (a_n) is called a *sequence associated with* f (briefly, *associated sequence*). Denote the sets of uniformly virtually stable maps on X and of uniformly virtually stable maps on X with respect to (a_n) by $\mathrm{UVS}(X)$ (briefly, UVS) and $\mathrm{UVS}(X; a_n)$ (briefly, $\mathrm{UVS}(a_n)$), respectively. Thus

$$f \in \mathrm{UVS} \iff \exists (a_n) \forall x \in \mathrm{F}(f) \forall U \in \mathfrak{N}(x) \exists V \in \mathfrak{N}(x) \forall n \in \mathbb{N} \, [f^{a_n}(V) \subseteq U] \, .$$

Remark 2.1.2. In the case that X is a metric space, uniform virtual stability of f with respect to (a_n) is equivalent to the equicontinuity of $\{f^{a_n} : n \in \mathbb{N}\}$.

Proposition 2.1.3.

(1) $\mathrm{UVS} \subseteq \mathrm{VS}$.

(2) *Every virtually nonexpansive map is uniformly virtually stable with respect to* (n).

Proof. (1) trivial; (2) see [3]. □

The following are examples of uniformly virtually stable maps with and without continuity and of a virtually stable map which is not uniformly virtually stable.

Example 2.1.4. The identity map is always uniformly virtually stable with respect to (n).

Example 2.1.5. A selfmap f on $\mathbb{C}$ defined by

$$f(z) = \begin{cases} z & \text{if } |z| = 1; \\ \overline{z} & \text{otherwise,} \end{cases}$$

is uniformly virtually stable with respect to $(2n)$ because $f^{2n} = \mathrm{id}_{\mathbb{C}} \in \mathrm{UVS}(n)$ for all $n \in \mathbb{N}$ (which implies that $f \in \mathrm{UVS}(2n)$).

Example 2.1.6. A selfmap f on $X := \bigcup_{r\in\mathbb{N}}\{z \in \mathbb{C} : |z| = r\}$ defined by

$$f(z) = f(re^{i\phi}) = \begin{cases} r & \text{if } \phi = 2\pi n \text{ for some } n \in \mathbb{Z}; \\ re^{i\left(\phi+\frac{2\pi}{r}\right)} & \text{otherwise,} \end{cases}$$

is virtually stable but not uniformly virtually stable. To see this, notice that X is disconnected, $\mathrm{F}(f) = \mathbb{N} \cup S^1$ and $f|_{S^1} = \mathrm{id}_{S^1}$. The virtual stability of f is obtained from the fact that for any $n > 1$, by setting $Y = \{z \in X : |z - n| < 0.5\}$, then $f^n(Y) = \mathrm{id}_Y(Y) = Y$. Suppose $f \in \mathrm{UVS}(a_n)$ for some (a_n). We already know that $2a_1 \in \mathrm{F}(f)$ and $U := \{z \in X : |z - 2a_1| < 0.5\} \in \mathfrak{N}(2a_1)$. Thus

$$f^{a_1}(2a_1e^{i\phi}) = 2a_1e^{i\left(\phi+\sum_{i=1}^{a_1}\frac{\pi}{a_1}\right)} = 2a_1e^{i(\phi+\pi)} = -2a_1e^{i\phi} \notin U$$

for all $2a_1e^{i\phi} \in U - \{2a_1\}$. Hence there does not exist $V \in \mathfrak{N}(2a_1)$ such that $f^{a_1}(V) \subseteq U$, which is a contradiction.

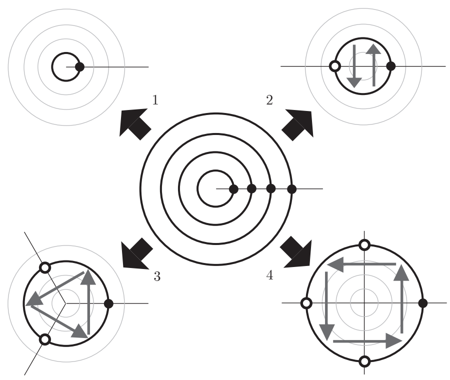

Figure 2.1: Illustrate the map f in Example 2.1.6

Example 2.1.7. Define a selfmap f on $\mathbb{C}$ by

$$f(z) = f(re^{i\phi}) = \begin{cases} r & \text{if } r \in \mathbb{N} \text{ and } \phi = 2\pi n \text{ for some } n \in \mathbb{Z}; \\ \lceil r - 0.1\rceil\, e^{i\left(\phi+\frac{2\pi}{\lceil r-0.1\rceil}\right)} & r \in (0.1, \infty) - \mathbb{N}; \\ 0 & \text{otherwise.} \end{cases}$$

This map is indeed adapted from Example 2.1.6 but its fixed point set is $\mathbb{N}_0 \cup S^1$; so it is easy to see that it is virtually stable but not uniformly virtually stable. The difference between these two examples is the connectedness of spaces.

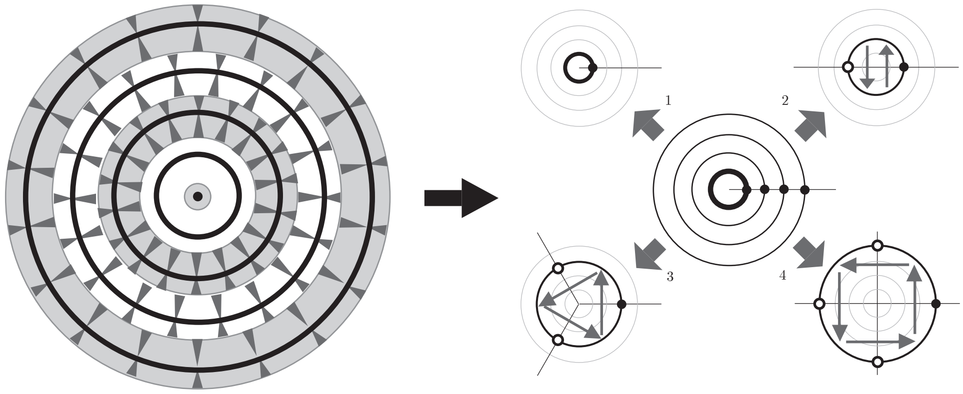

Figure 2.2: Illustrate the map in Example 2.1.7

Basic properties of virtual stability with respect to some sequence are now given in the following proposition.

Proposition 2.1.8. *Let* $f \in \mathrm{UVS}(a_n) \cap \mathrm{UVS}(b_n)$. *The following are sequences associated with* f*:*

(1) *every subsequence* (a_{n_k}) *of* (a_n)*;*

(2) $(a_n) \cup (b_n)$*;*

(3) $(a_n) \oplus (b_n)$*, where* $(a_n) \oplus (b_n) := \{a_n + b_m : n, m \in \mathbb{N}\}$*;*

Proof. (1) is trivial. For (2) and (3), let $x \in \mathrm{F}(f)$ and $U \in \mathfrak{N}(x)$.

