จุดตรึงของการส่งแบบหดตัวเชิงนัยทั่วไปบนปริภูมิอิงระยะทางรูปกรวย

นายอาลี ศาสนพิทักษ์

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต

สาขาวิชาคณิตศาสตร์ ภาควิชาคณิตศาสตร์และวิทยาการคอมพิวเตอร์

คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย

ปีการศึกษา 2554

FIXED POINTS OF GENERALIZED CONTRACTIVE-TYPE MAPPINGS ON CONE METRIC SPACES

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A Thesis Submitted in Partial Fulfillment of the Requirements
for the Degree of Master of Science Program in Mathematics
Department of Mathematics and Computer Science
Faculty of Science
Chulalongkorn University
Academic Year 2011

Thesis Title	FIXED POINTS OF GENERALIZED CONTRACTIVE-TYPE MAPPINGS ON CONE METRIC SPACES
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Field of Study	Mathematics
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อาลี ศาสนพิทักษ์ : จุดตรึงของการส่งแบบหดตัวเชิงนัยทั่วไปบนปริภูมิอิงระยะทางรูปกรวย. (FIXED POINTS OF GENERALIZED CONTRACTIVE-TYPE MAPPINGS ON CONE METRIC SPACES) อ. ที่ปรึกษาวิทยานิพนธ์หลัก : รศ.ดร.อิ่มจิตต์ เติมวุฒิพงษ์, 34 หน้า.

เราสร้างทฤษฎีบทซึ่งรับประกันการมีเพียงหนึ่งเดียวของจุดตรึงของการส่งแบบหดตัวเชิงนัยทั่วไปบนปริภูมิอิงระยะทางรูปกรวย และให้ลักษณะเฉพาะที่บ่งชี้ความบริบูรณ์ของปริภูมิอิงระยะทางรูปกรวย โดยใช้การมีอยู่ของจุดตรึงของการส่งแบบหดตัวบนปริภูมินั้น

ภาควิชา คณิตศาสตร์และวิทยาการคอมพิวเตอร์ ลายมือชื่อนิสิต..

ลายมือชื่อ อ.ที่ปรึกษาวิทยานิพนธ์หลัก........................

สาขาวิชา คณิตศาสตร์.

ปีการศึกษา 2554.

5272633723: MAJOR MATHEMATICS
KEYWORDS:FIXED POINT/COMPLETENESS/CONE METRIC SPACE/CONTRACTION /KANNAN

ALI SASSANAPITAX : FIXED POINTS OF GENERALIZED CONTRACTIVE-TYPE MAPPINGS ON CONE METRIC SPACES. ADVISOR : ASSOC. PROF. IMCHIT TERMWUTTIPONG, Ph.D., 34 pp.

Theorems that guarantee the existence and the uniqueness of fixed points of generalized contractive mappings on cone metric spaces are given. A characterization of the completeness of cone metric spaces by the existence of fixed point of some contractive mapping is also given.

Department : Mathematics and Computer Science Student's Signature ....

Field of Study : Mathematics Advisor's Signature ....

Academic Year : 2011

ACKNOWLEDGEMENTS

I am greatly indebted to Associate Professor Dr.Imchit Termwuttipong, my thesis advisor, for her willingness to sacrifice her time to suggest and advise me in preparing and writing this thesis. I am also sincerely grateful to Associate Professor Dr.Phichet Chaoha, Assistant Professor Dr.Wacharin Wichiramala and Dr.Gun Sunyeekhan, my thesis committee, for their valuable suggestions on this thesis. Moreover, I would like to thank all of my teachers and all the lecturers during my study.

I also owe my gratitude to Mrs. Tammatada Pongsriiam who suggests me the idea for the main results.

I would like to acknowledge Human Resource Development in Science Project for financial support throughout my graduate study.

Finally, I would like to express my deep gratitude to my beloved family especially my parents and my brother for their love and encouragement throughout my graduate study.

CONTENTS

CHAPTER I

INTRODUCTION

The fixed point theory of self-mappings in metric spaces has been widely studied since almost a century ago, when S. Banach [1] presented a famous theorem which becomes a powerful tool in nonlinear analysis known as *Banach contraction principle*. Later in 1969, R. Kannan [2] proposed a fixed point theorem called *Kannan fixed point theorem*. It is noted that the theorem is not an extension of Banach contraction principle.

There are many generalizations of Banach contraction principle, one of those was shown by T. Suzuki [3] in 2008. He proved that the conclusion is still true if the contraction criteria of T in Banach contraction principle is replaced by a certain weaker condition. Also M. Kikkawa and T. Suzuki [4] proved a theorem which can be applied to a wider class of mappings than that of Kannan.

In 1975, P.V. Subrahmanyam [5] proved that Kannan fixed point theorem characterizes the completeness of metric spaces, which is state that a metric space X is complete if and only if every Kannan mapping on X has a fixed point. On the other hand, E.H. Connell [6] showed that Banach contraction principle cannot characterize completeness of metric spaces by giving an example of a metric space which is not complete and every contraction mapping on the space has a fixed point. However, in 2008 T. Suzuki [3] established a theorem that characterize the completeness of the metric space.

In this work, we investigate the conditions of Suzuki and Subrahmanyam in cone metric spaces.

CHAPTER II

PRELIMINARIES

In this chapter, we review some notations, terminologies, and fundamental facts that will be used thoughtout our work. The symbols $\mathbb{R}, \mathbb{R}^+, \mathbb{R}_0^+$ stand for the set of real numbers, the set of positive real numbers and the set of nonnegative real numbers, respectively.

2.1 Metric Spaces

Definition 2.1.1. A *metric* on a nonempty set X is a map $d : X \times X \to \mathbb{R}$ such that the followings hold for any $x,\ y,\ z \in X$:-

(i) $d(x, y) = 0$ if and only if $x = y$,

(ii) $d(x, y) = d(y, x)$,

(iii) $d(x, z) \leq d(x, y) + d(y, z)$.

If d is a metric on X, then the space X with the metric d is called a *metric space* and denoted by (X, d).

Remark 2.1.2. From the conditions above, we have for each $x, y \in X$,

$$0 = d(x, x) \leq d(x, y) + d(y, x) = d(x, y) + d(x, y) = 2d(x, y);$$

i.e. $d(x, y) \geq 0$. Therefore $d : X \times X \to \mathbb{R}_0^+$.

Definition 2.1.3. Let (X, d) be a metric space, $p \in X$ and $E \subseteq X$.

(a) A neighborhood of p is a set $N_r(p)$ consisting of all $q \in X$ such that $d(p, q) < r$ for some $r > 0$. The number r is called the radius of $N_r(p)$.

(b) p is a limit point of E if every neighborhood of p contains $q \in E$ such that $q \neq p$.

(c) E is closed if every limit point of E is in E.

(d) p is an interior point of E if there is a neighborhood N of p such that $N \subseteq E$.

(e) The interior of E, int$E$, is the set of interior points of E.

Definition 2.1.4. A sequence on a nonempty set X is a function from $\mathbb{N}$ into X. The sequence $\{(n, x_n) : n \in \mathbb{N}\}$ will be denoted by (x_n).

Definition 2.1.5. A sequence (x_n) in a metric space (X, d) is said to converges if there is $x \in X$ with the following property: For every $\epsilon > 0$ there is $N \in \mathbb{N}$ such that $n \geq N$ implies that $d(x_n, x) < \epsilon$.

In this case we also say that (x_n) converges to x, or that x is the limit of (x_n). and we write

$$x_n \to x \quad \text{or} \quad \lim_{n\to\infty} x_n = x.$$

If (x_n) does not converge, it is said to diverge.

Definition 2.1.6. Let (x_n) be a sequence in X. A subsequence (x_{n_k}) of (x_n) is a mapping $k \mapsto x_{n_k}$ where (n_k) is a strictly increasing sequence in $\mathbb{N}$.

