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VAGUE PRIME IDEALS AND VAGUE PRIMARY IDEALS OF VAGUE RINGS

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การดำเนินการคลุมเครือบนเซต X เทียบกับ $E_{X\times X}$ และ E_X คือ ฟังก์ชันวิภัชนัยอย่างเข้ม $\tilde{\circ}$ เทียบกับภาวะเท่ากันวิภัชนัย $E_{X\times X}$ บน $X\times X$ และภาวะเท่ากันวิภัชนัย E_X บน X ยิ่งไปกว่านั้น S. Sezer ได้นิยามริงคลุมเครือเทียบกับ $E_{H\times H}$ และ E_H ขึ้นในปี ค.ศ. 2007 ซึ่งคือสามสิ่งอันดับ $\langle H, \tilde{\circ}, \tilde{\bullet} \rangle$ ซึ่งสอดคล้องเงื่อนไขที่แน่ชัดบางประการ นอกจากนี้ในปี ค.ศ. 2007 S. Sezer ยังได้นิยามไอดีลคลุมเครือและไอดีลเฉพาะคลุมเครืออีกด้วย ในวิทยานิพนธ์ฉบับนี้ นำเสนอไอดีลปฐมภูมิคลุมเครือรวมทั้งสมบัติพื้นฐานบางประการของไอดีลเฉพาะคลุมเครือ ไอดีลปฐมภูมิคลุมเครือและไอดีลคลุมเครือที่เกี่ยวข้อง ยิ่งไปกว่านี้เราให้เงื่อนไขที่เพียงพอของริงคลุมเครือของเมทริกซ์ที่ไม่มีไอดีลเฉพาะคลุมเครือ ในท้ายสุด เราให้เงื่อนไขที่เพียงพอซึ่งทำให้ไอดีลเฉพาะคลุมเครือ ไอดีลปฐมภูมิคลุมเครือและไอดีลคลุมเครือที่สัมพันธ์กันบางแบบกลายเป็นสิ่งเดียวกัน

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A vague binary operation on a set X with respect to $E_{X \times X}$ and E_X is a strong fuzzy function $\tilde{\circ}$ with respect to a fuzzy equality $E_{X \times X}$ on $X \times X$ and a fuzzy equality E_X on X. Moreover, a vague ring with respect to $E_{\mathcal{H} \times \mathcal{H}}$ and $E_{\mathcal{H}}$ defined in 2007 by Sezer is a 3-tuple $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ satisfying some certain conditions. Vague ideals and vague prime ideals were defined in 2007 by the same author. In this thesis, we present vague primary ideals and some elementary properties of vague prime ideals, vague primary ideals and their related vague ideals. In addition, we give the sufficient condition of the vague ring of matrices to have no vague prime ideals. Eventually, we give some sufficient conditions which vague prime ideals, vague primary ideals and some related vague ideals are coincide.

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Notation

The following notation are used frequently throughout this thesis.

$\mathbb{R}$	the set of real numbers,
$\mathbb{R}^+$	the set of positive real numbers,
$\mathbb{Z}$	the set of integers,
$\mathbb{Z}_0^+$	the set of non-negative integers,
$\mathbb{N}$	the set of natural numbers (positive integers),
$\mathbb{Q}$	the set of rational numbers,
$\mathbb{Q}^+$	the set of positive rational numbers,
$\mathbb{Z}_n$	the set of integers modulo a natural number n,
$\mathbb{M}_n(R)$	the $n \times n$ matrix rings over a ring R,
(a, b)	a real open interval,
$[a, b]$	a real close interval,
$\tilde{\circ}$	a vague binary operation,
$\mu_{\tilde{\circ}}(a, b, c)$	a characteristic functions of (a, b, c) corresponding the vague binary operation $\tilde{\circ}$,
E_X	a fuzzy equality on X,
$\min(a, b)$ or $a \wedge b$	the minimum of a and b,
$\max(a, b)$ or $a \vee b$	the maximum of a and b,
$\langle G, \tilde{\circ} \rangle$	a vague group,
$\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$	a vague ring,
$\langle G, \circ \rangle$	a classical group,
R or $\langle \mathcal{H}, \circ, \bullet \rangle$	classical rings,
$\left\langle \mathcal{V}(A : B), \tilde{\heartsuit}, \tilde{\spadesuit} \right\rangle$	a vague ideal quotient.

CHAPTER I

Introduction and Preliminaries

In 1999, Demirci defined operations of a group to be compatible with a given fuzzy equality which led to vague algebraic structures :- vague semigroups, vague groups, vague rings which are related to semigroups, groups and rings, respectively. Later on, general results of vague algebraic notions have been established by Demirci and Sezer.

In 2007, Sezer defined in [7] a vague prime ideal of a vague ring as follows : a proper vague ideal $\left\langle P, \tilde{\oplus}, \tilde{\odot} \right\rangle$ of a vague ring $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ is said to be a vague prime ideal if $\tilde{\bullet}$ is a vague binary operation on $\mathcal{H} \setminus P$. He also provided many interesting results relating to vague algebraic notions.

In rings, ideals are crucial notion and primary ideals are directly closed to prime ideals. This inspired us to define a vague primary ideal of a vague ring which is similar to a primary ideal of a ring. We prove that every vague prime ideal of a vague ring $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ is a vague primary ideal of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$. Moreover, we are interested in studying some properties of vague prime ideals and vague primary ideals which are parallel to those of prime ideals and primary ideals of classical rings. Furthermore, we investigate properties of the followings: vague left ideals, vague right ideals, vague ideals, vague prime left ideals, vague prime right ideals and vague prime ideals.

In classical sense, Noetherian rings are well known rings which are beneficial to investigate some properties of vague primary ideals. This led us to define a vague Noetherian ring in the last chapter. We also define some other vague ideals such as a vague irreducible ideal, a vague semiprime ideal in order to look for some related properties among them. Eventually, we give some sufficient conditions which vague prime ideals and vague primary ideals are coincide.

In this chapter, we gather some elementary concepts, some properties and some results on vague semigroups, vague groups and vague rings.

In Chapter 2, we begin with some elementary properties of vague ideals. We have an inspiration from the classical sense that the sum of ideals of a ring is also an ideal of R. Next, we give some sufficient conditions of a vague ring such that its maximal vague ideal is a vague prime ideal. Then we give the sufficient conditions of the vague ring of matrices to have no any vague prime ideals. Ultimately, for any vague prime ideal P, I, J of a vague ring $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ and for each $a \in \mathcal{H}$, we define $(P : a)$, $(I : a)_r$ and $(J : a)_l$ and give some elementary attributes of them.

In last chapter, because ideals are crucial notion and primary ideals are directly closed to prime ideals, this inspired us to define a vague primary ideal of a vague ring which is similar to a primary ideal of a ring. We prove that every vague prime ideal of a vague ring is a vague primary ideal . Consequently, we are interested in studying some properties of vague prime ideals and vague primary ideals which are parallel to those of prime ideals and primary ideals of classical rings. Next, we define and give some elementary properties of vague Noetherian rings and also define some crucial vague ideals such as vague irreducible ideals, vague semiprime ideals to look for some of their related properties. Eventually, we give some sufficient conditions which vague prime ideals and vague primary ideals are coincide.

1.1 Fuzzy Sets

In this section, we introduce some concepts and elementary properties of fuzzy sets that we use in this thesis.

Definition 1.1.1. [1] Let X be the universe. A fuzzy set is a pair (A, μ_A) where A is a subset of X and μ_A is a map from X to the real closed interval $[0, 1]$ which is called a ***membership function of*** A (or a characteristic function of A).

We may write "A is a fuzzy set" instead of "(A, μ_A) is a fuzzy set".

Example 1.1.2. [1] Let X be the universe. Basic examples of fuzzy sets are as follows:

1. The empty fuzzy set ϕ with the membership function $\mu_\phi(x) = 0$ for all $x \in X$.

2. The universal set X is characterised by the memberhip function $\mu_X(x) = 1$ for all $x \in X$.

3. The fuzzy set A of all reals which are nearly equal to 10. Then, of course, one shall consider the universe $X = \mathbb{R}^+$. One possibility for the membership function of A now is to take it as

$$\mu_A(x) = \max\left\{0, 1 - \frac{(10 - x^2)}{2}\right\}.$$

Definition 1.1.3. [1] A ***crisp set*** $\mathcal{M}$ is a fuzzy set with usual characteristic function $\mu_\mathcal{M} = \chi_\mathcal{M}$ as its membership function, i.e., we consider crisp sets as special cases of fuzzy sets, viz. those ones with 0 and 1 as membership degrees.

Definition 1.1.4. [1] Let A and B be fuzzy sets. Then we call A ***a fuzzy subset of*** B, denoted by $A \subseteq B$, if $\mu_A(x) \leq \mu_B(x)$ for all $x \in A$.

The above definition shows that if A is a fuzzy subset of B then A is a classical subset of B.

We give a definition of fuzzy relation in the following.

Definition 1.1.5. [1] Let X and Y be nonempty crisp sets. If R is a fuzzy subset of a fuzzy set $X \times Y$, then R is called a ***fuzzy relation from*** X ***to*** Y where $\mu_R(x, y)$ is interpreted as the characteristic function of the ordered pair (x, y) in R.

