# การทำให้ง่ายเชิงขั้นตอนวิธีของทฤษฎีบทที่ให้ผลเฉลยของทรงกลมของไหลสมบูรณ์ ในสัมพัทธภาพทั่วไป

นายพนิต เสือวรรณศรี



วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต

สาขาวิชาคณิตศาสตร์ ภาควิชาคณิตศาสตร์

คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย

ปีการศึกษา 2553

# ALGORITHMIC SIMPLIFICATION OF SOLUTION GENERATING THEOREMS OF PERFECT FLUID SPHERES IN GENERAL RELATIVITY

Mr. Panit Suavansri



A Thesis Submitted in Partial Fulfillment of the Requirements
for the Degree of Master of Science Program in Mathematics
Department of Mathematics
Faculty of Science
Chulalongkorn University
Academic Year 2010

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|---|---|
| Thesis Title | ALGORITHMIC SIMPLIFICATION OF SOLUTION GENERATING THEOREMS OF PERFECT FLUID SPHERES IN GENERAL RELATIVITY |
| By | Mr. Panit Suavansri |
| Field of Study | Mathematics |
| Thesis Advisor | Petarpa Boonserm, Ph.D. |

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Accepted by the Faculty of Science, Chulalongkorn University in Partial Fulfillment of the Requirements for the Master's Degree

........................................................ Dean of the Faculty of Science

(Professor Supot Hannongbua, Dr.rer.nat.)

THESIS COMMITTEE

........................................................ Chairman

(Associate Professor Paisan Nakmahachalasint, Ph.D.)

........................................................ Thesis Advisor

(Petarpa Boonserm, Ph.D.)

........................................................ Examiner

(Assistant Professor Nataphan Kitisin, Ph.D.)

........................................................ External Examiner

(Khamphee Karwan, Ph.D.)

พนิต เสือวรรณศรี : การทำให้ง่ายเชิงขั้นตอนวิธีของทฤษฎีบทที่ให้ผลเฉลยของทรงกลมของไหลสมบูรณ์ในสัมพัทธภาพทั่วไป. (ALGORITHMIC SIMPLIFICATION OF SOLUTION GENERATING THEOREMS OF PERFECT FLUID SPHERES IN GENERAL RELATIVITY) อ. ที่ปรึกษาวิทยานิพนธ์หลัก : ดร. เพชรอาภา บุญเสริม, 109 หน้า.

ทฤษฎีบทการแปลง คือ ตัวส่งซึ่งสร้างทรงกลมของไหลสมบูรณ์ต้นแบบไปยังทรงกลมของไหลสมบูรณ์ใหม่ มีหลายทางหรือกระบวนวิธีที่เป็นทฤษฎีบทการแปลงเพื่อที่จะได้ผลเฉลยแน่นอนโดยการแก้สมการอนุพันธ์จำเพาะจากสมการไอน์สไตน์ เป้าหมายในงานวิจัยนี้คือทำอย่างไรที่จะหาทางหรือกระบวนวิธีที่ดีที่สุดที่จะได้ผลเฉลยแน่นอนใหม่ในรูปแบบที่ทำให้ง่ายแล้ว เช่น มีจำนวนพจน์ในการคำนวณไม่มาก สร้างผลเฉลยได้ง่ายและซับซ้อนน้อยกว่าเดิม นอกจากนี้เรายังได้สร้างโปรแกรมย่อยของโปรแกรมคำนวณทางคณิตศาสตร์ เรียกว่า แมพเพลต ในโปรแกรมเมเปิล โปรแกรมย่อยนี้จะช่วยลดขั้นตอนการทำงาน เพิ่มความแม่นยำและง่ายต่อการหาผลเฉลยแน่นอนที่ได้จากทฤษฎีบทการแปลงก่อนหน้านี้แทนการคำนวณด้วยมือ



ภาควิชา............คณิตศาสตร์........ลายมือชื่อนิสิต..........................................
สาขาวิชา..........คณิตศาสตร์........ลายมือชื่ออ.ที่ปรึกษาวิทยานิพนธ์หลัก..........................
ปีการศึกษา.............2553.............

# # 5272437023 : MAJOR MATHEMATICS
KEYWORDS : PERFECT FLUID SPHERES / GENERAL RELATIVITY / TRANSFORMATION / ALGORITHMIC SIMPLIFICATION / MATHEMATICAL PROGRAM

PANIT SUAVANSRI: ALGORITHMIC SIMPLIFICATION OF SOLUTION GENERATING THEOREMS OF PERFECT FLUID SPHERES IN GENERAL RELATIVITY. ADVISOR: PETARPA BOONSERM, Ph.D., 109 pp.

Transformation theorems are the maps that construct the original perfect fluid spheres into new ones. There are many ways or algorithms that are transformation theorems to derive the exact solutions by solving specific differential equations from Einstein's equations. The object of this research is about how to find the best way to derive the new exact solution in the simplified forms such as not many terms in calculation, easy for construct and less complex than the original transformation theorem. Furthermore, we construct the subprogram of mathematical program called maplet in Maple. This subprogram will help us to reduce methods in algorithm, raise our accuracy and make easier to derive new exact solutions from the original transformation theorems instead of calculating by hand.



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|---|---|
| Department : Mathematics | Student's Signature PANIT SUAVANSRI |
| Field of Study : Mathematics | Advisor's Signature B. Petarpa |
| Academic Year : 2010 | |

# ACKNOWLEDGEMENTS

I would like to thank Dr. Petarpa Boonserm, my supervisor, for supporting me along time. My supervisor works industriously and seriously encourages me to do this thesis completely and finish on time.

I also gratefully thank Associate Professor Dr. Paisan Nakmahachalasint, Assistant Professor Dr. Nataphan Kitisin and Dr. Khamphee Karwan being my committees and giving all comments for my thesis.

I am very grateful to study in mathematics and thanks to Chulalongkorn university for 1st year scholarship.

Finally, I hope that this thesis will be useful for mathematicians, physicists , engineers, students, etc.

I very gratefully thank Professor Dr. Matt Visser and Dr. Narit Pidokrajt for his many advices, new points, and a lot of comments, which make me succeed this thesis and research. I am grateful in all their ways and hope that we will continue our work together in the future.

Specially, thanks to Kanokwan Thairatana, without her, I would not know a lot of news such as the place and the date to test my proposal and thesis.

Finally, I am very grateful to my mother and my father supporting me both my intention and my mind. If there are no both of them, I will not succeed.

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# CHAPTER I
# INTRODUCTION

## 1.1 Special relativity

This chapter is about "special relativity", a branch of theory of relativity. It is necessary for us to learn this relativity before we study general relativity. The reason is that special relativity can be described by simple equations so that we can visualize real-life situations using equations of Galilean and Lorentz transformations. Later on we will generalize special relativity to general relativity. In this chapter, we will discuss the content of special relativity such as transformations, connections between special relativity, and general relativity.

### 1.1.1 Background

Special relativity was established in 1905 by Albert Einstein in his journal about electrodynamics. About 300 years ago, Galileo stated that all motions with constant velocity were relative, and that their absolute static status could not be defined. For example, person A, who is on a boat thinks that he is at rest on the boat, however person B, who is standing on the beach, perceives person A as moving. Einstein's theory consists of Galilean relativity and hypothesis that all observers estimate the velocity of light to be about the same no matter what motion they are in as long as they have constant velocity [1–5].

Newton believed that objects had dimension, and mass. These can be observed in daily life. However, Einstein stated that it was not so since dimension, and mass were related depending on their velocity. More precisely, he said that if they moved as fast as the velocity of light, then the new and original time, length and mass would be different to observers. In this chapter, we will discuss concepts of time dilation, length contraction, and mass increasing.

Indeed, since we can never move as fast as the velocity of light, we therefore see that time, dimension and mass remain as they are [1–5]. Special relativity is just the special case of general relativity used in the situation for any observer that does not move with acceleration. That is, when they move with constant velocity. On the other hand, general relativity is used for describing accelerated motions, and objects in the gravitational field [1–5].

### 1.1.2 Frames of reference

In order to study the motion of objects, we have to specify their location at each instant. However, to identify the object's location, we have to compare it with a reference point. We can define the reference point to be the origin point in any coordinate system. Moreover, we have to identify time $t$ of the object at a given location [1–5].

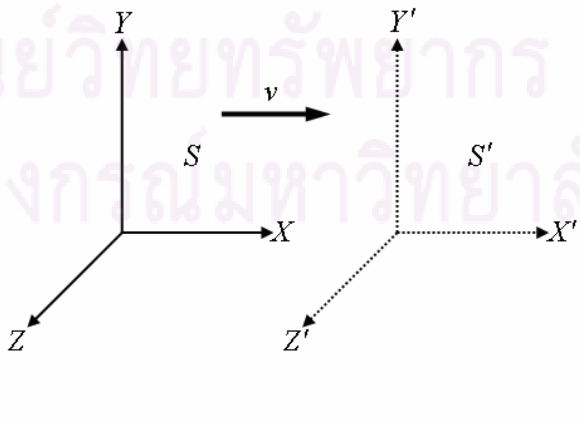


Figure 1.1: Example of two reference frames $S$ and $S'$ moving with velocity $\nu$ [4].

### 1.1.3 Inertial frame

The law of inertia states that

1. If an object is stationary, it will remain static unless there is an external forces applied upon it.

2. If an object moves with constant velocity, it will remain in such a motion unless an external force is applied upon it (see also [1]).

Assuming that there is a frame that satisfies these two conditions, we call it **inertial frame**. There can be more than one inertial frame. Hence, any frame moving with constant velocity is an inertial frame [1].

A frame that is not inertial and accelerated is called a non-inertial frame. Inertial frames are important for the physical rules such as Newtonian movement, conservation of momentum, etc [1, 2, 5].

### 1.1.4 Postulates of special relativity

1. In every inertial frame, the law of physics always remains the same.

2. The speed of light in vacuum, $c = 3 \times 10^8$ m/s, is always the same in all inertial frames which do not depend on the observers' velocity or the velocity of light sources (see also [1–3, 5]).

### 1.1.5 Concepts of special relativity

1. **Time dilation**: Time interval for a given event depends on the velocity of observers in a reference frame. For example, consider the problem of twin paradox. When the first twin went up into space with speed near the speed

of light and came back to earth, we would find that the second twin older than the first one.

2. **Simultaneity of events**: Two exact same events which happen in different frames under the same period of time may appear to be different for a given observer.

3. **Length (Lorentz) contraction**: Dimension (e.g. length) of the object which is measured by one observer may appear to be different for another observer. For example, in case of ladder paradox, a ladder moving at the speed of light will appear to be shorter than the original one.

4. **Combination of velocities**: The velocities and speeds cannot always be combined directly.

5. **Inertia and momentum**: when the object's speed is close to the speed of light, its mass will be greater.

6. **Equivalence of mass and energy**: Mass and energy can be transformed into each other. For example, gravitational force of apple falling can be divided into mass and kinetic energy (see also [1–3, 5]).

## 1.2 Newtonian relativity

Newton's laws are applicable to objects moving in all inertial frames. This inertial frame is the coordinates' axis or the reference frame with zero acceleration. Other reference frames with constant relative velocity are also inertial frames. For example, all experiments conducted on constant-velocity vehicle yield the same outcome as those conducted on zero-velocity or static vehicle [1–3, 5].

### 1.2.1 Galilean transformations

Galilean transformations deal with the reference frames moving with constant-velocity and the rest reference frames. This type of transformation applies only to non-relativistic situations. Moreover, the length with the observer in the measurable reference frame with constant-velocity is equal to the length with the observer in the measurable reference frame at rest ($L' = L$).

Consider figure 1.2

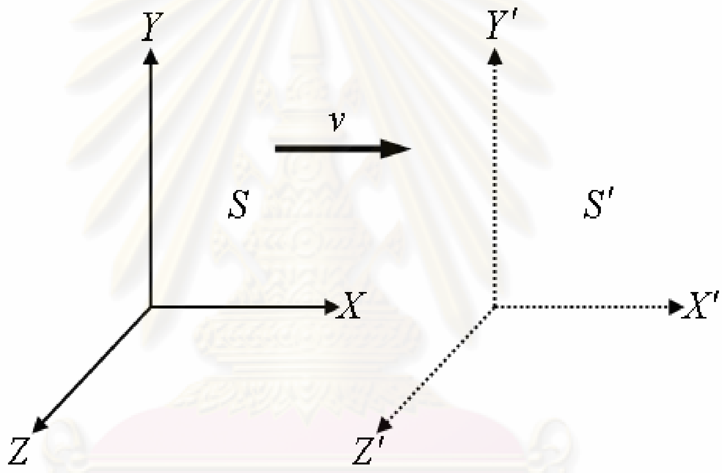


Figure 1.2: A reference frame $S$ or $(t, x, y, z)$ of the observer and the frame $S'$ or $(t', x', y', z')$ moving with its velocity $\nu$ [6].

When the reference frame $(t, x, y, z)$ of the observer noting the frame $(t', x', y', z')$

moving with its velocity $v$, we derive

$$x = x' + vt', \tag{1.1}$$

$$y = y', \tag{1.2}$$

$$z = z', \tag{1.3}$$

$$t = t'. \tag{1.4}$$

So the differences of dimension and time are

$$\Delta x' = \Delta x - v\Delta t, \tag{1.5}$$

$$\Delta y' = \Delta y, \tag{1.6}$$

$$\Delta z' = \Delta z, \tag{1.7}$$

$$\Delta t' = \Delta t. \tag{1.8}$$

Thus, velocities associated with all axes are given by

$$\frac{\mathrm{d}x'}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t} - v, \tag{1.9}$$

$$\frac{\mathrm{d}y'}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}t}, \tag{1.10}$$

$$\frac{\mathrm{d}z'}{\mathrm{d}t} = \frac{\mathrm{d}z}{\mathrm{d}t}, \tag{1.11}$$

i.e.



$$u_{x'} = u_x - v, \tag{1.12}$$

$$u_{y'} = u_y, \tag{1.13}$$

$$u_{z'} = u_z, \tag{1.14}$$

And their accelerations are

$$a_{x'} = a_x, \tag{1.15}$$

$$a_{y'} = a_y, \tag{1.16}$$

$$a_{z'} = a_z. \tag{1.17}$$

Similarly, from 2nd law of Newton, we obtain

$$F_{x'} = m\frac{\mathrm{d}^2x'}{\mathrm{d}t^2} = m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = F_x, \tag{1.18}$$

$$F_{y'} = m\frac{\mathrm{d}^2y'}{\mathrm{d}t^2} = m\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = F_y, \tag{1.19}$$

$$F_{z'} = m\frac{\mathrm{d}^2z'}{\mathrm{d}t^2} = m\frac{\mathrm{d}^2z}{\mathrm{d}t^2} = F_z. \tag{1.20}$$

Hence, the basic laws of physics such as laws of motion, conservation laws of momentum and energy will be invariant for two systems in different frames. That means the laws of physics or the form of equations are the same in both systems although they are at rest or moving with constant velocity. Galilean transformations is only applicable for systems moving with velocity far less than the velocity of light ($v \ll c$).

For all objects moving with velocity close to the velocity of light, it is necessary to use Einstein's relativity to describe their motions [1–3, 5].

### 1.2.2 Lorentz transformations

Lorentz transformations must be used instead of Galilean transformations when the object moves near the speed of light.

Once we know the coordinates of the event in the reference frame, we can obtain the coordinates in another reference frame by using Lorentz transformations and postulates of special relativity as follows:

In Galilean transformations

$$x = x' + vt', \tag{1.21}$$

$$y = y', \tag{1.22}$$

$$z = z', \tag{1.23}$$

$$t = t'. \tag{1.24}$$

The first postulate of special relativity states that "for any inertial frames, the laws or equations of physics are invariant." Thus, we can derive the following equations

$$x = x' + vt', \tag{1.25}$$

$$x' = \gamma(x - vt), \tag{1.26}$$

$$x = \gamma(x' + vt'), \tag{1.27}$$

$$y = y', \tag{1.28}$$

$$z = z', \tag{1.29}$$

If we put $x'$ from (1.27) in (1.26), then we derive

$$x = \gamma\left[\gamma\left(x - vt\right) + vt'\right], \tag{1.30}$$

$$t' = \gamma t + \left(\frac{1-\gamma^2}{\gamma v}\right)x. \tag{1.31}$$

The second postulate of special relativity implies that "the speed of light in vacuum is equal in every inertial frame and does not depend on the observer's velocity or the light source's velocity". Hence, this postulate allows the frame $(t, x, y, z)$ and the frame $(t', x', y', z')$ to have the same velocity of light. That is

$$x = ct, \tag{1.32}$$

$$x' = ct'. \tag{1.33}$$

Substituting $x'$ from (1.27) and $t'$ from (1.31), then we derive

$$\gamma\left(x - vt\right) = c\gamma t + \left(\frac{1-\gamma^2}{\gamma v}\right)cx, \tag{1.34}$$

$$x = ct\left[\frac{1+\frac{v}{c}}{1-(\frac{1}{\gamma^2}-1)\frac{c}{v}}\right]. \tag{1.35}$$

Using (1.32) and (1.36), we derive

$$\frac{1+\frac{v}{c}}{1-(\frac{1}{\gamma^2}-1)\frac{c}{v}} = 1. \tag{1.36}$$

Thus we derive the parameter $\gamma$ called "Lorentz factor" which is

$$\gamma = \frac{1}{\sqrt{1-(\frac{v^2}{c^2})}}, \tag{1.37}$$

And Lorentz transformations read

$$x= \frac{x' + vt'}{\sqrt{1-(v^2/c^2)}}, \tag{1.38}$$

$$y= y', \tag{1.39}$$

$$z= z', \tag{1.40}$$

$$t= \frac{t' + (vx'^2)}{\sqrt{1-(v^2/c^2)}}. \tag{1.41}$$

(See also [6–8])

**Time dilation**

The verification of time dilation was done by using "atomic clock" made from Cesium-133 heated in control stove which makes it run through the magnetic atoms enabled to absorb the Cesium-133 energy. This experiment was performed by Britain's National Physical Laboratory in 1955. When magnetic atoms pass through the microwave, they will release its energy with frequency at 9,192,631,770 Hz. In 1971, Haefelle and Keating used Helium atomic clock in the jet and let the plane fly around the earth about 45 hours and then they compared it with the atomic clock on earth. After they ignored some errors, they found that both atomic clocks showed different time. The atomic clock in the jet ran slower than the one on the earth which confirmed the prediction of the theory of relativity [1–5].

**Calculation for proper time**

For theory of relativity, the proper time can be calculated for each frame by substituting the time change. That is

$$T' = t_2' - t_1', \tag{1.42}$$

$$T = t_2 - t_1. \tag{1.43}$$

In order to calculate proper time $T'$, we will put $T$ and $v$ into the equation (1.44)

$$T' = \frac{T}{\sqrt{1 - (v^2/c^2)}}. \tag{1.44}$$

So we derive $T'$ from $T$ [5–7].

**Length contraction**


Figure 1.3: Length contraction of an object from a low velocity to a very high velocity.

Figure 1.3 illustrates that when an object moves with its velocity near the speed of light, its length is shorter and its mass increases which makes it heavier and heavier. As a result, it will never be able to move at the speed of light. Length contraction is also a consequence of the theory of relativity. When the object moves with relativistic speed near the speed of light, its length is shorter and its mass increases which makes it heavier and thus it cannot move at the speed of light.

To calculate the proper length, we use length contraction formulae together with the Lorentz transformations. That is we substitute

$$L = x_2 - x_1, \tag{1.45}$$

$$L' = x_2' - x_1', \tag{1.46}$$

into

$$L' = L\sqrt{1 - (v^2/c^2)}, \tag{1.47}$$

to obtain the length contraction of an object of interest [1, 2].

### 1.2.3 Minkowski space

#### Minkowski's diagram

To understand Minkowski's space we will use the system $S$ of graph presented by H. Minkowski in 1908. The coordinates consist of $(ct, x)$, where $ct$ is time coordinate and $x$ is 3-space coordinate. And we put $ct$ on the vertical axis whereas $x$ on the horizontal axis. Worldline, which is the trajectory in the spacetime is the series of events. We define the lightcone diagram to be $s^2 \equiv c^2t^2 - x^2 = 0$ and divide spacetime into three parts, past, now, future and elsewhere [3, 5].

#### Moving Coordinate Systems

Consider the body located at a point $(x', t')$ moving with a velocity $v$ to the right-hand side relative to the frame $(x, t)$, we derive $x' = 0$ iff $x = vt$ as shown in figure 1.4.

Consider two frames $(x, t)$ and $(x', t')$, we can see the spacetime diagram as shown in figure 1.5.

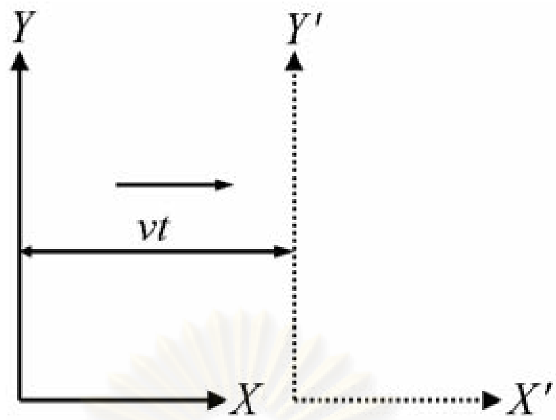


Figure 1.4: The movement of the frame $(x', t')$ with a velocity $v$ in the $x$-direction relative to the frame $(x, t)$ [6].

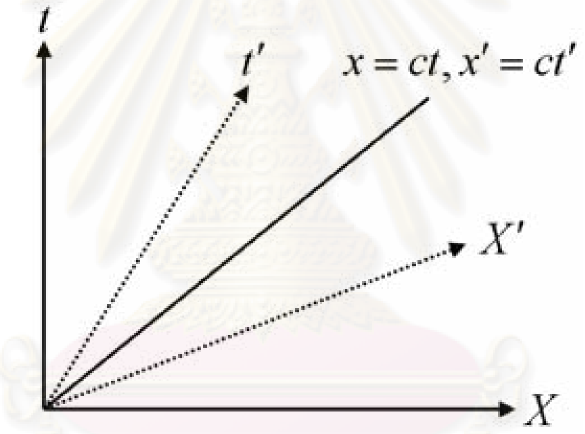


Figure 1.5: Graph of cone-shape, lightcone, in first quadrant [5].



## 1.3 General relativity

In the previous chapter, we learnt about special relativity. In this chapter, we will be dealing with general relativity which also constitute the concepts of special relativity. This subject is about the theory of gravitation built upon differential geometry established by Einstein. Equipped with knowledge about general relativity, we will be able to understand the next chapter which is the main part of this thesis.

General relativity is based on the principle of equivalence. That is, in the reference frame, freely falling object in gravitational field can be compared with the inertial reference frame without gravitation or its acceleration is equal to the result of gravitation [1–5].