(2): By virtual stability of f, there are $V, W \in \mathfrak{N}(x)$ such that $f^{a_n}(V) \subseteq U$ and $f^{b_n}(W) \subseteq U$ for all $n \in \mathbb{N}$. Therefore $V \cap W \in \mathfrak{N}(x)$ and $f^c(V \cap W) \subseteq U$ for all $c \in (a_n) \cup (b_n)$.

(3): By virtual stability of f, there are $V, W \in \mathfrak{N}(x)$ such that $f^{a_n}(V) \subseteq W$ and $f^{b_n}(W) \subseteq U$ for all $n \in \mathbb{N}$. Hence $f^{a_n+b_m}(V) \subseteq f^{b_m}(W) \subseteq U$ for all $n, m \in \mathbb{N}$ as desired. □

Corollary 2.1.9. *Let* $f \in \mathrm{UVS}$ *and* $\{(a_{n,i}) : i \in \mathbb{N}\}$ *be a collection of sequences associated with* f*. Then both* $\bigcup_{i<\infty}(a_{n,i})$ *and* $\bigoplus_{i<\infty}(a_{n,i})$ *are sequences associated with* f*.*

We end this section with some applications of virtual stability of maps. The first application was proved by P. Chaoha and W. Atiponrat in 2009 as stated in the following proposition.

Proposition 2.1.10. [2] *If f is a continuous uniformly virtually stable map on a regular space, then* $\mathrm{F}(f)$ *is a retract of* $\mathrm{C}(f)$*; also,* $\mathrm{C}(f)$ *is a G_δ-set if such space is a complete metric space and there is a sequence associated with f of the form (kn) for some $k \in \mathbb{N}$.*

In 2011, a generalization of previous proposition was shown by P. Chaoha and S. Iampiboonvatana, which is stated as follows:

Proposition 2.1.11. [3] *If f is a uniformly virtually stable on a regular space such that $f(\mathrm{C}(f)) \subseteq \mathrm{F}(f)$ and f^n is continuous for some $n \in \mathbb{N}$, then* $\mathrm{F}(f)$ *is a retract of* $\mathrm{C}(f)$.

2.2 Some Consequences from Number and Set Theories

The goal of this section is to introduce some necessary properties of integers and sets for this dissertation. Let us first recall the definition of the greatest common divisor of a finite subset of integers.

Definition 2.2.1. Let $a_1, \ldots, a_n$ be nonzero integers. The positive integer d is called the *greatest common divisor of* $a_1, \ldots, a_n$, denoted by $\gcd(a_1, \ldots, a_n)$ or $(a_1, \ldots, a_n)$, if

(1) $d \mid a_i$ for all $1 \leq i \leq n$;

(2) if $c \mid a_i$ for all $1 \leq i \leq n$, then $c \leq d$.

Remark 2.2.2. We sometimes denote $\gcd(a_1, \ldots, a_n)$ by $\gcd\{a_1, \ldots, a_n\}$ (or $\gcd A$ where $A = \{a_1, \ldots, a_n\}$).

Example 2.2.3. $\gcd\{24, 36\} = 12$, $\gcd\{24, 36, 18\} = 6$, $\gcd\{24, 36, 18, 14\} = 2$.

From Definition 2.2.1, one of its equivalence definitions can be given in the next proposition.

Proposition 2.2.4. [4] *For any natural numbers $a_1, \ldots, a_n$,*

$$\gcd\{a_1, \ldots, a_n\} = \min\left\{\sum_{i=1}^{n} m_i a_i \in \mathbb{N} : m_i \in \mathbb{Z} \text{ for all } 1 \leq i \leq n\right\}.$$

The following are some results on set theory which will be used later (for more details see [5]). Let ω_1 be the least uncountable ordinal.

Definition 2.2.5. For each ordinals α and β, we say that $f : \alpha \to \beta$ *maps* α *cofinally into* β if $f(\alpha)$ is unbounded in β.

The cofinality of β, denoted by $\operatorname{cf}\beta$, is the least ordinal α such that there is a map from α cofinally into β.

Remark 2.2.6. $\operatorname{cf}\omega = \omega$ and $\operatorname{cf}\omega_1 = \omega_1$.

Definition 2.2.7. An ordinal α is said to be *regular* if it is a limit ordinal and $\operatorname{cf}\alpha = \alpha$.

In fact, we have from the previous definition that α is a cardinal if it is regular. The following lemma is a consequence obtained from the property of regularity of ordinals.

Lemma 2.2.8. *For each ordinals α and $\beta \leq \alpha$, if α is regular and β is unbounded in α, then $|\beta| = \alpha$.*

Remark 2.2.9. ω is bounded in ω_1 because ω_1 is regular and $|\omega| = \omega < \omega_1$.

CHAPTER III
MAXIMAL SEQUENCES

From the previous chapter, the notions of uniformly virtually stable maps and their associated sequences have been introduced. In this chapter, we concentrate on associated sequences. The following notation will be used throughout this chapter (and also, for chapter IV).

Notation. For each uniformly virtually stable map $f : X \to X$, let

$$\mathscr{S}_f = \{(a_n) : f \in \mathrm{UVS}(a_n)\}.$$

Recall that Proposition 2.1.8(1) yields a result: every subsequence of associated sequence is also an associated sequence. Then, to investigate all sequences associated with a given map, it is sufficient to consider the sequence whose every sequence associated with f is its subsequence. Since our definition of sequences is good enough, the notion as required can be defined as follows.

Definition 3.1. For each uniformly virtually stable map f, a maximal element in $(\mathscr{S}_f, \subseteq)$ is called a *maximal sequence associated with* f (briefly, *maximal sequence*). That is,

$$(a_n) \text{ is maximal in } \mathscr{S}_f \iff \forall (b_n) \in \mathscr{S}_f [(a_n) \subseteq (b_n) \implies (a_n) = (b_n)].$$

Example 3.2. (n) is always a maximal sequence associated with every virtually nonexpansive map. This result follows directly from Proposition 2.1.3(2) and the fact that $(a_n) \subseteq (n)$ for all sequence (a_n) associated with that map.

Example 3.3. A selfmap on $\mathbb{C}$ defined in Example 2.1.5 has $(2n)$ as a maximal sequence. To see this, consider $\mathrm{i} \in \mathbb{C}$. Notice that $U = \{z : |z - \mathrm{i}| < 1\} \in \mathfrak{N}(\mathrm{i})$ and that $\mathrm{i} \in \mathrm{F}(f)$. For each $n \in \mathbb{N}$ and $r\mathrm{i} \in U$, since $f^{2n-1}(r\mathrm{i}) = f(r\mathrm{i}) = -r\mathrm{i} \notin U$, there is no $V \in \mathfrak{N}(\mathrm{i})$ such that $f^{2n-1}(V) \subseteq U$. Hence $(2n)$ is maximal as desired.