Theorem 2.1.7. *Let (x_n) be a sequence in a metric space X and $x \in X$. Then (x_n) converges to x if and only if every subsequence of (x_n) converges to x.*

Definition 2.1.8. Let (X,d) be a metric space. A sequence (x_n) in X is called a Cauchy sequence if for each $\epsilon > 0$ there is $N_\epsilon \in \mathbb{N}$ such that $d(x_m, x_n) < \epsilon$ for any $m, n \geq N_\epsilon$.

Definition 2.1.9. A metric space (X,d) is said to be complete if every Cauchy sequence in X converges to a point in X.

Theorem 2.1.10. *Every closed subset of a complete metric space is complete.*

Definition 2.1.11. Let X be a real vector space. A function $\|\cdot\| : X \to [0,\infty)$ is said to be a norm on X if the followings hold for any $x, y \in X$ and $c \in \mathbb{R}$:-

(i) $\|x\| = 0$ if and only if $x = 0$,

(ii) $\|cx\| = |c|\|x\|$,

(iii) $\|x + y\| \leq \|x\| + \|y\|$.

A vector space equipped with a norm is called a normed (linear) space.

Theorem 2.1.12. *Let X be a normed linear space. Then the function $d : X \times X \to [0,\infty)$ defined by*

$$d(x,y) = \|x - y\|$$

for any $x, y \in X$ is a metric on X. The metric d defined above is called the metric induced from the norm $\|\cdot\|$.

Definition 2.1.13. A Banach space is a normed linear space which is complete with respect to the metric induced from the norm.

2.2 Fixed Point Theorems on Metric Spaces

Definition 2.2.1. Let X be a set and $A \subseteq X$. A point $a \in A$ is called a fixed point of a mapping $f : A \to X$ if $a = f(a)$.

Definition 2.2.2. Let (X, d) be a metric space. A self-mapping $T : X \to X$ is said to be

(i) a contraction map if there is $r \in [0, 1)$ such that for each $x, y \in X$,

$$d(Tx, Ty) \leq rd(x, y).$$

(ii) a contractive map if for each $x, y \in X$ such that $x \neq y$,

$$d(Tx, Ty) < d(x, y).$$

(iii) a Kannan map if there is $r \in [0, \frac{1}{2})$ such that for each $x, y \in X$,

$$d(Tx, Ty) \leq r\left(d(x, Tx) + d(y, Ty)\right).$$

There are some well-known results that guarantee the existence of the fixed point of a certain kind of self-maps on a metric space.

Theorem 2.2.3. (Banach contraction principle [1]) *Let (X, d) be a complete metric space and T a contraction on X. Then T has a unique fixed point.*

Theorem 2.2.4. ([1]) *Let (X, d) be a compact metric space and T a contractive on X. Then T has a unique fixed point.*

Theorem 2.2.5. (Kannan fixed point theorem [2]) *Let (X, d) be a complete metric space and T a Kannan mapping on X. Then T has a unique fixed point.*

Theorem 2.2.6. [5] *Let (X, d) be a metric space in which every Kannan mapping on X has a fixed point. Then X is complete.*

Theorem 2.2.7. [3] *Let (X, d) be a metric space. Define a nonincreasing function $\theta : [0, 1) \to (\frac{1}{2}, 1]$ by*

$$\theta(r) = \begin{cases} 1 & , 0 \leq r \leq \frac{\sqrt{5}-1}{2}, \\ \frac{1-r}{r^2} & , \frac{\sqrt{5}-1}{2} \leq r \leq \frac{1}{\sqrt{2}}, \\ \frac{1}{1+r} & , \frac{1}{\sqrt{2}} \leq r < 1. \end{cases}$$

Then the followings are equivalent:

(i) X *is complete,*

(ii) every self-mapping T *on* X *satisfying the condition that there exists* $r \in (0, 1]$ *such that for each* $x, y \in X$,

$$\theta(r)d(x, Tx) \leq d(x, y) \quad \textit{implies} \quad d(Tx, Ty) \leq rd(x, y),$$

has a fixed point.

Theorem 2.2.8. [4] *Let* (X, d) *be a complete metric space. Define a nonincreasing function* $r : [0, \frac{1}{2}) \rightarrow (\frac{1}{2}, 1]$ *by*

$$\varphi(r) = \begin{cases} 1 & , 0 \leq r < \frac{1}{1+\sqrt{2}}, \\ 1 - r & , \frac{1}{1+\sqrt{2}} \leq r < \frac{1}{2}. \end{cases}$$

Let T *be a self-mapping on* X *satisfying the condition that there exists* $r \in [0, \frac{1}{2})$ *such that for each* $x, y \in X$,

$$\varphi(r)d(x, Tx) \leq d(x, y) \quad \textit{implies} \quad d(Tx, Ty) \leq rd(x, Tx) + rd(y, Ty).$$

Then T *has a unique fixed point.*

2.3 Cone Metric Spaces

The notion of cone metric spaces was first introduced by L.G. Huang and Z. Xian [7] in 2007. The definition of cone metric space is precisely stated as follows.

Definition 2.3.1. Let E be a real Banach space. A subset P of E is called a *cone* if the followings hold:

(i) P is closed, nonempty and $P \neq \{0\}$;

(ii) $a, b \in \mathbb{R}, a, b \geq 0$ and $x, y \in P$ imply that $ax + by \in P$;

(iii) $x \in P$ and $-x \in P$ imply that $x = 0$.

The followings are examples of cones in some well-known Banach spaces.

Example 2.3.2. (i) $P_1 = \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n : x_n \geq 0, \forall n\}$ and $P_2 = \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n : x_n \leq 0, \forall n\}$ are cones in $\mathbb{R}^n$.

(ii) $P = \{f \in C_{\mathbb{R}}([0, 1]) : f \geq 0\}$ is a cone in $C_{\mathbb{R}}([0, 1])$ with the supremum norm.

(iii) $P = \{(x_n) \in l^1 : x_n \geq 0, \forall n\}$ is a cone in l^1 with l^1 norm $\|x\| = \sum_{n=1}^{\infty} |x_n|$.

Given a cone $P \subseteq E$, we define a partial ordering $\preceq$ on E with respect to P by

$$x \preceq y \quad \text{if and only if} \quad y - x \in P.$$

We write $x \prec y$ to indicate that $x \preceq y$ and $x \neq y$, while $x \prec\prec y$ will stand for $y - x \in \text{int}P$, where $\text{int}P$ denotes the interior of P.

In our work we consider a cone P with nonempty interior. To study the properties of a cone metric, the following properties of cone P are useful. We state here as a remark. The proof is straightforwardly.

Remark 2.3.3. For any cone P of a Banach space,

(i) $\text{int}P + \text{int}P \subseteq \text{int}P$,

(ii) $\lambda\text{int}P \subseteq \text{int}P$, for all $\lambda > 0$,

(iii) $P + \text{int}P \subseteq \text{int}P$.

Definition 2.3.4. The cone P is said to be *normal* if there is a number $K > 0$ such that for all $x, y \in E$,

$$0 \preceq x \preceq y \text{ implies } \|x\| \leq K\|y\|.$$

The least positive number K satisfying the above condition is called the *normal constant* of P.

Example 2.3.5. The set $P = \{(x, y) \in \mathbb{R}^2 : x, y \geq 0\}$ is a normal cone in $\mathbb{R}^2$ with normal constant $K = 1$.

Example 2.3.6. $P = \{f \in C_{\mathbb{R}}([0,1]) : f \geq 0\}$ is a normal cone in $C_{\mathbb{R}}([0,1])$ with normal constant $K = 1$.

Proposition 2.3.7. [8] *For every normal cone, its normal constant must not be less than 1.*

Proposition 2.3.8. [8] *For each $m > 1$, there is a normal cone with normal constant $K > m$.*

The following example shows that there exists a non-normal cone.