## 1.4 Concepts and tools in general relativity

In this section, we present mathematical tools and notations used in general relativity from coordinate transformation to curvature calculation to stress-energy tensor, geodesic equations, covariant and contravariant derivatives, Christoffel's symbol, and Einstein's tensor.

**Space**

There are many types of space such as

1. **Euclidean space** is the $n$-dimension space which contains Euclidean or flat plane altogether to grid form with $n$-dimension called Euclidean geometry.

2. **Minkowski spacetime** is a set which is used in theory of relativity. This space composes of 3 space dimensions and one time $t$. For example, in Cartesian coordinates, the Minkowski spacetime is given by

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = diag\left(-1, 1, 1, 1\right) \tag{1.48}$$

3. **Manifold** is essentially a mathematical space. It is a set of points which can locally be mapped into $\mathbb{R}^n$, where $n$ is the dimension of the manifold. For example, real line $\mathbb{R}^1$ is one dimensional manifold. Euclidean space $\mathbb{R}^3$ is

3-dimensional manifold. Minkowski space $\mathbb{R}^{3+1}$ is four dimensional manifold [10, 11].

### Worldline

Worldline is the trajectory in spacetime which is a parametric curve depending on $t$. In special relativity, worldline is the path that traces out spacetime. The worldline can be written as $C = C(t)$ that tells us about the point or location in the spacetime whether it is an event or a series of events [3, 4, 10].

### Light cone

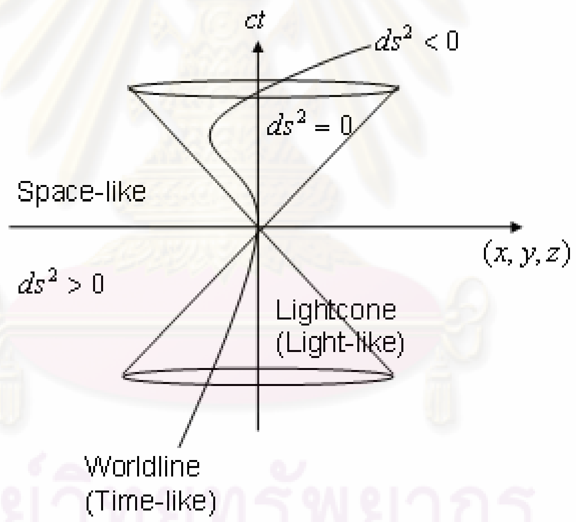


Figure 1.6: Lightcone and worldline [3].

Hence, the object's worldline is the unique path that indicates the movement of object in spacetime. From figure 1.6, we see an example of a 3D light cone which tells us about the motion of an object in time. In general, spacetime can be classified into three types as follows:

1. **Light-like**:consists of points or locations and that their velocity are equal

to the light's velocity and all of them are formed into cone-shape classified into two parts, future and past.

2. **Time-like**:consists of points or locations and that their velocities are less than the light's velocity. All objects except light itself move in the time-like region of spacetime.

3. **Space-like**:describes the region outside the cone. There is no physical object that can travel outside the lightcone. There is an imaginary object that can travel in the spacetime region known as Tachyon which is a type of theoretical particle and moves faster than the speed of light. The word "tachyon" was named by Gerald Feinberg in the 1960s [4, 10, 12].

In the next part, we discuss mathematical tools of general relativity.

## Geodesic

In mathematics, geodesic is defined by the shortest path between two points in space, in metric form. In affine connection, geodesic is defined by a curve whose tangent vector is still parallel when it is transported along this curve [10].

The word "geodesic" comes from "geodesy" which means the science of measuring the earth's size and shape. In navigation, geodesic is the shortest route between two points on the surface of the earth or the arch length of great circle. For example, geodesic in figure 1.15 is the arch between two vertexes. Worldline of free particles without external forces is an example of geodesic i.e. particles move freely along geodesic.

In special relativity, geodesic on Lorentz manifold can be classified into three categories by norm of tangent vector, $\mathrm{d}s^2$, using metric $(-+++)$ as follows:

1. **Time-like geodesic** has a tangent vector whose norm is negative.

2. **Null-geodesic** has a tangent vector whose norm is zero.

3. **Space-like geodesic** has a tangent vector whose norm is positive [10].

**Tangent vector**

Tangent vector is a vector which has its direction on curve or surface at a point. For example, $\frac{\partial}{\partial x^\alpha}$ is a tangent vector on some manifold [13, 14].

### 1.4.1 Tensor

Tensor is the transformation of coordinates from a space to another and is non-singular [4]. There are three types as follows:

1. Contravariant tensor has only an upper index such as velocity $U^\mu$ or acceleration $a^\mu$.

2. Covariant tensor has only lower index such as metric tensor $g_{\mu\nu}$.

3. Mixed tensor has both upper and lower indexes such as Riemann curvature tensor $R^\mu_{\nu\alpha\beta}$ [4].

### 1.4.2 Rank of tensor

Rank of tensor is a combination of both upper and lower indices of tensor such as $A^{\beta_1 \ldots \beta_s}_{\beta_{s+1} \ldots \beta_N}$ has $N^{th} rank$ [4].

### 1.4.3 Contraction

Contraction is the operation of both upper and lower indexes which are the same symbol confute such as $g_\mu g^{\mu\nu} = g^\nu$ [4].

### 1.4.4 Christoffel symbols

Christoffel symbols, established by Elwin Bruno Christoffel (1829–1900), are numerical arrays such as matrix implying location and using the result of parallel transportation in curved surface [4]. These Christoffel symbols tell us about the coordinates of Levi-Civita's continuity space which derives from a metric tensor. Christoffel symbols may be used to calculate in differential geometry. For example, Riemann curvature tensor can be written in terms of Christoffel symbols and first partial differentiation. If $(x_i), i = 1, 2, ..., n$ is the local specific coordinate in manifold $M$, then tangent vectors $e_i = \frac{\partial}{\partial x^i}, i = 1, 2, \ldots, n$ defined to be basis of tangent space in $M$. Thus we derive Christoffel symbols $\Gamma^k_{ij}$, which satisfies $\nabla_i e_j = \Gamma^k_{ij} e_k$, where $\nabla_i$ Levi-Civita continuity on $M$ [13, 14]. Thus if we write Christoffel symbols $\Gamma^k_{ij}$ in terms of metric tensor $g_{ik}$, then we derive

$$0 = \nabla_\ell g_{ik} = \frac{\partial g_{ik}}{\partial x^\ell} - g_{mk}\Gamma^m_{i\ell} - g_{im}\Gamma^m_{k\ell} = \frac{\partial g_{ik}}{\partial x^\ell} - 2g^m_{m[k}\Gamma^m_{i]l}. \tag{1.49}$$

These nabla $\nabla$ and partial differential $\partial$ are written in lower position with ; and ,. Thus the equation (1.49) can be arranged to the new form

$$0 = g_{ik;\ell} = g_{ik,\ell} - g_{mk}\Gamma^m_{i\ell} - g_{im}\Gamma^m_{k\ell}. \tag{1.50}$$

By permutation of index and rearrangement, we derive this equation below for finding Christoffel symbols in the form of metric tensors. That is

$$\Gamma^i_{k\ell} = \frac{1}{2} g^{im}\left(\frac{\partial g_{mk}}{\partial x^\ell} + \frac{\partial g_{m\ell}}{\partial x^k} - \frac{\partial g_{k\ell}}{\partial x^m}\right) = \frac{1}{2} g^{im}(g_{mk,\ell} + g_{m\ell,k} - g_{k\ell,m}) \tag{1.51}$$

(See details in [13, 14]).

## 1.4.5 Curvature

### Definition

1. Curvature refers to any number of loosely related concepts in different areas of geometry.

2. Curvature is the amount by which a geometric object deviates from being flat, or straight in the case of a line, but this is defined in different ways depending on the context.

3. The (signed) curvature of a curve parameterized by its arc length is the rate of change of direction of the tangent vector.

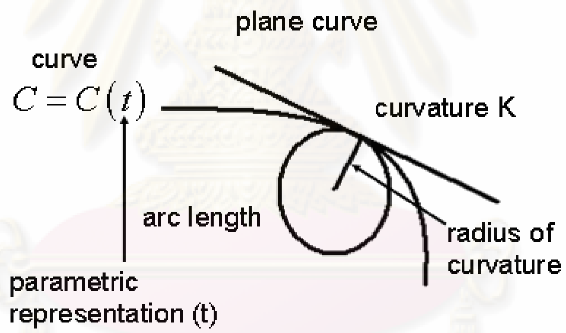


Figure 1.7: The components of curve $C = C\,(t)$ [15–17].

## 1.4.6 Calculating

To introduce the definition of curvature, in this section we consider that $\alpha(s)$ is a unit-speed curve, where $s$ is the arc length. The **tangential angle** $\varphi$ is measured counterclockwise from the x-axis to the unit tangent $T = \alpha(s)$, as shown in figure 2.1.

The **curvature** $\kappa$ of curve $\alpha$ is the rate of change of direction at that point of the tangent line with respect to arc length, that is,

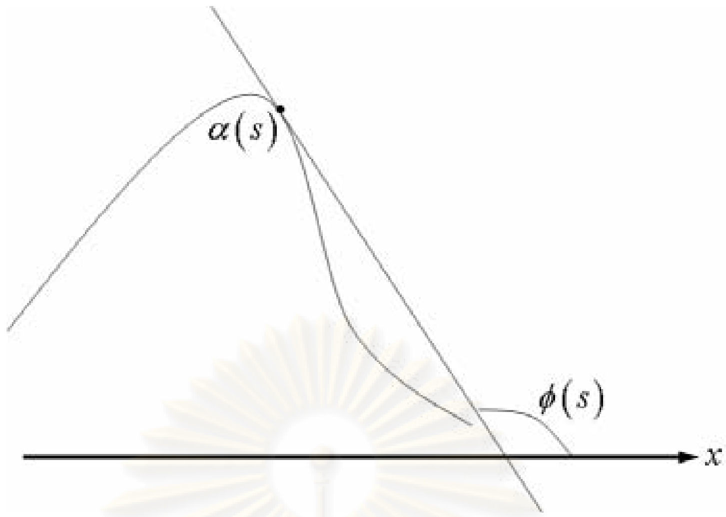


Figure 1.8: Curvature and tangential angle

$$\kappa = \frac{d\varphi}{ds}. \quad (1.52)$$

The **absolute curvature** of the curve at the point is the absolute value $|\kappa|$. Since $\alpha$ has unit speed, $T \cdot T = 1$. Differentiating this equation yields

$$T' \cdot T = 0. \quad (1.53)$$

The change of $T(s)$ is orthogonal to the tangential direction, so it must be along the normal direction. The curvature is also defined to measure the turning of $T(s)$ along the direction of the unit normal $N(s)$ where $T(s) \times N(s) = 1$. That is,

$$T' = \frac{dT}{ds} = \kappa N. \quad (1.54a)$$

We can easily derive one of the curvature definitions (1.51) and (1.54a) from the other. For instance, if we start (1.51) with (1.54a), then

$$
\begin{aligned}
\kappa &= T' \cdot N, &(1.55)\\
&= \frac{\mathrm{d}T}{\mathrm{d}s} \cdot N, &(1.56)\\
&= \lim_{\triangle s \to 0} \frac{T(s + \triangle s) - T(s)}{\triangle s} \cdot N, &(1.57)\\
&= \lim_{\triangle s \to 0} \frac{\triangle \varphi \cdot \|T\|}{\triangle s}, &(1.58)\\
&= \lim_{\triangle s \to 0} \frac{\triangle \varphi}{\triangle s}, &(1.59)\\
&= \frac{\mathrm{d}\varphi}{\mathrm{d}s}. &(1.60)
\end{aligned}
$$

**Types of curvature $\kappa$ by its sign**

1. **Positive** with $\kappa > 0$.

2. **Negative** with $\kappa < 0$.

3. **Spatial flat** with $\kappa = 0$.

The curvature affects the curve that bends or flexes the curve as shown in figure 1.9 (see also [15–17]).

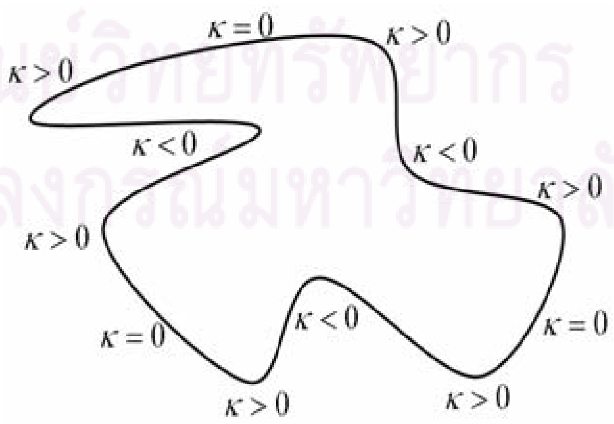


Figure 1.9: Types of curvature affect a curve [15–17].

### 1.4.7 Riemann curvature tensor

Before we learn about the Riemann curvature tensor, we shall consider the parallel transportation of vector around closed curve in figure 1.10.

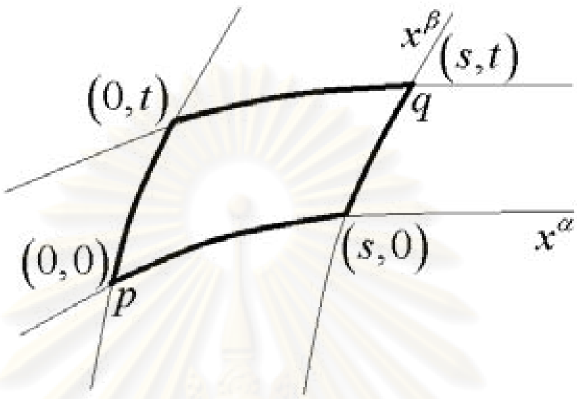


Figure 1.10: Paths from point $p$ to point $q$ [13].

We need to know the first route, from point $p = (0,0) \to (0,t) \to (s,t) = q$ as follows:

Firstly, consider $(0,0) \to (0,t)$ transferring the vector $V$, then we derive

$$V^\rho|_p \to V^\rho|_p - \Gamma^\rho_{\beta\sigma}|_p V^\sigma|_p \left.\frac{\mathrm{d}x^\beta}{\mathrm{d}t}\right|_p t + O\left(t^2\right) \equiv V^\rho\big|_{(0,t)} . \tag{1.61}$$

Next, consider $(0,t) \to (s,t)$ transferring the vector $V^\rho\big|_{(0,t)}$, then we derive

$$V^\rho\big|_{(0,t)} \to V^\rho\big|_{(0,t)} - \Gamma^\rho_{\alpha\sigma}\big|_{(0,t)} V^\sigma\big|_{(0,t)} \left.\frac{\mathrm{d}x^\alpha}{\mathrm{d}s}\right|_{(0,t)} .s + O\left(s^2\right) \equiv V^\rho\big|_{(s,t)} . \tag{1.62}$$

Similarly for the second route, consider point $p = (0,0) \to (s,0) \to (s,t) = q$, which is simply the permutation of $\alpha \leftrightarrow \beta$. Let vector from parallel transportation in the second route be $V'^\rho\big|_{(s,t)}$ against $V^\rho\big|_{(s,t)}$. Then we derive

$$V'^{\rho}\big|_{(s,t)} - V^{\rho}\big|_{(s,t)} = -\left\{\partial_\alpha \Gamma^{\rho}_{\beta\gamma} - \partial_\beta \Gamma^{\rho}_{\alpha\gamma} + \Gamma^{\rho}_{\alpha\sigma}\Gamma^{\sigma}_{\beta\gamma} - \Gamma^{\rho}_{\beta\sigma}\Gamma^{\sigma}_{\alpha\gamma}\right\} V^{\gamma}\big|_p \left.\frac{\mathrm{d}x^\alpha}{\mathrm{d}t}\right|_p \left.\frac{\mathrm{d}x^\beta}{\mathrm{d}s}\right|_p st + \ldots . \tag{1.63}$$

So we will define Riemann curvature tensor to be

$$R^{\rho}_{\gamma\alpha\beta} \equiv \partial_\alpha \Gamma^{\rho}_{\beta\gamma} - \partial_\beta \Gamma^{\rho}_{\alpha\gamma} + \Gamma^{\rho}_{\alpha\sigma}\Gamma^{\sigma}_{\beta\gamma} - \Gamma^{\rho}_{\beta\sigma}\Gamma^{\sigma}_{\alpha\gamma} \tag{1.64}$$

(See details in [13, 14]).

**Properties of Riemann curvature tensor**

1. $R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} = -R_{\alpha\beta\delta\gamma} = R_{\gamma\delta\alpha\beta}$ (**skew or anti-symmetry and two-pair symmetry**).

2. $R_{\alpha\beta\gamma\delta} + R_{\beta\delta\alpha\gamma} + R_{\alpha\delta\beta\gamma} = 0$ (**cyclic permutation sum or $1^{st}$ Bianchi identity**).

Hence, Riemann curvature tensor is calculated from Christoffel symbols ([13, 14]). Next, we will use this tensor to find another tensor, called Ricci curvature tensor.



### 1.4.8 Ricci curvature tensor

In differential geometry, Ricci curvature tensor, Ricci scalar or scalar curvature, named by Gregorio Ricci-Curbastro, means the amount which implies deviation of the particle's volume from Euclidean space to Riemann manifold. This method is a measurement of curvature or degree of geometry by Riemann curvature tensor or Riemann metric, which is used in Euclidean space and Riemann manifold [4, 13]. However, Ricci tensor is often used in pseudo-Riemannian manifold which is Lorentz manifold.

**Ricci tensor**

Ricci tensor is also a curvature in Riemann manifold defined by

$$R_{\sigma\nu} = R^{\rho}_{\sigma\rho\nu} = \Gamma^{\rho}_{\nu\sigma,\rho} - \Gamma^{\rho}_{\rho\sigma,\nu} + \Gamma^{\rho}_{\rho\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\rho\sigma} = 2\Gamma^{\rho}_{\sigma[\nu,\rho]} + 2\Gamma^{\rho}_{\lambda[\rho}\Gamma^{\lambda}_{\nu]\sigma}. \quad (1.65)$$

It is also a Riemann curvature tensor with simple contractions over two indices [4, 13, 14].

**Ricci scalar**

In Riemann manifold, Ricci scalar or scalar curvature is a simplest measurement of curvature. Define

$$R \equiv g^{\alpha\beta} R_{\alpha\beta} = R^{\alpha}_{\alpha}. \quad (1.66)$$

Ricci scalar is important in calculating **Einstein's tensor** [4, 10, 13, 14].

**4-vector**

Define 4-vector in spacetime or Minkowski space with four coordinates as shown below.

$$\mathrm{x} = x^{\mu} = \left(x^0, x^i\right) = \left(x^0, x^1, x^2, x^3\right) = (ct, x, y, z) = (t, x, y, z)\,, \quad (1.67)$$

where $\mu$ is the Greek letter meaning $\mu = 0, 1, 2, 3$ and $\vec{x} = x^i = (x, y, z)$ is the coordinates in Cartesian. Generally, we will use the natural unit, that is $c = 1$. Moreover, we can write this 4-vector in the linear algebraic form which is Cartesian coordinates as shown below.

$$\mathrm{x} = x^{\mu} = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}. \tag{1.68}$$

Thus the reason that $\mathrm{x} = x^{\mu} = (x^0, x^1, x^2, x^3)$ in the coordinates of spacetime is that it consists of the space $x^i = \vec{x} = (x^1, x^2, x^3)$ and time $x^0 = ct$. That is $\left(x^0, \vec{x}\right)$. This vector $\left(x^0, \vec{x}\right)$ points to a location called "**event**" which identifies a particular space and time [10].

We define 4-vector $x^{\mu}$, which has an upper index $\mu$ to be a "contravariant vector" and 4-vector $x_{\mu} = (x_0, x_1, x_2, x_3)$, which has a lower index $\mu$ to be a "covariant vector". Consider the length between two events, $x_1^{\mu} = (x_1^0, x_1^1, x_1^2, x_1^3)$ and $x_2^{\mu} = (x_2^0, x_2^1, x_2^2, x_2^3)$. By Pythgorean theorem, we derive this length

$$\mathrm{d}s^2 = (\mathrm{d}s)^2, \tag{1.69}$$

$$= \left(x_1^0 - x_2^0\right)^2 + \left(x_1^1 - x_2^1\right)^2 + \left(x_1^2 - x_2^2\right)^2 + \left(x_1^3 - x_2^3\right)^2, \tag{1.70}$$

$$= \left(\mathrm{d}x^0\right)^2 + \left(\mathrm{d}x^1\right)^2 + \left(\mathrm{d}x^2\right)^2 + \left(\mathrm{d}x^3\right)^2, \tag{1.71}$$

$$= \mathrm{dx} \cdot \mathrm{dx} = \mathrm{d}x^{\mu}\mathrm{d}x_{\mu}, \tag{1.72}$$

$$= \sum_{\mu,\nu=0}^{3} \eta_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu}, \tag{1.73}$$

$$\equiv \eta_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu}, \tag{1.74}$$

where $ds$ is the length between two events, $x^{\mu}$ and $x_{\mu} \equiv x^{\nu}$ ($dx^{\mu}$ is a differential value of $x^{\mu}$). Thus this value is the event in differential Manifold. Einstein ignored the symbol $\sum_{\mu,\nu=0}^{3}$to write in equations. This is called Einstein's summation convention for convenience to calculate. Moreover, we have defined $\eta_{\mu\nu}$ to be a metric tensor in the spacetime in this equation as shown below.

$$\begin{aligned} \mathrm{d}s^2 &= -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2, & (1.75) \\ &= \begin{pmatrix} \mathrm{d}t \\ \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z \end{pmatrix}^T \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{d}t \\ \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z \end{pmatrix}, & (1.76) \\ &= (\mathrm{dx})^T \eta_{\mu\nu} \mathrm{dx} = \eta_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu, & (1.77) \end{aligned}$$

Then $\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ can also be rewritten as $\eta_{\mu\nu} = \begin{pmatrix} \eta_{00} & 0 \\ 0 & \delta_{\mu\nu} \end{pmatrix}$ or $\eta_{\mu\nu} \equiv sign\,(-,+,+,+)$ which is the signature and satisfies $x_\mu = \eta_{\mu\nu} x^\nu$ [10]. The scalar product of two vectors, in general, can be written as $x \cdot y = x^\mu y_\mu = g_{\mu\nu} x^\mu y^\nu$ . Before we use this metric tensor $g_{\mu\nu} = \eta_{\mu\nu}$, we have to understand the meaning of $\mathrm{d}s^2$.

### 1.4.9 Differential geometry

Differential geometry is concerned with the theory of differentiable, Riemannian manifolds and calculations for curves and surfaces [11, 18].