Example 3.4. For a fixed $m \in \mathbb{N}$, a selfmap f on S^1 defined by

$$f(z) = f(\mathrm{e}^{\mathrm{i}\phi}) = \begin{cases} \mathrm{e}^{\mathrm{i}\phi} & \text{if } \phi = \frac{2n\pi}{m} \text{ for some } n \in \mathbb{Z}; \\ \mathrm{e}^{\mathrm{i}\left(\phi + \frac{2\pi}{m}\right)} & \text{otherwise,} \end{cases}$$

is uniformly virtually stable with respect to a maximal sequence (nm). To see this, observe that $\mathrm{F}(f) = \{z : \arg z = \frac{2k\pi}{n} \text{ for some } k \in \mathbb{N}\}$ and that a map f is in fact a $(\frac{2\pi}{m})$-rotation of $S^1 - \mathrm{F}(f)$. Thus $f^m = \mathrm{id}_{S^1} \in \mathrm{UVS}(n)$, i.e., $f \in \mathrm{UVS}(nm)$. To verify the maximality of (nm), set

- $A = \{z \in S^1 : 2k\pi < \arg z < 2k\pi + \frac{2\pi}{m} \text{ for some } k \in \mathbb{N}\}$;
- $B = \{z \in S^1 : 2k\pi - \frac{2\pi}{m} < \arg z < 2k\pi \text{ for some } k \in \mathbb{N}\}$.

Then $U = A \cup B \cup \{1\} \in \mathfrak{N}(1)$. However $f^i(A) \cap U = \varnothing = f^j(B) \cap U$ for all $mk < i, j < m(k+1) - 1$ where $k \in \mathbb{N}_0$. This implies that (nm) is maximal in $\mathscr{S}_f$.

After Definition 3.1 is given, our investigation is on the following two problems: what is the structure of maximal sequences? And do the maximal sequences exist?. Let us start at the first one.

3.1 Structure of Maximal Sequences

In this section, we give the structure of maximal sequences via the notion of subsemigroup of natural numbers. Before we start this, the following proposition states an interesting property of maximal sequence, the uniqueness.

Proposition 3.1.1. *The maximal sequence of a uniformly virtually stable map is unique.*

Proof. Let $f \in \mathrm{UVS}$. Assume (a_n) and (b_n) are maximal in $\mathscr{S}_f$. By Proposition 2.1.8(2), $(a_n) \cup (b_n) \in \mathscr{S}_f$. Since $(a_n) \subseteq (a_n) \cup (b_n)$, it follows from the maximality of (a_n) that $(a_n) = (a_n) \cup (b_n)$. Similarly, $(b_n) = (a_n) \cup (b_n)$. Thus $(a_n) = (b_n)$. $\square$

Notice that Proposition 3.1.1 is in fact a simple consequence of the directed set. Let us recall, now, the definition of the subsemigroup generated by a subset of natural numbers.

Definition 3.1.2. For each $A \subseteq \mathbb{N}$, define a subsemigroup of $\mathbb{N}$ generated by A to be

$$\langle A \rangle = \left\{ \sum_{i=1}^{l} n_i a_i \in \mathbb{N} : l \in \mathbb{N}, n_i \in \mathbb{N}_0 \text{ and } a_i \in A \right\}.$$

Example 3.1.3.

(1) $\langle 4, 10 \rangle := \langle \{4, 10\} \rangle = \{4, 8, 10, 12, 14, 16, 18, 20, \dots\}$.

(2) $\langle 4, 10, 11 \rangle := \langle \{4, 10, 11\} \rangle = \{4, 8, 10, 11, 12, 14, 15, 16, 18, 19, 20, 21, 22, \dots\}$

Apply the notion of subsemigroups generated by subsets of natural numbers to associated sequences, we obtain some properties of these sequences as follows.

Lemma 3.1.4. *Let f be a uniformly virtually stable map with respect to (a_n) and $c \in \langle a_n \rangle$. Then*

(1) $(a_n + c) \in \mathscr{S}_f$.

(2) $(a_n) \cup \{c\} \in \mathscr{S}_f$;

Proof. Let $c = \sum_{i=1}^{l} n_i a_i \in \langle a_n \rangle$. Then $c \in \bigoplus_{i=1}^{n_1+\cdots+n_l}(a_n)$. By Corollary 2.1.9 and Proposition 2.1.8(1), we finish the proof because

(1): $(a_n + c) = (a_n + n_1 a_1 + \cdots + n_l a_l) \subseteq \bigoplus_{i=1}^{1+n_1+\cdots+n_l}(a_n) \in \mathscr{S}_f$;

(2): $(a_n) \cup \{c\} \subseteq \bigcup_{i=1}^{n_1+\cdots+n_l} \bigoplus_{j=1}^{i}(a_n) \in \mathscr{S}_f$. □

Corollary 3.1.5. *Let f be a uniformly virtually stable map with respect to (a_n) and $c_1, \dots, c_m \in \langle a_n \rangle$. Then $(a_n) \cup \{c_1, \dots, c_m\} \in \mathscr{S}_f$.*

The next definition is a generalization of the concept of the greatest common divisor of finite subset of natural numbers, by consider that of an arbitrary nonempty subset of natural numbers.

Definition 3.1.6. For $\varnothing \neq A \subseteq \mathbb{N}$, the positive integer d is said to be the *greatest common divisor of A* if

(1) $d \mid a$ for all $a \in A$;

(2) if $c \mid a$ for all $a \in A$, then $c \leq d$.

We simply denote by $\gcd A$ the greatest common divisor of A.

Example 3.1.7. For any $a \in \mathbb{N}$, $\gcd a\mathbb{N} = a$.

One of the important results in this section will be shown in Lemma 3.1.8. It gives additional tools for finding the structure of maximal sequences via a notion of the greatest common divisor of sequences.

Lemma 3.1.8. *For each sequence* (a_n)*,*

(1) $\gcd(a_n) = \gcd \langle a_n \rangle = \gcd\{a_i : i \leq m\}$ *for some* $m \in \mathbb{N}$*;*

(2) *there are* $b \in \langle a_n \rangle$ *and* $m \in \mathbb{N}$ *such that* $k \gcd(a_n) + b \in \langle a_1, \ldots, a_m \rangle$ *for all* $k \geq 0$*;*

(3) *if* $(a_n) = \langle a_n \rangle$*, then exists* $l \in \mathbb{N}$ *such that* $a_i - a_l = (i - l) \gcd(a_n)$ *for all* $i \geq l$*.*

Proof. Let (a_n) be a sequence.