Example 2.3.9. Let $E = C^1_{\mathbb{R}}([0,1])$ with the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$, where $\|g\|_\infty = \sup\limits_{t\in[0,1]} |g(t)|$ and consider the cone $P = \{f \in E : f \geq 0\}$. For each $k \geq 1$, put $f(x) = x$ and $g(x) = x^{2k}$. Then $0 \leq g \leq f$, $\|f\| = 2$ and $\|g\| = 2k+1$. Hence $k\|f\| < \|g\|$. Therefore P is a non-normal cone.

By the notion of cone defined above, L.G. Huang and Z. Xian [7] introduced the cone metric spaces.

Definition 2.3.10. Let E be a Banach space with a cone P, $\preceq$ the partial ordering on E with respect to P. Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfying the following properties for each $x, y, z \in X$:

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, y) \preceq d(x, z) + d(z, y)$.

Then d is called a *cone metric* on X (with respect to the cone P) and the pair (X, d) is called a *cone metric space*.

Remark 2.3.11. From the above definition, we have for each $x, y \in X$,

$$0 = d(x, x) \preceq d(x, y) + d(y, x) = d(x, y) + d(x, y) = 2d(x, y);$$

i.e. $0 \preceq d(x, y)$. Therefore $d : X \times X \to P$.

Remark 2.3.12. If $E = \mathbb{R}$ and $P = \mathbb{R}_0^+$, then cone metric space is obviously a metric space with metric d.

Example 2.3.13. Let $P = \{(x, y) \in \mathbb{R}^2 : x, y \geq 0\}, X = \mathbb{R}, d : X \times X \to \mathbb{R}^2$ defined by $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space.

Example 2.3.14. Let $E = l^1, P = \{(x_n) \in E : x_n \geq 0, \forall n \in \mathbb{N}\}$, (X, ρ) a metric space and $d : X \times X \to E$ defined by $d(x, y) = \left(\frac{\rho(x,y)}{2^n}\right)$. Then (X, d) is a cone metric space.

Next, the definitions of convergence in cone metric spaces are presented. Some properties of convergent sequences in the spaces are proved by Huang and Xian [7].

Definition 2.3.15. Let (X, d) be a cone metric space, $x \in X$ and $\{x_n\}$ a sequence in X. Then

(i) $\{x_n\}$ converges to x whenever for every $c \in E$ with $0 \prec\prec c$ there is $N \in \mathbb{N}$ such that $d(x_n, x) \prec\prec c$ for all $n \geq N$. If this is the case, we denote by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$. The point x is called a limit of $\{x_n\}$.

(ii) $\{x_n\}$ is a Cauchy sequence whenever for every $c \in E$ with $0 \prec\prec c$ there is $N \in \mathbb{N}$ such that $d(x_n, x_m) \prec\prec c$ for all $n, m \geq N$.

(iii) (X, d) is complete if every Cauchy sequence in (X, d) is convergent to a point in X.

Lemma 2.3.16. [7] *Let P be a normal cone with normal constant K and (X, d) a cone metric space. Let $\{x_n\}, \{y_n\}$ be two sequences in X. Then*

(i) $\lim_{n\to\infty} x_n = x$ *if and only if* $\lim_{n\to\infty} d(x_n, x) = 0$.

(ii) If $\{x_n\}$ is convergent, then the limit of $\{x_n\}$ is unique.

(iii) If $\{x_n\}$ is convergent, then $\{x_n\}$ is a Cauchy sequence.

(iv) $\{x_n\}$ *is a Cauchy sequence if and only if* $\lim_{m,n\to\infty} d(x_m, x_n) = 0$.

(v) If $\lim_{n\to\infty} x_n = x$ *and* $\lim_{n\to\infty} y_n = y$, *then* $\lim_{n\to\infty} d(x_n, y_n) = d(x, y)$.

Definition 2.3.17. Let (X, d) be a cone metric space. If every sequence in X has a convergent subsequence, then X is called a sequentially compact cone metric space.

The Banach contraction principle and Kannan fixed point theorem in cone metric spaces were also established in [7].

Theorem 2.3.18. [7] *Let P be a normal cone with normal constant K and (X, d) a complete cone metric space. Suppose the mapping $T : X \to X$ satisfies the condition that there exists $r \in [0, 1)$ such that for each $x, y \in X$,*

$$d(Tx, Ty) \preceq rd(x, y).$$

Then T has a unique fixed point in X.

Theorem 2.3.19. [7] *Let P be a normal cone with normal constant K and (X, d) a complete cone metric space. Suppose the mapping $T : X \to X$ satisfies the condition that there exists $r \in [0, \frac{1}{2})$ such that for each $x, y \in X$,*

$$d(Tx, Ty) \preceq r\left(d(x, Tx) + d(y, Ty)\right).$$

Then T has a unique fixed point in X.

They also proved the fixed point theorem for nonexpansive mappings in cone metric spaces, which is stated as follows.

Theorem 2.3.20. [7] *Let P be a normal cone with normal constant K and (X, d) a sequentially compact cone metric space. Suppose the mapping $T : X \to X$ satisfies the condition that for each $x, y \in X$ and $x \neq y$,*

$$d(Tx, Ty) \prec d(x, y).$$

Then T has a unique fixed point in X.

In 2009, S. Radenovic [9] presented the fixed point theorems for the class of mappings satisfying a contractive condition in cone metric spaces.

Theorem 2.3.21. [9] *Let P be a normal cone with normal constant K and (X,d) a complete cone metric space. Suppose the mapping $T : X \to X$ satisfies the condition that there exists $r \in [0,1)$ such that for each $x, y \in X$,*

$$\|d(Tx,Ty)\| \leq r\,\|d(x,y)\|\,.$$

Then T has a unique fixed point in X.

Theorem 2.3.22. [9] *Let P be a normal cone with normal constant K and (X,d) a complete cone metric space. Suppose the mapping $T : X \to X$ satisfies the condition that there exists $r \in [0,\frac{1}{2})$ such that for each $x, y \in X$,*

$$\|d(Tx,Ty)\| \leq r\left(\|d(x,Tx)\| + \|d(y,Ty)\|\right).$$

Then T has a unique fixed point in X.

CHAPTER III

MAIN RESULTS

In this chapter, we present the fixed point theorems in cone metric spaces which are analogous to Theorem 2.2.7 and Theorem 2.2.8. A characterization of completeness in cone metric spaces is also given.

3.1 Fixed Point Theorems

The results in this section were inspired by Theorem 2.2.7 which stated that in any complete metric spaces X every self-mapping T on X satisfying a certain weaker contraction condition always has a fixed point. To prove the theorem, we need the following lemma.

Lemma 3.1.1. *Let P be a normal cone with normal constant K and (X, d) a complete cone metric space. Let T be a self-mapping on X satisfying the condition that for each $x \in X$,*

$$d(Tx, T^2x) \preceq rd(x, Tx)$$

for some $r > 0$. Then for every $x, y \in X$, either

$$\frac{1}{1+Kr}\|d(x,Tx)\| \leq K\|d(x,y)\| \quad \text{or} \quad \frac{1}{1+Kr}\left\|d(Tx,T^2x)\right\| \leq K\|d(Tx,y)\|$$

holds.

Proof. Let $x \in X$ be arbitrary. Suppose there exists $y \in X$ such that

$$\frac{1}{1+Kr}\|d(x,Tx)\| > K\|d(x,y)\| \quad \text{and} \quad \frac{1}{1+Kr}\left\|d(Tx,T^2x)\right\| > K\|d(Tx,y)\|.$$

Then

$$\begin{aligned}\|d(x,Tx)\| &\le K\,\|d(x,y)\| + K\,\|d(y,Tx)\| \\ &< \frac{1}{1+Kr}\,\|d(x,Tx)\| + \frac{1}{1+Kr}\,\left\|d(Tx,T^2x)\right\| \\ &< \frac{1}{1+Kr}\,\|d(x,Tx)\| + \frac{Kr}{1+Kr}\,\|d(x,Tx)\| \\ &= \|d(x,Tx)\|\,.\end{aligned}$$

This is a contradiction. □

Now we prove the theorem, which give a sufficient condition for any self-mapping T on a complete cone metric space with respect to a normal cone, to has a unique fixed point.