In curvature manifold, we define $\mathrm{d}s^2$ to be

$$\left[ \mathrm{d}s^2 \begin{cases} > 0 & \Rightarrow \text{spacelike interval} \\ = 0 & \Rightarrow \text{lightlike (null) interval} \\ < 0 & \Rightarrow \text{timelike interval} \end{cases} \right]$$

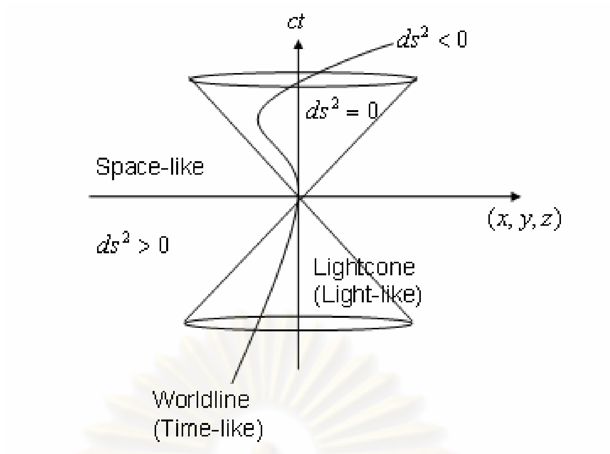


Figure 1.11: Light cone and worldline [3].

### 1.4.10 Stress energy tensor or energy momentum tensor

In order to understand the meaning of momentum and energy we need to know the meaning of stress energy tensor.

A momentum of an object is the scalar product between 4-velocity and the object's mass. We can define $m$ to be any object's mass and $U^\mu = (u^0, u^1, u^2, u^3)$ to be the 4-velocity. Define 4-momentum to be

$$p^\mu = mU^\mu, \tag{1.78}$$

$$= \left(mu^0, mu^1, mu^2, mu^3\right), \tag{1.79}$$

$$= \left(p^0, p^1, p^2, p^3\right), \tag{1.80}$$

in spacetime [13].

So stress energy tensor $T^{\mu\nu}$ is the $\mu^{th}$-component of the momentum $p^\mu$ which moves on the surface in the direction.

Before we define a perfect fluid, we need to know the definition of "**perfect**" . There are three fundamental properties to consider, namely isotropic property, no viscosity and no heat conduction. Isotropy means all directions are independent of each other and they are equal. Hence, if there exists an independent component

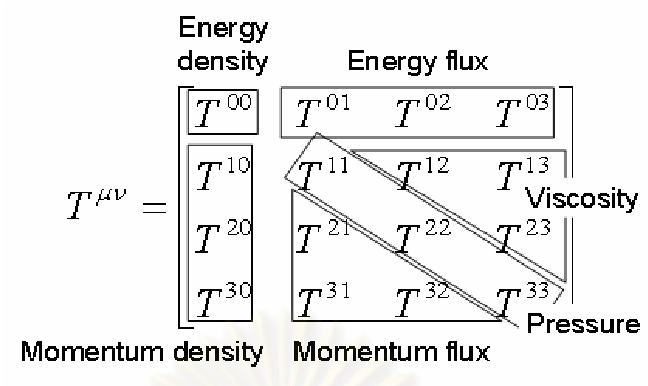


Figure 1.12: Stress energy tensor [19, 21].

$P^{ij}$, then the value of $P^{ij}$ is equal for $i, j = 1, 2, 3$ [19–21].

### 1.4.11 Viscosity

Viscosity is the force that is parallel to the interface between particles. So no viscosity means no such force i.e. the force and the component of the object are perpendicular. Mathematically, $T^{ij} = 0$ for all $i, j = 1, 2, 3$ and $i \neq j$. But there is still the force that are parallel to the component of the object that are nonzero. That is $T^{ij} = P^{ij}$ when $i = j = 1, 2, 3$ [20, 21].

### 1.4.12 Heat conduction



Heat conduction is the energy that transfers from a place or an object of higher temperature to a lower temperature [21, 22]. Because of the energy, that is $E = cp$, so heat conduction means the flux of the component $\nu$ of the energy into the direction $\mu$. So no heat conduction means $T^{0j} = 0$ for all $j = 1, 2, 3$ and $T^{i0} = 0$ for all $i = 1, 2, 3$ which is the symmetry of tensor ($T^{\mu\nu} = T^{\nu\mu}$) [20]. So stress energy tensor is

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P^{11} & 0 & 0 \\ 0 & 0 & P^{22} & 0 \\ 0 & 0 & 0 & P^{33} \end{pmatrix}. \tag{1.81}$$

Since spacetime has the property of isotropy, each coordinates are independent. So pressures in all direction are $P^{11} = P^{22} = P^{33} \equiv P$. Thus, stress energy tensor is

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}. \tag{1.82}$$

### 1.4.13 Curvature and relativity

Einstein attempted to use special relativity to explain gravitation which generalized Minkowski flat spacetime to curved spacetime through the use of curvature tensor analysis. We know that the earth is flat by our visual sense since we see only the locality of the earth. Indeed, the earth is spherical.



Since the earth's surface is a manifold, what we observe is therefore local which means that we see it as flat.

### 1.4.14 Geodesics

#### Classification by norm

In general relativity, geodesic is a straight line (shortest path) on a spacetime manifold. For example, worldline of free particle without external forces is a geodesic. Such a particle can move freely along geodesic.

In special relativity on Lorentz manifold, we can classify geodesics by norm as follows:

1. Time-like geodesic has a negative norm of tangent vector.
2. Null-geodesic has a zero norm of tangent vector.
3. Space-like geodesic has a positive norm of tangent vector.

(See details in [10, 13])

**Classification by shape**

In general geometry it is useful to distinguish the three definitions above as three different types of lines. Events of two geodesics in the hyperbolic plane are

1. Equidistant lines
2. Parallel geodesics
3. Geodesics sharing a common perpendicular

Two lines in a plane that do not intersect or meet are called **parallel lines.** Two geodesics can be either

1. **intersecting**: they intersect in a common point in the plane.
2. **parallel**: they do not intersect in the plane, but do in the limit to infinity.
3. **ultra parallel**: they do not even intersect in the limit to infinity.

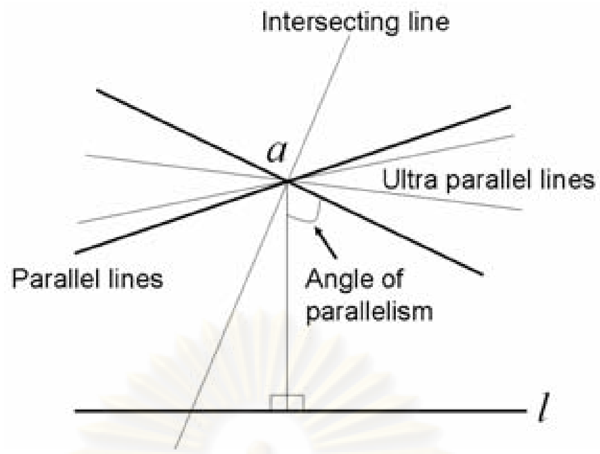


Figure 1.13: Types of line on a geometry [8].

**Three geodesics in the hyperbolic plane**

The three geodesics are intersecting, parallel, and ultra parallel lines through $a$ with respect to $l$ in the hyperbolic plane. The parallel lines appear to intersect $l$ just off the image. This is an artifact of the visualization. It is not possible to isometrically embed the hyperbolic plane in three dimensions. In a real hyperbolic space the line will get closer to each other and "touch" at infinity.

On the spherical plane there is no such thing as a parallel line. Line $a$ is a great circle, the equivalence of a straight line in the spherical plane. Line $c$ is equidistant to line $a$ but is not a great circle. It is a parallel of latitude. Line $b$ is another geodesic which intersects a in two antipodal points. They share two common perpendiculars.

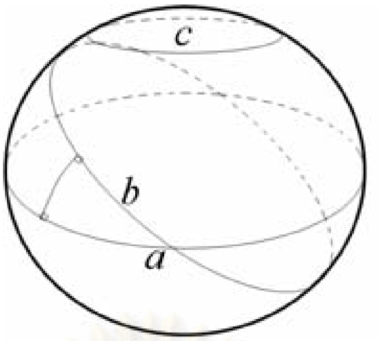


Figure 1.14: Perpendicular line from two great circles $a$ and $b$ [8].

## 1.5 Geodesic equations

### 1.5.1 Extremization of curve

From definition of geodesics, using $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$, the total length can be obtained as follows:

$$
\begin{aligned}
s &= \int \mathrm{d}s, &(1.83)\\
&= \int \frac{\mathrm{d}s}{\mathrm{d}\sigma}\mathrm{d}\sigma, &(1.84)\\
&= \int \sqrt{\frac{\mathrm{d}s^2}{\mathrm{d}\sigma^2}}\mathrm{d}\sigma, &(1.85)\\
&\equiv \int L\mathrm{d}\sigma, &(1.86)
\end{aligned}
$$

where

$$L = \sqrt{g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma}\frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma}} \equiv L\left(x, \dot{x}\right), \tag{1.87}$$

is a Lagrangian and

$$\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}\sigma}, \tag{1.88}$$

as shown in [21].

Now, by using Hamilton's principle, we can extremize the length to be $\delta s = \delta \int L\left(x, \dot{x}\right) \mathrm{d}\sigma = 0$ [10]. Then we will use this in the next section.

### 1.5.2 Extremization of curve length

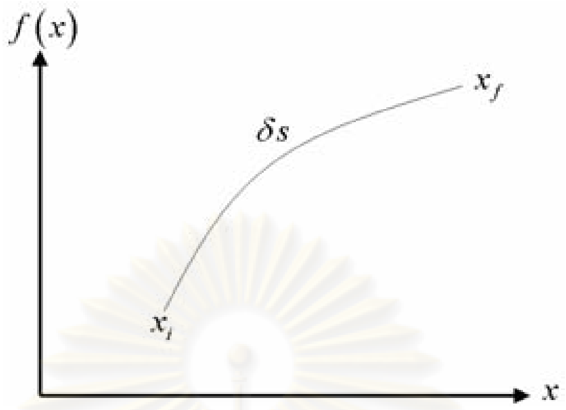


Figure 1.15: Geodesics between two points.

$$0= \delta s, \tag{1.89}$$

$$= \delta \int_{\sigma_i}^{\sigma_f} L(x,\dot{x})\mathrm{d}\sigma, \tag{1.90}$$

$$= \int_{\sigma_i}^{\sigma_f} \left(\frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial \dot{x}}\delta \dot{x}\right)\mathrm{d}\sigma, \tag{1.91}$$

$$= \int_{\sigma_i}^{\sigma_f} \frac{\partial L}{\partial x}\delta x \mathrm{d}\sigma + \int_{\sigma_i}^{\sigma_f} \frac{\partial L}{\partial \dot{x}}\delta \dot{x}\mathrm{d}\sigma, \tag{1.92}$$

$$= \int_{\sigma_i}^{\sigma_f} \frac{\partial L}{\partial x}\delta x \mathrm{d}\sigma + \int_{\sigma_i}^{\sigma_f} \frac{\partial L}{\partial \dot{x}}\frac{\mathrm{d}}{\mathrm{d}\sigma}\delta x\mathrm{d}\sigma, \tag{1.93}$$

$$= \int_{\sigma_i}^{\sigma_f} \frac{\partial L}{\partial x}\delta x \mathrm{d}\sigma + \left[\frac{\partial L}{\partial \dot{x}}\delta x\right]_{\sigma_i}^{\sigma_f} - \int_{\sigma_i}^{\sigma_f} \frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\frac{\partial L}{\partial \dot{x}}\right)\delta x\mathrm{d}\sigma. \tag{1.94}$$

So we derive

$$0 = \int_{\sigma_i}^{\sigma_f} \left[\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\frac{\partial L}{\partial \dot{x}}\right)\right](\delta x)\,\mathrm{d}\sigma. \tag{1.95}$$

Lagrange's equation

$$\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}\sigma}\frac{\partial L}{\partial \dot{x}} = 0. \tag{1.96}$$

Using

$$L = \frac{1}{2} g_{\gamma\delta}\, \dot{x}^{\gamma}\, \dot{x}^{\delta}, \tag{1.97}$$

then

$$\frac{1}{2} g_{\gamma\delta,\alpha}\, \dot{x}^{\gamma}\, \dot{x}^{\delta} - \frac{\mathrm{d}}{\mathrm{d}\lambda}\left[ g_{\gamma\delta}\, \dot{x}^{\gamma} \right] = 0, \tag{1.98}$$

where

$$g_{\gamma\delta,\alpha} \equiv \frac{\partial g_{\gamma\delta}}{\partial x^{\alpha}} \tag{1.99}$$

but we have

$$\frac{\mathrm{d}}{\mathrm{d}\sigma} g_{\gamma\alpha} = \frac{\partial g_{\gamma\alpha}}{\partial x^{\delta}}\, \dot{x}^{\delta}, \tag{1.100}$$

we obtain

$$\begin{aligned} \frac{1}{2} g_{\gamma\delta,\alpha}\, \dot{x}^{\gamma}\, \dot{x}^{\delta} - g_{\gamma\alpha,\delta}\, \dot{x}^{\delta}\, \dot{x}^{\gamma} - g_{\gamma\alpha}\, \ddot{x}^{\gamma} &= \left( \frac{1}{2} g_{\gamma\delta,\alpha} - g_{\gamma\alpha,\delta} \right) \dot{x}^{\gamma}\, \dot{x}^{\delta} - g_{\gamma\alpha}\, \ddot{x}^{\gamma}, \\ &= 0. \end{aligned} \tag{1.101}$$

Multiplying by $g^{\nu\alpha}$, we get

$$g^{\nu\alpha} \left( \frac{1}{2} g_{\gamma\delta,\alpha} - g_{\gamma\alpha,\delta} \right) \dot{x}^{\gamma}\, \dot{x}^{\delta} - \ddot{x}^{\gamma} = 0. \tag{1.102}$$

Rearranging it

$$\ddot{x}^{\gamma} = -g^{\nu\alpha} \left( g_{\gamma\alpha,\delta} - \frac{1}{2} g_{\gamma\delta,\alpha} \right) \dot{x}^{\gamma}\, \dot{x}^{\delta}\,. \tag{1.103}$$

This resembles Newton's laws. In terms of the Christoffel symbol, it reads

$$\ddot{x}^{\gamma} = -\Gamma^{\nu}_{\gamma\delta}\, \dot{x}^{\gamma}\, \dot{x}^{\delta} \tag{1.104}$$

which is a "**Geodesic equation**" [10, 13, 21].

### 1.5.3 Einstein's field equations and Einstein tensor

Einstein's equations or Einstein's field equations are equations that govern the relation between the curvature of spacetime and mass-energy. Einstein equations are in the form

$$G_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{1.105}$$

where $G_{\mu\nu}$ is Einstein's tensor which tells us about the gravitation through the curvature, while stress energy tensor $T_{\mu\nu}$ is the amount of the energy or mass that causes the gravitational force.

By considering Einstein's concept, we can state the following: The Einstein's equations describe how particles are affected by gravitational fields around them when they move. The use of the geodesic equation is similar to the use of the second law of Newtonian gravitation on the left-hand side. There is also a similarity with the Poisson equation, $\nabla^2\Phi(\vec{x}) = 4\pi G\rho(\vec{x})$, it resembles the right-hand part of this equation.

Next, we will discuss a mathematical structure of Einstein's equations. Equations in general relativity are normally given in terms of metric tensors. The rationales of Einstein equations are interesting due to the arguments based on physical grounds.

In Newton's gravitation, static mass creates a gravitational force. But in special relativity, we know that static mass and energy are equivalent or similar. So we aim that everything in general relativity, both energy and momentum, create the curvature of spacetime. Similarly, the mass-density $\rho$ is the result of the gravitational potential energy $\Phi$. Thus we can use the right side of Poisson equation that should be $\kappa T^{\alpha\beta}$ , where $\kappa$ is a constant. The left side of the Poisson equation is directly proportional to the potential energy. If we apply gradient $\nabla$ to both sides, then we derive the derivative of metric. Again, if we apply gradient $\nabla$ to both sides, then we derive the second order derivative of metric. We know that Riemann tensor and the contraction of Ricci tensor and scalar consist of the second-order metrics. Thus, this will make them become the left-hand side of the Einstein's equation. So the equation can be presented as

$$R_{\alpha\beta} = \kappa T_{\alpha\beta}. \tag{1.106}$$

But it is not completely right because of the followings. Due to the law of energy conservation, that is $T^{\alpha\beta}_{;\alpha} = 0$ which depends on Ricci tensor but $R^{\alpha\beta}_{;\alpha} \neq 0$. So Einstein rearranged this equation by transferring it to the same side and defined it to be a new tensor which satisfies the condition, called Einstein's tensor. Hence, the equation becomes

$$G_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = \kappa T_{\alpha\beta}. \tag{1.107}$$

If we compare the Einstein equation in a Newtonian system with a Poisson equation, then the constant $\kappa$ is $8\pi G/c^4$. So the Einstein's equation becomes

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = 8\pi G T_{\alpha\beta}/c^4 = 8\pi T_{\alpha\beta}, \tag{1.108}$$

where $c = 1$, and $G = 1$ is a universal gravitational constant [21]. Einstein tensor or trace-reversed Ricci tensor (defined by $G_{\alpha\beta}$), named by Albert Einstein, is a tensor that identifies the curvature of a Riemannian manifold. This tensor derived from Einstein's equation and is used to find the curvature of space-time [21].

### 1.5.4 Some analytical calculations in general relativity

#### Metric tensors and Christoffel's symbols

As an example, let $\theta$ be an opposite angle to latitude angle and $\varphi$ be azimuthal coordinate. Then the component of covariant metric tensor is

$$g_{\mu\nu} = \begin{bmatrix} g_{\theta\theta} & g_{\theta\varphi} \\ g_{\varphi\theta} & g_{\varphi\varphi} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{bmatrix}. \tag{1.109}$$

Christoffel's symbol and geodesic equation are

$$\Gamma^{\alpha}_{\mu\nu} \equiv \frac{1}{2} g^{\alpha\beta} \left( g_{\beta v,\mu} + g_{\beta v,\mu} - g_{\mu v,\beta} \right), \tag{1.110}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}x^{\alpha}}{\mathrm{d}t}\right) + \Gamma^{\alpha}_{\beta\mu}\frac{\mathrm{d}x^{\beta}}{\mathrm{d}t}\frac{\mathrm{d}x^{\mu}}{\mathrm{d}t} = 0. \tag{1.111}$$

From $g^{\mu\nu}g_{\mu\nu} = I$ or $\delta^{\nu}_{\mu}$ where $g^{\mu\nu}$ is contravariant metric tensor, $I$ is an identity matrix and $\delta^{\nu}_{\mu}$ is a kronecker delta function. So $g^{\mu\nu}$ is an inverse of $g_{\mu\nu}$. By calculation, we derive $g^{\mu\nu} = \begin{bmatrix} 1 & 0 \\ 0 & 1/\sin^2\theta \end{bmatrix}$, which are $g^{\varphi\varphi}$, $g^{\varphi\theta}$, $g^{\theta\varphi}$, $g^{\theta\theta}$, $g_{\varphi\varphi}$, $g_{\varphi\theta}$, $g_{\theta\varphi}$, $g_{\theta\theta}$ to calculate Christoffel's symbols. For example,

$$g^{\mu\nu}g_{\mu\nu} = I\left(\delta^{\nu}_{\mu}\right), \tag{1.112}$$

$$\begin{bmatrix} g^{\theta\theta} & g^{\theta\varphi} \\ g^{\varphi\theta} & g^{\varphi\varphi} \end{bmatrix} \begin{bmatrix} g_{\theta\theta} & g_{\theta\varphi} \\ g_{\varphi\theta} & g_{\varphi\varphi} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1/\sin^2\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{bmatrix}, \tag{1.113}$$

$$= I\left(\delta^{\nu}_{\mu}\right). \tag{1.114}$$

To calculate Christoffel's symbols, we can find it from its definition

$$\Gamma^{\alpha}_{\mu\nu} \equiv \frac{1}{2} g^{\alpha\beta} \left( g_{\beta v,\mu} + g_{\beta v,\mu} - g_{\mu v,\beta} \right). \tag{1.115}$$

There are $2^3$ Christoffel's symbols in this coordinate because this coordinate has two index, $\theta$ and $\varphi$. So Christoffel's symbols are

$$\Gamma^{\theta}{}_{\theta\theta} = \frac{1}{2} g^{\theta\theta} \left( g_{\theta\theta,\theta} + g_{\theta\theta,\theta} - g_{\theta\theta,\theta} \right) = 0, \tag{1.116}$$

$$\Gamma^{\theta}{}_{\theta\varphi} = \frac{1}{2} g^{\theta\theta} \left( g_{\theta\theta,\varphi} + g_{\theta\varphi,\theta} - g_{\theta\varphi,\theta} \right) = 0, \tag{1.117}$$

$$\Gamma^{\theta}{}_{\varphi\theta} = \Gamma^{\theta}{}_{\theta\varphi}, \tag{1.118}$$

$$\Gamma^{\theta}{}_{\varphi\varphi} = \frac{1}{2} \left( 0 + 0 - \frac{\partial \sin^2\theta}{\partial\theta} \right) = -\sin\theta\cos\theta, \tag{1.119}$$

$$\Gamma^{\varphi}{}_{\theta\varphi} = \Gamma^{\varphi}{}_{\varphi\theta} = \frac{\cos\theta}{\sin\theta}, \tag{1.120}$$

$$\Gamma^{\varphi}{}_{\varphi\varphi} = \frac{\cos\phi}{\sin\phi}. \tag{1.121}$$

To find the geodesic equation, from definition

$$\frac{\mathrm{d}}{\mathrm{d}t} \left( \frac{\mathrm{d}x^{\alpha}}{\mathrm{d}t} \right) + \Gamma^{\alpha}_{\beta\mu} \frac{\mathrm{d}x^{\beta}}{\mathrm{d}t} \frac{\mathrm{d}x^{\mu}}{\mathrm{d}t} = 0. \tag{1.122}$$

If we let the coordinate be $x^{\alpha} = \theta$ and substitute in the above equation, then we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \left( \frac{\mathrm{d}\theta}{\mathrm{d}t} \right) + \Gamma^{\theta}_{\theta\theta} \dot{\theta}\dot{\theta} + \Gamma^{\theta}_{\theta\varphi} \dot{\theta}\dot{\varphi} + \Gamma^{\theta}_{\varphi\theta} \dot{\varphi}\dot{\theta} + \Gamma^{\theta}_{\varphi\varphi} \dot{\varphi}\dot{\varphi} = 0. \tag{1.123}$$

That is

$$\ddot{\theta} - \sin\theta\cos\theta\dot{\varphi}^2 = 0. \tag{1.124}$$

Similarly, if we let the coordinate $x^{\alpha} = \varphi$ and substitute in the above equation, then we obtain

$$\ddot{\varphi} + 2\frac{\cos\theta}{\sin\theta} \dot{\theta}\dot{\varphi} = 0. \tag{1.125}$$

To calculate Riemann curvature tensor $R^{\alpha}_{\beta\mu\nu}$, there are $2^4$ components because this coordinate has two indices. Furthermore, we have a definition of Riemann curvature tensor in terms of Christoffel's symbols

$$R^{\alpha}_{\beta\mu\nu} \equiv \Gamma^{\alpha}_{\beta\nu,\mu} - \Gamma^{\alpha}_{\beta\mu,\nu} + \Gamma^{\alpha}_{\sigma\mu}\Gamma^{\sigma}_{\beta\nu} - \Gamma^{\alpha}_{\sigma\nu}\Gamma^{\sigma}_{\beta\mu}, \tag{1.126}$$

and the two properties of antisymmetry

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} = -R_{\alpha\beta\delta\gamma} = R_{\gamma\delta\alpha\beta}, \tag{1.127}$$

and cyclicity

$$R_{\alpha\beta\gamma\delta} + R_{\beta\delta\alpha\gamma} + R_{\alpha\delta\beta\gamma} = 0, \tag{1.128}$$

which implies that

$$R_{\theta\theta\varphi\varphi} = -R_{\theta\theta\varphi\varphi} = R_{\varphi\varphi\theta\theta}. \tag{1.129}$$

A contraction with a metric tensor $g^{\mu\nu}$, we arrived at

$$R_{\theta\varphi\theta\varphi} = g_{\theta\alpha}R^{\alpha}_{\varphi\theta\varphi} = g_{\theta\theta}R^{\theta}_{\varphi\theta\varphi}. \tag{1.130}$$

Finally, substituting all Christoffel's symbols in the equation (1.126)

$$R_{\theta\varphi\theta\varphi} = \Gamma^{\theta}_{\varphi\varphi,\theta} - \Gamma^{\theta}_{\varphi\theta,\varphi} + \Gamma^{\theta}_{\alpha\theta}\Gamma^{\sigma}_{\varphi\varphi} - \Gamma^{\theta}_{\sigma\varphi}\Gamma^{\sigma}_{\varphi\theta}, \tag{1.131}$$

then we obtain

$$R_{\theta\varphi\theta\varphi} = \left(-\cos^2\theta + \sin^2\theta\right) + 0 + 0 + \left(\cos^2\theta\right) = \sin^2\theta \tag{1.132}$$

(See details in [23])

| Transformations | Galilean | Lorentzian |
|---|---|---|
| constant $\gamma$ | 1 | $\gamma = \frac{1}{\sqrt{1-(\nu/c)^2}}$ |
| velocity | $\nu \ll c$ | $\nu < c$ |

Table 1.1: Comparison between Galilean and Lorentzian transformations

## 1.6 Conclusion

Einstein's special relativity tells us that the laws of physics remain invariant in all inertial frames which are reference frames that move at constant velocities. Therefore, studying a system in one inertial frame is equivalent to studying it in another one despite the big difference in velocities of the two frames.