(1): The fact that $\gcd(a_n) = \gcd \langle a_n \rangle$ follows directly from the definition. Now, let $d = \min\{\gcd\{a_i : i \leq k\} : k \in \mathbb{N}\}$, then $d = \gcd\{a_i : i \leq m\}$ for some $m \in \mathbb{N}$. If $d \nmid a_n$ for some $n \in \mathbb{N}$, then $\gcd\{a_1, \ldots, a_m, a_n\} < d$, a contradiction. Also, for any $c > 0$ satisfying $c \mid a_n$ for all $n \in \mathbb{N}$, we have $c \mid a_i$ for all $1 \leq i \leq m$, and hence, $c \leq d$. This implies $\gcd(a_n) = \gcd\{a_i : i \leq m\}$.

(2): If $\gcd(a_n) = a_1$, the proof is clear. So, we assume that $\gcd(a_n) \neq a_1$. By (1), $\gcd(a_n) = \gcd\{a_i : i \leq m\}$ for some $m \in \mathbb{N}$. Then, for some partition $\{P, N\}$ of $\{1, \ldots, m\}$,

$$\gcd(a_n) = \sum_{i \in P} n_i a_i - \sum_{i \in N} n_i a_i$$

where $n_i \geq 0$. Set $A = \sum_{i \in P} n_i a_i$ and $B = \sum_{i \in N} n_i a_i$. Notice that $B \neq 0$ because $\gcd(a_n) \neq a_1$, and $lB \in \langle a_1, \ldots, a_m \rangle$ for all $l \in \mathbb{N}$. Then for each $l \in \mathbb{N}$ and $0 \leq r \leq B$,

$$r \gcd(a_n) + B^2 + lB = rA + (B - k + l)B \in \langle a_1, \ldots, a_m \rangle .$$

Since for $k \geq 0$, we have $k = qB + r$ for some $q \geq 0$ and $0 \leq r < B$,

$$k \gcd(a_n) + (B^2 + B) = r \gcd(a_n) + B^2 + (1 + q \gcd(a_n))B \in \langle a_1, \ldots, a_m \rangle .$$

Therefore, the result follows by letting $b = B^2 + B$.

(3): Suppose $(a_n) = \langle a_n \rangle$. Then, by (2), there are $l, m \in \mathbb{N}$ such that for any $k \geq 0$, $k \gcd(a_n) + a_l \in \langle a_1, \ldots, a_m \rangle$. By induction on i, we show that $a_i - a_l = (i-l) \gcd(a_n)$ for all $i \geq l$. Recall that $a_l - a_l = 0 \gcd(a_n)$. Assume $a_i - a_l = (i-l) \gcd(a_n)$ for some $i \geq l$. Then

$$a_i + \gcd(a_n) = (i+1-l) \gcd(a_n) + a_l \in \langle a_1, \ldots, a_m \rangle \subseteq \langle a_n \rangle = (a_n).$$

If $a_i < a_{i+1} < a_i + \gcd(a_n)$, $\gcd(a_n) \mid a_{i+1} - a_i$ which yields a contradiction: $\gcd(a_n) \leq a_{i+1} - a_i < a_i + \gcd(a_n) - a_i = \gcd(a_n)$. Then

$$a_{i+1} = a_i + \gcd(a_n) = (i+1-l) \gcd(a_n) + a_l$$

and this finishes the proof. □

After all necessary tools are introduced, we are now ready to give the structure of maximal sequences.

Theorem 3.1.9. *Let f be uniformly virtually stable and (a_n) the maximal sequence associated with f. Then*

(1) $(a_n) = \langle a_n \rangle$;

(2) $(a_n) = \langle A \rangle$ *for some finite subset A of (a_n);*

(3) *if f is continuous, then $(a_n) = (n)$.*

Proof. (1): It suffices to show that $\langle a_n \rangle \subseteq (a_n)$. Let $c \in \langle a_n \rangle$. By Lemma 3.1.4, $(a_n) \cup \{c\} \in \mathscr{S}_f$, and hence, $c \in (a_n)$ by maximality of (a_n).

(2): From (1), we have $(a_n) = \langle a_n \rangle$. Then, by Lemma 3.1.8(2), there exist $l, m \in \mathbb{N}$ such that $k \gcd(a_n) + a_l \in \langle a_1, \ldots, a_m \rangle$ for all $k \geq 0$. By setting

$$A = \{a_i : i \leq \max\{l, m\}\},$$

we clearly have $\langle A \rangle \subseteq \langle a_n \rangle = (a_n)$. On the other hand, for each $i \in \mathbb{N}$, notice that

- if $i \leq \max\{l, m\}$, then $a_i \in A$, and
- if $i > \max\{l, m\}$, then $a_i = a_l + k \gcd(a_n) \in \langle a_1, \ldots, a_m \rangle \subseteq \langle A \rangle$ for some $k \in \mathbb{N}$.

Therefore, $(a_n) \subseteq \langle A \rangle$.

(3): Assume f is continuous. Then for each $x \in \mathrm{F}(f)$ and $U \in \mathfrak{N}(x)$, there are $V, W \in \mathfrak{N}(x)$ such that $f(V) \subseteq U$ and $f^{a_n}(W) \subseteq U$ for all $n \in \mathbb{N}$. Thus $V \cap W \in \mathfrak{N}(x)$ and $f^c(V \cap W) \subseteq U$ for all $c \in \{1\} \cup (a_n)$, i.e., $(a_n) \cup \{1\} \in \mathscr{S}_f$. Then, by maximality of (a_n), we have $1 \in (a_n)$ and hence $(n) = \langle 1 \rangle \subseteq \langle a_n \rangle = (a_n)$. □

3.2 Existence of Maximal Sequences

The previous section informs us what structures of maximal sequences should be. There is, however, no condition that guarantees the existence of maximal sequences. In fact, there are some uniformly virtually stable maps which have no maximal sequences, but these maps are quite complicated and we will see them in the next chapter. Thus, in this section, we are going to find conditions that guarantee the existence of maximal sequences. Two situations will be considered, one is a condition on maps and another is a condition on spaces.

3.2.1 A Condition on Maps

Definition 3.2.1.1. A sequence (a_n) is said to *satisfy the sup-finite condition* if $\sup_{n \in \mathbb{N}}(a_{n+1} - a_n) < \infty$.

