CHAPTER IV

ABRIKOSOV'S SOLUTION

In this chapter we will find the solutions of the Ginzburg-Landau equations in reduced units.

First, we start from the Gibbs free energy density

$$g = a|\Psi|^2 + \frac{1}{2}b|\Psi|^4 + \frac{1}{2m^*}\left|\left(-i\hbar\nabla + \frac{e^*\mathbf{A}}{c}\right)\Psi\right|^2 + \frac{(\mathbf{h} - \mathbf{H})^2}{8\pi} \qquad (4.1)$$

and change to reduced units by scaling according to

$$g \rightarrow \frac{|a|^2}{b}g$$

$$\mathbf{A} \rightarrow \sqrt{2}\,H_c\lambda\mathbf{A}$$

$$\mathbf{h} \rightarrow \sqrt{2}\,H_c\kappa\mathbf{h}$$

$$\Psi \rightarrow \sqrt{\frac{|a|}{b}}\,\Psi$$

$$\mathbf{r} \rightarrow \xi\mathbf{r}$$

Here, the definitions of the penetration depth λ, the coherence length ξ, and the Ginzburg-Landau parameter κ, are as following;

$$\lambda = \sqrt{\frac{m^* c^2 b}{4\pi e^{*2} |a|}}$$

$$\xi = \sqrt{\frac{\hbar^2}{2m^* |a|}}$$

$$\kappa = \frac{\lambda}{\xi} = \frac{m^* c}{\hbar e^*} \sqrt{\frac{b}{2\pi}}$$

The critical field H_c is given by eq.(3.9), $H_c = \sqrt{(4\pi\ |a|^2/b)}$. In reduced units the upper critical field $H_{c2} = 1$. Note that in these units $|\Psi| = 1$ in the bulk Meissner state. However, in normal units $|\Psi| << 1$ and we can neglect terms order $|\Psi|^6$ in the free energy.

Now, eq.(4.1) becomes

$$g = -|\Psi|^2 + \frac{1}{2}|\Psi|^4 + |(-i\nabla + \mathbf{A})\Psi|^2 + \kappa^2(\mathbf{h} - \mathbf{H})^2 \qquad (4.2)$$

Since

$$\Psi = |\Psi| e^{i\phi}$$

we have that

$$\left|\left(-i\nabla + \mathbf{A}\right)\Psi\right|^2 = \left|\left(-i\nabla + \mathbf{A}\right)|\Psi|\, e^{i\phi}\right|^2$$

$$= \left|-ie^{i\phi}\nabla|\Psi| - i|\Psi|\, ie^{i\phi}\nabla\phi + \mathbf{A}\,|\Psi|\, e^{i\phi}\right|^2 \qquad (4.3)$$

If we set $\mathbf{Q} = \mathbf{A} + \nabla\phi$, eq. (4.3) becomes

$$\left|\left(-i\nabla + \mathbf{A}\right)\Psi\right|^2 = \left|-ie^{i\phi}\,\nabla|\Psi| + |\Psi|\, e^{i\phi}\,\mathbf{Q}\right|^2$$

$$= \left(\nabla|\Psi|\right)^2 + |\Psi|^2\,\mathbf{Q}^2 \qquad (4.4)$$

Q is the so-called *supervelocity*, which is a gauge invariant quantity. It came from the definition of the supercurrent as shown in chapter II and $\nabla\times\mathbf{Q} = \mathbf{h}$. Since **A** and ϕ appear only in the combination of **Q**, the form of ϕ itself is irrelevant. In fact, an arbitrary function can be added to $\phi(r)$ if its gradient is also subtracted from **A**. We will now only be interested in the amplitude $|\Psi|$, and for convinience we will denote the amplitude as Ψ.

Now eq. (4.2) becomes

$$g = -\Psi^2 + \frac{1}{2}\Psi^4 + \left(\nabla\Psi\right)^2 + \mathbf{Q}^2\Psi^2 + \kappa^2\left(\nabla\times\mathbf{Q} - \mathbf{H}\right)^2 \qquad (4.5)$$

Setting the variation of g with respect to Ψ to zero gives

$$-\Psi + \Psi^3 + \mathbf{Q}^2\Psi - \nabla^2\Psi = 0 \qquad (4.6)$$

This is the first Ginzburg - Landau equation in reduced units.

Minimizing $G = \int g\, dr$ with respect to $\mathbf{Q}(\mathbf{r})$, by the calculus of variations (Arfken, 1970), gives

$$\frac{\partial g_s}{\partial Q_i} - \sum_j \frac{\partial}{\partial r_j}\left[\frac{\partial g_s}{\partial(\partial Q_i / \partial r_j)}\right] = 0 \qquad (4.7)$$

as a necessary condition.