Theorem 3.1.2. *Let P be a normal cone with normal constant K and (X,d) a complete cone metric space. Define a function $\theta : [0,1) \to \mathbb{R}$ by*

$$\theta(r) = \begin{cases} 1 & , 0 \le r \le \frac{\sqrt{K^2+4}-K}{2K}, \\ \frac{1-K^2r}{K^2r^2} & , \frac{\sqrt{K^2+4}-K}{2K} \le r \le \frac{1}{\sqrt{2K}}, \\ \frac{1}{1+Kr} & , \frac{1}{\sqrt{2K}} < r < 1. \end{cases}$$

Let T be a self-mapping on X satisfying the condition that there exists $r \in [0,1)$ such that for each $x,y \in X$,

$$\theta(r)\,\|d(x,Tx)\| \le K\,\|d(x,y)\| \quad \textit{implies} \quad d(Tx,Ty) \preceq rd(x,y). \tag{3.1}$$

Then there exists a unique fixed point z of T. Moreover $\lim\limits_{n\to\infty} T^n x = z$ for all $x \in X$.

Proof. Note that for every $r \in [0,1)$, $\theta(r) \le 1 \le K$, so

$$\theta(r)\,\|d(x,Tx)\| \le K\,\|d(x,Tx)\|\,.$$

From the assumption (3.1), we have

$$d(Tx, T^2x) \preceq rd(x, Tx), \tag{3.2}$$

for all $x \in X$. Let $u_0 \in X$ be arbitrary. Set $u_n = T^n u_0$ for each $n \in \mathbb{N}$. From (3.2), we have

$$d(u_n, u_{n+1}) \preceq r^n d(u_0, u_1).$$

So for $n > m$,

$$\begin{aligned} d(u_n, u_m) &\preceq d(u_n, u_{n-1}) + d(u_{n-1}, u_{n-2}) + ... + d(u_{m+1}, u_m) \\ &\preceq (r^{n-1} + r^{n-2} + ... + r^m) d(u_0, u_1) \\ &\preceq \left(\frac{r^m}{1-r}\right) d(u_0, u_1). \end{aligned}$$

Since K is the normal constant of P, $\|d(u_n, u_m)\| \leq \frac{Kr^m}{1-r} \|d(u_0, u_1)\|$. Then $\lim_{n,m\to\infty} d(u_n, u_m) = 0$. Hence (u_n) is a Cauchy sequence. By the completeness of X, there is $z \in X$ such that $\lim_{n\to\infty} u_n = z$. Next, we will show that

$$d(Tx, z) \preceq rd(x, z) \quad \text{for all} \quad x \in X \setminus \{z\}. \tag{3.3}$$

Let $x \in X \setminus \{z\}$. Since $\lim_{n\to\infty} u_n = z$, there exists $N \in \mathbb{N}$ such that $\|d(u_n, z)\| \leq \frac{\|d(x,z)\|}{3K^2}$, for $n \geq N$. Let $n \geq N$. Then

$$\begin{aligned} \theta(r) \|d(u_n, Tu_n)\| &\leq K \|d(u_n, u_{n+1})\| \\ &\leq K^2 (\|d(u_n, z)\| + \|d(u_{n+1}, z)\|) \\ &\leq K^2 \left(\frac{2}{3K^2} \|d(x, z)\|\right) \\ &= \|d(x, z)\| - \frac{1}{3} \|d(x, z)\| \\ &\leq \|d(x, z)\| - \|d(u_n, z)\| \\ &\leq K \|d(u_n, x)\|. \end{aligned}$$

By the assumption (3.1), we have

$$d(u_{n+1}, Tx) \preceq rd(u_n, x).$$

Then $d(z, Tx) = \lim\limits_{n\to\infty} d(u_{n+1}, Tx) \preceq \lim\limits_{n\to\infty} rd(u_n, x) = rd(z, x)$.

Next we show that $T^j z = z$ for some $j \in \mathbb{N}$. Suppose on the contrary that $T^j z \neq z$ for all $j \in \mathbb{N}$. Then (3.3) yields for each $j \in \mathbb{N}$,

$$d(T^{j+1}z, z) \preceq r^j d(Tz, z).$$

Case 1: $0 \leq r \leq \frac{\sqrt{K^2+4}-K}{2K}$ and $\theta(r) = 1$.

We note that

$$r^2 + r \leq \left(\frac{\sqrt{K^2+4}-K}{2K}\right)^2 + \frac{\sqrt{K^2+4}-K}{2K} = \frac{1}{K^2}$$

and

$$2Kr^2 \leq 2K\left(\frac{\sqrt{K^2+4}-K}{2K}\right)^2 < 2K\left(\frac{1}{\sqrt{2K}}\right)^2 = 1.$$

Suppose $\|d(T^2z, z)\| < \|d(T^2z, T^3z)\|$. Then

$$\begin{aligned}
\|d(z, Tz)\| &\leq K \left\|d(z, T^2z)\right\| + K \left\|d(T^2z, Tz)\right\| \\
&< K \left\|d(T^2z, T^3z)\right\| + K \left\|d(T^2z, Tz)\right\| \\
&\leq K^2(r^2 + r) \|d(z, Tz)\| \\
&\leq \|d(z, Tz)\|,
\end{aligned}$$

which is not true. So we have

$$\theta(r) \left\|d(T^2z, T^3z)\right\| = \left\|d(T^2z, T^3z)\right\| \leq \left\|d(T^2z, z)\right\| \leq K \left\|d(T^2z, z)\right\|.$$

From the assumption (3.1), we get

$$d(T^3z, Tz) \preceq rd(T^2z, z) \preceq r^2 d(Tz, z).$$

Then

$$d(z, Tz) \preceq d(z, T^3z) + d(T^3z, Tz) \preceq 2r^2 d(T^2z, z).$$

Therefore

$$\|d(z,Tz)\| \leq 2Kr^2 \|d(z,Tz)\| < \|d(z,Tz)\| .$$

This is a contradiction.

Case 2: $\frac{\sqrt{K^2+4}-K}{2K} \leq r \leq \frac{1}{\sqrt{2K}}$ and $\theta(r) = \frac{1-K^2r}{K^2r^2}$.

In this case $2Kr^2 < 1$. Suppose $\theta(r)\|d(T^2z,T^3z)\| > K\|d(T^2z,z)\|$. Then

$$\begin{aligned}
\|d(z,Tz)\| &\leq K\left\|d(z,T^2z)\right\| + K\left\|d(T^2z,Tz)\right\| \\
&< \theta(r)\left\|d(T^2z,T^3z)\right\| + K\left\|d(T^2z,Tz)\right\| \\
&\leq \big(Kr^2\theta(r) + Kr\big)\|d(z,Tz)\| \\
&= \frac{1}{K}\|d(z,Tz)\| \\
&\leq \|d(z,Tz)\| ,
\end{aligned}$$

which is not true. So we have

$$\theta(r)\left\|d(T^2z,T^3z)\right\| \leq K\left\|d(T^2z,z)\right\| .$$

Similar to the previous case, it implies that

$$\|d(z,Tz)\| \leq 2Kr^2 \|d(z,Tz)\| < \|d(z,Tz)\| .$$

This is a contradiction.

Case 3: $\frac{1}{\sqrt{2K}} < r < 1$ and $\theta(r) = \frac{1}{1+Kr}$.