Lorentzian and Galilean transformations convert the inertial (rest) reference frames into one another, see table 1.1.

Spacetime is not always flat, it can be curved because of the existence of a gravitational field due to the presence of mass or energy. One consequence is the bending of light rays in the vicinity of massive objects such as the sun. This effect means that to an observer, the star light that appears in the line of sight does not imply that it is located precisely in the line of sight. We will discuss more about this in the next chapter.

General relativity is the geometric theory of gravitation first published by Albert Einstein in 1915. This is thus called Einstein's theory of gravitation. It generalizes special relativity and Newton's law of universal gravitation into a unified theory of gravitation based on a spacetime concept. Special relativity, which does not describe gravity, corresponds to a flat spacetime called Minkowski space. Spacetime is a concept that already occurs in special relativity, a theory which does not incorporate gravity. Especially, the curvature of Minkowski space is related to the four-momentum of matter and radiation. In this chapter, we have described mathematical structures of general relativity. The basic equations of general relativity are the Einstein field equations or Einstein equations, a system of partial differential equations. These constitute a system of nonlinear partial differential equations which indicate the interaction between the geometry of spacetime and the ambient mass-energy. The main idea of this theory of general relativity is the principle of equivalence. The equivalence principle is involved in the calculation of gravity by considering reference frames with accelerations.

General relativity is a concept of relativity in general cases so that it can be considered in any reference frame with metric tensor. In special relativity, we only consider inertial reference frames; meaning no acceleration or deceleration and a specific metric tensor namely the Minkowski metric. Both special relativity and general relativity apply to relativistic situations, namely when objects travel at very high speeds. For the other two theories, that is, Newtonian and non-Newtonian gravitation, they are applicable when velocities are much less than the velocity of light based on Galilean relativity. Thus, to learn general relativity, one should start from special relativity.

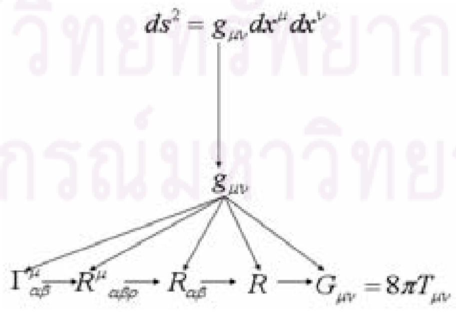




Figure 1.16: Algorithm for computing geometrical quantities in general relativity.

This figure 1.16 illustrates how to calculate geometrical quantities such as Christoffel's symbols, Riemann curvature tensors, Ricci tensors, Ricci scalars, and

Einstein's tensors in general relativity.

# CHAPTER II

# TRANSFORMATION THEOREMS

## 2.1 Introduction

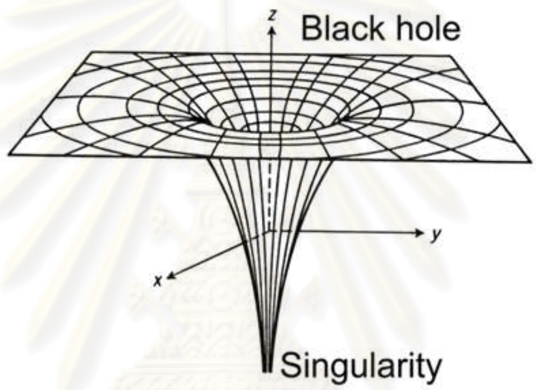


Figure 2.1: The main form of black hole consists of the large amount of gravitational field [24].

We already know that a black hole is a region of space from which nothing, not even light, can escape. But we do not know a few things about black holes. For instance, whether black holes really exist, or if they exist, what could their shape be. So we just assume their shape by using coordinates to discover their properties such as if using perfect fluid spheres (Schwarzchild's coordinates, Gaussian Polar coordinates, etc.) to be the model of black holes, then we derive a few differential equation to generate transformation between perfect fluid spheres in this thesis. Before using perfect fluid spheres, we need to know their definition and examples, then we can differentiate black holes and perfect fluid spheres.

### 2.1.1 Definition of black holes

A black hole is a region of spacetime with an extremely strong force of gravitation from which nothing, including light, can escape. Hence, we cannot see the center of a black hole. There are areas around the center of a black hole called "event horizon". At the Schwarzchild's radius, when an object comes into the event horizon, it must accelerate its velocity to more than the velocity of light to escape from the event horizon. However, it is possible that an object can generate a velocity more than the velocity of light (there has been suggestions by Einstein and by other scientists in 1967 that it is possible that, indeed, Einstein's relativity does not prohibit that any particle or signal cannot move with their velocity faster than the velocity of light [25, 27, 28]).

### 2.1.2 Classification of black holes by size

We can classify black holes in many different ways. In general, different types of black holes vary significantly in sizes, which can be classified into 4 types as follows:

**Supermassive black holes**

Supermassive black holes consist of one hundred billion folds of solar masses and it is believed that they really exist in the center of most galaxies including Milky Ways. It is believed that they are important in the occurrence of nucleus of galaxies and may occur from the combination of many small back holes or accumulation of stars and gases in space [25].

### Intermediate-mass black holes

Intermediate-mass black holes consist of several thousand folds of solar masses. It is believed that they are the energetic sources of the high-density X-rays but there is no evidence where these black holes come from. It is supposed that they occur from collision of small-size black holes in the center of stars in groups of spherical stars or galaxies. This event results in the creation of an extremely strong gravitational field. These two classifications are merely ordinary occurences. Other information such that smallest-size mass or largest-size mass black holes can establish from collapsing of supermassive stars is not well known but it is believed that this type of black holes may be smaller than two hundred folds of solar masses [25].

### Stellar-mass black holes

Stellar-mass black holes consist of a few folds of solar masses (from Tolman–Oppenheimer–Volkoff limit for the maximal mass of neutron stars) to twenty folds. These black holes occur from collapsing of the single stars or combination of the dual neutron stars which cannot be separated due to gravitational radiation. Initially, these stars may consist of one hundred folds of solar masses or more but they lose their outer mass during the initial evolution. For instance, the losing of mass of a star during its stage of existence as red huge stars or the explosion of a supernova can change a star into a neutron star or a black hole. In the last step of the model in the theory of stars' evolution, we still do not know the maximal size of stars that could change into a black hole. If the center of a star is clear, then it will become a white dwarf [25].

**Micro black holes**

Micro black holes have a smaller size than stellar-mass black holes. Since they have this size, they are highly dependant on quantum mechanics. No known mechanics can explain the general occurrence of these black holes from stars' evolution. But the assumption of extensive galaxies shows that black holes have been occurring since first period of the universe. By considering some theorem about gravitation in quantum physics, these black holes may occur from the high-energetic reaction which comes from the reaction between cosmic rays and atmosphere or a particle's accelerator. The theory of Hawking's rays predicts that these types of black holes will evaporate into a bright light during the radiation of gamma rays [25].

### 2.1.3 Properties and components of black holes

**Event horizon**

Event horizon is the surface of spacetime which identifies coordinates such that anything cannot escape when they enter into this region. Thus, anything in this surface cannot be seen by any outer observer. Besides, event horizon is unified together with the general space, but without any distinctive observable freature. Event horizon is not matter (in physics terms), or a solid obstacle or slow moving radioactivity [25].

Outside of Event horizon, the gravitational field is established by the symmetrical spheres that have equal masses. This tells us that the idea of black holes absorbing everything is incorrect because there are still chains of materials around black holes, outside the photon sphere, not affecting the gravitation radioactivity. Then it makes the losing of energy from running around resembling the effect from electromagnet radioactivity [25].

### Singularity

In theory of general relativity, within the center of black holes, there is a singularity of spacetime. The black holes will be pressured until their volume becomes zero. Thus, the black holes' density becomes infinity when their volume becomes zero. This density, which is at the center of these black holes, is called "singularity" [25].

### Photon sphere

Photon sphere is the boundary of zero thickness when photons move along the tangent line of circle, which makes orbit circular. In nonspinning black holes, their photon spheres have their radius around 1.5 times of Schwarzchild's radius. Their orbits are not constant. Thus, anything that comes into photon sphere will grow across the time although it is fixed in orbit to escape from black holes or Event horizon [25].

### Ergosphere

Spinning black holes will be surrounded by spacetime that can not be stationery. This is called Ergosphere which results from moving of frames [25].

### Schwarzschild radius

Schwarzschild radius is the boundary region of nonspinning black holes. Its length is about 3 kilometres calculated from $R_s = 2GM/c^2$, where $R_s$ is Schwarzschild radius, $G$ is the universal gravitational constant and $c$ is the speed of light [25].

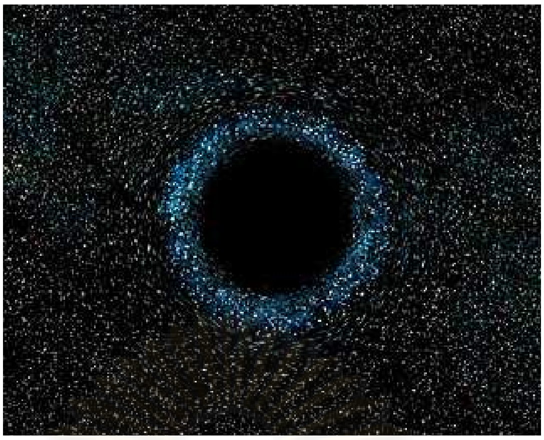

Figure 2.2: This figure shows the black hole [28].

**Escape velocity**

Escape velocity is the minimal speed that makes any object escape from black holes calculated by $v_{esc} = \sqrt{2GM/R_s} = c$, where $v_{esc}$ is escape velocity [25].

## 2.2 Black hole solutions

In this thesis, to learn about black holes, we first need to learn about Einstein's field equation. This equation implies that singularities of black holes really exist. Thus, before we study in Einstein's field equation, we have to learn about relativity by starting from special relativity to general relativity and Einstein's field equation.

Consider the spherical mass $M$ with its Einstein equation $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0$. In this spherical symmetric coordinate, when we get the solutions, they are called "Schwarzschild solution" found by German physicist, Karl Schwarzschild in 1916, which greatly surprised Eienstein because he did not think that anyone could solve the Einstein equation in such a short period of time. From $G_{\mu\nu} = 0$

and contraction of Ricci tensor and metric tensor, that is $0 = g^{\mu\nu}G_{\mu\nu} = g^{\mu\nu}R_{\mu\nu} - \frac{1}{2}g^{\mu\nu}g_{\mu\nu}R = R - 2R$. So $R_{\mu\nu} = 0$.

The Schwarzschild's solution in spherical coordinate is given by

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2, \tag{2.1}$$

where

$$\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2, \tag{2.2}$$

$\mathrm{d}\Omega^2$ is the metric on 2-sphere ($S^2$).

Before we study perfect fluid sphere and compare it with black hole, we need to know about the definition of perfect fluid sphere. The next section will explain about fluid, sphere and perfect fluid in physics, respectively [26].

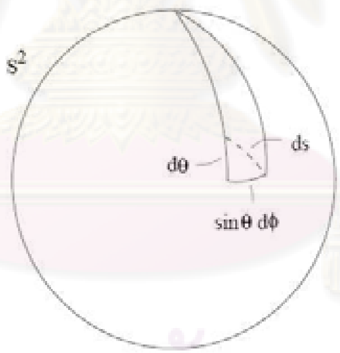


Figure 2.3: Two-sphere coordinate



## 2.3 Fluid Mechanics

Fluid mechanics is the branch of mechanics of liquid or gas which is static or dynamic. The study of fluid mechanics and fluid dynamics is necessary for the fundamental understanding of fluid mechanics. However, we need to know the basic definition and the properties.

## 2.4 Spherical symmetry

Spherically symmetry means "having the same symmetry as a sphere". That is a metric of $S^2$ is $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. Indeed, "sphere" or "2-sphere" means $S^2$. In the metric on a differentiable manifold, we can consider those metrics that have such symmetries. We can also use the characteristics of symmetries of the metric. By spherical symmetry, we can simplify the equations of motion considerably [26].

## 2.5 Perfect fluid spheres

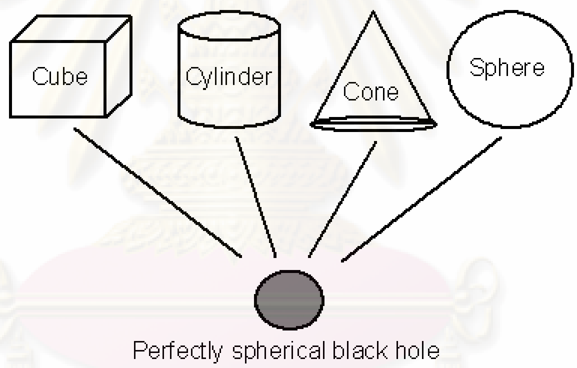


Figure 2.4: A few model of black hole [24].

Perfect fluid spheres are well known in this research field because of their first approximations to construct a realistic model for a relativistic star in general relativity. Though they illustrate a real importance in astrophysics, they are full of the general solutions of the perfect fluid in general relativity and have been gradually developed for other solutions. The first Static Spherically Symmetric Perfect Fluid solution, abbreviated to be **SSSPF**, with constant density was found by Karl Schwarzschild in 1918. He discovered the two exact solutions in

Einstein's equation, the "exterior solution" and the "interior solution". Especially, Schwarzschild solutions are easily available in view of Einstein's equations, which are very complex. Schwarzschild solution means that anyone can explain most general relativistic effects in the planetary system. The reason is that this statement can be described by the gravitational field outside spherically symmetric body like the planets and the sun, which are quite spherical [26].

For almost a hundred years, there had been a confusion about the specific perfect fluid spheres which had been discovered with most of these examples that seemed independent from each other [26].

Many algorithms of solving differential equations together with the field equations have been explored, often by choosing the special coordinate systems, or making simple ansatzë for one or other of the metric components. The evolution over the last several years has introduced many algorithmic techniques that allow us to generate large classes of perfect fluid spheres in a purely mechanical way. Perfect fluid spheres can be simplified, but they still provide a amazingly mathematical and physical structure [26].

In this chapter, we will extend these algorithmic ideas, by proving several solution generating theorems of varying levels of complexity. Then, we shall explore the formal properties of these solution-generating theorems and then will use these theorems to classify some of the previously known exact solutions. In addition, we will generate several previously unknown perfect fluid solutions by the original perfect fluid spheres.

### 2.5.1 Static spherically symmetric perfect fluid

In metric theory of gravitation, especially in general relativity, static spherically symmetric perfect fluid, is the fluid with isotropic pressure and static sphere in the spacetime created by stress energy tensor.

These solutions are often used to construct the model of stars, especially impact object such as White dwarf star or neutron stars. In general relativity, the models of single stars consisting of fluid are the solutions of perfect fluid from Einstein's equation and in external area are flat vacuum solutions. Both areas must be satisfied on the earth's surface spherically and pressure is zero [26].

### 2.5.2 Spheres

Perfect fluid spheres are the model for general relativistic stars that collapse to be black holes. Firstly, in geometry of physics and mathematics, sphere can be classified into various forms depending on its dimension. Sphere is a round object in three-dimensional space such as the shape of a ball in three-dimensional Euclidean space (or 2-sphere) or the shape of a circle in two-dimensional Euclidean space (or 1-sphere). Sphere consists of center and radial distance from center to spherical surface. Perfect sphere implies that sphere is completely symmetrical around its center that it is not quite elliptic such as the earth. For example, terrestrial sphere is a model or a coordinate system for the earth or the sun, and celestial sphere is a model or a coordinate system for the solar system.

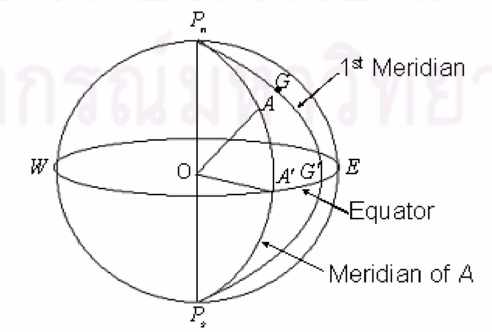


Figure 2.5: Terrestrial sphere is the model for the Earth and Sun with 1st meridian at $G$ and its constituents [29].

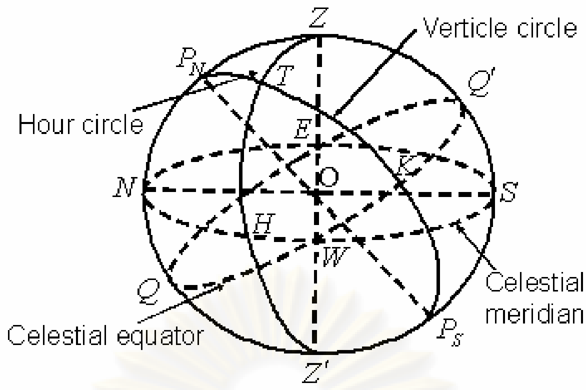


Figure 2.6: Celestial sphere is the model for the Solar system in cosmology and astronomy [29].

### 2.5.3 Fluid

In physics, fluid is a substance that has the "continuum" property, being a continuous material that flows under shear stress. The "continuum" is a collection of particles so numerous that the dynamics of individual particles cannot be followed. Fluids are a subset of the phases of matter that include liquids, gases, plasmas and, to some extent, plastic solids. Fluid mechanics can be divided into Newtonian and Non-Newtonian fluid mechanics.

### 2.5.4 Perfect fluid

Previously, we already know the stress energy tensor $T_{\mu\nu}$ of perfect fluid, that is

$$T_{\mu\nu} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & p_r & 0 & 0 \\ 0 & 0 & p_t & 0 \\ 0 & 0 & 0 & p_t \end{bmatrix}, \tag{2.3}$$

where $p_r = p_t$. Perfect fluid spheres are not only a completely spherical object, but also have the other three properties; no heat conduction which implies that $T_{0i} = T_{i0} = 0$, no viscosity which implies that only diagonal entries are not zero and isotropy that is $p_r = p_t$ [26, 30–32, 37, 38].

## 2.6 Generating theorems of perfect fluid spheres

### 2.6.1 Introduction

It is well known that perfect fluid spheres in general relativity, both static and non-static, are realistic models for a general relativistic star [26, 30, 33–37]. In this thesis, we develop several new transformation theorems that map perfect fluid spheres into perfect fluid spheres. Firstly, we need to know the definition of the stress energy tensor $T_{\mu v}$, which is defined by

$$T_{\mu v} = (\rho + p)\, u_\mu u_\nu + p g_{\mu\nu}, \tag{2.4}$$

$$= \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & p_r & 0 & 0 \\ 0 & 0 & p_t & 0 \\ 0 & 0 & 0 & p_t \end{bmatrix}, \tag{2.5}$$

where $\rho$ is the energy density, $u_\mu$ and $u_\nu$ are the four-velocity as measured by an observer moving together with the fluid, and $p_r, p_t$ are the radial pressure and the transverse pressure, respectively.

In the conditions of perfect fluid spheres, the word "perfect" means it has the isotropic property, independent of all coordinates in the stress energy tensor and equality, that is

$$p_r = p_t. \tag{2.6}$$

The Einstein's equations

$$G_{\mu\nu} = 8\pi T_{\mu\upsilon}, \tag{2.7}$$

and (2.6) give us

$$G_{\hat{\theta}\hat{\theta}} = G_{\hat{r}\hat{r}} = G_{\hat{\phi}\hat{\phi}}. \tag{2.8}$$

Note that Einstein's tensor, $G^r_r = G^\theta_\theta = G^\phi_\phi$, are the version that are more traditional in the older literature, but it only works for diagonal metrics, whereas the hatted version , $G_{\hat{r}\hat{r}} = G_{\hat{\theta}\hat{\theta}} = G_{\hat{\phi}\hat{\phi}}$ , can in principle be generalized to arbitrary metrics.