Since

$$\frac{\partial g}{\partial \mathbf{Q}} = 2\mathbf{Q}\Psi^2 \qquad (4.8)$$

and by the same method used in the previous chapter (eqs.(2.20) - (2.24)), we obtain

$$\sum_j \frac{\partial}{\partial r_j}\left[\frac{\partial g_s}{\partial(\partial Q_i / \partial r_j)}\right] = -2\,\kappa^2\,\nabla \times (\mathbf{h} - \mathbf{H}) \qquad (4.9)$$

and eq.(4.7) becomes

$$\kappa^2 \nabla \times (\mathbf{h} - \mathbf{H}) + \mathbf{Q}\Psi^2 = 0 \qquad (4.10)$$

This is the second Ginzburg-Landau equation in reduced units.

When the external magnetic field H is just below H_{c2} (which has magnetude 1 in these units), the internal magnetic field $\mathbf{h}$ is very close to $\mathbf{H}$ ($\mathbf{Q} \approx \mathbf{Q}_0$) and the Ginzburg - Landau order parameter Ψ is very small. Therefore, in eq. (4.6) we can neglect terms of order Ψ^3, yielding.

$$- \Psi_0 + Q_0^2 \Psi_0 = \nabla^2 \Psi_0 \qquad (4.11)$$

This is the linearized equation approximation, where Ψ_0 is real and $\nabla \times \mathbf{Q}_0 = \mathbf{H}$.

We try,

$$\mathbf{Q}_0 = -\hat{\mathbf{z}} \times \frac{\nabla \Psi_0}{\Psi_0} + \sqrt{1 - H}\, \hat{\mathbf{z}} \qquad (4.12)$$

and we have

$$\mathbf{H} = -\nabla \times \left(\hat{\mathbf{z}} \times \frac{\nabla \Psi_0}{\Psi_0} \right)$$

$$= -\hat{\mathbf{z}} \nabla \cdot \frac{\nabla \Psi_0}{\Psi_0}$$

$$= -\hat{\mathbf{z}} \left[\left(\frac{\nabla^2 \Psi_0}{\Psi_0} \right) - \left(\frac{\nabla \Psi_0}{\Psi_0} \right)^2 \right] \qquad (4.13)$$

So, since $\mathbf{H} = H\hat{\mathbf{z}}$,

$$H = \left(\frac{\nabla \Psi_0}{\Psi_0} \right)^2 - \frac{\nabla^2 \Psi_0}{\Psi_0} \qquad (4.14)$$

and

$$Q_0^2 = \left(\frac{\nabla \Psi_0}{\Psi_0} \right)^2 + 1 - H \qquad (4.15)$$

We now calculate $-\Psi_0 + Q_0{}^2 \Psi_0$ and find that

$$-\Psi_0 + Q_0^2\Psi_0 = \left(\left(\frac{\nabla\Psi_0}{\Psi_0}\right)^2 - H\right)\Psi_0 = \nabla^2\Psi_0$$

in agreement with eq.(4.11).

If we multiply eq.(4.6) by Ψ, we find that

$$-\Psi^2 + \Psi^4 + Q^2\Psi^2 = \Psi\nabla^2\Psi \tag{4.16}$$

Using eq.(4.16), eq.(4.5) becomes

$$g = -\frac{1}{2}\Psi^4 + (\nabla\Psi)^2 + \Psi\nabla^2\Psi + \kappa^2(\mathbf{h} - \mathrm{H})^2 \tag{4.17}$$

Consider, now the identity

$$\int_V d\mathbf{r}\left((\nabla\Psi)^2 + \Psi\nabla^2\Psi\right) = \int_V d\mathbf{r}\ \nabla\cdot(\Psi\nabla\Psi)$$

$$= \int_S \Psi\nabla\Psi\cdot\hat{\mathbf{n}}\,ds \tag{4.18}$$

Since $\hat{\mathbf{n}}\cdot\nabla\Psi = 0$ is the surface condition (Landau and Lifshitz, 1980), the average of eq.(4.17) has only two terms remaining, and

$$\langle g\rangle = -\frac{1}{2}\langle\Psi^4\rangle + \kappa^2\langle(\mathbf{h} - \mathbf{H})^2\rangle \tag{4.19}$$

where the average value of a quantity x is defined as

$$\langle x \rangle = \frac{1}{V}\int d\mathbf{r}\, x$$

Consider now $\Psi = \alpha\Psi_0$, where α is a constant, and we can choose, for H close to H_{c2},

$$\mathbf{Q} = -\hat{\mathbf{z}} \times \frac{\nabla\Psi}{\Psi} + \text{small terms} \tag{4.20}$$