Lemma 3.2.1.2. *For each $f \in \mathrm{UVS}$, if $(a_n) \in \mathscr{S}_f$ satisfies the sup-finite condition, then $\langle a_n \rangle \in \mathscr{S}_f$.*

Proof. Let $f \in \mathrm{UVS}(a_n)$ be such that $d := \sup_n(a_{n+1} - a_n) < \infty$. By Lemma 3.1.8, there is $b \in \langle a_n \rangle$ for which $k \gcd(a_n) + b \in \langle a_n \rangle$ for all $k \in \mathbb{N}_0$. Then, by Lemma 3.1.4 and Corollary 3.1.5,

$$(b_n) := (a_n) \cup (a_n + \gcd(a_n) + b) \cup (a_n + 2\gcd(a_n) + b) \cup \cdots \cup (a_n + d\gcd(a_n) + b) \in \mathscr{S}_f,$$

and hence $(c_n) := (b_n) \cup \{c \in \langle a_n \rangle : c \leq a_1 + b\} \in \mathscr{S}_f$. The proof is immediately followed from Proposition 2.1.8(1), once we can show that $\langle a_n \rangle \subseteq (c_n)$. To see this, let $c \in \langle a_n \rangle$. The case that $c \leq a_1 + b$ is clear. Otherwise, $a_m + b \leq c < a_{m+1} + b$ for some $m \geq 1$. Since $\gcd(a_n) \mid c - a_m - b$, we have $c = a_m + b + r\gcd(a_n)$ for some $0 \leq r < a_{m+1} - a_m \leq d$, and hence $c \in (a_n + r\gcd(a_n) + b) \subseteq (b_n) \subseteq (c_n)$. □

The following theorem is the main result in this subsection which gives an equivalence condition on maps of the existence of maximal sequence.

Theorem 3.2.1.3. *For each* $f \in \mathrm{UVS}$, *its maximal sequence exists if and only if there is* $(a_n) \in \mathscr{S}_f$ *satisfying the sup-finite condition.*

Proof. Let $f \in \mathrm{UVS}$.

($\Rightarrow$): If (a_n) is maximal, then $(na_1) \subseteq (a_n)$ and hence

$$\sup_n (a_{n+1} - a_n) \leq \sup_n ((n+1)a_1 - na_1) = a_1 < \infty.$$

($\Leftarrow$): Assume $d := \sup_n (a_{n+1} - a_n) < \infty$ for some $(a_n) \in \mathscr{S}_f$. By Lemma 3.2.1.2, $\langle a_n \rangle \in \mathscr{S}_f$. If $a_1 = 1$, we are done. Now suppose $a_1 > 1$, and let, for $1 \leq i < a_1 = \min \langle a_n \rangle$,

$$A_i = \{m : m \equiv i \bmod a_1 \text{ and } \langle a_n \rangle \cup \{m\} \in \mathscr{S}_f\},$$

and $m_i = \min A_i$ if $A_i \neq \varnothing$, otherwise $m_i = a_1$. Then

$$\{c : \langle a_n \rangle \cup \{c\} \in \mathscr{S}_f\} = A_0 \cup \cdots \cup A_{a_1 - 1}.$$

By Corollary 3.1.5, $\langle a_n \rangle \cup \{m_i : 0 \leq i < a_1\} \in \mathscr{S}_f$ and again by Lemma 3.2.1.2, since $\sup_n (b_{n+1} - b_n) \leq \sup_n (a_{n+1} - a_n) < \infty$ where

$$(b_n) := \langle \langle a_n \rangle \cup \{m_i : 0 \leq i < a_1\} \rangle,$$

$(b_n) \in \mathscr{S}_f$. Notice that (b_n) is maximal because whenever $(b_n) \cup \{c\} \in \mathscr{S}_f$ for some c, there is $0 \leq i < a_1$ such that $c \in A_i \subseteq A_0 \cup \cdots \cup A_{a_1 - 1} \subseteq (b_n)$. □

3.2.2 A Condition on Spaces

Definition 3.2.2.1. A Hausdorff space X is said to *satisfy the property* (P) if for each $x \in X$ and each countable collection $\mathscr{C}$ of neighborhoods of x, there is a neighborhood U of x such that $U \subseteq \bigcap \mathscr{C}$.

The next theorem gives a sufficient condition guaranteeing the existence of maximal sequence.

Theorem 3.2.2.2. *Every uniformly virtually stable selfmap on a space which satisfies the property* (P) *has a maximal sequence.*

Proof. Let $f \in \mathrm{UVS}(X)$ where X satisfies the property (P), $(a_n) \in \mathscr{S}_f$, $(b_n) = \langle a_n \rangle$, $x \in \mathrm{F}(f)$ and $U \in \mathfrak{N}(x)$. Then, by Corollary 3.1.5, there exists a collection $\mathscr{C} = \{V_n : n \in \mathbb{N}\}$ of neighborhoods of x such that $f^i(V_m) \subseteq U$ for all $i \in (a_n) \cup \{b_1, \ldots, b_m\}$ and $m \in \mathbb{N}$. Since X satisfies the property (P), there is $W \in \mathfrak{N}(x)$ such that $W \subseteq \bigcap \mathscr{C}$. Hence $f^{b_n}(W) \subseteq f^{b_n}(V_n) \subseteq U$ for all $n \in \mathbb{N}$. This means that $(b_n) \in \mathscr{S}_f$. Since $\sup_n(b_{n+1} - b_n) \leq \sup_n((n+1)a_1 - na_1) = a_1$, f has a maximal sequence by Theorem 3.2.1.3. □

Example 3.2.2.3. The discrete space is a Hausdorff space which satisfies the property (P). Thus every uniformly virtually stable selfmap on this space has maximal sequence.

By using some knowledge in set theory, the following is an interesting example of Hausdorff space which satisfies the property (P).

Example 3.2.2.4. A space $X = \{0,1\}^{\omega_1}$ together with a topology generated by base:

$$\mathscr{B} := \left\{ \prod_{\gamma<\alpha} A_\gamma \times \prod_{\gamma \geq \alpha} \{0,1\} : A_\gamma \subseteq \{0,1\} \text{ and } \alpha < \omega_1 \right\}$$

is a Hausdorff space and satisfies the property (P). Thus every uniformly virtually stable selfmap on X has a maximal sequence.