Example 1.1.6. [1] Let $\varphi_1 = \{a_1, a_2, a_3\}$ and $\varphi_2 = \{b_1, b_2, b_3, b_4\}$. Consider $R = \{(x, y) \,|x \in \varphi_1 \text{ and } y \in \varphi_2\}$ as a fuzzy relation from φ_1 to φ_2. Next we give an example of characteristic functions of the ordered pairs (x, y) in R which can be represented by a matrix with elements in $[0, 1]$, e.g., by

$$\begin{bmatrix} \mu_R & b_1 & b_2 & b_3 & b_4 \\ a_1 & 0.8 & 0.3 & 0 & 0 \\ a_2 & 1 & 0.7 & 1 & 0.2 \\ a_3 & 0.6 & 0.9 & 1 & 0.5 \end{bmatrix}$$

evidently, for examples, $\mu_R(a_1, b_2) = 0.3$ and $\mu_R(a_3, b_4) = 0.5$.

1.2 Vague binary operations and vague semigroups

The notions of Fuzzy equalities, strong fuzzy functions and their fundamental properties were introduced by Sezer [7]. Our aim in this section is to recall these notions and some of their elementary properties which will be needed in this thesis.

The symbols "$\wedge$"and "$\vee$"stand for the minimum and the maximum operations between finitely many real numbers, respectively ; and X, Y, Z always stand for nonempty crisp sets in this thesis.

Definition 1.2.1. [2] A mapping $E_X : X \times X \to [0, 1]$ is called a ***fuzzy equality on*** X if the following conditions are satisfied :

$(E1)$ $E_X(x, y) = 1 \Leftrightarrow x = y$ for all $x, y \in X$,

$(E2)$ $E_X(x, y) = E_X(y, x)$ for all $x, y \in X$,

$(E3)$ $E_X(x, y) \wedge E_X(y, z) \leq E_X(x, z)$ for all $x, y, z \in X$.

One can define a fuzzy equality on X with respect to the equality of the elements of X. Indeed, the mapping $E_X^c : X \times X \to [0,1]$, defined by

$$E_X^c(x,y) = \begin{cases} 1, & \text{if } x = y, \\ 0, & \text{if } x \neq y, \end{cases}$$

is obviously a fuzzy equality on X.

Definition 1.2.2. [4] Let E_X and E_Y be two fuzzy equalities on X and Y, respectively. Then a fuzzy relation $\tilde{\circ}$ from X to Y, denoted by $\tilde{\circ} : X \rightsquigarrow Y$ is called a ***strong fuzzy function from*** X ***to*** Y ***with respect to the fuzzy equalities*** E_X ***and*** E_Y, or a ***strong fuzzy function from*** X ***to*** Y for short, if the characteristic function $\mu_{\tilde{\circ}} : X \times Y \to [0,1]$ of $\tilde{\circ}$ satisfies the following two conditions :
$(F1)$ for each $x \in X$, there exists $y \in Y$ such that $\mu_{\tilde{\circ}}(x,y) = 1$.
$(F2)$ for each $x_1, x_2 \in X$ and $y_1, y_2 \in Y$,

$$\mu_{\tilde{\circ}}(x_1,y_1) \wedge \mu_{\tilde{\circ}}(x_2,y_2) \wedge E_X(x_1,x_2) \leq E_Y(y_1,y_2).$$

Sezer also provided an example of a strong fuzzy function.

Example 1.2.3. [7] Let $X = \mathbb{R}^+$ and $Y = \mathbb{Z}_0^+$. Let $\alpha, \beta \in [0,1]$ be such that $\alpha < \beta$ and $n \in \mathbb{N}$. Define E_X and E_Y as follows:

$$E_X(x_1,x_2) = \begin{cases} 1, & \text{if } x_1 = x_2, \\ \alpha, & \text{if } x_1 \neq x_2, \end{cases}$$

$$E_Y(y_1,y_2) = \begin{cases} 1, & \text{if } y_1 = y_2, \\ \beta, & \text{if } y_1 \neq y_2. \end{cases}$$

Finally, define $\mu_{\tilde{\circ}} : X \times Y \to [0,1]$ by

$$\mu_{\tilde{\circ}}(x,y) = \begin{cases} 1, & \text{if } y = 0, \\ \min\left\{\dfrac{1}{x}, \dfrac{\alpha}{n}\right\}, & \text{if } y \neq 0. \end{cases}$$

Therefore, $\tilde{\circ}$ is a strong fuzzy function from X to Y with respect to the fuzzy equalities E_X and E_Y.

The concept of a vague binary operation on a set X are defined as follows.

Definition 1.2.4. [3, 4]
A strong fuzzy function $\tilde{\circ} : X \times X \rightsquigarrow X$ with respect to a fuzzy equality $E_{X\times X}$ on $X \times X$ and a fuzzy equality E_X on X is called a ***vague binary operation on*** X ***with respect to*** $E_{X\times X}$ ***and*** E_X or a ***vague binary operation on*** X for short. $(\forall (x_1, x_2) \in X \times X \forall x_3 \in X, \mu_{\tilde{\circ}}((x_1, x_2), x_3)$ will be denoted by $\mu_{\tilde{\circ}}(x_1, x_2, x_3)$ for the sake of simplicity.)

For the construction of vague groups, we first introduce the following well-known definitions in the classical group theory.

Definition 1.2.5. A nonempty set X together with a binary operation $\circ$, denoted by $(X, \circ)$ is a ***semigroup*** if the associative property is satisfied :
(S) $a \circ (b \circ c) = (a \circ b) \circ c$ for all $a, b, c \in X$.

A semigroup $(X, \circ)$ is a ***monoid*** if
(M) there exists an element $e \in X$, called the ***identity element*** of X, such that $e \circ a = a$ and $a \circ e = a$ for each $a \in X$.

A monoid $(X, \circ)$ is a ***group*** if
(G) for each $a \in X$, there exists an element of X, denoted by a^{-1} and called the ***inverse element*** of a, such that $a^{-1} \circ a = e$ and $a \circ a^{-1} = e$.

A semigroup $(X, \circ)$ is said to be ***abelian (commutative)*** if the binary operation $\circ$ has the following property :
(A) $a \circ b = b \circ a$ for all $a, b \in X$.

The conditions (S) and (A) can be written in the following equivalent statements, respectively :

$\left(S^{/}\right) \forall a,b,c,d,m,q,w \in X$, $\Big(b \circ c = d$ and $a \circ d = m$ and $a \circ b = q$ and $q \circ c = w\Big) \Rightarrow m = w.$

$\left(A^{/}\right) \forall a,b,m,w \in X$, $\Big(a \circ b = m$ and $b \circ a = w \Rightarrow m = w\Big).$

The binary operation $\circ$ can be conceivable as a special vague binary operation $\tilde{\circ}$ on X with respect to $E_{X\times X}$ and E_X satisfying the condition $\mu_{\tilde{\circ}}\left(X \times X\right) \subseteq \{0,1\}$. Then for each $a,b,m \in X$, the classical notation $a \circ b = m$ means that $\mu_{\tilde{\circ}}\left(a,b,m\right) = 1$, or equivalently, $\mu_{\tilde{\circ}}\left(a,b,m\right) > 0$. Therefore, regarding $\left(S^{/}\right)$ and $\left(A^{/}\right)$ instead of (S) and (A), respectively, we observe that $(S)\,,(M)\,,(G)$ and (A) can be respectively represented in the following definition.

Definition 1.2.6. [3] Let $\tilde{\circ}$ be a vague binary operation on a nonempty set G with respect to a fuzzy equality $E_{G\times G}$ on $G \times G$ and a fuzzy equality E_G on G. Then
1. G together with $\tilde{\circ}$, denoted by $\langle G, \tilde{\circ}, E_{G\times G}, E_G\rangle$ or simply $\langle G, \tilde{\circ}\rangle$, is called a ***vague semigroup*** if the characteristic function $\mu_{\tilde{\circ}} : G \times G \times G \to [0,1]$ of $\tilde{\circ}$ fulfills the condition: for all $a,b,c,d,m,q,w \in G$,

$$\mu_{\tilde{\circ}}\left(b,c,d\right) \wedge \mu_{\tilde{\circ}}\left(a,d,m\right) \wedge \mu_{\tilde{\circ}}\left(a,b,q\right) \wedge \mu_{\tilde{\circ}}\left(q,c,w\right) \leq E_G\left(m,w\right).$$

2. A vague semigroup $\langle G, \tilde{\circ}\rangle$ is called a ***vague monoid*** if there exists an identity $e_{\tilde{\circ}} \in G$, that is an element $e_{\tilde{\circ}} \in G$ satisfying the followings: for all $a \in G$,

$$\mu_{\tilde{\circ}}\left(e_{\tilde{\circ}},a,a\right) \wedge \mu_{\tilde{\circ}}\left(a,e_{\tilde{\circ}},a\right) = 1.$$

3. A vague monoid $\langle G, \tilde{\circ}\rangle$ is called a ***vague group*** if for each $a \in G$ there exists an inverse $a^{-1} \in G$, that is an element $a^{-1} \in G$ satisfying

$$\mu_{\tilde{\circ}}\left(a^{-1},a,e_{\tilde{\circ}}\right) \wedge \mu_{\tilde{\circ}}\left(a,a^{-1},e_{\tilde{\circ}}\right) = 1.$$

4. A vague semigroup $\langle G, \tilde{\circ} \rangle$ is said to be ***commutative (abelian)*** if

$$\mu_{\tilde{\circ}}(a, b, m) \wedge \mu_{\tilde{\circ}}(b, a, w) \leq E_G(m, w) \text{ for all } a, b, m, w \in G.$$

Next, we introduce concept of vague semigroups and show that for each classical semigroup $\langle S, \circ \rangle$, there exists a non-trivial vague binary operation such that $\langle S, \tilde{\circ} \rangle$ is a vague semigroup.