The second Ginzburg-Landau equation , eq.(4.10), gives

$$\kappa^2 \nabla \times \mathbf{h} \approx \hat{\mathbf{z}} \times \Psi\nabla\Psi \tag{4.21}$$

When

$$\mathbf{h} = h(x, y)\,\hat{\mathbf{z}} \tag{4.22}$$

we have that

$$\nabla \times \mathbf{h} = -\hat{\mathbf{z}} \times \nabla h$$

and

$$\nabla h = -\frac{1}{2\kappa^2}\nabla\Psi^2 \tag{4.23}$$

The solution is

$$h = H - \frac{1}{2\kappa^2}\Psi^2$$

and

$$(h - H)^2 = \frac{\Psi^4}{4\kappa^4} \tag{4.24}$$

Substituting eq. (4.24) in eq. (4.19) we get

$$\langle g \rangle = \frac{1 - 2\kappa^2}{4\kappa^2}\left\langle \Psi^4 \right\rangle$$

$$= \frac{1 - 2\kappa^2}{4\kappa^2}\beta\left\langle \Psi^2 \right\rangle^2 \tag{4.25}$$

where, β is defined according to

$$\beta = \frac{\left\langle \Psi^4 \right\rangle}{\left\langle \Psi^2 \right\rangle^2} \tag{4.26}$$

We can write

$$Q = Q_H + Q' \tag{4.27}$$

where

$$Q_H = -\hat{\mathbf{z}} \times \frac{\nabla\Psi}{\Psi} \tag{4.28}$$

From the linearized equation eq. (4.11)

$$-\Psi + Q_0^2\Psi = \nabla^2\Psi \tag{4.29}$$

(remember that $\Psi = \alpha\Psi_0$, where α is a constant), and multiplying by Ψ we get

$$-\Psi^2 + Q_0^2\Psi^2 = \Psi\nabla^2\Psi \tag{4.30}$$

Subtracting eq. (4.16) from eq. (4.30) yields.

$$\Psi^4 + \left(Q^2 - Q_0^2\right)\Psi^2 = 0 \tag{4.31}$$

From the second of the Ginzburg-Landau equations, eq.(4.10) we get

$$Q\Psi^2 = \kappa^2\nabla \times (\mathbf{H} - \mathbf{h}) \tag{4.32}$$

Putting eq. (4.27) into eq. (4.32) and then multiplying by $\mathbf{Q}'$, we have

$$\left(Q_H \cdot Q' + Q'^2\right)\Psi^2 = \kappa^2 Q' \cdot \left(\nabla \times (\mathbf{H} - \mathbf{h})\right)$$

$$= \kappa^2\nabla \cdot \left((\mathbf{H} - \mathbf{h}) \times \mathbf{Q}'\right) + \kappa^2(\mathbf{H} - \mathbf{h}) \cdot \nabla \times \mathbf{Q}' \tag{4.33}$$

By taking the curl of eq (4.27), we obtain

$$\nabla \times \mathbf{Q}' = -(\mathbf{H} - \mathbf{h}) \tag{4.34}$$

so that

$$\left(Q_H \cdot Q' + Q'^2\right)\Psi^2 = \kappa^2 \nabla \cdot \left((\mathbf{H} - \mathbf{h}) \times \mathbf{Q}'\right) - \kappa^2 (\mathbf{H} - \mathbf{h})^2 \tag{4.35}$$

Consider the first term of the right hand side of eq.(4.35), and by the divergence theorem, we have

$$\int_V dr \ \nabla \cdot \left((\mathbf{H} - \mathbf{h}) \times \mathbf{Q}'\right) = \int_S ds \, \widehat{\mathbf{n}} \cdot \left((\mathbf{H} - \mathbf{h}) \times \mathbf{Q}'\right)$$

$$= \int_S ds \ \mathbf{Q}' \cdot \left(\widehat{\mathbf{n}} \times (\mathbf{H} - \mathbf{h})\right) = 0 \tag{4.36}$$

since $\widehat{\mathbf{n}} \times (\mathbf{H} - \mathbf{h}) = 0$ on the surface (Fetter and Walecka, 1971).