From Lemma 3.1.1, we have for each $x, y \in X$,

$$\theta(r)\|d(x,Tx)\| \leq K\|d(x,y)\| \quad \text{or} \quad \theta(r)\left\|d(Tx,T^2x)\right\| \leq K\|d(Tx,y)\| .$$

Then $\theta(r)\|d(u_n,u_{n+1})\| \leq K\|d(u_n,z)\|$ or $\theta(r)\|d(u_{n+1},u_{n+2})\| \leq K\|d(u_{n+1},z)\|$.

From the assumption (3.1), we have for all $n \in \mathbb{N}$

$$d(u_{n+1},Tz) \preceq rd(u_n,z) \quad \text{or} \quad d(u_{n+2},Tz) \preceq rd(u_{n+1},z).$$

Thus there exists a subsequence (u_{n_j}) of (u_n) such that

$$d(u_{n_j+1}, Tz) \preceq rd(u_{n_j}, z)$$

for all $j \in \mathbb{N}$. Then we have

$$d(z, Tz) = \lim_{j\to\infty} d(u_{n_j+1}, Tz) \preceq \lim_{j\to\infty} rd(u_{n_j}, z) = d(z, z) = 0.$$

Hence $z = Tz$. This is a contradiction.

Therefore in every case, $T^j z = z$ for some $j \in \mathbb{N}$. Since $(T^n z)$ is a Cauchy sequence, for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for each $m, n \geq N$,

$$d(T^n z, T^m z) < \epsilon.$$

Then

$$d(z, Tz) = d(T^{Nj} z, T^{Nj+1} z) < \epsilon.$$

Hence $d(z, Tz) = 0$. This implies $Tz = z$. So z is a fixed point of T.

Now if w is another fixed point of T, by (3.3) we have

$$d(w, z) = d(Tw, z) \preceq rd(w, z).$$

Hence $d(w, z) = 0$. So $w = z$. Therefore the fixed point of T is unique. $\square$

Remark 3.1.3. If $K \geq \sqrt[3]{2}$ and $\frac{\sqrt{K^2+4}-K}{2K} \leq r \leq \frac{1}{\sqrt{2K}}$, then $\theta(r) \leq 0$ and the Theorem 3.1.2 is the Theorem 2.3.18.

Remark 3.1.4. We note that Theorem 3.1.2 is a generalization of Theorem 2.2.7 when we consider $[0, \infty)$ as a cone in $\mathbb{R}$ and it is obvious that the Theorem 3.1.2 holds if we substitute the funcion θ by any function $\eta : [0, 1) \to \mathbb{R}$ such that $\eta(r) \leq \theta(r)$ for all $r \in [0, 1)$.

It is obvious that the class of the mappings satisfying the contractive condition in Theorem 2.3.18 contains that of the usual condition of contractions. However, the following example shows that the contractive condition in Theorem 2.3.18 is certainly weaker than the usual contraction condition.

Example 3.1.5. Let P be the cone $\{(x, y) \in \mathbb{R}^2 : x, y \geq 0\}$ and

$$X = \{(0, 0), (4, 0), (0, 4), (4, 5), (5, 4)\}.$$

Define $d : X \times X \to \mathbb{R}^2$ by

$$d((x_1, y_1), (x_2, y_2)) = (|x_1 - x_2| + |y_1 - y_2|, 0).$$

Then (X, d) is a complete cone metric space because every Cauchy sequence in X is a constant sequence. Define a self-mapping T on X by

$$T(x_1, x_2) = \begin{cases} (x_1, 0) & , x_1 \leq x_2, \\ (0, x_2) & , x_1 > x_2. \end{cases}$$

Note that if $(x, y) \neq ((4, 5), (5, 4))$ and $(y, x) \neq ((4, 5), (5, 4))$, then

$$d(Tx, Ty) \leq \frac{4}{5} d(x, y).$$

Since $\theta(\frac{4}{5})\|d((5, 4), T(5, 4))\| = \theta(\frac{4}{5})\|d((4, 5), T(4, 5))\| = \frac{25}{9} > 2 = \|d((4, 5), (5, 4))\|$, T satisfies the condition (3.1). But $d(T(4, 5), T(5, 4)) \not\preceq rd((4, 5), (5, 4))$ for every $r \in [0, 1)$ because $rd((4, 5), (5, 4)) - d(T(4, 5), T(5, 4)) = (2r - 8, 0) \notin P$. This means that T is not contraction.

Note that if z is a fixed point of T^n, then z is also a fixed point of T. The following corollary is immediately obtained from Theorem 3.1.2.

Corollary 3.1.6. *Let P be a normal cone with normal constant K and (X,d) a complete cone metric space. Define a function θ as in Theorem 3.1.2. Let T be a self-mapping on X satisfying the condition that there exists $r \in [0,1)$ and $n \in \mathbb{N}$ such that for each $x, y \in X$,*

$$\theta(r)\,\|d(x,T^n x)\| \;\leq\; K\,\|d(x,y)\| \quad \textit{implies} \quad d(T^n x, T^n y) \;\preceq\; r d(x,y).$$

Then T has a unique fixed point.

Proof. From Theorem 3.1.2, T^n has a unique fixed point z. Then

$$T^n(Tz) = T(T^n z) = Tz,$$

so Tz is also a fixed point of T^n. Hence $Tz = z$, z is a fixed point of T. Since the fixed point of T is also fixed point of T^n, the fixed point of T is unique. $\square$

The following theorem gives a condition which guarantee the existence and uniqueness for generalized contractive mappings in cone metric spaces.

Theorem 3.1.7. *Let P be a normal cone with normal constant K and (X,d) a sequentially compact cone metric space. Let T be a self-mapping on X such that for each $x, y \in X$ and $x \neq y$,*

$$\frac{1}{1+K}\,\|d(x,Tx)\| \;\leq\; K\,\|d(x,y)\| \quad \textit{implies} \quad d(Tx,Ty) \prec d(x,y). \qquad (3.4)$$

Then T has a unique fixed point.

Proof. Suppose on the contrary that $x \neq Tx$ for all $x \in X$. Since $\frac{1}{1+K} \leq K$, $\frac{1}{1+K}\,\|d(x,Tx)\| \leq K\,\|d(x,Tx)\|$. From assumption (3.4), we have

$$d(Tx, T^2x) \prec d(x,Tx). \qquad (3.5)$$

for all $x \in X$. Let $u_0 \in X$ be arbitrary. Set $u_n = T^n u_0$ for each $n \in \mathbb{N}$. We note that $u_{n+1} \neq u_n$ for all $n \in \mathbb{N}$. From the sequential compactness of X, there exists a subsequnece (u_{n_j}) of (u_n) such that $\lim_{j\to\infty} u_{n_j} = z$. By (3.5) and Lemma 3.1.1, we have,

$$\frac{1}{1+K}\|d(x,Tx)\| \le K\|d(x,y)\| \quad \text{or} \quad \frac{1}{1+K}\left\|d(Tx,T^2x)\right\| \le K\|d(Tx,y)\|$$

for all $x, y \in X$. Then

$$\frac{1}{1+K}\left\|d(u_{n_j},u_{n_j+1})\right\| \le K\left\|d(u_{n_j},z)\right\| \quad \text{or} \quad \frac{1}{1+K}\left\|d(u_{n_j+1},u_{n_j+2})\right\| \le K\left\|d(u_{n_j+1},z)\right\|.$$

From the assumption (3.4), we have

$$d(u_{n_j+1},Tz) \prec d(u_{n_j},z) \quad \text{or} \quad d(u_{n_j+2},Tz) \prec d(u_{n_j+1},z)$$

for all $j \in \mathbb{N}$. Thus there exists a subsequence (n_k) of (n_j) such that

$$d(u_{n_k+1},Tz) \prec d(u_{n_k},z)$$

for all $k \in \mathbb{N}$. Then we have

$$d(z,Tz) = \lim_{k\to\infty} d(u_{n_k+1},Tz) \prec \lim_{k\to\infty} d(u_{n_k},z) = d(z,z) = 0.$$

Hence $z = Tz$, This is a contradiction. Therefore there exists $v \in X$ such that $v = Tv$.