Proposition 1.2.7. *Let S be any semigroup. Let $c, d \in (0, 1)$ be such that $d < c$. First, we define a fuzzy equality $E^c_{S\times S}$, i.e.,*

$$E^c_{S\times S}[(x_1, y_1), (x_2, y_2)] = \begin{cases} 0, & \text{if } (x_1, y_1) \neq (x_2, y_2), \\ 1, & \text{if } (x_1, y_1) = (x_2, y_2), \end{cases}$$

for all $(x_1, y_1), (x_2, y_2) \in S \times S$.
Next, we define the characteristic function $\mu_{\tilde{\circ}}(x, y, z)$ with the following properties :

1. *$\forall x, y \in S$, $\exists! z_{x,y} \in S$ such that $\mu_{\tilde{\circ}}(x, y, z_{x,y}) = 1$,*
2. *$\forall x, y \in S$, $\forall z \in S \setminus \{z_{x,y}\}$, $\mu_{\tilde{\circ}}(x, y, z) = d$.*

Finally, for all $x, y \in S$, we define

$$E_S(x, y) = \begin{cases} 1, & \text{if } x = y, \\ c, & \text{if } x \neq y. \end{cases}$$

Then $\left\langle S, \tilde{\circ}, E^c_{S\times S}, E_S \right\rangle$ is a vague semigroup .

Proof. Evidently, $E^c_{S\times S}$ and E_S are fuzzy equalities on $S \times S$ and S, respectively. Next, we are going to prove that $\tilde{\circ}$ is a vague binary operation by showing that $\tilde{\circ}$ is a strong fuzzy function from $S \times S$ to S. We separate the proof into two steps as follows:

1. for all $(x, y) \in S \times S$, there exists $z \in S$ such that $\mu_{\tilde{\circ}}(x, y, z) = 1$,

2. for all $(x_1, y_1), (x_2, y_2) \in S \times S$ for all $z_1, z_2 \in S$,

$$\mu_{\tilde{\circ}}(x_1, y_1, z_1) \wedge \mu_{\tilde{\circ}}(x_2, y_2, z_2) \wedge E^c_{S\times S}[(x_1, y_1), (x_2, y_2)] \leq E_S(z_1, z_2).$$

Notice that the first step is obvious from the given properties of the characteristic function. In the second step, $E^c_{S\times S}[(x_1, y_1), (x_2, y_2)]$ has possible values either 0 or 1. It suffices to show that

- if $(x_1, y_1) \neq (x_2, y_2)$, done.
- if $(x_1, y_1) = (x_2, y_2)$, we just show that

$$\mu_{\tilde{\circ}}(x_1, y_1, z_1) \wedge \mu_{\tilde{\circ}}(x_2, y_2, z_2) \leq E_S(z_1, z_2).$$

Therefore,
if $z_1 = z_2$, then $E_S(z_1, z_2) = 1$,
if $z_1 \neq z_2$, then $\mu_{\tilde{\circ}}(x_1, y_1, z_1) \wedge \mu_{\tilde{\circ}}(x_2, y_2, z_2) = d$ but $E_S(z_1, z_2) = c$, and $c > d$. Thus the result is proved.

Hence, $\tilde{\circ}$ is a strong fuzzy relation so that $\tilde{\circ}$ is a vague binary operation . Finally, for the same reason as above, we obtain that $\left\langle S, \tilde{\circ}, E^c_{S\times S}, E_S \right\rangle$ is a vague semigroup as desired. □

1.3 Vague Groups

The notions of vague groups, generalized vague subgroups and their fundamental properties were introduced by Sezer [7]. Our purpose of this section is to recall these notions and some of their elementary properties which will be needed in this thesis.

For the rest of this thesis, if G is a nonempty set, then the notation $\langle G, \tilde{\circ} \rangle$ always stands for the vague group $\langle G, \tilde{\circ} \rangle$ with respect to a fuzzy equality $E_{G\times G}$

on $G \times G$ and a fuzzy equality E_G on G.

Proposition 1.3.1. [3] *For a given vague group $\langle G, \tilde{\circ} \rangle$, there exists a unique binary operation in the classical sense, denoted by $\circ$, on G such that $\langle G, \circ \rangle$ is a group in the classical sense.*

From now on, if $\tilde{\circ}$ is a vague binary operation on a nonempty set X with respect to a fuzzy equality $E_{X \times X}$ and a fuzzy equality E_X, then from [4, 5],

$$\mu_{\tilde{\circ}}(a, b, c) \leq E_X(a \circ b, c) \quad \text{for all } a, b, c \in X. \tag{1.3.2}$$

The binary operation "$\circ$" in Proposition 1.3.1 is explicitly given by the equivalence

$$a \circ b = c \Leftrightarrow \mu_{\tilde{\circ}}(a, b, c) = 1 \quad \text{for all } a, b, c \in G. \tag{1.3.3}$$

In the next theorem, we show how a vague binary operation on X is related to a binary operation mentioned in (1.3.3)

Theorem 1.3.4. *Let X be a nonempty set and $\tilde{\bullet}$ be a vague binary operation on X with respect to the fuzzy equalities $E_{X \times X}$ and E_X. Then there exists a binary operation $\bullet$ on X such that*

$$a \bullet b = c \quad \textit{if and only if} \quad \mu_{\tilde{\bullet}}(a, b, c) = 1 \ \textit{ for all } a, b, c \in X.$$

Proof. Let $a, b, c \in X$. Assume that $\mu_{\tilde{\bullet}}(a, b, c) = 1$. Utilizing (1.3.2) and the fact that $1 = \mu_{\tilde{\bullet}}(a, b, c) \leq E_X(a \bullet b, c)$, we see that $a \bullet b = c$. Conversely, assume that $a \bullet b = c$. Suppose that $\mu_{\tilde{\bullet}}(a, b, c) \neq 1$. Since $\tilde{\bullet}$ is a vague binary operation on X, there exists $d \in X$ such that $c \neq d$ and $\mu_{\tilde{\bullet}}(a, b, d) = 1$. Therefore $a \bullet b = d$ contradicts the fact that $\bullet$ is a binary operation.

Hence $a \bullet b = c$ if and only if $\mu_{\tilde{\bullet}}(a, b, c) = 1$ for all $a, b, c \in X$. $\square$

The binary operation "$\circ$", defined by the equivalence (1.3.3), is called **the ordinary description** of $\tilde{\circ}$ see [4, 5].

We show in the following proposition that if $\tilde{\bullet}$ is a vague binary operation on X with respect to the fuzzy equalities $E_{X\times X}$ and E_X, then $\tilde{\bullet}$ is commutative if and only if $\bullet$ is commutative.

Proposition 1.3.5. *Let X be a nonempty set and $\tilde{\bullet}$ be a vague binary operation on X with respect to the fuzzy equalities $E_{X\times X}$ and E_X. Then*

$$\tilde{\bullet} \text{ is commutative if and only if } \bullet \text{ is commutative.}$$

Proof. Let $\tilde{\bullet}$ be a commutative vague binary operation on X. By Theorem 1.3.4, we see that

$$\mu_{\tilde{\bullet}}\left(a, b, a \bullet b\right) = 1 = \mu_{\tilde{\bullet}}\left(b, a, b \bullet a\right) \qquad \text{for all } a, b \in X$$

and apply the commutativity, we observe that

$$1 = \mu_{\tilde{\bullet}}\left(a, b, a \bullet b\right) \wedge \mu_{\tilde{\bullet}}\left(b, a, b \bullet a\right) \leq E_X\left(a \bullet b, b \bullet a\right).$$

Therefore $E_X\left(a \bullet b, b \bullet a\right) = 1$, i.e., $a \bullet b = b \bullet a \quad$ for all $a, b \in X$.

Conversely, suppose that $\bullet$ is a commutative binary operation. Applying (1.3.2) and the commutativity of $\bullet$, we see that

$$\begin{aligned} \mu_{\tilde{\bullet}}\left(a, b, m\right) \wedge \mu_{\tilde{\bullet}}\left(b, a, w\right) &\leq E_X\left(a \bullet b, m\right) \wedge E_X\left(b \bullet a, w\right) \\ &= E_X\left(m, a \bullet b\right) \wedge E_X\left(a \bullet b, w\right) \\ &\leq E_X\left(m, w\right) \end{aligned}$$

for all $a, b, m, w \in X$. Hence $\tilde{\bullet}$ is commutative as desired. □

For a given fuzzy equality E_G on a set G and for a crisp subset A of G, the restriction of the mapping from E_G to $A \times A$, denoted by E_A, is obviously a fuzzy equality on A.

Definition 1.3.6. [6] Let $\langle G, \tilde{\circ} \rangle$ be a vague group and A be a nonempty crisp subset of G. Let $\tilde{\odot}$ be a vague binary operation on A such that

$$\mu_{\tilde{\odot}}(a, b, c) \leq \mu_{\tilde{\circ}}(a, b, c) \text{ for all } a, b, c \in A.$$

If $\langle A, \tilde{\odot} \rangle$ is itself a vague group with respect to the fuzzy equalities $E_{A \times A}$ on $A \times A$ and E_A on A, then $\langle A, \tilde{\odot} \rangle$ is said to be a ***generalized vague subgroup of*** $\langle G, \tilde{\circ} \rangle$, denoted by $\langle A, \tilde{\odot} \rangle \overset{v.s.}{\leq} \langle G, \tilde{\circ} \rangle$.

For a given vague group $\langle G, \tilde{\circ} \rangle$, because of the uniqueness of the identity and the inverse of an element of $\langle G, \tilde{\circ} \rangle$, it can be easily seen that if $\langle A, \tilde{\odot} \rangle \overset{v.s.}{\leq} \langle G, \tilde{\circ} \rangle$, then the identity e_A of $\langle A, \tilde{\odot} \rangle$ and the inverse x_A^{-1} of $x \in A$ with respect to $\langle A, \tilde{\odot} \rangle$ are the identity e_G of $\langle G, \tilde{\circ} \rangle$ and the inverse x_G^{-1} of $x \in G$ with respect to $\langle G, \tilde{\circ} \rangle$, i.e., $e_A = e_G$ and $x_A^{-1} = x_G^{-1}$, respectively.