Over the last 90 years, there are many algorithmic methods to solve this differential equation, which have been explored, often by picking special coordinate systems, or making simple ansatzë which is the metric in the form of

$$\mathrm{d}s^2 = -B(r)\,\mathrm{d}t^2 + A(r)\,\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2, \tag{2.9}$$

where $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2$ is the metric of a two-sphere and for one or other of the metric components [38–41]. (For recent overviews see [33–35].) The outcome over the last several years has been the "algorithmic" techniques that allow one to generate large classes of perfect fluid spheres in a purely mechanical way [42–44]. In this thesis, we will present and extend these new algorithmic ideas, by proving several solution-generating theorems of complex different levels. Then we shall demonstrate their properties of these solution-generating theorems and use these transformation to group some of the previously known exact solutions. Moreover, we will construct several new previously unknown perfect fluid solutions. Besides, we use the mathematical programming, “maplet”, which is an easy and compact way to construct the program of perfect fluid sphere, to find and classify the set of all perfect fluid spheres instead of calculating by hand.

### 2.6.2 Schwarzschild solution

The gravitational field of a spherically symmetric particle such as black hole firstly appears in Newtonian as well as in the Einsteinian theory. This gravitational field of the Eienstein model of stars consists of the exterior and the interior Schwarzschild solutions. They are joined together at its surface.

The use of arbitrary coordinates is allowed in general relativity. Indeed, the physical significance of statements about tensors or vectors and other quantities are not always obvious. Nevertheless, there exist some situations where the interpretation is nearly as straightforward as in special relativity. The center of the point of a local inertial coordinate system is the most common example that is easy to understand [26].

## 2.7 Coordinates system in perfect fluid spheres

Generally, there are several coordinates for us to apply with our object that we want to measure. For example, Euclidean system is useful for a straight line or graph. Cylindrical and spherical systems are useful for a cylindrical or circular-shaped curve. Polar system is useful for the graph with constant radial distance and angle.

In special relativity, the metric of perfect fluid sphere often appears in the form of

$$A\left(r\right)\mathrm{d}r^2 + B\left(r\right)\mathrm{d}\Omega^2, \tag{2.10}$$

where $\mathrm{d}\Omega^2 = \mathrm{d}r^2 + \sin^2\theta\mathrm{d}\phi^2$. There are still several coordinates for us to use.

### 2.7.1 Schwarzchild's coordinates

This coordinate system is the most well-known coordinate system for studying perfect fluid spheres, approximately 55% while other coordinates such as isotropic coordinates is estimated to be 35%. Schwarzchild's metric is in the form of

$$\mathrm{d}s^2 = -\zeta\left(r\right)^2\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{\mathrm{B}\left(r\right)} + r^2\mathrm{d}\Omega^2, \tag{2.11}$$

For a perfect sphere, there is no arbitrary parameter in term of $r^2\mathrm{d}\Omega^2$. This coordinate system is the model for the object in the gravitational field outside a spherical, non-rotating mass (a non-rotating star, planet, or black hole). It is good for estimating a slowly rotating body like the earth or the sun.

### 2.7.2 Isotropic coordinates

With the approximate usage of 35%, this coordinate system is the second most well-known coordinate system for studying perfect fluid spheres. It is different from the radial coordinate of Schwarschild's coordinates. It is defined for light cones to appear round which means that (except for the trivial case of a locally flat space), the angular isotropic coordinates do not represent distances within the nested spheres, nor does the radial coordinate represent radial distances. Conversely, angles in the constant time exist without distortion, that follows from the name of their coordinates [26].

This system is often used in static spherically symmetric spacetimes in gravitational theories such as general relativity. Moreover, they can also be applied to a model of a spherically pulsating fluid 2-sphere object. The metric of isotropic coordinates is

$$\mathrm{d}s^2 = -\zeta\left(r\right)^2 \mathrm{d}t^2 + \frac{1}{\zeta\left(r\right)^2\ \mathrm{B}\left(r\right)^2}\left\{\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\right\}. \tag{2.12}$$

The co-parameter $1/\zeta\left(r\right)^2 \mathrm{B}\left(r\right)^2$ between $\mathrm{d}r^2$ and $r^2\mathrm{d}\Omega^2$ tells us that both of them cannot be distorted individually [30–32].

### 2.7.3 Gaussian polar coordinates

A third alternative is the Gaussian polar coordinates, which correctly represent radial distances, but distorts transverse distances and angles. In all three possibilities, the nested geometric spheres are represented by coordinate spheres, so we can say that their roundness is correctly represented. The metric of Gaussian polar coordinates is

$$\mathrm{d}s^2 = -\zeta\left(r\right)^2 \mathrm{d}t^2 + \mathrm{d}r^2 + R\left(r\right)^2 \mathrm{d}\Omega^2. \tag{2.13}$$

The parameter $R\left(r\right)^2$ in $\mathrm{d}\Omega^2$ means that there is some distortion between transverse distances and angles [30–32].



### 2.7.4 Synge isothermal coordinates

This coordinates system was first introduced by Gauss Korn and Lichtenstein. They have proved that these coordinates exist around any point on a two-dimensional Riemannian manifold. The metric of Synge isothermal coordinates is

$$\mathrm{d}s^2 = -\zeta\left(r\right)^{-2}\ \left\{\mathrm{d}t^2 - \mathrm{d}r^2 + R\left(r\right)^2 \mathrm{d}\Omega^2\right\} \tag{2.14}$$

(see also [30–32]).

### 2.7.5 General diagonal coordinates

This coordinate system can be represent in the metric in the form of

$$\mathrm{d}s^2 = -\zeta\,(r)^2\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{\mathrm{B}\,(r)} + R\,(r)^2\,\mathrm{d}\Omega^2. \tag{2.15}$$

Generally, this form is used to calculate various manifold but quite inefficient in specific cases.

### 2.7.6 Buchdahl's coordinates

This coordinate system is a sort of a mixture between Synge isothermal (tortoise) coordinates and Gaussian polar (proper radius) coordinates [30–32]. This coordinate system represents the metric in the form

$$\mathrm{d}s^2 = -\zeta\,(r)^2\;\mathrm{d}t^2 + \zeta\,(r)^{-2}\left\{\mathrm{d}r^2 + R(r)^2\mathrm{d}\Omega^2\right\}. \tag{2.16}$$

### 2.7.7 Solution generating theorems

In Schwarzschild (curvature) coordinates [26, 30, 37, 43, 44], the metric is

$$\mathrm{d}s^2 = -\zeta(r)^2\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{\mathrm{B}(r)} + r^2\mathrm{d}\Omega^2, \tag{2.17}$$

whose parameters are $\{\zeta(r), \mathrm{B}(r)\}$. Then, its ODE is

$$\left[r(r\zeta)'\right]\mathrm{B}' + \left[2r^2\zeta'' - 2(r\zeta)'\right]\mathrm{B} + 2\zeta = 0, \tag{2.18}$$

which reduces the freedom to choose the two functions in equation (2.17) to one. This equation (2.18) is a first-order non-homogeneous linear equation in $\mathrm{B}(r)$.

Thus, once you have chosen a $\zeta(r)$ this equation (2.18) can always be solved for $\mathrm{B}(r)$. Solving for $\mathrm{B}(r)$ in terms of $\zeta(r)$ is the basis of [43, 44], (and is the basis for 1st BVW theorem and integrating factor below) [26]. On the other hand, we can also re-group this same equation as

$$2r^2\zeta'' + \left[r^2\mathrm{B}' - 2r\mathrm{B}\right]\zeta' + \left[r\mathrm{B}' - 2\mathrm{B} + 2\right]\zeta = 0, \tag{2.19}$$

which is a linear homogeneous second-order ODE for $\zeta(r)$, which will become the basis for 2nd BVW theorem [26] and a new convenient transformation theorem below. Our objective in this section is, how to systematically "deform" the geometry (2.17) while still maintaining the perfect fluid spheres. We start with the Schwarzschild's metric, defined by

$$\mathrm{d}s^2 = -\zeta_0(r)^2\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{\mathrm{B}_0(r)} + r^2\mathrm{d}\Omega^2, \tag{2.20}$$

and assume that it represents a perfect fluid sphere. We need to show how to "deform" this solution, from $\{\zeta_0(r), \mathrm{B}_0(r)\}$ to another, by applying our different transformation theorems on $\{\zeta_0(r), \mathrm{B}_0(r)\}$, such that the outcome still presents a perfect fluid sphere. The result of this process will depend on one or more free parameters that are $\sigma$, $\varepsilon$ or $\gamma$, and so automatically produce the entire family of perfect fluid spheres of which the original starting point is only one member. Additionally, we analyze what is going on if we apply these theorems more than once, iterating them in various ways.

Similarly, in general diagonal coordinates, the metric is

$$\mathrm{d}s^2 = -\zeta(r)^2\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{\mathrm{B}(r)} + R(r)^2\,\mathrm{d}\Omega^2, \tag{2.21}$$

whose parameters are $\{\zeta(r), \mathrm{B}(r), R(r)\}$. Then, its ODE is

$$\left[R\left(R\zeta\right)'\right]\mathrm{B}' + \left[2RR''^2\zeta'' - 2RR'\zeta' - 2\left(R'\right)^2\zeta\right]\mathrm{B} + 2\zeta = 0. \tag{2.22}$$

This is the first-order non-homogeneous linear in $\mathrm{B}(r)$, and the second-order homogeneous linear in $\zeta(r)$. Also in exponential coordinates, the metric is

$$\mathrm{d}s^2 = -\exp\left(-2z\right)\mathrm{d}t^2 + \exp\left(+2z\right)\left\{\frac{\mathrm{d}z^2}{\mathrm{B}\left(z\right)} + R\left(z\right)^2\mathrm{d}\Omega^2\right\}, \tag{2.23}$$

whose parameters are $\{\mathrm{B}(z), R(z)\}$. Then, its ODE is

$$\mathrm{B}'\left[RR'\right] + \mathrm{B}\left[4R^2 - 2\left(R'\right)^2 + 2RR''\right] + 2 = 0. \tag{2.24}$$

This is the first-order non-homogeneous linear in $\mathrm{B}(z)$.

### 2.7.8 New technique

In this thesis, we present a new technique to derive the first theroem which is the same as transformation in [26]. This new technique of the following theorem uses an integrating factor technique to solve an ODE (2.18), then we derive a new Beta. This method is not too complicated because it consists of only one factor. Thus, it can easily be used to generate a new metric.

**Theorem 1** (Integrating Factor [45])**.** *For a perfect fluid sphere* $\{\zeta(r), \mathrm{B}_0(r)\}$, *define* $\gamma\mathrm{K}(r)$, *where* $\gamma$ *is an arbitrary constant, and*

$$\mathrm{K}\left(r\right) \equiv \exp\left(\int -\frac{2r^2\zeta'' - 2r\zeta' - 4\zeta}{r\left(r\zeta\right)'}\mathrm{d}r\right), \tag{2.25}$$

*then* $\{\zeta(r), \mathrm{B}_0(r) + \gamma\mathrm{K}(r)\}$ *is still a perfect fluid sphere. That is a transformation*

$$\mathrm{T}_1 : \{\zeta\left(r\right), \mathrm{B}_0\left(r\right)\} \mapsto \{\zeta\left(r\right), \mathrm{B}_0\left(r\right) + \gamma\mathrm{K}\left(r\right)\}, \tag{2.26}$$

*map a perfect fluid sphere to a perfect fluid sphere and this transformation is idempotent.*

*Proof.* We know that for the 1st order differential equation

$$M(x,y)\mathrm{d}x + N(x,y)\mathrm{d}y = 0, \tag{2.27}$$

and we need to find an integrating factor $\mu = \mu(v)$ for exact condition, that is

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N). \tag{2.28}$$

Simplify and rearrange them, then we derive

$$\frac{1}{\mu}\frac{\mathrm{d}\mu}{\mathrm{d}v} = \frac{\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)}{\left(N\frac{\partial}{\partial x} - M\frac{\partial}{\partial y}\right)v} \equiv F(v). \tag{2.29}$$

We assume that $v = v(x)$, and check $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \equiv F = F(x)$, then

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \equiv F(x) = \frac{1}{\mu}\frac{\mathrm{d}\mu}{\mathrm{d}x}, \tag{2.30}$$

and we derive $\mu = \exp\left(\int F(x)\,\mathrm{d}x\right)$ as desired.

Now, we can change this 1st order differential equation

$$M(x,y)\mathrm{d}x + N(x,y)\mathrm{d}y = 0 \tag{2.31}$$

to

$$y' + P(x)y = Q(x), \tag{2.32}$$

where

$$M(x,y) = P(x)y - Q(x), \tag{2.33}$$

and

$$N(x,y) = 1. \tag{2.34}$$

Therefore,

$$F(x) = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = P(x), \tag{2.35}$$

and

$$\mu = \mu(x) = \exp\left(\int F(x)\,\mathrm{d}x\right) = \exp\left(\int P(x)\,\mathrm{d}x\right). \tag{2.36}$$

Thus, we can solve this differential equation which we derive

$$y = \frac{1}{\mu(x)}\left(\int \mu(x)\, Q(x)\, \mathrm{d}x\right). \tag{2.37}$$

Finally, we compare this differential equation $y' + P(x)y = Q(x)$ with our 1st order ODE

$$\left[r\,(r\zeta)'\right]\mathrm{B}' + \left[2r^2\zeta'' - 2\,(r\zeta)'\right]\mathrm{B} + 2\zeta = 0, \tag{2.38}$$

where

$$x = r, \tag{2.39}$$

$$y = \mathrm{B}(r), \tag{2.40}$$

$$P(x) = \frac{2r^2\zeta'' - 2\,(r\zeta)'}{r\,(r\zeta)'}, \tag{2.41}$$

$$Q(x) = -\frac{2\zeta}{r\,(r\zeta)'}, \tag{2.42}$$

(recall that $\zeta = \zeta(r)$).

By calculating, we define $\mu = \exp\left(\int \frac{2r^2\zeta'' - 2r\zeta' - 2\zeta}{r(r\zeta)'}\mathrm{d}r\right)$ and

$$\begin{aligned}\mathrm{B}(r) &= \left[\exp\left(\int -\frac{2r^2\zeta'' - 2r\zeta' - 2\zeta}{r\,(r\zeta)'}\mathrm{d}r\right)\right] \\ &\quad \times\left(\int -2\frac{\exp\left(\int \frac{2r^2\zeta'' - 2r\zeta' - 2\zeta}{r(r\zeta)'}\mathrm{d}r\right)}{r\,(r\zeta)'}\mathrm{d}r + \gamma\right), \qquad (2.43)\\ &= \mathrm{B}_0(r) + \gamma \mathrm{K}_1(r), \qquad (2.44)\end{aligned}$$

where $\gamma$ is an arbitrary constant, and

$$\begin{aligned}\mathrm{B}_0(r) &= \left[\exp\left(\int -\frac{2r^2\zeta'' - 2r\zeta' - 2\zeta}{r\,(r\zeta)'}\mathrm{d}r\right)\right] \\ &\quad \times\left(\int -2\frac{\exp\left(\int \frac{2r^2\zeta'' - 2r\zeta' - 2\zeta}{r(r\zeta)'}\mathrm{d}r\right)}{r\,(r\zeta)'}\mathrm{d}r\right), \qquad (2.45)\\ \mathrm{K}(r) &= \exp\left(\int -\frac{2r^2\zeta'' - 2r\zeta' - 2\zeta}{r\,(r\zeta)'}\mathrm{d}r\right). \qquad (2.46)\end{aligned}$$

Hence, this transformation is $\mathrm{T}_1 : \{\zeta, \mathrm{B}\} \mapsto \{\zeta, \mathrm{B}_0(r) + \gamma \mathrm{K}_1(r)\}$.

To prove this transformation is idempotent. We see that

$$
\begin{aligned}
\mathrm{T}_1^2 &= \mathrm{T}_1 \circ \mathrm{T}_1 = \{\zeta(r), \mathrm{B}_0(r)\} && (2.47)\\
&\mapsto \{\zeta(r), \mathrm{B}_1(r)\} = \{\zeta(r), \mathrm{B}_0(r) + \gamma \mathrm{K}(r)\} && (2.48)\\
&\mapsto \{\zeta(r), \mathrm{B}_2(r)\} = \{\zeta(r), \mathrm{B}_1(r) + \gamma \mathrm{K}(r)\} && (2.49)\\
&= \{\zeta, \mathrm{B}_0(r) + 2\gamma \mathrm{K}(r)\} \triangleq \{\zeta(r), \mathrm{B}(r)\} = \mathrm{T}_1. && (2.50)
\end{aligned}
$$

Thus, this transformation is idempotent. □

* the notation "J1*, J2*" means that they are new metrics, which are discovered by Panit, they are definitely perfect fluid solutions, where

$$
F_1(m, r, R) \equiv -2 \int \frac{\left[\begin{array}{c} (mr^3 + 2r^4) R^2 - 2r^6 \\ + (m^2 + rm - r^2) R^4 \end{array}\right]}{\left[\begin{array}{c} (2mR^2 + r^3 - rR^2) r \\ \times (mR^2 + 2r^3 - rR^2) \end{array}\right]} \mathrm{d}r, \qquad (2.51)
$$

$$
F_2(A, B, r) \equiv -\frac{A}{8Br^4} - \frac{1}{2} \ln r. \qquad (2.52)
$$

**Note that** $\mathrm{B}(r)$ of Wyman IIb (n=2) in [33] is wrong. We can check that $\{\zeta(r), \mathrm{B}(r)\}$ is exactly a perfect fluid sphere by using maplets (program for perfect fluid spheres (Theorem 1-4)) in chapter 5. The correct $\mathrm{B}(r)$ of Wyman IIb (n=2) is from $ar \to \frac{a}{r}$.

| Name | Parameters | | | Name |
|---|---|---|---|---|
| | $\{\zeta(r), \mathrm{B}(r)\}$ | $\gamma K(r)$ | $\gamma$ | |
| Minkowski | $\{1,1\}$ | $\gamma r^2$ | $-\frac{1}{R^2}$ | Einstein static |
| Exterior Schwarz-schild | $\left\{\begin{matrix}\sqrt{1-2m/r}, \\ 1-2m/r\end{matrix}\right\}$ | $\gamma(m-r)^2 \times \left(1-\frac{2m}{r}\right)$ | $4C$ | Kuch68 II |
| Tolman V (A=0) | $\left\{\begin{matrix}Br^{1+n}, \\ (2-n^2)^{-1}\end{matrix}\right\}$ | $\gamma r^{\frac{2\left(2-n^2\right)}{2+n}}$ | $\frac{-A}{(2-n^2)}$ | Tolman V |
| M–W III | $\left\{\begin{matrix}\sqrt{Ar(r-a)}, \\ \left(\frac{7/4}{1-r^2/a^2}\right)^{-1}\end{matrix}\right\}$ | $4\gamma\frac{(r-a)2^{2/3}r^{7/3}}{(4r-3a)^{4/3}}$ | $-B/4$ | Martin 2 |
| Heint IIa (C=0) | $\left\{\begin{matrix}(1+ar^2)^{3/2}, \\ \left[\frac{(1+ar^2)}{(1-ar^2/2)}\right]^{-1}\end{matrix}\right\}$ | $\left(\frac{-2\sqrt{5}\gamma r^2}{15\frac{\sqrt{1+4r^2a}}{1+r^2a}}\right)$ | $-\frac{\sqrt{3}}{2}C$ | Heint IIa |
| B-L | $\left\{\sqrt{A}\frac{r}{a}, \frac{1+\frac{r^2}{a^2}}{2}\right\}$ | $\gamma r^2$ | arbitrary | B-L |
| K-O III | $\{A+Br^2, 1\}$ | $\frac{\gamma r^2}{(A+3Br^2)^{\frac{2}{3}}}$ | $-C$ | Martin 3 |
| Kuch1 Ib | $\left\{Ar+Br\ln r, \frac{1}{2}\right\}$ | $\frac{\gamma r^2}{2A+2B\ln r+B}$ | $\frac{-C}{2}$ | Martin 1 |
| Kottler | $\left\{\begin{matrix}c\sqrt{1-\frac{2m}{r}-\frac{r^2}{R^2}}, \\ 1-\frac{2m}{r}-\frac{r^2}{R^2}\end{matrix}\right\}$ | $\gamma\exp\left(F_1(m,R,r)\right)$ | $\gamma$ | J1* |
| Wyman IIb (n=2) | $\left\{\begin{matrix}\frac{A}{r}-Br^3, \\ -\frac{1}{2}+\frac{a}{r}\exp\left(-\frac{A}{4Br^4}\right)\end{matrix}\right\}$ | $\gamma\exp\left(F_2(A,B,r)\right)$ | $\gamma$ | J2* |

Table 2.1: Examples of metrics implied by theorem 1.

### 2.7.9 Additivity theorems in various coordinate systems

**Schwarzschild coordinates**

In Schwarzschild coordinates (also called curvature coordinates) [30–32, 43, 44], the metric is most usefully rearranged in the form

$$\mathrm{d}s^2 = -\zeta(r)^2 \mathrm{d}t^2 + \frac{\mathrm{d}r^2}{\mathrm{B}(r)} + r^2 \mathrm{d}\Omega^2, \tag{2.53}$$

which is a functional of the two functions $\{\zeta(r), \mathrm{B}(r)\}$. Then, the ODE arising from the isotropy condition is

$$\left[r(r\zeta)'\right] \mathrm{B}' + \left[2r^2\zeta'' - 2(r\zeta)'\right] \mathrm{B} + 2\zeta = 0, \tag{2.54}$$

which reduces the freedom to choose the two functions in equation (2.53) to one. This equation (2.54) is a first-order linear non-homogeneous equation in $\mathrm{B}(r)$. The solution space is one-dimensional, and since the ODE is inhomogeneous, this one-dimensional solution space is uniquely determined by any two distinct solutions $\mathrm{B}_1(r)$ and $\mathrm{B}_2(r)$.

However, in this thesis we will find it more useful to rearrange equation (2.54) into the form:

$$2r^2\zeta'' + \left[r^2\mathrm{B}' - 2r\mathrm{B}\right] \zeta' + \left[r\mathrm{B}' - 2\mathrm{B} + 2\right] \zeta = 0, \tag{2.55}$$

This equation is a linear homogeneous second-order ODE for $\zeta(r)$ and is the basis for 2nd BVW theorem in [26]. The solution space is two-dimensional, and since the ODE is homogeneous, this two-dimensional solution space is uniquely determined by any two distinct solutions $\zeta_1(r)$ and $\zeta_2(r)$.

**Definition 2** (Solution space — Schwarzschild)**.** *Let* $\Upsilon(\mathrm{B})$ *be the set of all solutions to equation* (2.55) *for fixed* $\mathrm{B}(r)$.

**Theorem 3** (Summation — Schwarzschild)**.** *Let* $\{\zeta_1(r), \mathrm{B}(r)\}, \{\zeta_2(r), \mathrm{B}(r)\}$ *represent perfect fluid spheres. Then for any arbitrary linear combination*

$$\zeta(r) = c_1\zeta_1(r) + c_2\zeta_2(r), \tag{2.56}$$

*the pair* $\{\zeta(r), \mathrm{B}(r)\}$ *also represents a perfect fluid sphere. Furthermore since* $\Upsilon(\mathrm{B})$ *is a two-dimensional vector space, any element of* $\Upsilon(\mathrm{B})$ *can be put in this form for suitable constants* $c_1$ *and* $c_2$*.*

The proof is immediate from the fact that equation (2.55) is a 2nd-order linear homogeneous ODE.