To prove the above assertion, let $\mathscr{C} := \{V_n : n \in \mathbb{N}\}$ be a countable collection of neighborhoods of $x \in X$. Then, by setting $B_n = \bigcap_{i \leq n} V_i$ for any $n \in \mathbb{N}$, a collection $\mathscr{B} := \{B_n : n \in \mathbb{N}\} \subseteq \mathfrak{N}(x)$ satisfies $B_1 \supseteq B_2 \supseteq \ldots$. Without loss of generality, suppose all B_n's are distinct and $B_n = \prod_{\gamma<\alpha_n} \pi_\gamma(x) \times \prod_{\gamma \geq \alpha_n}\{0,1\}$ for some $\alpha_n < \omega_1$; here π_γ is the γth-projection map. Thus $\alpha_n < \alpha_{n+1}$ for all $n \in \mathbb{N}$. For each $\alpha < \omega_1$, set $D_\alpha = \prod_{\gamma<\alpha} \pi_\gamma(x) \times \prod_{\gamma \geq \alpha}\{0,1\}$. Then $\mathscr{D} := \{D_\alpha : \alpha < \omega_1\}$ is a collection of neighborhoods of x such that $D_\alpha \supseteq D_\beta$ if and only if $\alpha \leq \beta$. Since both $(\mathscr{C}, \supseteq)$ and $(\mathscr{D}, \supseteq)$ are well-ordered, $|\mathscr{B}| = \omega < \omega_1$ which is an order-type of $(\mathscr{D}, \supseteq)$ and ω_1 is regular, it follows that $\mathscr{B}$ is $\supseteq$-bounded in $\mathscr{D}$ by Lemma 2.2.8. So, there exists $\alpha < \omega_1$ such that $D_\alpha \subseteq \bigcap \mathscr{B} \subseteq \mathscr{C}$. That is, X satisfies the property (P) as desired.

CHAPTER IV
EXAMPLES OF UNIFORMLY VIRTUALLY STABLE MAPS WITH RESPECT TO PRESCRIBED SEQUENCES

In this chapter, some interesting examples of uniformly virtually stable maps are given. One is an example of uniformly virtually stable map with respect to a possible prescribed maximal sequence and other are uniformly virtually stable maps having no maximal sequence. Let us start with an example of a uniformly virtually stable map with respect to a prescribed maximal sequence. Notice from above that there is the word: *possible* for a prescribed sequence. This is indeed from Theorem 3.1.9(1): if (a_n) is the maximal sequence (associated to some uniformly virtually stable map), then $(a_n) = \langle a_n \rangle$. Thus we will consider only the sequence $(a_n) = \langle a_n \rangle$.

Recall that the identity map is already itself uniformly virtually stable map with respect to (n). For the case that the prescribed maximal sequence is not (n), consider the following example.

Example 4.1. For $k \in \mathbb{N} \cup \{0\}$, let $\theta_k = \pi \sum_{i=0}^{k} \frac{1}{2^i}$. Then $\pi \le \theta_k < 2\pi$ for all k. Consider the sequence $(a_n) = \langle a_n \rangle$. Set $\delta_0 = a_1$ and $\delta_n = a_{n+1} - a_n$ for any $n \in \mathbb{N}$. Define $f : \mathbb{C} \to \mathbb{C}$ by

$$f(z) = f(r\mathrm{e}^{\mathrm{i}\phi}) = \begin{cases} (r+1)\mathrm{e}^{\mathrm{i}\phi} & \text{if } \phi = \theta_k, 0 < r < \delta_k - 1 \text{ for some } k; \\ (r - \lfloor r \rfloor)\mathrm{e}^{\mathrm{i}(\pi + \frac{\phi}{2})} & \text{if } \phi = \theta_k, r \ge \delta_k - 1 \text{ for some } k; \\ 0 & \text{otherwise.} \end{cases}$$

Then f is uniformly virtually stable with respect to the maximal sequence (a_n). To see this, note that $\mathrm{F}(f) = \{0\}$,

$$\pi + \frac{\theta_k}{2} = \pi(1 + \sum_{i=1}^{k+1} \frac{1}{2^i}) = \pi \sum_{i=0}^{k+1} \frac{1}{2^i} = \theta_{k+1}$$

for all $k \geq 0$, and for each $n \in \mathbb{N}$ and $0 < r < 1$,

$$f^{a_n}(re^{i\theta_0}) = f^{\delta_{n-1}+\cdots+\delta_0}(re^{i\theta_0}) = f^{\delta_{n-1}+\cdots+\delta_1}(re^{i\theta_1}) = \cdots = re^{i\theta_n}.$$

To show the virtual stability of f with respect to (a_n), it suffices to prove that $f^{a_n}(\mathrm{B}(0;r)) \subseteq \mathrm{B}(0;r)$ for all $0 < r < 1$. Let $0 < r < 1$. For each $z = se^{i\phi} \in \mathrm{B}(0;r)$,

$$|f^{a_n}(z)| \leq |f^{a_n}(se^{i\theta_k})| = |f^{a_n}f^{a_k}(se^{i\theta_0})| = |f^{a_m}(se^{i\theta_0})| = |se^{i\theta_m}| = s < r$$

for some k, and $a_m = a_n + a_k$. Thus we get the claim. Moreover (a_n) is maximal because whenever $b \notin (a_n)$, either $b < a_1$ or $a_m < b < a_{m+1}$ for some $m \in \mathbb{N}$ (and so $b = a_m + k$ for some $1 \leq k < \delta_m$) and the first case implies that $f^b(re^{i\theta_0}) = (r+b)e^{i\theta_0} \notin \mathrm{B}(0;1)$ for all $0 < r < 1$, while the latter implies that

$$|f^b(re^{i\theta_0})| = |f^k f^{a_m}(re^{i\theta_0})| = |f^k(re^{i\theta_m})| = |(r+k)e^{i\theta_m}| = r+k > 1$$

for all $0 < r < 1$, i.e., $f^b(re^{i\theta_0}) \notin \mathrm{B}(0;1)$. Thus $(a_n) \cup \{b\} \notin \mathscr{S}_f$ for all $b \notin (a_n)$, which induces the maximality of (a_n) as desired.

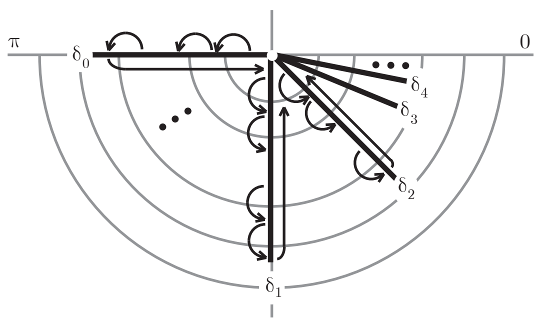

Figure 4.1: The map f in Example 4.1

Next, an example of uniformly virtually stable map without maximal sequence is given. To do this, we recall some properties of a partition of the set of positive even numbers.

Lemma 4.2. *Let* $A_1 = \{2^n : n \in \mathbb{N}\}$, *and for each* $n > 1$,

$$A_n = \left\{ \sum_{i=1}^{n} 2^{n_i} : n_i \in \mathbb{N} \right\} - \bigcup_{i<n} A_i.$$

Then $\{A_n : n \in \mathbb{N}\}$ *forms a partition of* $2\mathbb{N} := \{2n : n \in \mathbb{N}\}$ *and each* A_n *is infinite for all* $n \in \mathbb{N}$.