Example 1.3.7. [7] Let $\alpha \in [0, 1)$ be fixed, and set $x^{\bullet} = \dfrac{1}{\max(x, 1)} \wedge \min(x, 1)$ for any $x \in \mathbb{R}^+$. For $x, y, u, v, z \in \mathbb{R}^+$, define the fuzzy equalities

$$E_{\mathbb{R}^+}(x, y) = \begin{cases} 1, & \text{if } x = y, \\ \alpha \vee \left(\dfrac{1}{x^{\bullet}} \wedge \dfrac{1}{y^{\bullet}} \right), & \text{otherwise,} \end{cases}$$

on $\mathbb{R}^+$ and

$$E_{\mathbb{R}^+ \times \mathbb{R}^+}[(x, y), (u, v)] = \begin{cases} 1, & \text{if } (x, y) = (u, v), \\ \alpha \vee \left[(x^2)^{\bullet} \wedge (y^2)^{\bullet} \wedge (u^2)^{\bullet} \wedge (v^2)^{\bullet} \right], & \text{otherwise,} \end{cases}$$

on $\mathbb{R}^+ \times \mathbb{R}^+$, and for $x, y, u, v, z \in \mathbb{Q}^+$, also define the fuzzy equalities $E_{\mathbb{Q}^+}(x, y) = E_{\mathbb{R}^+}(x, y)$ on $\mathbb{Q}^+$ and

$$E_{\mathbb{Q}^+ \times \mathbb{Q}^+}[(x, y), (u, v)] = E_{\mathbb{R}^+ \times \mathbb{R}^+}[(x, y), (u, v)]$$

on $\mathbb{Q}^+ \times \mathbb{Q}^+$. For $n \in \mathbb{N}$, we obtain that the fuzzy relations $\tilde{\circ}$ and $\tilde{\odot}_n$ on $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ and $\mathbb{Q}^+ \times \mathbb{Q}^+ \times \mathbb{Q}^+$, defined by

$$\mu_{\tilde{\circ}}(x, y, z) = \begin{cases} 1, & \text{if } z = xy, \\ \alpha(x^\bullet \wedge y^\bullet \wedge z^\bullet), & \text{otherwise,} \end{cases}$$

and

$$\mu_{\tilde{\odot}_n}(x, y, z) = \begin{cases} 1, & \text{if } z = xy, \\ \dfrac{\alpha}{n}(x^\bullet \wedge y^\bullet \wedge z^\bullet), & \text{otherwise,} \end{cases}$$

are vague binary operations on $\mathbb{R}^+$ and $\mathbb{Q}^+$, respectively; furthermore, $\langle \mathbb{R}^+, \tilde{\circ} \rangle$ and $\left\langle \mathbb{Q}^+, \tilde{\odot}_n \right\rangle$ are vague groups. Due to the definitions of $\tilde{\odot}_n$ and $\tilde{\circ}$, we have $\mu_{\tilde{\odot}_n}(x, y, z) \leq \mu_{\tilde{\circ}}(x, y, z)$ for each $x, y, z \in \mathbb{Q}^+$, i.e., $\left\langle \mathbb{Q}^+, \tilde{\odot}_n \right\rangle \overset{v.s.}{\leq} \langle \mathbb{R}^+, \tilde{\circ} \rangle$.

Proposition 1.3.8. [6] *Let $\langle G, \tilde{\circ} \rangle$ be a vague group. If $\left\langle A, \tilde{\odot} \right\rangle \overset{v.s.}{\leq} \langle G, \tilde{\circ} \rangle$ and $\langle B, \tilde{\bullet} \rangle \overset{v.s.}{\leq} \left\langle A, \tilde{\odot} \right\rangle$, then $\langle B, \tilde{\bullet} \rangle \overset{v.s.}{\leq} \langle G, \tilde{\circ} \rangle$.*

We give another equivalent conditions of generalized vague subgroup as follows:

Proposition 1.3.9. [6] *Let $\langle G, \tilde{\circ} \rangle$ be a vague group, A be a nonempty, crisp subset of G and let $\tilde{\odot}$ be a vague binary operation on A. Then*

$$\left\langle A, \tilde{\odot} \right\rangle \overset{v.s.}{\leq} \langle G, \tilde{\circ} \rangle \Longleftrightarrow \begin{array}{l} 1.\ \mu_{\tilde{\odot}}(a, b, c) \leq \mu_{\tilde{\circ}}(a, b, c) \text{ for each } a, b, c \in A, \text{ and} \\ 2.\ x^{-1} \in A \text{ for each } x \in A. \end{array}$$

Corollary 1.3.10. [6] *Let $\langle G, \tilde{\circ} \rangle$ be a vague group and $\tilde{\odot}$ be a vague binary opearation on G such that $\mu_{\tilde{\odot}}(a, b, c) \leq \mu_{\tilde{\circ}}(a, b, c)$ for all $a, b, c \in G$. Moreover, let $e_{\tilde{\circ}}$ be the identity element of G. Then $\left\langle \{e_{\tilde{\circ}}\}, \tilde{\circ} \right\rangle \overset{v.s.}{\leq} \langle G, \tilde{\circ} \rangle$ and $\left\langle G, \tilde{\tilde{\odot}} \right\rangle \overset{v.s.}{\leq} \langle G, \tilde{\circ} \rangle$.*

In this thesis, we denote the minimum of $\{x_j\}_{j \in J}$ by $\bigwedge\limits_{j \in J} x_j$.

Corollary 1.3.11. [6] *Let* $\langle G, \tilde{\circ} \rangle$ *be a vague group and let* $\langle A_j, \tilde{\odot}_j \rangle \overset{v.s.}{\leq} \langle G, \tilde{\circ} \rangle$ *for all* $j \in J$. *If* $\tilde{\star}$ *is a vague binary operation on* $\bigcap\limits_{j \in J} A_j$ *such that*

$$\mu_{\tilde{\star}}(x, y, z) \leq \bigwedge_{j \in J} \mu_{\tilde{\odot}_j}(x, y, z) \text{ for all } x, y, z \in \bigcap_{j \in J} A_j,$$

then $\left\langle \bigcap\limits_{j \in J} A_j, \tilde{\star} \right\rangle \overset{v.s.}{\leq} \langle A_k, \tilde{\odot}_k \rangle \overset{v.s.}{\leq} \langle G, \tilde{\circ} \rangle$ *for all* $k \in J$.

1.4 Vague Rings

In this section, similar fashion to classical algebra, we give the notion of vague rings and vague subrings which are introduced by Sezer [7]. We also provide some elementary properties and some examples.

Definition 1.4.1. [7] Let $E_{\mathcal{H} \times \mathcal{H}}$ and $E_{\mathcal{H}}$ be fuzzy equalities on $\mathcal{H} \times \mathcal{H}$ and $\mathcal{H}$, respectively. Let $\tilde{\circ}$ and $\tilde{\bullet}$ be two vague binary operations on $\mathcal{H}$. Then, the 3-tuple $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ is called a ***vague ring with respect to*** $E_{\mathcal{H} \times \mathcal{H}}$ ***and*** $E_{\mathcal{H}}$, or a ***vague ring*** for short, if the following three conditions are satisfied :
$(VR1)$ $\langle \mathcal{H}, \tilde{\circ} \rangle$ is a commutative vague group,
$(VR2)$ $\langle \mathcal{H}, \tilde{\bullet} \rangle$ is a vague semigroup,
$(VR3)$ $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ satisfies the ***distributive laws***, i.e., for each $a, b, c, d, t, x, y, z \in \mathcal{H}$,

$$\mu_{\tilde{\bullet}}(x, y, a) \wedge \mu_{\tilde{\bullet}}(x, z, b) \wedge \mu_{\tilde{\circ}}(a, b, c) \wedge \mu_{\tilde{\circ}}(y, z, d) \wedge \mu_{\tilde{\bullet}}(x, d, t) \leq E_{\mathcal{H}}(t, c),$$

$$\mu_{\tilde{\bullet}}(x, z, a) \wedge \mu_{\tilde{\bullet}}(y, z, b) \wedge \mu_{\tilde{\circ}}(a, b, c) \wedge \mu_{\tilde{\circ}}(x, y, d) \wedge \mu_{\tilde{\bullet}}(d, z, t) \leq E_{\mathcal{H}}(t, c).$$

$(VR4)$ A vague ring $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ is said to be a ***vague ring with identity*** if there exists $e_{\tilde{\bullet}} \in \mathcal{H}$ such that $\mu_{\tilde{\bullet}}(x, e_{\tilde{\bullet}}, x) \wedge \mu_{\tilde{\bullet}}(e_{\tilde{\bullet}}, x, x) = 1$ for each $x \in \mathcal{H}$.

From now on, we write $x \bullet y$ and $x \circ y$ by xy and $x + y$, respectively. The notation $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ always stands for a vague ring $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ with respect to $E_{\mathcal{H} \times \mathcal{H}}$

and $E_{\mathcal{H}}$. If $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ is a vague ring, then we write $-a$ the inverse of a with respect to the vague group $\langle \mathcal{H}, \tilde{\circ} \rangle$; additionally, if $\langle \mathcal{H}, \tilde{\bullet} \rangle$ is a vague group, then we denote a^{-1} the inverse of a with respect to the vague group $\langle \mathcal{H}, \tilde{\bullet} \rangle$.