Now if w is another fixed point of T, we have $\frac{1}{1+K}\|d(w,Tw)\| = 0 \le K\|d(w,v)\|$. From the assumption (3.4), we have $d(Tw,Tv) \prec d(w,v)$ but v and w are fixed points of T, so $d(v,z) = d(Tv,Tz) \prec d(w,z)$. This is a contradiction. Therefore T has a unique fixed point. $\square$

Remark 3.1.8. The self-mapping on complete cone metric space defined as in Example 3.1.5 satisfies the condition (3.4) but not contractive.

Next we prove the following theorem, which is a generalization of Theorem 2.3.19.

Theorem 3.1.9. *Let P be a normal cone with normal constant K and (X, d) a complete cone metric space. Define a function $\varphi : [0, \frac{1}{2}) \to \mathbb{R}$ by*

$$\varphi(r) = \begin{cases} 1 & , 0 \le r < \frac{1}{1+\sqrt{2K}}, \\ \frac{1-r}{1+(K-1)r} & , \frac{1}{1+\sqrt{2K}} \le r < \frac{1}{2}. \end{cases}$$

Let T be a self-mapping on X satisfying the condition that there exists $r \in \left[0, \frac{1}{2}\right)$ such that for each $x, y \in X$,

$$\varphi(r)\, \|d(x, Tx)\| \le K\, \|d(x, y)\| \quad \textit{implies} \quad d(Tx, Ty) \preceq rd(x, Tx) + rd(y, Ty). \tag{3.6}$$

Then there exists a unique fixed point z of T. Moreover $\lim\limits_{n\to\infty} T^n x = z$ for all $x \in X$.

Proof. Since $\varphi(r) \le 1 \le K$, $\varphi(r)\, \|d(x, Tx)\| \le K\, \|d(x, Tx)\|$. From the assumption (3.6), we have

$$d(Tx, T^2x) \preceq rd(x, Tx) + rd(Tx, T^2x).$$

Hence

$$d(Tx, T^2x) \preceq \frac{r}{1-r} d(x, Tx) \tag{3.7}$$

for all $x \in X$. Let $u_0 \in X$. Set $u_n = T^n u_0$. As in the proof of Theorem 3.1.2, we can show that (u_n) is a Cauchy sequence. By the completeness of X, there is $z \in X$ such that $\lim\limits_{n\to\infty} u_n = z$. Next, we will show that

$$d(z, Tx) \preceq rd(x, Tx) \quad \text{for all} \quad x \in X \setminus \{z\}. \tag{3.8}$$

Indeed, if $x \in X \setminus \{z\}$, by the same process as in Theorem 3.1.2, we have

$$\varphi(r)\, \|d(u_n, Tu_n)\| \le K\, \|d(u_n, x)\|.$$

Then from the assumption (3.6), we have

$$d(u_{n+1}, Tx) \preceq rd(u_n, u_{n+1}) + rd(x, Tx).$$

Thus $d(z,Tx) = \lim_{n\to\infty} d(u_{n+1},Tx) \preceq \lim_{n\to\infty} (rd(u_n,u_{n+1}) + rd(x,Tx)) = rd(x,Tx)$.

Next we show that z is a fixed point of T. Suppose $Tz \neq z$. Then (3.7), (3.8) yield

$$d(z,T^2z) \preceq \frac{r^2}{1-r} d(z,Tz).$$

Case 1: $0 \le r < \frac{1}{1+\sqrt{2K}}$ and $\varphi(r) = 1$.

We note that $\frac{r^2+r}{1-r} < 1$. Then

$$\begin{aligned} d(z,Tz) &\preceq d(z,T^2z) + d(Tz,T^2z) \\ &\preceq \frac{r^2}{1-r} d(z,Tz) + \frac{r}{1-r} d(z,Tz) \\ &= \frac{r^2+r}{1-r} d(z,Tz) \\ &\prec d(z,Tz). \end{aligned}$$

This is a contradiction.

Case 2: $\frac{1}{1+\sqrt{2K}} \le r < \frac{1}{2}$ and $\varphi(r) = \frac{1-r}{1+(K-1)r}$.

From Lemma 3.1.1, we have for each $x, y \in X$,

$$\varphi(r) \|d(x,Tx)\| \le K \|d(x,y)\| \quad \text{or} \quad \varphi(r) \left\|d(Tx,T^2x)\right\| \le K \|d(Tx,y)\|.$$

Then

$$\varphi(r) \|d(u_n,u_{n+1})\| \le K \|d(u_n,z)\| \quad \text{or} \quad \varphi(r) \|d(u_{n+1},u_{n+2})\| \le K \|d(u_{n+1},z)\|.$$

Therefore, we have

$$d(u_{n+1},Tz) \preceq rd(u_n,u_{n+1}) + rd(z,Tz) \quad \text{or} \quad d(u_{n+2},Tz) \preceq rd(u_{n+1},u_{n+2}) + rd(z,Tz).$$

Thus there exists a subsequence (u_{n_j}) of (u_n) such that

$$d(u_{n_j+1},Tz) \preceq rd(u_{n_j},u_{n_j+1}) + rd(z,Tz).$$

Then

$$d(z,Tz) = \lim_{n\to\infty} d(u_{n_j+1},Tz) \preceq \lim_{n\to\infty} \big(rd(u_{n_j},u_{n_j+1}) + rd(z,Tz)\big) = rd(z,Tz).$$

Hence $d(z, Tz) = 0$. So $z = Tz$. This is a contradiction.

Now if w is any fixed point of T, by (3.8) we have

$$d(w, z) = d(Tw, z) \preceq rd(w, Tw) = 0.$$

Hence $w = z$. Therefore z is a unique fixed point of T. □

It is immediate that the following result is obtained from Theorem 3.1.9.

Corollary 3.1.10. *Let P be a normal cone with normal constant K and (X, d) a complete cone metric space. Define a function φ as in Theorem 3.1.9. Let T be a mapping on X satisfying the condition that there exists $r \in \left[0, \frac{1}{2}\right)$ and $n \in \mathbb{N}$ such that for each $x, y \in X$,*

$$\varphi(r) \|d(x, T^n x)\| \leq K \|d(x, y)\| \quad \textit{implies} \quad d(T^n x, T^n y) \preceq rd(x, T^n x) + rd(y, T^n y).$$

Then T has a unique fixed point.

We also establish the existence fixed point result for a generalization of Theorem 2.3.22

Theorem 3.1.11. *Let P be a normal cone with normal constant K and (X, d) a complete cone metric space. Suppose the mapping $T : X \to X$ satisfies condition that there exists $\gamma \in [0, 1)$ such that for each $x, y \in X$,*

$$\|d(Tx, Ty)\| \leq \gamma max\{\|d(x, Tx)\|, \|d(y, Ty)\|\}. \tag{3.9}$$

Then T has a unique fixed point in X.

Proof. Let $u_0 \in X$. Set $u_n = T^n u_0$ for all $n \in \mathbb{N}$. We have

$$\begin{aligned}
\|d(u_{n+1}, u_n)\| &= \|d(Tu_n, Tu_{n-1})\| \\
&\leq \gamma max\{\|d(u_n, Tu_n)\|, \|d(u_{n-1}, Tu_{n-1})\|\} \\
&= \gamma max\{\|d(u_{n+1}, u_n)\|, \|d(u_n, u_{n-1})\|\}
\end{aligned}$$

That is

$$\|d(u_{n+1}, u_n)\| \leq \gamma max\{\|d(u_{n+1}, u_n)\|, \|d(u_n, u_{n-1})\|\}. \tag{3.10}$$

If there exists $n \in N$ such that $max\{\|d(u_{n+1}, u_n)\|, \|d(u_n, u_{n-1})\|\} = \|d(u_{n+1}, u_n)\|$, then from (3.10) we have, $\|d(u_{n+1}, u_n)\| \leq \gamma\|d(u_{n+1}, u_n)\|$, but $\gamma \in [0, 1)$, so we get $d(u_{n+1}, u_n) = 0$. This means $Tu_n = u_{n+1} = u_n$. Hence u_n is a fixed point of T.