**Theorem 4** (2nd BVW Theorem [26])**.** *If* $\{\zeta\left(r\right), \mathrm{B}\left(r\right)\}$ *represents a perfect fluid sphere, then* $\{\zeta\left(r\right)\mathrm{Z}\left(r\right), \mathrm{B}\left(r\right)\}$ *also represents a perfect fluid sphere, where* $\sigma, \varepsilon$ *are arbitrary constants and*

$$Z\left(r\right) = \sigma + \varepsilon \int \frac{r\mathrm{d}r}{\zeta^2\left(r\right)\sqrt{\mathrm{B}\left(r\right)}}. \tag{2.57}$$

*That is, the map or transformation*

$$\mathrm{T}_2 : \{\zeta\left(r\right), \mathrm{B}\left(r\right)\} \mapsto \{\zeta\left(r\right)\mathrm{Z}\left(r\right), \mathrm{B}\left(r\right)\}. \tag{2.58}$$

*takes perfect fluid spheres to perfect fluid spheres. Furthermore since* $\Upsilon(\mathrm{B})$ *is a two-dimensional vector space, any element of* $\Upsilon(\mathrm{B})$ *can be put in this form for an arbitrary choice of* $\zeta(r) \in \Upsilon(\mathrm{B})$ *and for suitable constants* $\sigma$ *and* $\epsilon$*.*

The proof is immediate from an application of reduction of order to the (assumed known) solution $\zeta(r)$ of the ODE (2.55). For one of these theorems chose to apply, it depends on the amount of background information which one has. If for instance, one has a table or list of perfect fluid spheres [33, 34] that already contain distinct perfect fluid spheres $\{\zeta_1(r), \mathrm{B}(r)\}$ and $\{\zeta_2(r), \mathrm{B}(r)\}$ then summation is the easiest course. If after diligent searching one only finds a single solution $\{\zeta(r), \mathrm{B}(r)\}$, then the 2nd BVW theorem is indicated — though it might be difficult or impossible to perform the integration hiding inside the factor Z in a simple closed form. If for some specified B$(r)$ no previously known perfect fluid spheres are found, then one has to resort to attempting to directly solve the ODE (2.55). This may or may not be possible. In any of these cases once two distinct solutions have been found, then the general solution is automatic via the summation theorem — there is no point to trying to apply the summation theorem and 2nd BVW theorem in tandem, as either one of them will yield the full solution space.

**Isotropic coordinates**

Isotropic coordinates are quite common in the study of perfect fluid spheres. These coordinates are used in about 35% of the relevant literature [44]. In isotropic coordinates the metric is most conveniently given by

$$\mathrm{d}s^2 = -\zeta(r)^2\mathrm{d}t^2 + \frac{1}{\zeta(r)^2 B(r)^2}\{\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\}. \tag{2.59}$$

The Einstein tensor components are

$$G_{\hat{r}\hat{r}} = -\frac{2BB'^2}{r} + B'^2\zeta^2 - \zeta'^2 B^2, \tag{2.60}$$

$$G_{\hat{\theta}\hat{\theta}} = G_{\hat{\phi}\hat{\phi}} = -\frac{BB'^2}{r} + B'^2\zeta^2 - BB''^2 + B^2\zeta'^2, \tag{2.61}$$

$$G_{\hat{t}\hat{t}} = 2B^2\zeta\zeta'' + \frac{4B^2\zeta\zeta'}{r} - 3B^2\zeta'^2 - 2BB'\zeta\zeta' + 2BB''\zeta^2 - 3B'^2\zeta^2 + \frac{4BB'\zeta^2}{r}. \tag{2.62}$$

The ODE coming from the isotropy demand is either

$$\left(\frac{\zeta'}{\zeta}\right)^2 = \frac{B'' - B'/r}{2B}; \qquad \frac{\zeta'}{\zeta} = \pm\sqrt{\frac{B'' - B'/r}{2B}}, \tag{2.63}$$

or

$$B'' - \frac{B'}{r} - 2\left(\frac{\zeta'}{\zeta}\right)^2 B = 0. \tag{2.64}$$

The ODE for $\zeta(r)$ can either be viewed as a first-order nonlinear ODE, or as a *pair* of first-order linear ODEs. In counterpoint to the situation for Schwarzschild coordinates, here it is the ODE for $\mathrm{B}(r)$ that is second-order linear homogeneous. This the additivity theorem will now be a theorem for $\mathrm{B}(r)$.

**Definition 5** (Solution space — isotropic)**.** *Let $\Upsilon(\zeta)$ be the set of all solutions to equation* (2.64) *for fixed $\zeta(r)$.*

**Theorem 6** (Summation — isotropic)**.** *Let $\{\zeta(r), \mathrm{B}_1(r)\}$ and $\{\zeta(r), \mathrm{B}_2(r)\}$ represent perfect fluid spheres. Then for any arbitrary linear combination*

$$\mathrm{B}(r) = c_1\mathrm{B}_1(r) + c_2\mathrm{B}_2(r), \tag{2.65}$$

*the pair $\{\zeta(r), \mathrm{B}(r)\}$ also represents a perfect fluid sphere. Furthermore since $\Upsilon$ is a two-dimensional vector space, any element of $\Upsilon(\mathrm{B})$ can be put in this form for suitable constants $c_1$ and $c_2$.*

The proof is immediate from the fact that equation (2.64) is a 2nd-order linear homogeneous ODE.

**Theorem 7** (8th BVW Theorem [26])**.** *Let $\{\zeta(r), \mathrm{B}(r)\}$ be a perfect fluid sphere and let*

$$Z\left(r\right) = \sigma + \varepsilon\int\frac{r\mathrm{d}r}{\mathrm{B}\left(r\right)^2}. \tag{2.66}$$

*Then for all $\sigma$ and $\varepsilon$, with fixed $\zeta(r)$, the pair $\{\zeta(r), \mathrm{Z}(r)\mathrm{B}(r)\}$ also represents a perfect fluid sphere. That is, in isotropic coordinates, if*

$$\mathrm{d}s^2 = -\zeta(r)^2\mathrm{d}t^2 + \frac{1}{\zeta(r)^2\mathrm{B}(r)^2}\{\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\} \tag{2.67}$$

*is a perfect fluid sphere then*

$$\mathrm{d}s^2 = -\zeta(r)^2\mathrm{d}t^2 + \frac{1}{\zeta(r)^2\mathrm{B}(r)^2\mathrm{Z}(r)^2}\{\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\} \tag{2.68}$$

*is also a perfect fluid sphere. That is, the mapping*

$$\mathrm{T}_8 : \{\zeta\,(r)\,, \mathrm{B}\,(r)\} \mapsto \{\zeta\,(r)\,, \mathrm{B}\,(r)\,\mathrm{Z}(\mathrm{B}\,(r))\} \tag{2.69}$$

*maps perfect fluid spheres into perfect fluid spheres. Furthermore since* $\Upsilon(\zeta)$ *is a two-dimensional vector space, any element of* $\Upsilon(\zeta)$ *can be put in this form for an arbitrary choice of* $\mathrm{B}(r) \in \Upsilon(\zeta)$ *and for suitable constants* $\sigma$ *and* $\epsilon$*.*

The proof is immediate from an application of reduction of order to the (assumed known) solution $\mathrm{B}(r)$ of the ODE (2.64). Note that in comparing Schwarzschild coordinates with isotropic coordinates the roles of $\zeta(r)$ and $\mathrm{B}(r)$ have effectively changed. Note also that the precise nature of the ODEs one encounters depend not only on the specific coordinate system one adopts but also on the choice of functional form of the spacetime metric. Analogously to the case of Schwarzschild coordinates, either the summation theorem or the 8th BVW theorem is enough to explore the entire two-dimensional solution space, now for $\mathrm{B}(r)$ at fixed $\zeta(r)$ — there is no point to trying to apply the summation theorem and 8th BVW theorem in tandem, as either one of them will yield the full solution space.

**Gaussian polar coordinates**

In Gaussian polar coordinates the metric can be put in the form

$$\mathrm{d}s^2 = -\zeta(r)^2 \mathrm{d}t^2 + \mathrm{d}r^2 + R(r)^2 \mathrm{d}\Omega^2. \tag{2.70}$$

The Einstein tensor components are easily computed

$$G_{\hat{t}\hat{t}} = \frac{-R'^2 - 2RR'' + 1}{R^2}, \tag{2.71}$$

$$G_{\hat{r}\hat{r}} = \frac{\zeta R'^2 - \zeta + 2\zeta' RR'}{R^2\zeta}, \tag{2.72}$$

$$G_{\hat{\theta}\hat{\theta}} = G_{\hat{\phi}\hat{\phi}} = \frac{\zeta R'' + \zeta' R' + \zeta'' R}{R\zeta}. \tag{2.73}$$

The ODE arising from the demand of pressure isotropy is

$$\zeta'' - \zeta' \frac{R'}{R} + \zeta \left\{ \frac{1 - R'^2 + R''R}{R^2} \right\} = 0. \tag{2.74}$$

We note that this is a second-order linear homogeneous ODE for $\zeta(r)$, and proceed in the by now quite standard manner.

**Definition 8** (Solution space — Gaussian)**.** *Let $\Upsilon(R)$ be the set of all solutions to equation (2.74) for fixed $R(r)$.*

**Theorem 9** (Summation — Gaussian)**.** *Now in Gaussian coordinates, let $\{\zeta_1(r), R(r)\}$ and $\{\zeta_2(r), R(r)\}$ represent perfect fluid spheres. Then for any arbitrary linear combination*

$$\zeta(r) = c_1\zeta_1(r) + c_2\zeta_2(r), \tag{2.75}$$

*the pair $\{\zeta(r), R(r)\}$ also represents a perfect fluid sphere. Furthermore since $\Upsilon(R)$ is a two-dimensional vector space, any element of $\Upsilon(R)$ can be put in this form for suitable constants $c_1$ and $c_2$.*

The proof is immediate from the fact that equation (2.74) is a 2nd-order linear homogeneous ODE.

**Theorem 10** (9th BVW Theorem [26])**.** *Suppose we are in Gaussian polar coordinates, and that $\{\zeta(r), R(r)\}$ represents a perfect fluid sphere. Define*

$$\Lambda(r) = \sigma + \varepsilon \int \frac{R(r)\,\mathrm{d}r}{\zeta(r)^2}. \tag{2.76}$$

*Then for all $\sigma$ and $\varepsilon$, the geometry defined by holding $R(r)$ fixed and setting*

$$\mathrm{d}s^2 = -\zeta(r)^2\Lambda(r)^2\,\mathrm{d}t^2 + \mathrm{d}r^2 + R(r)^2\mathrm{d}\Omega^2, \tag{2.77}$$

*is also a perfect fluid sphere. That is, the mapping*

$$\mathrm{T}_9 : \{\zeta, R\} \mapsto \{\zeta\Lambda(\zeta, R), R\} \tag{2.78}$$

*takes perfect fluid spheres into perfect fluid spheres. Furthermore since $\Upsilon(R)$ is a two-dimensional vector space, any element of $\Upsilon(R)$ can be put in this form for an arbitrary choice of $\zeta(r) \in \Upsilon(R)$ and for suitable constants $\sigma$ and $\epsilon$.*

The proof is immediate from an application of reduction of order to the (assumed known) solution $\zeta(r)$ of the ODE (2.74). Analogously to the case of Schwarzschild or isotropic coordinates, either the summation theorem or the 9th BVW theorem is enough to explore the entire two-dimensional solution space, now for $\zeta(r)$ at fixed $R(r)$ — there is no point to trying to apply the summation theorem and 9th BVW theorem in tandem, as either one of them will yield the full solution space.

**Synge isothermal coordinates**

Similarly, consider the metric in Synge isothermal coordinates, that is

$$\mathrm{d}s^2 = -\zeta(r)^{-2}\left\{\mathrm{d}t^2 - \mathrm{d}r^2 + R(r)^2\mathrm{d}\Omega^2\right\}. \tag{2.79}$$

The Einstein's tensor components are

$$G_{\hat{t}\hat{t}} = \frac{-3R^2\zeta'^2 + 2R^2\zeta\zeta'' + 4RR'\zeta\zeta'^2 + \zeta^2 - R'^2\zeta^2 - 2RR''^2\zeta^2}{R^2}, \tag{2.80}$$

$$G_{\hat{r}\hat{r}} = \frac{3R^2\zeta'^2 - 4R\zeta R'\zeta' + R'^2\zeta^2 - \zeta^2}{R^2}, \tag{2.81}$$

$$G_{\hat{\theta}\hat{\theta}} = \frac{3R\zeta'^2 - 2\zeta R'\zeta' - 2R\,\zeta\zeta'' + \zeta^2 R''}{R}. \tag{2.82}$$

Demanding isotropy yields the ODE

$$\zeta'' - \zeta'\frac{R'}{R} - \zeta\left\{\frac{1 - R'^2 + RR''}{2R^2}\right\} = 0. \tag{2.83}$$

We now proceed as usual.

**Definition 11** (Solution space — Synge)**.** *Let $\Upsilon(R)$ be the set of all solutions to equation (2.83) for fixed $R(r)$.*

**Theorem 12** (Summation — Synge)**.** *Now in Synge isothermal coordinates let* $\{\zeta_1(r), R(r)\}$ *and* $\{\zeta_2(r), R(r)\}$ *represent perfect fluid spheres. Then for any arbitrary linear combination*

$$\zeta(r) = c_1\zeta_1(r) + c_2\zeta_2(r), \tag{2.84}$$

*the pair* $\{\zeta(r), R(r)\}$ *also represents a perfect fluid sphere. Furthermore since* $\Upsilon(R)$ *is a two-dimensional vector space, any element of* $\Upsilon(R)$ *can be put in this form for suitable constants* $c_1$ *and* $c_2$*.*

The proof is immediate from the fact that equation (2.83) is a 2nd-order linear homogeneous ODE.

**Theorem 13** (11th BVW Theorem [26])**.** *Suppose* $\{\zeta(r), R(r)\}$ *represents a perfect fluid sphere in Synge isothermal coordiantes. Define*

$$A(r) = \sigma + \varepsilon \int \frac{R(r)\,\mathrm{d}r}{\zeta(r)^2}. \tag{2.85}$$

*Then for all* $\sigma$ *and* $\varepsilon$*, the geometry defined by holding* $R(r)$ *fixed and setting*

$$\mathrm{d}s^2 = -\frac{1}{\zeta(r)^2 A(r)^2}\left\{\mathrm{d}t^2 - \mathrm{d}r^2\right\} + \frac{R(r)^2}{\zeta(r)^2 A(r)^2}\mathrm{d}\Omega^2 \tag{2.86}$$

*is also a perfect fluid sphere. That is, the mapping*

$$\mathrm{T}_{11} : \{\zeta, R\} \mapsto \{\zeta A(\zeta, R), R\} \tag{2.87}$$

*takes perfect fluid spheres into perfect fluid spheres. Furthermore since* $\Upsilon(R)$ *is a two-dimensional vector space, any element of* $\Upsilon(R)$ *can be put in this form for an arbitrary choice of* $\zeta(r) \in \Upsilon(R)$ *and for suitable constants* $\sigma$ *and* $\epsilon$*.*

The proof is immediate from an application of reduction of order to the (assumed known) solution $\zeta(r)$ of the ODE (2.83). Analogously to the previous cases, either the summation theorem or the 11th BVW theorem is enough to explore the entire two-dimensional solution space, now for $\zeta(r)$ at fixed $R(r)$ — there is no point to trying to apply the summation theorem and 11th BVW theorem in tandem, as either one of them will yield the full solution space.

### 2.7.10 Weighted Means

In this part, we will introduce this theorem called "**weighted means**". This theorem is used to generate new $\mathrm{B}(r)$ by finding weighted means of perfect fluid sphere with fixed $\zeta_0(r)$ for each $\mathrm{B}_i(r)$.

**Corollary 14** (Weighted Means)**.** *Let* $\{\{\zeta_0(r), \mathrm{B}_i(r)\}\}_{i=1}^n$ *be the family of perfect fluid spheres with fixed* $\zeta_0(r)$*, then* $\left\{\zeta_0(r), \sum_{i=1}^n p_i \mathrm{B}_i(r) \middle/ \sum_{i=1}^n p_i\right\}$ *is the perfect fluid sphere for all arbitrary constants* $p_1, p_2, ..., p_n$*.*

*Proof.* Let $\{\{\zeta_0(r), \mathrm{B}_i(r)\}\}_{i=1}^n$ be the family of perfect fluid spheres with fixed $\zeta_0(r)$, then

$$\left[r(r\zeta_0)'\right]\mathrm{B}_i' + \left[2r^2\zeta_0'' - 2(r\zeta_0)'\right]\mathrm{B}_i + 2\zeta_0 = 0, \tag{2.88}$$

for all $i = 1, .., n$ for some $n \in \mathbb{N}$.

When we derive (2.88) with $p_i$, we derive

$$\left[r(r\zeta_0)'\right](p_i\mathrm{B}_i)' + \left[2r^2\zeta_0'' - 2(r\zeta_0)'\right]p_i\mathrm{B}_i + 2p_i\zeta_0 = 0. \tag{2.89}$$

We now, take summation of (2.89) for all $i = 1, .., n$, so

$$\left[r(r\zeta_0)'\right]\left(\sum_{i=1}^n p_i\mathrm{B}_i\right)' + \left[2r^2\zeta_0'' - 2(r\zeta_0)'\right]\sum_{i=1}^n p_i\mathrm{B}_i + 2\sum_{i=1}^n p_i\zeta_0 = 0, \tag{2.90}$$

i.e.

$$\left[r(r\zeta_0)'\right]\left(\frac{\sum_{i=1}^n p_i\mathrm{B}_i}{\sum_{i=1}^n p_i}\right)' + \left[2r^2\zeta_0'' - 2(r\zeta_0)'\right]\left(\frac{\sum_{i=1}^n p_i\mathrm{B}_i}{\sum_{i=1}^n p_i}\right) + 2\zeta_0 = 0. \tag{2.91}$$

Therefore $\left\{\zeta_0(r), \sum_{i=1}^{n} p_i \mathrm{B}_i(r) \Big/ \sum_{i=1}^{n} p_i\right\}$ is also the perfect fluid sphere for all arbitrary constants $p_1, p_2, ..., p_n$. □

The proof of this theorem is easy but it give us more advantages and better than we expected. For instance, if we need to eliminate some constants that we do not know, then we can choose suitable weights to eliminate them as desired.

| Name | Parameters | | | |
|---|---|---|---|---|
| | $\zeta(r)$ | $B_1(r)$, $B_2(r)$ and $\mathrm{B}(r) = \frac{aB_1(r)+bB_2(r)}{a+b}$ | $a$ | $b$ |
| M-W III | $\sqrt{Ar(r-a)}$ | $\mathrm{B}_1(r) = \frac{4}{7}\left(1-\frac{r^2}{a^2}\right)$ | - | - |
| Martin 2 | $\sqrt{Ar(r-a)}$ | $B_2(r) = \frac{4}{7}\left(1-\frac{r^2}{a^2}-B\frac{(r-a)r^{\frac{7}{3}}}{(4r-3a)^{\frac{4}{3}}}\right)$ | - | - |
| *Martin 2 | $\sqrt{Ar(r-a)}$ | $B(r) = \frac{4}{7}\left(1-\frac{r^2}{a^2}+\frac{Br^{\frac{7}{3}}(r-a)}{(4r-3a)^{\frac{4}{3}}}\right)$ | 1 | $-\frac{1}{B}$ |

Table 2.2: This table shows new $B(r)$ generated by applying the weighted means theorem.

### 2.7.11 New convenient transformation theorem

In this part, the new convenient transformation theorem is similar to the new technique but this transformation needs an initial $\mathrm{B}_0(r)$ to calculate a new $\mathrm{B}(r)$. This is not a problem since we want to generate a new $\mathrm{B}(r)$ from the initial perfect fluid sphere $\{\zeta_0(r), \mathrm{B}_0(r)\}$ that already has $\mathrm{B}_0(r)$ to use together with $\zeta_0(r)$. Thus, calculation for finding a new $\mathrm{B}(r)$ is easier than the new technique.