Example 1.4.2. Let $\langle \mathcal{H}, \circ, \bullet \rangle$ be a ring. For $x, y, a, b \in \mathcal{H}$ and $\alpha, \beta, \gamma, \nu \in \mathbb{R}$ such that $0 \le \nu \le \gamma \le \beta \le \alpha < 1$, define the fuzzy equalities

$$E_{\mathcal{H}}(a, b) = \begin{cases} 1, & \text{if } a = b, \\ \alpha, & \text{otherwise,} \end{cases}$$

on $\mathcal{H}$ and

$$E_{\mathcal{H} \times \mathcal{H}}((a, b), (x, y)) = \begin{cases} 1, & \text{if } (a, b) = (x, y), \\ \beta, & \text{otherwise,} \end{cases}$$

on $\mathcal{H} \times \mathcal{H}$. Next, consider the vague binary operations

$$\tilde{\circ} : \mathcal{H} \times \mathcal{H} \rightsquigarrow \mathcal{H}, \;\; \text{with} \;\; \mu_{\tilde{\circ}}(a, b, c) = \begin{cases} 1, & \text{if } a \circ b = c, \\ \gamma, & \text{otherwise,} \end{cases}$$

and

$$\tilde{\bullet} : \mathcal{H} \times \mathcal{H} \rightsquigarrow \mathcal{H}, \;\; \text{with} \;\; \mu_{\tilde{\bullet}}(a, b, c) = \begin{cases} 1, & \text{if } a \bullet b = c, \\ \nu, & \text{otherwise.} \end{cases}$$

In this case, it is clearly seen that $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ is a vague ring from the inequality in (1.3.2) and the condition $(E3)$.

Proposition 1.4.3. [4] *If $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ is a vague ring, then $\langle \mathcal{H}, \circ, \bullet \rangle$ is a ring where $\circ$ and $\bullet$ are ordinary descriptions of $\tilde{\circ}$ and $\tilde{\bullet}$, respectively.*

Proposition 1.4.4. [7] *Let $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ be a vague ring and $e_{\tilde{\circ}}$ the identity element of the vague group $\langle \mathcal{H}, \tilde{\circ} \rangle$. Then, the following statements are satisfied for all $m, n, x, y \in \mathcal{H}$:*

1. $\mu_{\tilde{\bullet}}(x, e_{\tilde{\circ}}, m) \wedge \mu_{\tilde{\bullet}}(e_{\tilde{\circ}}, x, n) \le E_{\mathcal{H}}(m, n)$.

2. $\mu_{\tilde{\bullet}}(-x,-y,m) \wedge \mu_{\tilde{\bullet}}(x,y,n) \leq E_{\mathcal{H}}(m,n)$.

3. $\mu_{\tilde{\bullet}}(-e_{\tilde{\bullet}},x,-x) = 1 = \mu_{\tilde{\bullet}}(-e_{\tilde{\bullet}},-e_{\tilde{\bullet}},e_{\tilde{\bullet}})$ *if* $\langle \mathcal{H},\tilde{\circ},\tilde{\bullet}\rangle$ *is a vague ring with identity* $e_{\tilde{\bullet}}$*.*

Definition 1.4.5. [7] Let $\langle \mathcal{H},\tilde{\circ},\tilde{\bullet}\rangle$ be a vague ring and A be a nonempty crisp subset of $\mathcal{H}$. Let $\tilde{\oplus}$ and $\tilde{\odot}$ be two vague binary operations on A such that

$$\mu_{\tilde{\oplus}}(a,b,c) \leq \mu_{\tilde{\circ}}(a,b,c) \text{ and } \mu_{\tilde{\odot}}(a,b,c) \leq \mu_{\tilde{\bullet}}(a,b,c) \text{ for all } a,b,c \in A.$$

If $\left\langle A,\tilde{\oplus},\tilde{\odot}\right\rangle$ is itself a vague ring with respect to $E_{A\times A}$ and E_A, then $\left\langle A,\tilde{\oplus},\tilde{\odot}\right\rangle$ is said to be a ***vague subring*** of $\langle \mathcal{H},\tilde{\circ},\tilde{\bullet}\rangle$, denoted by $\left\langle A,\tilde{\oplus},\tilde{\odot}\right\rangle \overset{v.r.}{\leq} \langle \mathcal{H},\tilde{\circ},\tilde{\bullet}\rangle$.

The following proposition and corollaries assure that some results of classical rings also valid for vague rings.

Proposition 1.4.6. *Let* $\langle \mathcal{H},\tilde{\circ},\tilde{\bullet}\rangle$ *be a vague ring and* A *be a subset of* $\mathcal{H}$*. Let* $\tilde{\oplus}$ *and* $\tilde{\odot}$ *be two vague binary operations on* A*. Then the following equivalence is satisfied :*

$$\left\langle A,\tilde{\oplus},\tilde{\odot}\right\rangle \overset{v.r.}{\leq} \langle \mathcal{H},\tilde{\circ},\tilde{\bullet}\rangle \Leftrightarrow \begin{array}{l} 1.\left\langle A,\tilde{\oplus}\right\rangle \overset{v.s.}{\leq} \langle \mathcal{H},\tilde{\circ}\rangle, \textit{ and} \\ 2.\ \mu_{\tilde{\odot}}(a,b,c) \leq \mu_{\tilde{\bullet}}(a,b,c) \textit{ for all } a,b,c \in A. \end{array}$$

Next corollary explains that the intersection of vague subrings is also a vague subring.

Corollary 1.4.7. [7] *Let* $\langle \mathcal{H},\tilde{\circ},\tilde{\bullet}\rangle$ *be a vague ring and* $\left\langle A_j,\tilde{\oplus}_j,\tilde{\odot}_j\right\rangle \overset{v.r.}{\leq} \langle \mathcal{H},\tilde{\circ},\tilde{\bullet}\rangle$ *for all* $j \in J$*. Let* $A = \bigcap\limits_{j\in J} A_j$*, and let* $\tilde{\oplus}$ *and* $\tilde{\odot}$ *be two vague binary operations on* A *such that*

$$\mu_{\tilde{\oplus}}(a,b,c) \leq \bigwedge_{j\in J} \mu_{\tilde{\oplus}_j}(a,b,c) \textit{ and } \mu_{\tilde{\odot}}(a,b,c) \leq \bigwedge_{j\in J} \mu_{\tilde{\odot}_j}(a,b,c) \textit{ for all } a,b,c \in A.$$

Then $\left\langle A,\tilde{\oplus},\tilde{\odot}\right\rangle \overset{v.r.}{\leq} \langle \mathcal{H},\tilde{\circ},\tilde{\bullet}\rangle$.

Corollary 1.4.8. [7] *Let* $\langle \mathcal{H},\tilde{\circ},\tilde{\bullet}\rangle$ *be a vague ring and* $e_{\tilde{\circ}}$ *be the identity element of*

$\langle \mathcal{H}, \tilde{\circ} \rangle$. *Let* $\tilde{\oplus}$ *and* $\tilde{\odot}$ *be two vague binary operations on* $\mathcal{H}$ *such that*

$$\mu_{\tilde{\oplus}}(x, y, z) \leq \mu_{\tilde{\circ}}(x, y, z) \text{ and } \mu_{\tilde{\odot}}(x, y, z) \leq \mu_{\tilde{\bullet}}(x, y, z) \text{ for all } x, y, z \in \mathcal{H}.$$

Then $\langle \{e_{\tilde{\circ}}\}, \tilde{\circ}, \tilde{\bullet} \rangle \overset{v.r.}{\leq} \langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ *and* $\langle \mathcal{H}, \tilde{\oplus}, \tilde{\odot} \rangle \overset{v.r.}{\leq} \langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$.

1.5 Vague Ideals

In this section, we give some concepts and some fundamental properties of vague ideals which are introduced by Sezer [7]. Ultimately, we give some ideas of vague prime ideals and maximal vague ideals which are also introduced by the same author.

Definition 1.5.1. [7] Let $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ be a vague ring and $\langle A, \tilde{\oplus}, \tilde{\odot} \rangle \overset{v.r.}{\leq} \langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$. If for all $a \in A$ and for all $h, t, s \in \mathcal{H}$

$$\mu_{\tilde{\circ}}(a, h, t) = 1 \Rightarrow t \in A \quad \text{and} \quad \mu_{\tilde{\circ}}(h, a, s) = 1 \Rightarrow s \in A,$$

then $\langle A, \tilde{\oplus}, \tilde{\odot} \rangle$ is said to be a ***vague ideal*** of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$, denoted by $\langle A, \tilde{\oplus}, \tilde{\odot} \rangle \overset{v.i.}{\leq} \langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$.

It is clear from Definition 1.5.1 that if $\mu_{\tilde{\circ}}(\mathcal{H} \times \mathcal{H} \times \mathcal{H}) \subseteq \{0, 1\}$, $E_{\mathcal{H}} = E^c_{\mathcal{H}}$, $E_{\mathcal{H}\times\mathcal{H}} = E^c_{\mathcal{H}\times\mathcal{H}}$ and $\langle A, \tilde{\oplus}, \tilde{\odot} \rangle \overset{v.i.}{\leq} \langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$, then $\langle A, \oplus, \odot \rangle$ is an ideal of $\langle \mathcal{H}, \circ, \bullet \rangle$. Therefore, in this case, a vague ideal $\langle A, \tilde{\oplus}, \tilde{\odot} \rangle$ of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ is nothing but an ideal of the classical ring $\langle \mathcal{H}, \circ, \bullet \rangle$ in the classical sense.