If $max\{\|d(u_{n+1}, u_n)\|, \|d(u_n, u_{n-1})\|\} = \|d(u_n, u_{n-1})\|$ for all $n \in \mathbb{N}$, then from (3.10) we have,

$$\begin{aligned}
\|d(u_{n+1}, u_n)\| &\leq \gamma\|d(u_n, u_{n-1})\| \\
&\leq \gamma^2\|d(u_{n-1}, u_{n-2})\| \\
&\leq \ldots \leq \gamma^n\|d(u_1, u_0)\|
\end{aligned}$$

For $n > m$,

$$d(u_n, u_m) \preceq d(u_n, u_{n-1}) + d(u_{n-1}, u_{n-2}) + ... + d(u_{m+1}, u_m).$$

By the normality of cone, we get

$$\begin{aligned}
\|d(u_n, u_m)\| &\leq K\|d(u_n, u_{n-1}) + d(u_{n-1}, u_{n-2}) + ... + d(u_{m+1}, u_m)\| \\
&\leq K\left(\|d(u_n, u_{n-1})\| + \|d(u_{n-1}, u_{n-2})\| + ... + \|d(u_{m+1}, u_m)\|\right) \\
&\leq K\left(\gamma^{n-1} + \gamma^{n-2} + ... + \gamma^m\right)\|d(u_1, u_0)\| \\
&\leq K\left(\frac{\gamma^m}{1-\gamma}\right)\|d(u_1, u_0)\|.
\end{aligned}$$

That is

$$\lim_{n,m\to\infty} d(u_n, u_m) = 0.$$

Hence $\{u_n\}$ is a Cauchy sequence. By completeness of X, there is $z \in X$ such that $\lim_{n\to\infty} u_n = z$. Then, we note that

$$\|d(u_n, Tz)\| = \|d(Tu_{n-1}, Tz)\| \leq \gamma max\{\|d(u_{n-1}, Tu_{n-1})\|, \|d(z, Tz)\|\}. \tag{3.11}$$

So, we consider the following 2 possible cases.

Case 1: $\|d(z, Tz)\| \leq \|d(u_{n-1}, Tu_{n-1})\|$,

from (3.11), we have

$$\|d(u_n, Tz)\| \leq \gamma\|d(u_{n-1}, Tu_{n-1})\| = \gamma\|d(u_{n-1}, u_n)\|,$$

then by Lemma 2.3.16, we get

$$\|d(z, Tz)\| = \lim_{n\to\infty} \|d(u_n, Tz)\| \leq \gamma \lim_{n\to\infty} \|d(u_{n-1}, u_n)\| = 0.$$

That is $d(z, Tz) = 0$. Hence $z = Tz$.

Case 2: $\|d(z, Tz)\| \geq \|d(u_{n-1}, Tu_{n-1})\|$,

from (3.11), we have

$$\|d(u_n, Tz)\| \leq \gamma\|d(z, Tz)\|,$$

then by Lemma 2.3.16, we get

$$\|d(z, Tz)\| = \lim_{n\to\infty} \|d(u_n, Tz)\| \leq \gamma\|d(z, Tz)\|.$$

Since $\gamma \in [0, 1)$, $d(z, Tz) = 0$. This means $z = Tz$.

Now if w is another fixed point of T, then

$$\|d(w, z)\| = \|d(Tw, Tz)\| \leq \gamma max\{\|d(w, Tw)\|, \|d(z, Tz)\|\} = 0.$$

Then $d(w, z) = 0$, that is $w = z$. Hence z is a unique fixed point of T. □

Example 3.1.12. Let $X = \{-2, -1, 0, 1, 2\}$, with the usual metric in $\mathbb{R}$. We note that X is complete. Define a self-mapping T on X by

$$Tx = \begin{cases} 0 & , x \neq 2, \\ -2 & , x = 2. \end{cases}$$

Then T satisfies the condition (3.10) but it is not a Kannan mapping.

The next purpose is to establish a generalized Redenovic's Theorem, which can be proved in a similar way to Theorem 3.1.2.

Theorem 3.1.13. *Let P be a normal cone with normal constant K and (X,d) a complete cone metric space. Define a function θ as in Theorem 3.1.2. Let T be a self-mapping on X satisfying the condition that there exists $r \in [0,1)$ such that for each $x, y \in X$,*

$$\theta(r)\left\|d(x,Tx)\right\| \leq K\left\|d(x,y)\right\| \quad \textit{implies} \quad \left\|d(Tx,Ty)\right\| \leq r\left\|d(x,y)\right\|.$$

Then there exists a unique fixed point z of T. Moreover $\lim\limits_{n\to\infty} T^n x = z$ for all $x \in X$.

3.2 Characterization of Completeness

In this section, we give a necessary and sufficient condition for cone metric spaces to be complete.

Theorem 3.2.1. *Let P be a normal cone with normal constant $K < \sqrt[3]{2}$ and (X,d) a cone metric space and define a function θ as in Theorem 3.1.2. Then the followings are equivalent:*

(i) X is complete.

(ii) For each self-mapping T on X satisfying the condition that there exists $r \in [0,1)$ such that for each $x, y \in X$,

$$\theta(r)\left\|d(x,Tx)\right\| \leq K\left\|d(x,y)\right\| \quad \textit{implies} \quad \left\|d(Tx,Ty)\right\| \leq r\left\|d(x,y)\right\|$$

has a fixed point.

Proof. Note that (i) implies (ii) by Theorem 3.1.13. To show (ii) implies (i), assume (ii). Suppose X is not complete. Then there exists a Cauchy sequence (u_n) which does not converge in X. Define $f : X \to P$ by $f(x) = \lim_{n\to\infty} d(x, u_n)$, $\forall x \in X$. To show that f is well-defined, let $\epsilon > 0$ and choose $c \succ\succ 0$ such that $K^2 \|c\| < \epsilon$. Then there exists $M \in \mathbb{N}$ such that for each $m, n \geq M$,

$$\begin{aligned}
\|d(x, u_n) - d(x, u_m)\| &\leq K \|d(x, u_m) + d(u_n, u_m) - d(x, u_m)\| \\
&= K \|d(u_n, u_m)\| \\
&\leq K^2 \|c\| \\
&< \epsilon.
\end{aligned}$$

Hence $(d(x, u_n))$ is a Cauchy sequence in P, so $d(x, u_n)$ is a convergent sequence, therefore f is well-defined. Note that the followings hold:

1. $f(x) - f(y) \preceq d(x, y) \preceq f(x) + f(y)$.
2. $f(x) \succ 0$.
3. $\lim_{n\to\infty} f(u_n) = 0$.

Define a mapping T on X as follows: For each $x \in X$, since $f(x) > 0$ and $\lim_{n\to\infty} f(u_n) = 0$, there exists a smallest $N \in \mathbb{N}$ such that $\|f(u_N)\| \leq \frac{\theta(r)r}{(3+\theta(r)r)K^4} \|f(x)\|$. We put $Tx = u_N$. Then $\|f(Tx)\| = \|f(u_N)\| \leq \frac{\theta(r)r}{(3+\theta(r)r)K^4} \|f(x)\| < \|f(x)\|$, so for each $x \in X, Tx \neq x$. That is T does not have a fixed point. Next, we will show that for each $x, y \in X$,

$$\theta(r) \|d(x, Tx)\| \leq K \|d(x, y)\| \quad \text{implies} \quad \|d(Tx, Ty)\| \leq r \|d(x, y)\|.$$

Let $x, y \in X$ be such that $\theta(r) \|d(x, Tx)\| \leq K \|d(x, y)\|$.