Proof. Following from the fact that $2^n + 2^n = 2^{n+1}$ for all $n \in \mathbb{N}$, if $\sum_{i=1}^{m} 2^{n_i} \in A_m$, then $n_i \neq n_j$ for all $i \neq j$. Also for each $n \in \mathbb{N}$, since $2n = \sum_{i=1}^{n} 2 \in \bigcup_{i=1}^{n} A_i$, we have $2n \in A_i$ for some $i \leq n$. Hence, $\{A_n : n \in \mathbb{N}\}$ forms a partition of $2\mathbb{N}$. For the next assertion, observe that $A_1 = (2^n)$ is infinite. Assume that $n > 1$. We first claim that $2^n - 2 \in A_{n-1}$. Since $2^n - 2 \in 2\mathbb{N} = \langle 2^n \rangle$, $2^n - 2 \in A_k$ for some $k \in \mathbb{N}$. It follows that $k < n$ and if $2^n - 2 = \sum_{i=1}^{k} 2^{n_i}$, then $n_i < n$ and $\sum_{i=1}^{k} 2^{n_i} = 2^n - 2 = \sum_{i=1}^{n-1} 2^i$, and hence $k = n - 1$. Next, we claim that $2^{n+m} + 2^n - 2 \in A_n$ for all $m \in \mathbb{N}$. Let $m \in \mathbb{N}$. By the previous claim, we have $2^{n+m} + (2^n - 2) \in \bigcup_{i=1}^{n} A_i$. Hence $2^{n+m} + 2^n - 2 \in A_k$ for some $1 \leq k \leq n$, i.e., $2^{n+m} + 2^n - 2 = \sum_{i=1}^{k} 2^{n_i}$. Since

$$\sum_{i=1}^{n+m-1} 2^i = 2^{n+m} - 2 < 2^{n+m} + 2^n - 2 < 2^{n+m+1},$$

$2^{n_i} = 2^{n+m}$ for some $1 \leq i \leq k$. Thus, $2^{n+m} + 2^n - 2 \in A_n$ as claimed. Finally, since $\{2^{n+m} + 2^n - 2 : m \in \mathbb{N}\} \subseteq A_n$, A_n is infinite as desired. □

We are ready to give an example as claimed now.

Example 4.3. Let A_n be defined as in Lemma 4.2, $p_1 = 3$ and for each $n > 1$,

$$p_n := \text{the smallest prime number such that } p_n > \max\{n^2 + n, p_{n-1}\}.$$

For examples, $p_2 = 7$ and $p_3 = 13$. Clearly, $p_n \neq p_m$ whenever $n \neq m$. Now, let $X = [0,2] \times [0,2]$ be the subspace of $\mathbb{R}^2$ equipped with the maximum norm defined by $\|(x,y)\| = \max\{|x|, |y|\}$.

The goal of this example is to find a uniformly virtually stable having no maximal sequence, which can be shown by constructing a uniformly virtually stable map with respect to (2^n) but not to $(2n)$. If so, we have that whenever such a map has an associated sequence (a_n) satisfying the sup-finite condition, so are $(a_n) \cup (2^n)$ and $\langle (a_n) \cup (2^n) \rangle \supseteq (2^n)$, a contradiction. Therefore each associated sequence of this map does not satisfy the sup-finite condition, i.e., this map has no maximal sequence.

To construct such a desired map, named f, it suffices to prove the following:

(C1) $\mathrm{F}(f) = \{\mathbf{0} := (0,0)\}$;

(C2) For each $k \in \mathbb{N}$, $f^{2^n}(\mathrm{B}(\mathbf{0}; \frac{1}{k+1})) \subseteq \mathrm{B}(\mathbf{0}; \frac{1}{k})$ for all $n \in \mathbb{N}$.

(C3) For each $k \in \mathbb{N}$, there is $c \in 2\mathbb{N}$ such that $f^c(\frac{1}{k+2}, 0) \notin \mathrm{B}(\mathbf{0}; 1)$.

Notice that (C1) and (C2) imply the virtual stability of f with respect to (2^n), while (C1) and (C3) imply that, for each $k \in \mathbb{N}$, there exists $a \in 2\mathbb{N}$ such that $f^a\left(\mathrm{B}(\mathbf{0}; \frac{1}{k+1})\right) \nsubseteq \mathrm{B}(\mathbf{0}; 1)$, and hence f is not uniformly virtually stable with respect to $(2n)$.

We are now ready to define such a map f. Consider the following subsets of X:

- $T_1 = \{(\frac{1}{k+2}, 0) : k \in \mathbb{N}\}$,
- $T_2 = \{(1, \frac{1}{p_k^l}) : k, l \in \mathbb{N}$ and $l+1 \in A_m$ for some $m \leq k\}$,
- $T_3 = \{(\frac{1}{k+2-m}, \frac{1}{p_k^l}) : k, l \in \mathbb{N}$ and $l \in A_m$ for some $m \leq k\}$,
- $T_4 = \{(1, \frac{1}{p_k^l}) : k, l \in \mathbb{N}$ and either $l \in A_m$ or $l+1 \in A_m$ for some $m > k\}$.

Observe that

(O1) $T_i \cap T_j = \varnothing$ whenever $i \neq j$. All cases are trivial except for the case of $T_2 \cap T_4$, where it follows from the fact that $\{A_m : m \in \mathbb{N}\}$ forms a partition of $2\mathbb{N}$.

(O2) $\|(x, y)\| = x$ for all $(x, y) \in \bigcup_{i=1}^4 T_i$, because for each $k, m, l \in \mathbb{N}$ with $m \leq k$, we have $p_k^l \geq p_k > k^2 + k \geq k + 1$ and hence $\frac{1}{p_k^l} < \frac{1}{k+1} \leq \frac{1}{k+2-m}$.

Define $f : X \to X$ by

$$f(x,y) = \begin{cases} \left(1, \frac{1}{p_k}\right) & \text{if } (x,y) = (\frac{1}{k+2}, 0) \in T_1, \\ \left(\frac{1}{k+2-m}, \frac{1}{p_k^{l+1}}\right) & \text{if } (x,y) = (1, \frac{1}{p_k^l}) \in T_2 \text{ where } l+1 \in A_m, \\ \left(1, \frac{1}{p_k^{l+1}}\right) & \text{if } (x,y) = (\frac{1}{k+2-m}, \frac{1}{p_k^l}) \in T_3 \text{ or } (x,y) = (1, \frac{1}{p_k^l}) \in T_4, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Clearly, f is well-defined by (O1). The following properties are also satisfied :

(P1) $f(T_1) \subseteq T_2$, $f(T_2) \subseteq T_3$ and $f(T_3 \cup T_4) \subseteq T_2 \cup T_4$. So, $\bigcup_{i=1}^4 T_i$ is f-invaraint. Moreover, since p_i's are all distinct primes, we have $p_i^k = p_j^l$ iff $(i, k) = (j, l)$, and hence $(\pi_2 \circ f)|_{\bigcup_{i=1}^4 T_i}$ is injective, where π_2 denotes the second-coordinate projection $\pi_2(x, y) = y$.