Proposition 1.5.2. [7] *Let* $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ *be a vague ring. If* $\langle A, \tilde{\oplus}, \tilde{\odot} \rangle \overset{v.i.}{\leq} \langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$, *then* $\langle A, \oplus, \odot \rangle$ *is an ideal of* $\langle \mathcal{H}, \circ, \bullet \rangle$.

Proposition 1.5.3. [7] *Let* $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ *be a vague ring. Then*

$$\langle \{e_{\tilde{\circ}}\}, \tilde{\circ}, \tilde{\bullet} \rangle \overset{v.i.}{\leq} \langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle \text{ and } \langle \mathcal{H}, \tilde{\oplus}, \tilde{\odot} \rangle \overset{v.i.}{\leq} \langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$$

where $\tilde{\oplus}$ *and* $\tilde{\odot}$ *are vague binary operations on* $\mathcal{H}$ *such that* $\mu_{\tilde{\oplus}}(a,b,c) \leq \mu_{\tilde{\circ}}(a,b,c)$ *and* $\mu_{\tilde{\odot}}(a,b,c) \leq \mu_{\tilde{\bullet}}(a,b,c)$ *for all* $a,b,c \in \mathcal{H}$.

We give another equivalence conditions of vague ideals.

Proposition 1.5.4. [7] *Let* $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ *be a vague ring, A be a nonempty crisp subset of* $\mathcal{H}$ *and* $\tilde{\oplus}, \tilde{\odot}$ *be two vague binary operations on A. Then*

$$\left\langle A, \tilde{\oplus}, \tilde{\odot} \right\rangle \overset{v.i.}{\leq} \langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle \Longleftrightarrow 1.\ \left\langle A, \tilde{\oplus} \right\rangle \overset{v.s.}{\leq} \langle \mathcal{H}, \tilde{\circ} \rangle,$$

$$2.\ \mu_{\tilde{\odot}}(a,b,c) \leq \mu_{\tilde{\bullet}}(a,b,c) \text{ for all } a,b,c \in A,$$

$$3.\ \mu_{\tilde{\bullet}}(a,h,t) = 1 \Rightarrow t \in A \text{ for all } a \in A,\ h,t \in \mathcal{H}, \text{ and}$$

$$4.\ \mu_{\tilde{\bullet}}(h,a,s) = 1 \Rightarrow s \in A \text{ for all } a \in A,\ h,s \in \mathcal{H}.$$

Proposition 1.5.5. [7] *Let* $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ *be a vague ring and* $\left\langle A_j, \tilde{\oplus}_j, \tilde{\odot}_j \right\rangle \overset{v.i.}{\leq} \langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ *for all* $j \in J$. *If* $\tilde{\oplus}$ *and* $\tilde{\odot}$ *are vague binary operations on* $\bigcap\limits_{j \in J} A_j$ *such that*

$$\mu_{\tilde{\oplus}}(a,b,c) \leq \bigwedge_{j \in J} \mu_{\tilde{\circ}_j}(a,b,c) \text{ and } \mu_{\tilde{\odot}}(a,b,c) \leq \bigwedge_{j \in J} \mu_{\tilde{\bullet}_j}(a,b,c),$$

then $\left\langle \bigcap\limits_{j \in J} A_j, \tilde{\oplus}, \tilde{\odot} \right\rangle \overset{v.i.}{\leq} \langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$.

We give a crucial notion in vague structures called "vague prime ideal."

Definition 1.5.6. **[7]** Let $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ be a vague ring, A be a proper crisp subset of $\mathcal{H}$ and $\left\langle A, \tilde{\oplus}, \tilde{\odot} \right\rangle \overset{v.i}{\leq} \langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$. If $\tilde{\bullet}$ is a vague binary operation on $\mathcal{H} \setminus A$, then $\left\langle A, \tilde{\oplus}, \tilde{\odot} \right\rangle$ is said to be a ***vague prime ideal*** of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$.

Proposition 1.5.7. [7] *Let* $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ *be a vague ring and* $\left\langle A, \tilde{\oplus}, \tilde{\odot} \right\rangle \overset{v.i}{\leq} \langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$. *Then the following two statements are equivalent.*

1. $\left\langle A, \tilde{\oplus}, \tilde{\odot} \right\rangle$ *is a vague prime ideal of* $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$.
2. $\mu_{\tilde{\bullet}}(x,y,z) < 1$ *for each* $z \in A$ *and for each* $x, y \in \mathcal{H} \setminus A$.

Next, a relation between vague prime ideals of the vague ring $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ and prime ideals of the ring $\langle \mathcal{H}, \circ, \bullet \rangle$ is provided.

Proposition 1.5.8. *If $\langle A, \tilde{\oplus}, \tilde{\odot} \rangle$ is a vague prime ideal of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$, then $\langle A, \oplus, \odot \rangle$ is a prime ideal of $\langle \mathcal{H}, \circ, \bullet \rangle$.*

Proof. Let M and N be any ideals of A such that $M \nsubseteq A$ and $N \nsubseteq A$. Thus there exists $x \in M$, and $y \in N$ such that $x, y \in \mathcal{H} \setminus A$. But $\mu_{\tilde{\bullet}}(x, y, xy) = 1$ by Proposition 1.5.7, we have $xy \notin A$. Since $xy \in MN$, it follows that $MN \nsubseteq A$. Hence $\langle A, \oplus, \odot \rangle$ is a prime ideal of $\langle \mathcal{H}, \circ, \bullet \rangle$ as desired. □

Definition 1.5.9. [7] Let $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ be a vague ring, M be a nonempty proper crisp subset of $\mathcal{H}$ and $\langle M, \tilde{\oplus}, \tilde{\odot} \rangle \overset{v.i}{\leq} \langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$. If there are no vague ideals $\langle N, \tilde{\ominus}, \tilde{\star} \rangle$ such that

$$\langle M, \tilde{\oplus}, \tilde{\odot} \rangle \subsetneqq \langle N, \tilde{\ominus}, \tilde{\star} \rangle \overset{v.i}{\leq} \langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle,$$

then $\langle M, \tilde{\oplus}, \tilde{\odot} \rangle$ is said to be a ***maximal vague ideal*** of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$.

Note that for a vague ideal $\langle I, \tilde{\oplus}, \tilde{\odot} \rangle$ of a vague ring $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ where $\mu_{\tilde{\oplus}} \neq \mu_{\tilde{\circ}}$ and $\mu_{\tilde{\odot}} \neq \mu_{\tilde{\bullet}}$, we obtain that $\langle I, \tilde{\oplus}, \tilde{\odot} \rangle$ cannot be a maximal vague ideal since

$$\langle I, \tilde{\oplus}, \tilde{\odot} \rangle \subsetneqq \langle I, \tilde{\circ}, \tilde{\bullet} \rangle \subseteq \langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle.$$

Proposition 1.5.10. [7] *Let $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ be a vague ring. If $\langle M, \tilde{\oplus}, \tilde{\odot} \rangle$ is a maximal vague ideal of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$, then $\langle M, \oplus, \odot \rangle$ is a maximal ideal of $\langle \mathcal{H}, \circ, \bullet \rangle$.*

Proposition 1.5.11. [7] *Let $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ be a vague ring. Then, maximal vague ideals of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ are $\langle M, \tilde{\circ}, \tilde{\bullet} \rangle$ where M is one of the maximal ideals of $\langle \mathcal{H}, \circ, \bullet \rangle$.*

Finally, we give some examples about vague prime ideals and maximal vague ideals in the following.

Example 1.5.12. [7] For the following vague binary operations $\tilde{\oplus}, \tilde{\odot}, \tilde{\circ}$ and $\tilde{\bullet}$, we show that $\langle 2\mathbb{Z}, \tilde{\oplus}, \tilde{\odot} \rangle$ is a vague prime ideal of $\langle \mathbb{Z}, \tilde{\circ}, \tilde{\bullet} \rangle$ and $\langle 2\mathbb{Z}, \tilde{\oplus}, \tilde{\odot} \rangle$ is not a maximal vague ideal of $\langle \mathbb{Z}, \tilde{\circ}, \tilde{\bullet} \rangle$. Let $G = \mathbb{Z}$, $A = 2\mathbb{Z}$ and $\alpha, \beta, \gamma \in \mathbb{R}$ be such that $0 \leq \gamma \leq \beta \leq \alpha < 1$. We define

$$E_{\mathbb{Z}} : \mathbb{Z} \times \mathbb{Z} \to [0,1], \qquad E_{\mathbb{Z}}(u,v) = \begin{cases} 1, & \text{if } u = v, \\ \alpha, & \text{otherwise,} \end{cases}$$

$$E_{2\mathbb{Z}} : 2\mathbb{Z} \times 2\mathbb{Z} \to [0,1], \qquad E_{2\mathbb{Z}}(m,n) = E_{\mathbb{Z}}(m,n),$$

$E_{\mathbb{Z}\times\mathbb{Z}} = E^c_{\mathbb{Z}\times\mathbb{Z}}$, $E_{2\mathbb{Z}\times 2\mathbb{Z}} = E^c_{2\mathbb{Z}\times 2\mathbb{Z}}$,

$$\tilde{\circ} : \mathbb{Z} \times \mathbb{Z} \rightsquigarrow \mathbb{Z}, \qquad \mu_{\tilde{\circ}}(x,y,z) = \begin{cases} 1, & \text{if } x + y = z, \\ \beta, & \text{otherwise,} \end{cases}$$

and

$$\tilde{\oplus} : 2\mathbb{Z} \times 2\mathbb{Z} \rightsquigarrow 2\mathbb{Z}, \qquad \mu_{\tilde{\oplus}}(a,b,c) = \begin{cases} 1, & \text{if } a + b = c, \\ \gamma, & \text{otherwise.} \end{cases}$$