### 2.7.12 General diagonal coordinates

Consider the metric

$$\mathrm{d}s^2 = -\zeta\left(r\right)^2 \mathrm{d}t^2 + \frac{\mathrm{d}r^2}{\mathrm{B}\left(r\right)} + R\left(r\right)^2 \mathrm{d}\Omega^2, \tag{2.92}$$

and assume that it satisfies the condition of perfect fluid spheres, then

$$G_{\hat{r}\hat{r}} = G_{\hat{\theta}\hat{\theta}} = G_{\hat{\varphi}\hat{\varphi}}, \tag{2.93}$$

where

$$G_{\hat{r}\hat{r}} = \frac{\zeta \mathrm{B}\left(R'\right)^2 + 2\mathrm{B}R\zeta' R' - \zeta}{R^2\zeta}, \tag{2.94}$$

and

$$G_{\hat{\theta}\hat{\theta}} = -\frac{1}{2}\frac{2\mathrm{B}\zeta' R' + 2\zeta R''\mathrm{B} + \zeta\mathrm{B}' R' + 2R\zeta''\mathrm{B} + R\zeta'\mathrm{B}'}{R\zeta}. \tag{2.95}$$

By using $G_{\hat{r}\hat{r}} = G_{\hat{\theta}\hat{\theta}}$, which gives us an ODE

$$\left[R\left(R\zeta\right)'\right]\mathrm{B}' + \left[2RR''^2\zeta'' - 2RR'\zeta' - 2\left(R'\right)^2\zeta\right]\mathrm{B} + 2\zeta = 0, \tag{2.96}$$

we will construct the new transformation as the following:

**Theorem 15** (New convenience)**.** *In general diagonal coordinates, if we have a perfect fluid sphere* $\{\zeta_0(r), \mathrm{B}_0(r), R_0(r)\}$*, then*

$$\{\zeta_0(r), \mathrm{B}_0(r)\, Y_0(\zeta_0(r), \mathrm{B}_0(r), R_0(r)), R_0(r)\} \tag{2.97}$$

*is also a perfect fluid sphere, where*

$$Y_0\left(\zeta_0(r), \mathrm{B}_0(r)\right) \equiv 1 + k\exp\left(\int \frac{2\zeta_0(r)/\left[R_0(r)\left(R_0(r)\,\zeta_0(r)\right)'\right]}{\mathrm{B}_0(r)}\mathrm{d}r\right), \tag{2.98}$$

*and an arbitrary constant* $k$ *i.e. this transformation* $\mathrm{T}_g$ *maps a perfect fluid sphere to a perfect fluid sphere such that*

$$\mathrm{T}_g : \{\zeta_0(r), \mathrm{B}_0(r)\} \mapsto \{\zeta_0(r), \mathrm{B}_0(r)\, Y_0(\zeta_0(r), \mathrm{B}_0(r), R_0(r)), R_0(r)\}. \tag{2.99}$$

*Proof.* Let $\{\zeta_0(r), \mathrm{B}_0(r), R_0(r)\}$ be a perfect fluid sphere, then

$$\left[R_0 (R_0\zeta_0)'\right] \mathrm{B}_0' + \left[2R_0R_0''\zeta_0 + 2R_0^2\zeta_0'' - 2R_0R_0'\zeta_0' - 2(R_0')^2\zeta_0\right]\mathrm{B}_0 + 2\zeta_0 = 0, \tag{2.100}$$

i.e.

$$\mathrm{B}_0' + \left[\frac{2R_0R_0''\zeta_0 + 2R_0^2\zeta_0'' - 2R_0R_0'\zeta_0' - 2(R_0')^2\zeta_0}{R_0(R_0\zeta_0)'}\right]\mathrm{B}_0 + \frac{2\zeta_0}{R_0(R_0\zeta_0)'} = 0, \tag{2.101}$$

Conveniently, we will define

$$F_1(r) \equiv \frac{2R_0R_0''\zeta_0 + 2R_0^2\zeta_0'' - 2R_0R_0'\zeta_0' - 2(R_0')^2\zeta_0}{R_0(R_0\zeta_0)'}, \tag{2.102}$$

$$F_2(r) \equiv \frac{2\zeta_0}{R_0(R_0\zeta_0)'}. \tag{2.103}$$

Hence,

$$\mathrm{B}_0' + F_1(r)\mathrm{B}_0 + F_2(r) = 0. \tag{2.104}$$

Let $Y$ satisfy

$$(\mathrm{B}_0Y)' + F_1(r)\mathrm{B}_0Y + F_2(r) = 0. \tag{2.105}$$

Then

$$\begin{aligned}
(\mathrm{B}_0Y)' + F_1(r)\mathrm{B}_0Y + F_2(r) &= \mathrm{B}_0Y' + Y\mathrm{B}_0' + F_1(r)\mathrm{B}_0Y && (2.106)\\
&\quad + F_2(r), && (2.107)\\
&= \mathrm{B}_0Y' + Y\mathrm{B}_0' + F_1(r)\mathrm{B}_0Y && (2.108)\\
&\quad + F_2(r)Y - F_2(r)Y + F_2(r), && (2.109)\\
&= Y(\mathrm{B}_0' + F_1(r)\mathrm{B}_0 + F_2(r)) && (2.110)\\
&\quad + \mathrm{B}_0Y' - F_2(r)Y + F_2(r), && (2.111)\\
&= \mathrm{B}_0Y' - F_2(r)Y + F_2(r). && (2.112)
\end{aligned}$$

Now we derive the new first ODE

$$\mathrm{B}_0Y' - F_2(r)Y + F_2(r) = 0. \tag{2.113}$$

We know that

$$\frac{Y'}{Y-1} = \frac{(Y-1)'}{Y-1}, \tag{2.114}$$

$$= (\ln(Y-1))', \tag{2.115}$$

then

$$\frac{Y'}{Y-1} = \frac{F_2(r)}{\mathrm{B}_0}, \tag{2.116}$$

$$(\ln(Y-1))' = \frac{F_2(r)}{\mathrm{B}_0}, \tag{2.117}$$

$$\ln(Y-1) = C + \int \frac{F_2(r)}{\mathrm{B}_0}\mathrm{d}r, \tag{2.118}$$

$$Y-1 = \exp\left(C + \left(\int \frac{F_2(r)}{\mathrm{B}_0}\mathrm{d}r\right)\right), \tag{2.119}$$

$$= k\exp\left(\int \frac{F_2(r)}{\mathrm{B}_0}\mathrm{d}r\right), \tag{2.120}$$

$$Y = 1 + k\exp\left(\int \frac{F_2(r)}{\mathrm{B}_0}\mathrm{d}r\right). \tag{2.121}$$

By substituting $F_2(r) \equiv \frac{2\zeta_0}{R_0(R_0\zeta_0)'}$, we derive

$$Y = 1 + k\exp\left(\int \frac{2\zeta_0/\left[R_0(R_0\zeta_0)'\right]}{\mathrm{B}_0(r)}\mathrm{d}r\right), \tag{2.122}$$

where $k$ is an arbitrary constant which comes from integration.

Thus, this theorem reduces the several terms for calculation by using $\mathrm{B}_0(r)$ that we already know from the assumption.

Since this case is the same as in Schwarzschild coordinates, this transformation $\mathrm{T}_g$ is not idempotent. □

### 2.7.13 Schwarzschild coordinates

**Theorem 16** (New convenience)**.** *In Schwarzschild coordinates, if we have a perfect fluid sphere* $\{\zeta_0(r), \mathrm{B}_0(r)\}$*, then* $\{\zeta_0(r), \mathrm{B}_0(r)\Lambda(\zeta_0(r), \mathrm{B}_0(r))\}$ *is also a* *perfect fluid sphere, where*

$$\Lambda\left(\zeta_0\left(r\right),\mathrm{B}_0\left(r\right)\right)\equiv 1+k\exp\left(\int\frac{2\zeta_0\left(r\right)/\left[r\left(r\zeta_0\left(r\right)\right)'\right]}{\mathrm{B}_0\left(r\right)}\mathrm{d}r\right), \tag{2.123}$$

*and an arbitrary constant* $k$ *i.e. this transformation* $\mathrm{T}_s$ *maps a perfect fluid sphere to a perfect fluid sphere such that*

$$\mathrm{T}_s:\left\{\zeta_0\left(r\right),\mathrm{B}_0\left(r\right)\right\}\mapsto\left\{\zeta_0\left(r\right),\mathrm{B}_0\left(r\right)\Lambda\left(\zeta_0\left(r\right),\mathrm{B}_0\left(r\right)\right)\right\}. \tag{2.124}$$

*Proof.* Let $\left\{\zeta_0\left(r\right),\mathrm{B}_0\left(r\right)\right\}$ be a perfect fluid sphere, then

$$\left[r\left(r\zeta_0\right)'\right]\mathrm{B}_0'+\left[2r^2\zeta_0''-2\left(r\zeta_0\right)'\right]\mathrm{B}_0+2\zeta_0=0, \tag{2.125}$$

i.e.

$$\mathrm{B}_0'+\left[\frac{2r^2\zeta_0''-2\left(r\zeta_0\right)'}{r\left(r\zeta_0\right)'}\right]\mathrm{B}_0+\frac{2\zeta_0}{r\left(r\zeta_0\right)'}=0. \tag{2.126}$$

Conveniently, we will define

$$F_1\left(r\right)\equiv\frac{2r^2\zeta_0''-2\left(r\zeta_0\right)'}{r\left(r\zeta_0\right)'}, \tag{2.127}$$

$$F_2\left(r\right)\equiv\frac{2\zeta_0}{r\left(r\zeta_0\right)'}. \tag{2.128}$$

Hence,

$$\mathrm{B}_0'+F_1\left(r\right)\mathrm{B}_0+F_2\left(r\right)=0. \tag{2.129}$$

Let $\Lambda$ satisfy

$$\left(\mathrm{B}_0\Lambda\right)'+F_1\left(r\right)\mathrm{B}_0\Lambda+F_2\left(r\right)=0. \tag{2.130}$$

Then

$$(\mathrm{B}_0\Lambda)' + F_1(r)\,\mathrm{B}_0\Lambda + F_2(r) = \mathrm{B}_0\Lambda' + \Lambda\mathrm{B}_0' + F_1(r)\,\mathrm{B}_0\Lambda \tag{2.131}$$

$$+F_2(r)\,, \tag{2.132}$$

$$= \mathrm{B}_0\Lambda' + \Lambda\mathrm{B}_0' + F_1(r)\,\mathrm{B}_0\Lambda \tag{2.133}$$

$$+F_2(r)\,\Lambda - F_2(r)\,\Lambda + F_2(r)\,, \tag{2.134}$$

$$= \Lambda\left(\mathrm{B}_0' + F_1(r)\,\mathrm{B}_0 + F_2(r)\right) \tag{2.135}$$

$$+\mathrm{B}_0\Lambda' - F_2(r)\,\Lambda + F_2(r)\,, \tag{2.136}$$

$$= \mathrm{B}_0\Lambda' - F_2(r)\,\Lambda + F_2(r)\,. \tag{2.137}$$

Now we derive the new first ODE

$$\mathrm{B}_0\Lambda' - F_2(r)\,\Lambda + F_2(r) = 0. \tag{2.138}$$

We know that

$$\frac{\Lambda'}{\Lambda - 1} = \frac{(\Lambda - 1)'}{\Lambda - 1}, \tag{2.139}$$

$$= \left(\ln(\Lambda - 1)\right)'\,, \tag{2.140}$$

then

$$\frac{\Lambda'}{\Lambda - 1} = \frac{F_2(r)}{\mathrm{B}_0}, \tag{2.141}$$

$$\left(\ln(\Lambda - 1)\right)' = \frac{F_2(r)}{\mathrm{B}_0}, \tag{2.142}$$

$$\ln(\Lambda - 1) = C + \int \frac{F_2(r)}{\mathrm{B}_0}\mathrm{d}r, \tag{2.143}$$

$$\Lambda - 1 = \exp\left(C + \left(\int \frac{F_2(r)}{\mathrm{B}_0}\mathrm{d}r\right)\right), \tag{2.144}$$

$$= k\exp\left(\int \frac{F_2(r)}{\mathrm{B}_0}\mathrm{d}r\right), \tag{2.145}$$

$$\Lambda = 1 + k\exp\left(\int \frac{F_2(r)}{\mathrm{B}_0}\mathrm{d}r\right). \tag{2.146}$$

By substituting $F_2(r) \equiv \frac{2\zeta_0}{r(r\zeta_0)'}$, we derive

$$\Lambda = 1 + k\exp\left(\int \frac{2\zeta_0(r)}{r\left(r\zeta_0(r)\right)'\mathrm{B}_0(r)}\mathrm{d}r\right), \tag{2.147}$$

where $k$ is an arbitrary constant which comes from integration.

Thus, this theorem reduces the several terms for calculation by using $\mathrm{B}_0(r)$ that we already know from the assumption.This transformation $\mathrm{T}_s$ is not idempotent since

$$\mathrm{T}_s \circ \mathrm{T}_s : \{\zeta_0, \mathrm{B}_0\} \mapsto \{\zeta_0, \mathrm{B}_0\Lambda_0\} \mapsto \{\zeta_0, \mathrm{B}_0\Lambda_0\Lambda_1\}, \tag{2.148}$$

and

$$\begin{aligned}
\Lambda_0 \times \Lambda_1 \;&=\; \left(1 + k \exp\left(\int \frac{2\zeta_0(r)}{r\,(r\zeta_0(r))'\,\mathrm{B}_0(r)}\mathrm{d}r\right)\right) && (2.149)\\
&\times \left(1 + k \exp\left(\int \frac{2\zeta_0(r)}{r\,(r\zeta_0(r))'\,\mathrm{B}_0(r)\,\Lambda_0}\mathrm{d}r\right)\right), && (2.150)\\
&\neq \left[1 + k \exp\left(\int \frac{2\zeta_0(r)}{r\,(r\zeta_0(r))'\,\mathrm{B}_0(r)}\mathrm{d}r\right)\right] = \Lambda && (2.151)
\end{aligned}$$

for an arbitrary form of $\Lambda = 1 + k\exp\left(\int \frac{2\zeta_0(r)}{r(r\zeta_0(r))'\mathrm{B}_0(r)}\mathrm{d}r\right)$. □

$$\begin{aligned}
F_3(m, R, r) \;&\equiv\; \exp\left(-2R^2 \int \frac{1}{-rR^2 + mR^2 + 2r^3}\mathrm{d}r\right), && (2.152)\\
F_4(A, B, r) \;&\equiv\; \frac{\exp\left(-\frac{A}{4Br^4}\right)}{-r + 2a\exp\left(-\frac{A}{4Br^4}\right)}. && (2.153)
\end{aligned}$$

* the notation "J3*, J4*" means they are modified metrics by applying new convenient transformation theorem ($\mathrm{T}_s$) (Kottler for J3, Wyman IIb (n=2) for J4*). Indeed, J1*, J2* (from integrating factor) and J3*, J4* (from new convenient) form are equivalent but both of them are produced in different ways depending on each algorithm, respectively.

### 2.7.14 Exponential coordinates

Consider the metric

$$\mathrm{d}s^2 = -\exp(-2z)\,\mathrm{d}t^2 + \exp(+2z)\left\{\frac{\mathrm{d}z^2}{\mathrm{B}(z)} + R(z)^2\,\mathrm{d}\Omega^2\right\}, \tag{2.154}$$

| Name | Parameters | | | Theorem 4 |
|---|---|---|---|---|
| | $\{\zeta(r), B(r)\}$ | $\Lambda\left(\zeta(r), B(r)\right)$ | $k$ | |
| Minkowski | $\{1, 1\}$ | $1 + kr^2$ | $-\frac{1}{R^2}$ | Einstein static |
| Exterior Schwarz-schild | $\left\{\begin{matrix}\sqrt{1-2m/r}, \\ 1-2m/r\end{matrix}\right\}$ | $1 + k(r-m)^2$ | $4C$ | Kuch68 II |
| Tolman V (A=0) | $\left\{\begin{matrix}Br^{1+n}, \\ (2-n^2)^{-1}\end{matrix}\right\}$ | $1 + kr^{\left(\frac{2\left(2-n^2\right)}{2+n}\right)}$ | $-A$ | Tolman V |
| K-O III | $\{A + Br^2, 1\}$ | $1 + k\frac{r^2}{(A+3Br^2)^{2/3}}$ | $-C$ | Martin 3 |
| Kuch1 Ib | $\left\{Ar + Br\ln r, \frac{1}{2}\right\}$ | $1+\frac{kr^2}{2A+2B\ln r+B}$ | $-C$ | Martin 1 |
| Kottler | $\left\{\begin{matrix}c\sqrt{1-\frac{2m}{r}-\frac{r^2}{R^2}}, \\ 1-\frac{2m}{r}-\frac{r^2}{R^2}\end{matrix}\right\}$ | $1 + kF_3\left(m, R, r\right)$ | $k$ | J3* |
| Wyman IIb (n=2) | $\left\{\begin{matrix}\frac{A}{r} - Br^3, \\ -\frac{1}{2}+\frac{a}{r}\exp\left(-\frac{A}{4Br^4}\right)\end{matrix}\right\}$ | $1 + kF_4\left(A, B, r\right)$ | $k$ | J4* |

Table 2.3: This table shows new $B(r)$ generated by applying the new convenient transformation theorem ($\mathrm{T}_s$).

and assume that it satisfies the condition of perfect fluid spheres, then

$$G_{\hat{z}\hat{z}} = G_{\hat{\theta}\hat{\theta}} = G_{\hat{\varphi}\hat{\varphi}}, \tag{2.155}$$

where

$$G_{\hat{z}\hat{z}} = \frac{\left(\mathrm{B}R^2 - (R')^2\,\mathrm{B} + 1\right)\exp\left(-2z\right)}{R^2}, \tag{2.156}$$

and

$$G_{\hat{\theta}\hat{\theta}} = -\frac{1}{2}\frac{\left(\mathrm{B}'R' + 2R''\mathrm{B} + 2\mathrm{B}R\right)\exp\left(-2z\right)}{R}. \tag{2.157}$$

By using $G_{\hat{z}\hat{z}} = G_{\hat{\theta}\hat{\theta}}$, which gives us an ODE

$$[RR']\,\mathrm{B}' + \left[4R^2 - 2\,(R')^2 + 2RR''\right]\mathrm{B} + 2 = 0, \tag{2.158}$$

we will construct the new transformation as the following:

**Theorem 17** (New convenience)**.** *In exponential coordinates, if we have a perfect fluid sphere* $\{\mathrm{B}_0\,(z)\,, R_0\,(z)\}$*, then* $\{\mathrm{B}_0\,(z)\,V_0\,(\mathrm{B}_0\,(z)\,, R_0\,(z))\,, R_0\,(z)\}$ *is also a perfect fluid sphere, where*

$$V_0\,(\mathrm{B}_0\,(z)\,, R_0\,(z)) \equiv 1 + k\exp\left(\int \frac{2/[R_0R_0']}{\mathrm{B}_0\,(z)}\mathrm{d}z\right), \tag{2.159}$$

*and for an arbitrary constant* $k$ *i.e. this transformation* $\mathrm{T}_e$ *maps a perfect fluid sphere to a perfect fluid sphere such that*

$$\mathrm{T}_e : \{\mathrm{B}_0\,(z)\,, R_0\,(z)\} \mapsto \{\mathrm{B}_0\,(z)\,V_0\,(\mathrm{B}_0\,(z)\,, R_0\,(z))\,, R_0\,(z)\}\,. \tag{2.160}$$

*Proof.* Let $\{\mathrm{B}_0\,(z)\,, R_0\,(z)\}$ be a perfect fluid sphere, then

$$[R_0R_0']\,\mathrm{B}_0' + \left[4R_0^2 - 2\,(R_0')^2 + 2R_0R_0''\right]\mathrm{B}_0 + 2 = 0, \tag{2.161}$$

i.e.

$$\mathrm{B}_0' + \left[\frac{4R_0^2 - 2\,(R_0')^2 + 2R_0R_0''}{R_0R_0'}\right]\mathrm{B}_0 + \frac{2}{R_0R_0'} = 0, \tag{2.162}$$

Conveniently, we will define

$$F_1\,(r) \equiv \frac{4R_0^2 - 2\,(R_0')^2 + 2R_0R_0''}{R_0R_0'}, \tag{2.163}$$

$$F_2\,(r) \equiv \frac{2}{R_0R_0'}. \tag{2.164}$$

Hence,

$$\mathrm{B}_0' + F_1\,(r)\,\mathrm{B}_0 + F_2\,(r) = 0. \tag{2.165}$$

Let $V$ satisfy

$$(\mathrm{B}_0V)' + F_1\,(r)\,\mathrm{B}_0V + F_2\,(r) = 0. \tag{2.166}$$

Then

$$
\begin{aligned}
(\mathrm{B}_0 V)' + F_1(r)\,\mathrm{B}_0 V + F_2(r) &= \mathrm{B}_0 V' + V\mathrm{B}_0' + F_1(r)\,\mathrm{B}_0 V \\
&\quad + F_2(r)\,, && (2.167)\\
&= \mathrm{B}_0 V' + V\mathrm{B}_0' + F_1(r)\,\mathrm{B}_0 V + F_2(r)\,V \\
&\quad - F_2(r)\,V + F_2(r)\,, && (2.168)\\
&= V\left(\mathrm{B}_0' + F_1(r)\,\mathrm{B}_0 + F_2(r)\right) \\
&\quad + \mathrm{B}_0 V' - F_2(r)\,V + F_2(r)\,, && (2.169)\\
&= \mathrm{B}_0 V' - F_2(r)\,V + F_2(r)\,. && (2.170)
\end{aligned}
$$

Now we derive the new first ODE

$$\mathrm{B}_0 V' - F_2(r)\,V + F_2(r) = 0. \tag{2.171}$$

We know that

$$
\begin{aligned}
\frac{V'}{V-1} &= \frac{(V-1)'}{V-1}, && (2.172)\\
&= (\ln(V-1))'\,, && (2.173)
\end{aligned}
$$

then

$$
\begin{aligned}
\frac{V'}{V-1} &= \frac{F_2(r)}{\mathrm{B}_0}, && (2.174)\\
(\ln(V-1))' &= \frac{F_2(r)}{\mathrm{B}_0}, && (2.175)\\
\ln(V-1) &= C + \int \frac{F_2(r)}{\mathrm{B}_0}\mathrm{d}r, && (2.176)\\
V-1 &= \exp\left(C + \left(\int \frac{F_2(r)}{\mathrm{B}_0}\mathrm{d}r\right)\right), && (2.177)\\
&= k\exp\left(\int \frac{F_2(r)}{\mathrm{B}_0}\mathrm{d}r\right), && (2.178)\\
V &= 1 + k\exp\left(\int \frac{F_2(r)}{\mathrm{B}_0}\mathrm{d}r\right). && (2.179)
\end{aligned}
$$

By substituting $F_2(r) \equiv \frac{2}{R_0 R_0'}$, we derive

$$V = 1 + k\exp\left(\int \frac{2/[R_0 R_0']}{\mathrm{B}_0(r)}\mathrm{d}r\right), \tag{2.180}$$

where $k$ is an arbitrary constant which comes from integration.

Thus, this theorem reduces the several terms for calculation by using $\mathrm{B}_0(r)$ that we already know from the assumption.

Since this case is the same as in Schwarzschild coordinates. Thus, in the exponential coordinates, this transformation $\mathrm{T}_e$ is not idempotent. □

## 2.8 Relations of three transformations

Before starting, we will introduce the 1st BVW transformation ($\mathrm{T}_1$) as following [26]:

**Theorem 18** (1st BVW transformation ($\mathrm{T}_1$)[26]). *Suppose* $\{\zeta_0(r), \mathrm{B}_0(r)\}$ *represents a perfect fluid sphere. Define*

$$\Delta_0(r) = \left(\frac{\zeta_0(r)}{\zeta_0(r) + r\zeta_0'(r)}\right)^2 r^2 \exp\left\{2\int \frac{\zeta_0'(r)}{\zeta_0(r)} \frac{\zeta_0(r) - r\zeta_0'(r)}{\zeta_0(r) + r\zeta_0'(r)} \mathrm{d}r\right\}. \qquad (2.181)$$

*Then for all arbitrary constant* $\lambda$*, the geometry defined by holding* $\zeta_0(r)$ *fixed and setting*

$$\mathrm{d}s^2 = -\zeta_0(r)^2 \mathrm{d}t^2 + \frac{\mathrm{d}r^2}{\mathrm{B}_0(r) + \lambda\Delta_0(r)} + r^2\mathrm{d}\Omega^2 \qquad (2.182)$$

*is also a perfect fluid sphere. That is, the mapping*

$$\mathrm{T}_1(\lambda) : \{\zeta_0, \mathrm{B}_0\} \mapsto \{\zeta_0, \mathrm{B}_0 + \lambda\Delta_0(\zeta_0)\} \qquad (2.183)$$

*takes perfect fluid spheres into perfect fluid spheres. Furthermore, it has idempotence.*

Thus, we will find the relation of all three transformations as follows:

### 2.8.1 Algorithm and proof for calculation

In the 1st BVW transformation, its algorithm defines the summation's term ($\Delta_0(r)$). Then, we substitute this term into an ODE for $\mathrm{B}(r)$ and find its value which is above. In transformations from new technique, its algorithm uses the integrating factor method to directly find the solution $\mathrm{B}(r)$. Also, if we know an initial $\mathrm{B}_0(r)$, then we can take $\mathrm{B}(r)$ in the term of $\mathrm{B}_0(r)$. In the new convenient transformation theorem ($\mathrm{T}_0$), its algorithm adds the factor $\Lambda$ multiplying with an initial $\mathrm{B}_0(r)$, then find its value. The comparison of these transformations is how complex these algorithms are. The first and the third transformation theorems are just substituting for each the summation's term as we define an ODE for $\mathrm{B}(r)$ and find its value but for the second, we need to know the integrating factor method before finding the solution $\mathrm{B}(r)$ and define an initial $\mathrm{B}_0(r)$ to derive the summation's term. Hence, the first and the third transformation theorems will reduce the steps of algorithm.