Thus, the average of eq (4.35) is

$$\left\langle Q_H \cdot Q' \Psi^2 \right\rangle + \left\langle Q'^2 \Psi^2 \right\rangle + \kappa^2 \left\langle (\mathbf{H} - \mathbf{h})^2 \right\rangle = 0 \tag{4.37}$$

We now recall (eq. 4.15)

$$Q_0^2 = \left(\frac{\nabla \Psi_0}{\Psi_0}\right)^2 + 1 - H$$

Squaring eq (4.27) we have

$$Q^2 = Q_H^2 + 2Q_H \cdot Q' + Q'^2 \tag{4.38}$$

where

$$Q_H = -\hat{z} \times \frac{\nabla\Psi}{\Psi}$$

and we find that

$$Q_H^2 = \left(\frac{\nabla\Psi}{\Psi}\right)^2 \tag{4.39}$$

Then

$$Q^2 = \left(\frac{\nabla\Psi}{\Psi}\right)^2 + 2Q_H \cdot Q' + Q'^2 \tag{4.40}$$

so that

$$Q^2 - Q_0^2 = 2Q_H \cdot Q' + Q'^2 - 1 + H \tag{4.41}$$

Substituting eq. (4.41) into eq. (4.31) gives

$$\left\langle \Psi^4 \right\rangle + 2\left\langle Q_H \cdot Q' \Psi^2 \right\rangle + \left\langle Q'^2 \Psi^2 \right\rangle = (1 - H)\left\langle \Psi^2 \right\rangle \tag{4.42}$$

and by eq.(4.37), we get

$$\left\langle \Psi^4 \right\rangle - 2\kappa^2 \left\langle (\mathbf{H} - \mathbf{h})^2 \right\rangle - \left\langle Q'^2 \Psi^2 \right\rangle = (1 - H)\left\langle \Psi^2 \right\rangle \tag{4.43}$$

Next, from eq.(4.19) and eq.(4.25), we have that

$$-\frac{1}{2}\left\langle \Psi^4 \right\rangle + \kappa^2 \left\langle (\mathbf{h} - \mathbf{H})^2 \right\rangle = \frac{1 - 2\kappa^2}{4\kappa^2} \beta \left\langle \Psi^2 \right\rangle^2 \tag{4.44}$$

Then eq.(4.43) becomes

$$\beta\left(1 - \frac{1}{2\kappa^2}\right) \left\langle \Psi^2 \right\rangle^2 - \left\langle Q'^2 \Psi^2 \right\rangle = (1 - H)\left\langle \Psi^2 \right\rangle \tag{4.45}$$

Then, since the second term of eq.(4.45) is small ($\left\langle Q'^2 \Psi^2 \right\rangle \sim O\left(\Psi^6\right)$),

$$\left\langle \Psi^2 \right\rangle \approx \frac{(1 - H)\, 2\kappa^2}{\beta\left(2\kappa^2 - 1\right)} \tag{4.46}$$

Substituting eq.(4.46) into eq.(4.25), we get

$$\langle g \rangle = -\frac{\kappa^2 (1 - H)^2}{\beta\left(2\kappa^2 - 1\right)} \tag{4.47}$$

Also, from eq.(4.24)

$$h = H - \frac{1}{2\kappa^2} \Psi^2 \tag{4.48}$$

and we see that

$$B = H - \frac{(1 - H)}{\beta\left(2\kappa^2 - 1\right)} \tag{4.49}$$

where $B = \langle h(x,y) \rangle$ is the magnetic induction. $H_{c2} = 1$ in these units.

According to eqs.(4.47), (4.48), and (4.49), for type II superconductors, both the average Gibbs free energy density <g> and the magnetic induction B are dependent on β (the investigation of β will be described in detail in the next chapter). The value of β is approximately equal to 1.16 for a triangular lattice of flux tubes, and 1.18 for a rectangular lattice (Abrikosov, 1957). The Gibbs free energy levels of these two states are very similar, but the triangular lattice is more stable since it occupies a lower energy level. The type I solution ($\mathbf{h} = 0$, $\mathbf{Q} = 0$, $\Psi = 1$ in the bulk) has

$$\langle g \rangle = \kappa^2 H^2 - \frac{1}{2} \tag{4.50}$$

$$B = 0 \tag{4.51}$$

and the normal solution is <g> = 0.

In case of type I superconductors, the Ginzburg-Landau parameter $\kappa < 1/\sqrt{2}$. In the presence of an applied magnetic field H, from eq.(4.50) the average Gibbs free energy density would be positive if $\kappa > 1/\sqrt{2}$ and $H = 1$, and equal to zero if the external field $H = H_c$ ($H_c = 1/\kappa\sqrt{2}$ in our units).