We obtain that $\langle 2\mathbb{Z}, \tilde{\oplus} \rangle \overset{v.s}{\leq} \langle \mathbb{Z}, \tilde{\circ} \rangle$. Moreover, let $\nu, \eta \in \mathbb{R}$ be such that $0 \leq \nu < \eta < 1$, and we define

$$\tilde{\bullet} : \mathbb{Z} \times \mathbb{Z} \rightsquigarrow \mathbb{Z}, \quad \mu_{\tilde{\bullet}}(x,y,z) = \begin{cases} 1, & \text{if } x \cdot y = z, \\ \eta, & \text{otherwise,} \end{cases}$$

and

$$\tilde{\odot} : 2\mathbb{Z} \times 2\mathbb{Z} \rightsquigarrow 2\mathbb{Z}, \quad \mu_{\tilde{\odot}}(a,b,c) = \begin{cases} 1, & \text{if } a \cdot b = c, \\ \nu, & \text{otherwise.} \end{cases}$$

In this case, it is clearly seen that $\langle \mathbb{Z}, \tilde{\circ}, \tilde{\bullet} \rangle$ is a vague ring and $\tilde{\odot}$ is a vague binary operation on $2\mathbb{Z}$. Therefore, by using Proposition 1.5.4, we get $\langle 2\mathbb{Z}, \tilde{\oplus}, \tilde{\odot} \rangle \overset{v.i}{\leq}$

$\langle \mathbb{Z}, \tilde{\circ}, \tilde{\bullet} \rangle$. On the other hand, since $\mu_{\tilde{\odot}}(a, b, c) = \nu < 1$ for each $a, b \in \mathbb{Z} \setminus 2\mathbb{Z}$ and for each $c \in 2\mathbb{Z}$, it follows from Proposition 1.5.7 that $\left\langle 2\mathbb{Z}, \tilde{\oplus}, \tilde{\odot} \right\rangle$ is a vague prime ideal of $\langle \mathbb{Z}, \tilde{\circ}, \tilde{\bullet} \rangle$. Furthermore, since

$$\left\langle 2\mathbb{Z}, \tilde{\oplus}, \tilde{\odot} \right\rangle \subsetneqq \langle 2\mathbb{Z}, \tilde{\circ}, \tilde{\bullet} \rangle \overset{v.i}{\leq} \langle \mathbb{Z}, \tilde{\circ}, \tilde{\bullet} \rangle ,$$

$\left\langle 2\mathbb{Z}, \tilde{\oplus}, \tilde{\odot} \right\rangle$ is not a maximal vague ideal of $\langle \mathbb{Z}, \tilde{\circ}, \tilde{\bullet} \rangle$.

Next, we give a different example of vague prime ideals.

Example 1.5.13. Let p be a prime number. Consider $(p)_i$ as an principal ideal of $\langle \mathbb{Z}, \circ, \bullet \rangle$ containing p. We show that $\left\langle (p)_i, \tilde{\circ}, \tilde{\bullet} \right\rangle$ is a vague prime ideal of $\langle \mathbb{Z}, \tilde{\circ}, \tilde{\bullet} \rangle$ where $\tilde{\circ}$ and $\tilde{\bullet}$ are any vague binary operations of which $\circ$ and $\bullet$ are the ordinary descriptions, respectively. Suppose not, i.e.,

$$\exists x, y \notin (p)_i, \exists z \in (p)_i \text{ such that } \mu_{\tilde{\bullet}}(x, y, z) = 1.$$

Thus $xy = z \in (p)_i$. Therefore $p|xy$, that is $p|x$ or $p|y$ leads to a contradiction. Hence $\left\langle (p)_i, \tilde{\circ}, \tilde{\bullet} \right\rangle$ is a vague prime ideal of $\langle \mathbb{Z}, \tilde{\circ}, \tilde{\bullet} \rangle$.

CHAPTER II

Vague Prime Ideals

In this chapter, some elementary properties of vague ideals are first given. We have an inspiration from the classical sense that the sum of ideals of a ring is also an ideal. Next, we give some sufficient conditions of a vague ring that every maximal vague ideal is a vague prime ideal. Then we give the sufficient condition of the vague ring of matrices to have no vague prime ideals.

Eventually, for a vague ring $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$, we set up certain sets induced form a vague prime ideal, a vague prime right ideal and a vague prime left ideal of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ and also study their properties whether they are vague ideals.

2.1 Vague Ideals, Maximal Vague Ideals, Vague Prime Ideals and Vague Irreducible Ideals

In this section, first, we give some properties of a vague ideal. Then every maximal vague ideal implying a vague prime ideal of a certain vague ring is provided. Finally, we show that the commutativity of a vague ring in Theroem 2.1.2 is neccessary.

Lemma 2.1.1. *Let $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ be a vague ring, $\left\langle A, \tilde{\oplus}, \tilde{\odot} \right\rangle$ and $\left\langle B, \tilde{\boxplus}, \tilde{\boxdot} \right\rangle$ be vague ideals of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$. If $\tilde{\heartsuit}$ and $\tilde{\spadesuit}$ are vague binary operations of $\mathcal{H}$ such that for each $a, b, c \in \mathcal{H}$,*

$$\mu_{\tilde{\heartsuit}}(a, b, c) \leq \mu_{\tilde{\circ}}(a, b, c) \text{ and } \mu_{\tilde{\spadesuit}}(a, b, c) \leq \mu_{\tilde{\bullet}}(a, b, c),$$

then $\left\langle A + B, \tilde{\heartsuit}, \tilde{\spadesuit} \right\rangle$ is also a vague ideal of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ where $\tilde{\circ}, \tilde{\oplus}, \tilde{\boxplus}, \tilde{\heartsuit}$ and $\tilde{\bullet}, \tilde{\odot}, \tilde{\boxdot}, \tilde{\spadesuit}$

are any vague binary operations of which $+$ and $\cdot$ are the ordinary descriptions, respectively.

Proof. First, we show that $\left\langle A+B, \tilde{\heartsuit} \right\rangle \overset{v.s.}{\leq} \langle \mathcal{H}, \tilde{\circ} \rangle$. Let $x \in A+B$. Thus $x = a+b$ for some $a \in A$ and $b \in B$. Since A and B are vague ideals of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$, we obtain that $-a \in A$ and $-b \in B$. Therefore $-x = (-a) + (-b) \in A+B$. By assumption, for each $a, b, c \in A+B$, $\mu_{\tilde{\heartsuit}}(a,b,c) \leq \mu_{\tilde{\circ}}(a,b,c)$. Therefore Proposition 1.3.9 gives $\left\langle A+B, \tilde{\heartsuit} \right\rangle \overset{v.s.}{\leq} \langle \mathcal{H}, \tilde{\circ} \rangle$. Similarly, we have $\mu_{\tilde{\spadesuit}}(a,b,c) \leq \mu_{\tilde{\bullet}}(a,b,c)$ for each $a, b, c \in A+B$. Finally, let $k \in A+B$ and $h, t \in \mathcal{H}$ be such that $\mu_{\tilde{\bullet}}(k,h,t) = 1$. Thus $t = kh$. Since $k \in A+B$, we have $k = a+b$ for some $a \in A$ and $b \in B$. Since $\mu_{\tilde{\bullet}}(a,h,ah) = 1 = \mu_{\tilde{\bullet}}(b,h,bh)$, we obtain that $ah \in A$ and $bh \in B$. Therefore $t = kh = (a+b)h = ah + bh \in A+B$. Analogously, we can show that if $\mu_{\tilde{\bullet}}(h,k,t) = 1$, then $t \in A+B$. Hence by Proposition 1.5.4, we have $\left\langle A+B, \tilde{\heartsuit}, \tilde{\spadesuit} \right\rangle$ is a vague ideal of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ as desired. $\square$

Note that for any vague ring $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ and $x \in \mathcal{H}$, we denote $(x)_i$ the smallest vague ideal of $\mathcal{H}$ containing x. If $\mathcal{H}$ is also commutative and M is a nonempty crisp subset of $\mathcal{H}$, we denote $M + (x)_i$ the set

$$\{m + rx \mid m \in M \text{ and } r \in \mathcal{H}\}.$$

Since for each $a, b, c \in \mathcal{H}$,

$$\mu_{\tilde{\circ}}(a,b,c) \leq \mu_{\tilde{\circ}}(a,b,c) \text{ and } \mu_{\tilde{\bullet}}(a,b,c) \leq \mu_{\tilde{\bullet}}(a,b,c),$$

by Lemma 2.1.1, if M is a vague ideal of $\mathcal{H}$, then $\langle M + (x)_i, \tilde{\circ}, \tilde{\bullet} \rangle$ is also a vague ideal of $\mathcal{H}$. In a classical commutative ring R with identity, every maximal ideal of R is a prime ideal of R. The corrresponding result also holds in a vague ring.

Theorem 2.1.2. *Let $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ be a commutative vague ring with identity. If $\langle M, \tilde{\circ}, \tilde{\bullet} \rangle$ is a maximal vague ideal of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$, then $\langle M, \tilde{\circ}, \tilde{\bullet} \rangle$ is a vague prime ideal of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$.*

Proof. Assume that $\langle M, \tilde{\circ}, \tilde{\bullet} \rangle$ is a maximal vague ideal of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ such that $\mu_{\tilde{\bullet}}(x, y, z) = 1$ for some $x, y \in \mathcal{H} \smallsetminus M$ and $z \in M$, i.e., $xy = z \in M$. Consider the following sequence

$$\langle M, \tilde{\circ}, \tilde{\bullet} \rangle \subsetneq \left\langle M + (x)_i, \tilde{\circ}, \tilde{\bullet} \right\rangle \subseteq \langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle .$$

By the maximality of M and Proposition 1.5.7, we have $M + (x)_i = \mathcal{H}$ in the classical sense. Therefore, there exist $r \in \mathcal{H}$ and $m \in M$ such that $m + rx = 1$. Hence

$$y = 1y = (m + rx)\, y = my + r\,(xy) \in M$$

which is a contradiction. Consequently, $\langle M, \tilde{\circ}, \tilde{\bullet} \rangle$ is a vague prime ideal of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ as desired. □

Next example shows that the commutativity of a vague ring in Theroem 2.1.2 is neccessary.