### 2.8.2 The summation's term and its amounts of terms

By considering each of these three summation's term, the 1st BVW transformation has an easy way to compute since there is no second derivative of $\zeta(r)$. Also, it is easy to make the mathematical program such as this maplet for calculating new $\mathrm{B}(r)$ because the second derivative of $\zeta(r)$ does not sometimes exist. For the second, it has nearly the same amounts of terms in computation but it has the second derivative of $\zeta(r)$, then we need to find it before calculating. For the third, it has the least amounts of terms in computation although it has to know an initial $\mathrm{B}_0(r)$ for solving. Instead of its difficulty, it is useful to bring an initial $\mathrm{B}_0(r)$ for calculating. However, if we want to transform some perfect fluid sphere with an initial $\mathrm{B}_0(r)$, then it is preferred to the third transformation.

| Transformation | 1st BVW | Integrating factor | New convenient form |
|---|---|---|---|
| Number of terms | many | moderate | few |
| $\zeta''(r)$ exists | No | Yes | No |
| Using $B_0(r)$ | No | No | Yes |
| Idempotence | Yes | Yes | No |
| Complexity | Yes | Yes | No |

Table 2.4: Comparison between three transformations.

## 2.9 Conclusion

In this chapter, we have introduced many theorems, and corollary. These will help us generate the new metric of perfect fluid spheres easier. Instead of explaining the details of the analysis yet again, we would like to stress a few points that we believe are useful to understand the overall concept of these transformations as the followings:

- The first theorem (Integrating factor) can be used to generate new Beta with fixed Zeta where new Beta comes from summation of the initial Beta and a term that consists of an integrating factor with an arbitrary constant. This theorem is the same as the first theorem in [26] but less terms for calculating. This theorem uses the method of solving ODE by using an integrating factor.

- A binary operator on perfect fluid spheres called "summation" will be used with the second transformation in [26] to construct additivity of the second transformation to generate Zeta with fixed Beta.

- Weighted means can be used to generate new Beta by finding weighted means of perfect fluid sphere with fixed Zeta for each Beta. The proof of this theorem is easy but it gives us more advantages than we expect. For instance, if we need to eliminate some constants that we do not know, then we can choose suitable weights to eliminate them as desired.

- The third theorem (new convenient transformation theorem) can also be used to generate new Beta with fixed Zeta which is consistent with the first theorm. But the method to get new Beta is different. Instead of summation, this theorem uses a multiplicative factor with the initial Beta to generate new Beta. Again, this theorem is equivalent to the first theorem in [26] and the first theorem in this thesis. However, this theorem can be used with the least number of terms for calculating. Thus, it should be called a new convenient transformation theorem.

# CHAPTER III

# CONCLUSION

In general, there are many ways to find solutions to various problems. Indeed, we need the best way for deriving our solutions as required. For instance, if we can reduce the steps or modules used in calculating the solutions, then our tasks will be more conveniently completed. Moreover, we can reduce not only the algorithms but also complexity, variables and calculations. In physics, we would like to extend theorem from a few cases to general cases. In this thesis, there are not only theoretical physics but also applied mathematics and computational programs for calculation.

## 3.1 The main concepts and analysis of this thesis

Firstly, in chapters one and two, we highlighted the key features: special relativity and general relativity. Before we derived the solutions to the problems related to these relativities, we had to learn about the physical values such as tensors (stress energy tensor, metric tensor, Riemann tensor, Ricci tensor, etc.). All physical values used in this thesis come from relativity. However, we had to understand special relativity before learning general relativity since it is easier to comprehend special relativity. Then we extended this relativity to general relativity. These first three chapters were background information for the construction of our researches which appeared in chapter four.

In chapter three, we focused on the solutions of black holes in the form of perfect fluid spheres. Mainly, in this thesis, we have developed a few transformation theorems that map perfect fluid spheres to perfect fluid spheres using all related coordinates. Moreover, we have established other algorithms or different ways that are equivalent to the first BVW transformation theorem for upgrading methods and advantages such as reducing operation counting and complexity. Hence, this concept makes the title of this thesis "Algorithmic simplification of solution generating theorems for perfect fluid spheres in general relativity".

In chapter four "Mathematical programming with maplets", we also constructed a program for finding our solutions as required. Many times, the calculation process in physics can yield the solutions with some errors because of inaccurate input process, runtime error, and wrong codes in mathematical program, etc.

## 3.2 Additional information

In this thesis, there are also algorithms and examples for finding physical values such as Riemann curvature tensor, Christoffel's symbol, Ricci tensor and Einstein's tensor for each coordinates.



At last, we would like you to comprehend the meaning of "exact solutions" which satisfy the "exactness" conditions. In mathematics, there are many conditions of exactness such as integrating factor of first and second order differential equation. But, in physics, there are more and more conditions of exactness such as the 3D Navier-Stokes equations, Schrodinger equation for modified Kratzer's molecular potential, pendulum Differential Equations, etc. Finally, if we can develop this research further, we may further explore "non exactness" conditions which extend to general cases.

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# APPENDICES

## APPENDIX A

## ANOTHER ALGORITHM FOR FINDING SOLUTIONS

Consider 1st order linear differential equation

$$a(x)u\prime(x) + b(x)u(x) + c(x) = 0. \tag{2.184}$$

If one solution is known, say $u_0(x)$ then the ***useful*** trick for finding the general solution is to make the additive substitution

$$u(x) = u_0(x) + Q(x), \tag{2.185}$$

since then the differential equation for $Q(x)$ is very simple

$$a(x)Q\prime(x) + b(x)Q(x) = 0, \tag{2.186}$$

with solution

$$Q\left(x\right) = \exp\left(\int b\left(x\right)/a\left(x\right)\mathrm{d}x\right). \tag{2.187}$$

This was the basis of our (BVW) old theorems.

"**New convenient**" theorem is based in contrast on the multiplicative substitution

$$u(x) = u_0(x)Z(x), \tag{2.188}$$

since the differential equation for $Z(x)$ now is simply

$$a(x)u_0(x)Z'(x) - c(x)Z(x) + c(x) = 0. \tag{2.189}$$

This is a special first order linear ODE with special coefficients and with a particularly simple solution

$$Z(x) = 1 + k \exp\left(\int c(x)/[a(x)\, u_0(x)]\mathrm{d}x\right), \tag{2.190}$$

so that the general solution is

$$u(x) = u_0(x)\left[1 + k \exp\left(\int c(x)/[a(x)\, u_0(x)]\mathrm{d}x\right)\right]. \tag{2.191}$$

The point is that although you now need to know $u_0(x)$ explicitly to do the integral, sometimes it turns out to be an easier integral to do.

# APPENDIX B

# MATHEMATICAL PROGRAMMING WITH MAPLETS

### Introduction

In this section we introduce a mathematical programming tool, "maplets". The maplet software package resides on top of the Maple symbolic calculator, which is an advanced calculator for mathematics, that is, a program that has many packages that can be used to calculate any areas of mathematics. Moreover, Maple is also useful for creating teaching files which are effective, accurate, and faster than manual calculation. Additionally, these mathematical programs can be used to check correctness of analytic solutions, or verify numerical solutions that are extracted if analytic solutions do not exist.

A maplet is a powerful mathematical program for calculation and solving problems in all areas of modern mathematics using a Graphical User Interface (GUI, sometimes pronounced gooey) which is a type of user interface item that allows people to interact with programs in a direct manner. A maplet program running under Maple uses Maple commands to create a Graphical User Interface for using, for example, the viewer, popup menus, input and output dialog or check boxes. Given recent developments in general relativity, the evolution of an easily-used mathematical program for calculations in general relativity is important for convenience [46]. In this thesis, these maplet programs given are examples of finding the exact solutions of the perfect fluid spheres. A maplet consists of one or more windows which interact with the user by means of buttons, checkboxes, text fields, and other standard graphical controls [46–49].

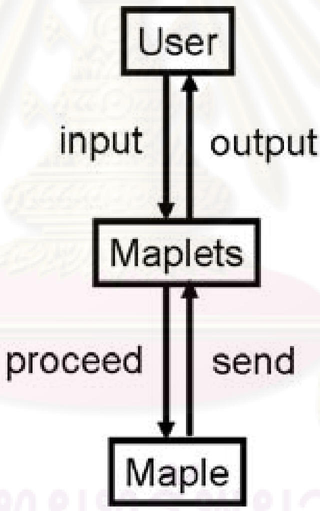



Figure 2.7: This figure shows the results when we insert input variables through maplets, these programs send input variables to Maple to calculate and send back the solutions to maplets to show us the output.

While Maple is a powerful problem-solving tool in advanced mathematics, the cost of this power is moderately high. Although Maplesoft products are relatively expensive, it gives us many more tools than we originally expected, especially with the latest version having user-interface builder so that we can make ourselves customized interfaces.

**How to construct maplets?**

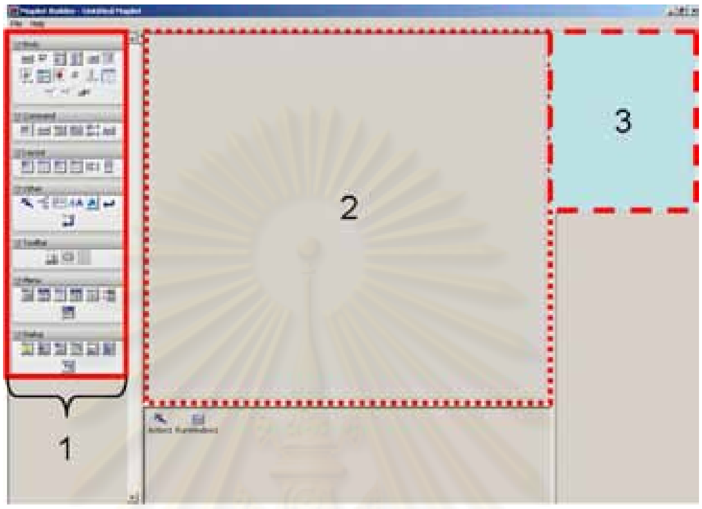



Figure 2.8: maplet builder

Briefly, the 1st box consists of tools such as bodies (buttons, viewers, boxes), commands, layouts, toolbars, menus, dialogs and others which are used to build user-friendly interfaces for clicking, inserting and viewing output. Next, the 2nd box is a panel for putting tools from 1st-order box into the 2nd box in their desired form. Finally, we have to insert commands (Maple code fragments) for calculation into the 3rd box. Then, saving to a maplet file (*.maplet), the dedicated program is ready to be used.

The particular maplet we developed is called "Program for Perfect Fluid Spheres (Theorem 1-4)", and is designed to check and find new exact solutions for perfect fluid spheres by applying theorems 1 to 4 of [26, 30, 37].

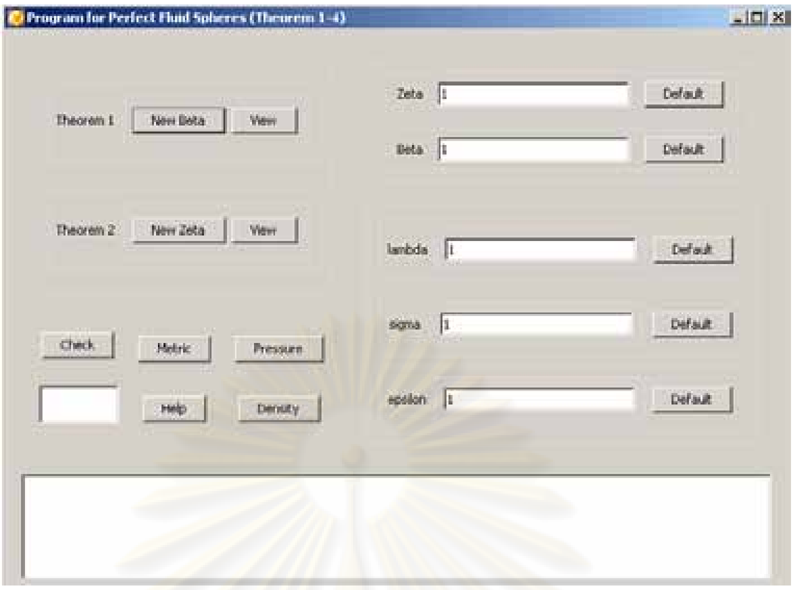



Figure 2.9: This is a maplet application for finding new exact solutions for perfect fluid spheres by applying theorems 1 to 4 of references [26, 30, 37].

**User guide**

To find the new exact solution of perfect fluid spheres by applying theorems 1 to 4, we need to understand the basic features of the buttons, textfields, and viewers. From figure 2.10, this textfield is used to enter the value of "Beta", $B_0(r)$, with default value "1". We can enter this Beta using standard Maple code as required. The "Default" button is used to reset value in textfield of "Beta" to the default value "1".

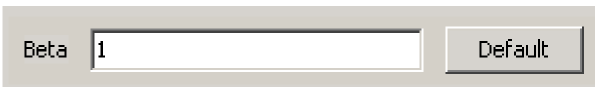



Figure 2.10: Textbox for input $B(r)$


Figure 2.11: Toggle to calculate new $B(r)$

By using theorem 1, we know that whenever $\{\zeta_0(r), B_0(r)\}$ is a perfect fluid sphere, then $\{\zeta_0(r), B_0\,(r) + \lambda\Delta_0(r)\}$ is also a perfect fluid sphere, where $\lambda$ is an arbitrary constant and $\Delta_0 = \Delta_0\,(r)$ such that

$$\Delta_0(r) = \left(\frac{\zeta_0(r)}{\zeta_0(r) + r\zeta_0'(r)}\right)^2 r^2 \exp\left\{2\int \frac{\zeta_0'(r)}{\zeta_0(r)}\frac{\zeta_0(r) - r\zeta_0'(r)}{\zeta_0(r) + r\zeta_0'(r)}\mathrm{d}r\right\}. \qquad (2.192)$$

Indeed, we do not need to know $\Delta_0(r)$ (since calculating $\Delta_0(r)$ is the job of this maplet). So we just input $B_0(r)$ in the textfield "Beta" in figure 2.10 and then click "view" button in figure 2.11. So new $B_1(r)$ is generated as in figure 2.12 as below.

$$1 + r^2$$

Figure 2.12: Output $B(r)$


Figure 2.13: Textbox for input parameter $\lambda$

When we want to specify $\lambda$, we can specify this by inserting $\lambda$ (or any other parameter name) in the textfield "lambda" in figure 2.13 and then click "view" button in figure 2.11. The new Beta is generated as in figure 2.14 as below.

$$1+\lambda\, r^2$$

Figure 2.14: Output $B(r)$

Using theorem 2, we know that if $\{\zeta_0(r), B_0(r)\}$ is a perfect fluid sphere, then $\{\zeta_0(r)Z_0(r), B_0(r)\}$ is also a perfect fluid sphere, where $Z_0 = Z_0(r)$ (see [26, 30, 37]). Considering figure 2.15 below which is the textfield "Zeta" for entering $(\zeta_0(r))$. To generate a new Zeta is similar to Beta as above, the user does not need to calculate $Z_0(r)$, the maplet does it for us.

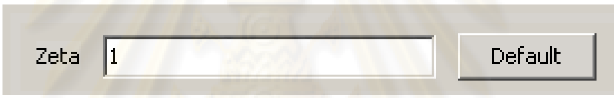



Figure 2.15: Textbox for input $\zeta(r)$




Figure 2.16: Toggle to calculate new $\zeta(r)$



By applying theorem 3 and 4 to the metric, since theorem 3 is the composition of theorem 1, then theorem 2, we can click button "New Beta" before "New Zeta" . Similarly, theorem 4 is composition of theorem 2 followed by theorem 1 — we just click button "New Zeta" before "New Beta" .

**Other features**

Several other features we built into the maplet are as follows.

1. **Check Button**

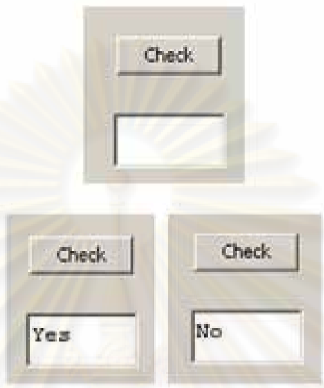



Figure 2.17: Toggle for verification purposes

The "check" button and textbox are used to verify that $\{\zeta_0(r), B_0(r)\}$ really is a perfect fluid sphere. After we input $\{\zeta_0(r), B_0(r)\}$ in their respective textfields and click the "check" button, the output is "yes" or "no" . "Yes" means $\{\zeta_0(r), B_0(r)\}$ is a perfect fluid sphere. In contrast, "No" means $\{\zeta_0(r), B_0(r)\}$ is not a perfect fluid sphere.

2. **Metric**

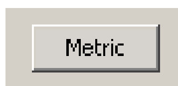


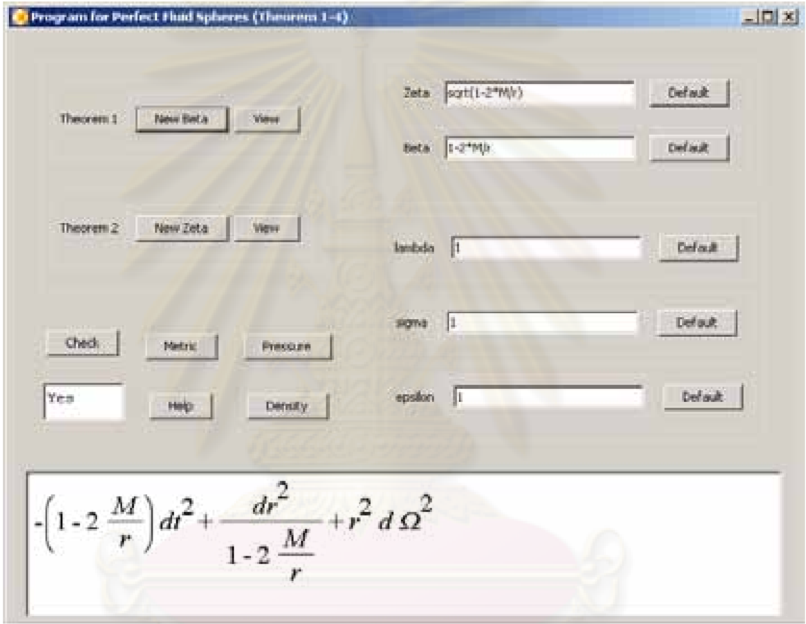

Figure 2.18: Toggle to generate output metric

The "Metric" button is used to view the spacetime metric

$$\mathrm{d}s^2 = -\zeta_0(r)^2 \mathrm{d}t^2 + \frac{1}{B_0(r)}\mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2, \tag{2.193}$$

when we insert $\zeta_0(r)$ and $B_0(r)$. The output is shown as below in figure 2.19.

Figure 2.19: Output metric when we insert $\zeta_0(r)$ and $\mathrm{B}_0(r)$ as shown




3. **Pressure and Density**


Figure 2.20: Toggles to calculate pressure and density

These "Pressure" and "Density" buttons are used to view their respective values as calculated by these formulae:

$$G_{tt} = 8\pi\rho, \quad (2.194)$$

$$G_{rr} = 8\pi p, \quad (2.195)$$

where $G_{tt}$ and $G_{rr}$ are components of the Einstein tensor (in Schwarzschild coordinates) with metric

$$\mathrm{d}s^2 = -\zeta_0(r)^2\mathrm{d}t^2 + \frac{1}{\mathrm{B}_0(r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2. \quad (2.196)$$

**Conclusion**

The mathematical program, called Maplet, in this research, will help us to find new parameters of the perfect fluid sphere with the first, second, third, and fourth transformations in [26]. Moreover, this program can be used to check that if parameters are perfect fluid spheres and also has tools that check their metrics whether they are. Finally, this maplet helps us to find the solutions of perfect fluid spheres easier and faster.

# APPENDIX C

# ALBERT EINSTEIN'S BIOGRAPHY

Albert Einstein was born on March the 14th , 1879, in a town called Ulm in southwest Germany. His father, Hermann Einstein, was in electrical equipment business. Later the business failed so his family moved to Italy, but he stayed at Munich, to pursue his studies at Luitpold Gymnasium.

When Albert was young, even though he had some speech difficulties, he was the top student at school. He was a slow talker, pausing to consider what he would say. As he grew older, he experimented by making models, mechanical devices and showed keen interest in mathematics.

In 1905, he received his PhD from the University of Zurich and had published four scientific papers. One introduced his special relativity and another his equation which related mass and energy.

In 1909, he became an associate professor of theoretical physics at Zurich and professor at the German University in Prague in 1911. Then he returned to the Institute of Technology in Zurich the following year.

In 1914, he was appointed Director of the Kaiser Wilhelm Institute for Physics and Professor in the University of Berlin. During that year, he became a German citizen and published his general theory of relativity later in 1916.

In 1921, Einstein received the Nobel Prize in Physics for his discovery of the law of the photoelectric effect and his work in the field of theoretical physics.

In 1920's, he had lectured in Europe, North and South America and Palestine, where he was involved in the foundation of the Hebrew University in Jerusalem.

In 1933, Einstein emigrated to America as Nazis took power in Germany. He accepted a position at the Institute of Advanced Study in Princeton and took US citizenship.

He retired from the institute in 1945 and continued to work towards a unified field theory to construct a merger between quantum theory and his general relativity. He also continued to be active in the peace movement and in support of Zionist causes and in 1952 he was offered the presidency of Israel, which he declined.

Albert Einstein died on April the 18th , 1955, in Princeton, New Jersey.

In his lifetime, he received honorary doctorate degrees in various fields from many European and American universities. The leading scientific academies throughout the world considered it a privilege to award fellowships of their institutes to Einstein. He gained numerous awards, some of the most important being the Nobel Prize, Copley Medal of the Royal Society of London and Franklin Medal of the Franklin Institute.

Albert Einstein, a German-born theoretical physicist, was considered the most famous scientist of the 20th century. He is not just a role model for teachers, but also an inspiring personality for students of science all over the world [53].

# VITA

| | |
|---|---|
| **Name** | Mr. Panit Suavansri |
| **Date of Birth** | 11 May 1983 |
| **Place of Birth** | Bangkok, Thailand |
| **Education** | B.Sc. (Medical Science), Thammasat University, 2009 |
| **Scholarship** | Chulalongkorn University, and Research Assistant Scholarship |