Case 1: $\|f(y)\| > 2K\|f(x)\|$. Then

$$\begin{aligned}
\|d(Tx,Ty)\| &\le K\left(\|f(Tx)\| + \|f(Ty)\|\right)\\
&\le K\left(\frac{\theta(r)r}{(3+\theta(r)r)\,K^4}\right)\left(\|f(x)\| + \|f(y)\|\right)\\
&\le \frac{r}{3}\|f(x)\| + \frac{r}{3K}\|f(y)\|\\
&\le \frac{r}{3}\|f(x)\| + \frac{r}{3K}\|f(y)\| + \frac{2r}{3K}\left(\|f(y)\| - 2K\|f(x)\|\right)\\
&= \frac{r}{K}\|f(y)\| - r\|f(x)\|\\
&\le r\|d(x,y)\|.
\end{aligned}$$

Case 2: $\|f(y)\| \le 2K\|f(x)\|$. Then

$$\begin{aligned}
K\|d(x,y)\| \ge \theta(r)\|d(x,Tx)\| &\ge \theta(r)\left(\frac{1}{K}\|f(x)\| - \|f(Tx)\|\right)\\
&\ge \theta(r)\left(\frac{1}{K} - \frac{\theta(r)r}{(3+\theta(r)r)K}\right)\|f(x)\|\\
&= \frac{3\theta(r)}{(3+\theta(r)r)K}\|f(x)\|.
\end{aligned}$$

Hence

$$\begin{aligned}
\|d(Tx,Ty)\| &\le K\left(\|f(Tx)\| + \|f(Ty)\|\right)\\
&\le K\left(\frac{\theta(r)r}{(3+\theta(r)r)\,K^4}\right)\|f(x)\| + K\left(\frac{\theta(r)r}{(3+\theta(r)r)\,K^4}\right)\|f(y)\|\\
&\le \frac{\theta(r)r}{(3+\theta(r)r)\,K^2}\|f(x)\| + \frac{\theta(r)r}{(3+\theta(r)r)\,K^3}2K\|f(x)\|\\
&= \frac{3\theta(r)r}{(3+\theta(r)r)\,K^2}\|f(x)\|\\
&\le r\|d(x,y)\|.
\end{aligned}$$

By the hypothesis, T has a fixed point. This is a contradiction. Hence X is complete. □

If the restriction on the normal constant K is omitted and a condition for mapping T is added, then the sufficient part of Theorem 3.2.1 is still true.

Theorem 3.2.2. *Let P be a normal cone with normal constant K and (X,d) a cone metric space and define a function θ as in Theorem 3.1.2. If for each self-mapping T on X satisfying the condition that there exists $r \in [0,1)$ such that for each $x, y \in X$,*

$$|\theta(r)| \, \|d(x,Tx)\| \; \leq \; K \, \|d(x,y)\| \quad \textit{implies} \quad \|d(Tx,Ty)\| \; \leq \; r \, \|d(x,y)\|$$

has a fixed point, then X is complete.

Theorem 3.2.3. *Let P be a normal cone with normal constant K and (X,d) a cone metric space. Then the followings are equivalent:*

(i) *X is complete.*

(ii) *For each self-mapping T on X satisfying the condition that there exists $\gamma \in [0, \frac{1}{2})$ such that for each $x, y \in X$,*

$$\|d(Tx,Ty)\| \leq \gamma max\{\|d(x,Tx)\|, \|d(y,Ty)\|\}$$

has a fixed point.

Proof. It is clear that (i) implies (ii) by Theorem 3.1.11. Next assume (ii) and suppose that X is not complete. Then there exists a Cauchy sequence $\{u_n\}$ which is not convergent in X. Without loss of generality, we may assume that u_n's are distinct for all $n \in \mathbb{N}$.

Let $x \in X$ and define $\alpha_x = inf\{\|d(x,u_n)\| \mid n \in \mathbb{N}\}$. We note that for any $x \notin \{u_n \mid n \in \mathbb{N}\}$, $\alpha_x > 0$. Since (u_n) is a Cauchy sequence, $\lim_{m,n\to\infty} d(u_m,u_n) = 0$. Then there exists a least positive integer N_x such that

$$\|d(u_m,u_n)\| < \gamma\alpha_x \leq \gamma\|d(x,u_l)\|,$$

for each $l \in \mathbb{N}$ and $m, n \geq N_x$. Therefore

$$\|d(u_m,u_{N_x})\| < \gamma\|d(x,u_l)\|, \tag{3.12}$$

for each $l \in \mathbb{N}$ and $m \geq N_x$. Then by a similar reasoning, for each $n \in \mathbb{N}$ there exists a least positive integer $n' > n$ such that

$$\|d(u_m, u_{n'})\| < \gamma \|d(u_n, u_{n'})\|, \tag{3.13}$$

for each $m \geq n'$.

Define $T : X \to X$ by

$$T(x) = \begin{cases} u_{N_x} & , x \notin \{u_n \mid n \in \mathbb{N}\}, \\ u_{n'} & , x \in \{u_n \mid n \in \mathbb{N}\}. \end{cases}$$

It is clearly that T has no fixed point.

To show that $\|d(Tx, Ty)\| \leq \gamma max\{\|d(x, Tx)\|, \|d(y, Ty)\|\}$, we consider the 3 possibilities. Let $x, y \in X$.

Case 1: $x, y \notin \{u_n \mid n \in \mathbb{N}\}$.

Then there exist positve integers N_x and N_y such that $T(x) = u_{N_x}$ and $T(y) = u_{N_y}$. By (3.12) we get,

$$\|d(Tx, Ty)\| < \gamma \|d(y, Ty)\| \quad \text{if} \quad N_x \geq N_y$$

and

$$\|d(Tx, Ty)\| < \gamma \|d(x, Tx)\| \quad \text{if} \quad N_x < N_y.$$

Case 2: $x, y \in \{u_n \mid n \in \mathbb{N}\}$.

Then there exist positve integer m and n such that $x = u_m$ and $y = u_n$. By the definition of T, there exist positive integers $m' > m$ and $n' > n$ such that $T(x) = u_{m'}$ and $T(y) = u_{n'}$. By (3.13) we get,

$$\|d(Tx, Ty)\| < \gamma \|d(y, Ty)\| \quad \text{if} \quad m' \geq n'$$

and

$$\|d(Tx, Ty)\| < \gamma \|d(x, Tx)\| \quad \text{if} \quad m' < n'.$$

Case 3: $x \in \{u_n \mid n \in \mathbb{N}\}$ and $y \notin \{u_n \mid n \in \mathbb{N}\}$.

So $x = u_m$ for some $m \in \mathbb{N}$, then by the definition of T there exist positive integers $m' > m$ and N_y such that $T(x) = u_{m'}$ and $T(y) = u_{N_y}$. By (3.12) we get,

$$\|d(Tx, Ty)\| < \gamma \|d(y, Ty)\| \quad \text{if} \quad m' \geq N_y$$

and by (3.13) we get,

$$\|d(Tx, Ty)\| < \gamma \|d(x, Tx)\| \quad \text{if} \quad m' < N_y.$$

By all of these cases, we have $\|d(Tx, Ty)\| \leq \gamma max\{\|d(x, Tx)\|, \|d(y, Ty)\|\}$, which contradicts to the hypothesis of the theorem. Therefore (X, d) is a complete cone metric space.

□

It is obvious that the following result is immediately obtained from Theorem 3.2.3.

Corollary 3.2.4. *Let P be a normal cone with normal constant K and (X, d) a cone metric space. Then the followings are equivalent:*

(i) X is complete.

(ii) For each self-mapping T on X satisfying the condition that there exists $r \in [0, \frac{1}{2})$ such that for each $x, y \in X$,

$$\|d(Tx, Ty)\| \leq r(\|d(x, Tx)\| + \|d(y, Ty)\|)$$

has a fixed point.

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