(P2) For each $k, l \in \mathbb{N}$, one can verify that

$$f^l\left(\frac{1}{k+2}, 0\right) = \begin{cases} \left(\frac{1}{k+2-m}, \frac{1}{p_k^l}\right) & \text{if } l \in A_m \text{ for some } m \leq k, \\ \left(1, \frac{1}{p_k^l}\right) & \text{otherwise.} \end{cases}$$

Thus we have the following:

- $f^{2^l}(\frac{1}{k+2}, 0) = (\frac{1}{k+2-1}, \frac{1}{p_k^{2^l}}) = (\frac{1}{k+1}, \frac{1}{p_k^{2^l}})$ since $2^l \in (2^n) = A_1$ and $1 \leq k$.
- $f^c(\frac{1}{k+2}, 0) = (1, \frac{1}{p_k^c})$, where $c = \min A_{k+1}$, since $c \in A_{k+1}$ and $k+1 > k$.

(P3) For each $k, l, m, n \in \mathbb{N}$ with $m < k$ and $l \in A_m$, we have $m+1 \leq k$, $f^l(\frac{1}{k+2}, 0) = (\frac{1}{k+2-m}, \frac{1}{p_k^l})$ by (P2), and $2^n + l \in A_j \subseteq \bigcup_{i=1}^{m+1} A_i$ for some $j \leq m+1$. Again by (P2), it follows that

$$\left(\frac{1}{k+2-j}, \frac{1}{p_k^{2^n+l}}\right) = f^{2^n+l}\left(\frac{1}{k+2}, 0\right) = f^{2^n}\left(\frac{1}{k+2-m}, \frac{1}{p_k^l}\right).$$

Clearly, (C1) is implied by (P1). Let $i, n \in \mathbb{N}$. To prove (C2), it suffices to consider the following two cases:

1. If $(x, y) = (\frac{1}{k+2}, 0) \in T_1 \cap \mathrm{B}(\mathbf{0}; \frac{1}{i+1})$, then $\frac{1}{k+2} < \frac{1}{i+1}$, and hence

$$\|f^{2^n}(x, y)\| = \left\|f^{2^n}\left(\frac{1}{k+2}, 0\right)\right\| \overset{(P2)}{=} \left\|\left(\frac{1}{k+1}, \frac{1}{p_k^{2^n}}\right)\right\| \overset{(O2)}{=} \frac{1}{k+1} < \frac{1}{i}.$$

2. If $(x, y) = (\frac{1}{k+2-m}, \frac{1}{p_k^l}) \in T_3 \cap \mathrm{B}(\mathbf{0}; \frac{1}{i+1})$, then $l \in A_m$, $m \leq k$ and

$$\frac{1}{k+2-m} \overset{(O2)}{=} \left\|(\frac{1}{k+2-m}, \frac{1}{p_k^l})\right\| < \frac{1}{i+1}.$$

If $m = k$, we have $\frac{1}{2} = \frac{1}{k+2-m} < \frac{1}{i+1}$, a contradiction. Thus $m < k$, and by (P3), there is some $j \leq m+1$ such that

$$\begin{aligned} \left\|f^{2^n}\left(\frac{1}{k+2-m}, \frac{1}{p_k^l}\right)\right\| &\overset{(P3)}{=} \left\|\left(\frac{1}{k+2-j}, \frac{1}{p_k^{2^n+l}}\right)\right\| \\ &\overset{(O2)}{=} \frac{1}{k+2-j} \\ &\leq \frac{1}{k+2-(m+1)} \\ &< \frac{1}{i}. \end{aligned}$$

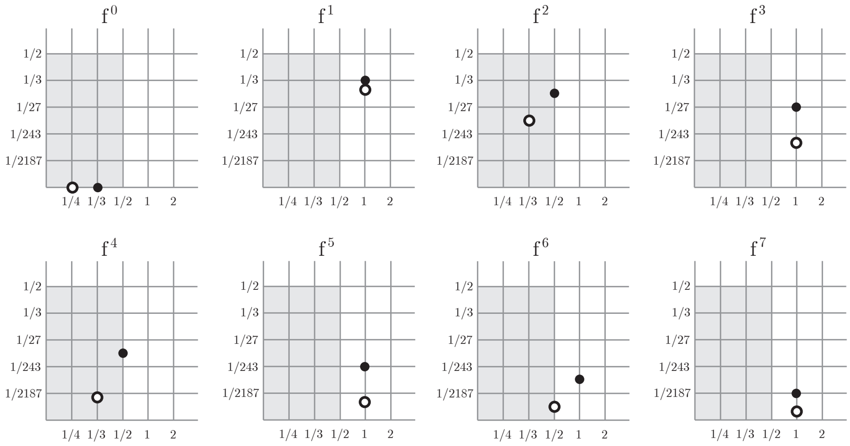

Figure 4.2: The map f in Example 4.3 at $(\frac{1}{3}, 0)$ and $(\frac{1}{4}, 0)$

From above cases, it follows that $f^{2^n}\left(\mathrm{B}(\mathbf{0}; \frac{1}{i+1})\right) \subseteq \mathrm{B}(\mathbf{0}; \frac{1}{i})$.

Finally, for (C3), let $k \in \mathbb{N}$ and $c = \min A_{k+1} \in 2\mathbb{N}$. Then by (P2), we have $f^c(\frac{1}{k+2}, 0) = (1, \frac{1}{p_k^{k+1}}) \notin \mathrm{B}(\mathbf{0}; 1)$.

Remark 4.4. The map f in the previous example is uniformly virtually stable with respect to $\bigcup_{j=1}^{l} \bigoplus_{i=1}^{j} (2^n)$, $l \in \mathbb{N}$, although $(2n) = \bigcup_{j=1}^{\infty} \bigoplus_{i=1}^{j} (2^n)$ is not its associated sequence. This is because for each $\mathrm{B}(\mathbf{0}; \frac{1}{k})$, $f^c(\mathrm{B}(\mathbf{0}; \frac{1}{k+l})) \subseteq \mathrm{B}(\mathbf{0}; \frac{1}{k})$ for all $l \in \bigcup_{j=1}^{l} \bigoplus_{i=1}^{j} (2^n)$.

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