Example 2.1.3. First observe that $\langle \mathbb{M}_n(\mathbb{Z}_6), +, \cdot \rangle$ is a non-commutative ring where $+$ and $\cdot$ are the usual addition and usual multiplication of matrices. By Proposition 1.3.5, any vague ring $\langle \mathbb{M}_n(\mathbb{Z}_6), \tilde{\circ}, \tilde{\bullet} \rangle$ is not commutative where $\tilde{\circ}$ and $\tilde{\bullet}$ are any vague binary operations of which $+$ and $\cdot$ are the ordinary descriptions, respectively. Since there is a one-to-one correspondence between the ideals of the ring $\mathbb{M}_n(\mathbb{Z}_6)$ and the ideals of the ring $\mathbb{Z}_6$, evidently, $\{\bar{0}, \bar{3}\}$ is a maximal ideal of $\mathbb{Z}_6$. Therefore

$$\mathcal{I} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \{\bar{0}, \bar{3}\} \right\}$$

is a maximal ideal of $\langle \mathbb{M}_n(\mathbb{Z}_6), +, \cdot \rangle$. Thus by Proposition 1.5.11, it follows that $\langle \mathcal{I}, \tilde{\circ}, \tilde{\bullet} \rangle$ is a maximal vague ideal of $\langle \mathbb{M}_n(\mathbb{Z}_6), \tilde{\circ}, \tilde{\bullet} \rangle$. But $\mathcal{I}$ is not a vague prime ideal because

$$\left(\begin{smallmatrix} \bar{1} & \bar{5} \\ \bar{2} & \bar{1} \end{smallmatrix}\right) \notin \mathcal{I} \text{ and } \left(\begin{smallmatrix} \bar{1} & \bar{1} \\ \bar{1} & \bar{4} \end{smallmatrix}\right) \notin \mathcal{I} \quad \text{but} \quad \mu_{\tilde{\bullet}}\left(\left(\begin{smallmatrix} \bar{1} & \bar{5} \\ \bar{2} & \bar{1} \end{smallmatrix}\right), \left(\begin{smallmatrix} \bar{1} & \bar{1} \\ \bar{1} & \bar{4} \end{smallmatrix}\right), \left(\begin{smallmatrix} \bar{0} & \bar{3} \\ \bar{3} & \bar{0} \end{smallmatrix}\right)\right) = 1,$$

and $\left(\begin{smallmatrix} \bar{0} & \bar{3} \\ \bar{3} & \bar{0} \end{smallmatrix}\right) \in \mathcal{I}$. Consequently, $\langle \mathcal{I}, \tilde{\circ}, \tilde{\bullet} \rangle$ is a maximal vague ideal of $\langle \mathbb{M}_n(\mathbb{Z}_6), \tilde{\circ}, \tilde{\bullet} \rangle$ but it is not a vague prime ideal of $\langle \mathbb{M}_n(\mathbb{Z}_6), \tilde{\circ}, \tilde{\bullet} \rangle$ as desired.

Next, we define and give some elementary properties of a considerable vague ideal called "vague irreducible ideal" which is related to vague primary ideals in Chapter 3.

Definition 2.1.4. Let $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ be a vague ring. A vague ideal $\left\langle A, \tilde{\oplus}, \tilde{\odot} \right\rangle$ of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ is called a ***vague irreducible ideal*** if $\left\langle A, \tilde{\oplus}, \tilde{\odot} \right\rangle = \left\langle J, \tilde{\oplus}, \tilde{\odot} \right\rangle \cap \left\langle K, \tilde{\oplus}, \tilde{\odot} \right\rangle$ implies either $\left\langle A, \tilde{\oplus}, \tilde{\odot} \right\rangle = \left\langle J, \tilde{\oplus}, \tilde{\odot} \right\rangle$ or $\left\langle A, \tilde{\oplus}, \tilde{\odot} \right\rangle = \left\langle K, \tilde{\oplus}, \tilde{\odot} \right\rangle$ for any vague ideals $\left\langle J, \tilde{\oplus}, \tilde{\odot} \right\rangle$ and $\left\langle K, \tilde{\oplus}, \tilde{\odot} \right\rangle$ of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$.

Next lemma provides some properties of a chain of vague ideals.

Lemma 2.1.5. *Let* $\left\{ \left\langle J_i, \tilde{\oplus}, \tilde{\odot} \right\rangle \mid i \in \Omega \right\}$ *be a chain of vague ideals of a ring* $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$. *Then* $\left\langle \bigcup J_i, \tilde{\oplus}, \tilde{\odot} \right\rangle$ *is a vague ideal of* $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$.

Proof. First, we show that $\left\langle \bigcup J_i, \tilde{\oplus} \right\rangle \overset{v.s.}{\leq} \langle \mathcal{H}, \tilde{\circ} \rangle$. Let $x \in \bigcup J_i$. Thus $x \in J_k$ for some $k \in \Omega$. Since J_k is a vague ideal of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$, we obtain that $-x \in J_k \subseteq \bigcup J_i$. Evidently, for each $a, b, c \in \bigcup J_i$, $\mu_{\tilde{\oplus}}(a, b, c) \leq \mu_{\tilde{\circ}}(a, b, c)$. Therefore, by Proposition 1.3.9, we have $\left\langle \bigcup J_i, \tilde{\oplus} \right\rangle \overset{v.s.}{\leq} \langle \mathcal{H}, \tilde{\circ} \rangle$. Similarly, $\mu_{\tilde{\odot}}(a, b, c) \leq \mu_{\tilde{\bullet}}(a, b, c)$ for each $a, b, c \in \bigcup J_i$.

Finally, let $a \in \bigcup J_i$ and $h, t \in \mathcal{H}$ be such that $\mu_{\tilde{\bullet}}(a, h, t) = 1$. Then $a \in J_k$ for some $k \in \Omega$. Since J_k is a vague ideal of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$, $t \in J_k \subseteq \bigcup J_i$. Analogously, we can show that if $\mu_{\tilde{\bullet}}(h, a, t) = 1$, then $t \in \bigcup J_i$.

Therefore, by Proposition 1.5.4, we have $\left\langle \bigcup J_i, \tilde{\oplus}, \tilde{\odot} \right\rangle$ is a vague ideal of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ as desired. □

Lemma 2.1.6. *Let* $\left\langle A, \tilde{\oplus}, \tilde{\odot} \right\rangle$ *be a vague ideal of* $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$. *If* $a \notin \left\langle A, \tilde{\oplus}, \tilde{\odot} \right\rangle$, *then there exists a vague irreducible ideal containing* $\left\langle A, \tilde{\oplus}, \tilde{\odot} \right\rangle$ *but not containing* a.

Proof. Let $\left\{ \left\langle J_i, \tilde{\oplus}, \tilde{\odot} \right\rangle \mid i \in \Omega \right\}$ be a chain of vague ideals of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ containing $\left\langle A, \tilde{\oplus}, \tilde{\odot} \right\rangle$ but not containing a. Then, by Lemma 2.1.5, $\left\langle \bigcup J_i, \tilde{\oplus}, \tilde{\odot} \right\rangle$ is a vague ideal of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ containing $\left\langle A, \tilde{\oplus}, \tilde{\odot} \right\rangle$ but not containing a. Therefore, by Zorn's Lemma, the set of all vague ideals of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ containing $\left\langle A, \tilde{\oplus}, \tilde{\odot} \right\rangle$ but not containing a has a maximal element, say $\left\langle J, \tilde{\oplus}, \tilde{\odot} \right\rangle$. Suppose that $\left\langle J, \tilde{\oplus}, \tilde{\odot} \right\rangle = \left\langle M, \tilde{\oplus}, \tilde{\odot} \right\rangle \cap \left\langle N, \tilde{\oplus}, \tilde{\odot} \right\rangle$ where $\left\langle M, \tilde{\oplus}, \tilde{\odot} \right\rangle$ and $\left\langle N, \tilde{\oplus}, \tilde{\odot} \right\rangle$ are both vague ideals of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ properly containing J. We see that if

$$\left\langle M, \tilde{\oplus}, \tilde{\odot} \right\rangle, \left\langle N, \tilde{\oplus}, \tilde{\odot} \right\rangle \in \left\{ \left\langle J_i, \tilde{\oplus}, \tilde{\odot} \right\rangle \mid i \in \Omega \right\},$$

then they are contrary to the maximality of $\left\langle J, \tilde{\oplus}, \tilde{\odot} \right\rangle$. Since $A \subseteq J \subseteq M$ and $A \subseteq J \subseteq N$, we obtain $a \in M$ and $a \in N$. Consequently, $a \in M \cap N = J$ which is a contradiction. Hence $\left\langle J, \tilde{\oplus}, \tilde{\odot} \right\rangle$ is a vague irreducible ideal of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$. $\square$

Theorem 2.1.7. *A vague ideal* $\left\langle A, \tilde{\oplus}, \tilde{\odot} \right\rangle$ *of* $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ *is the intersection of all vague irreducible ideals of* $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ *containing* $\left\langle A, \tilde{\oplus}, \tilde{\odot} \right\rangle$.

Proof. Let $\left\langle A, \tilde{\oplus}, \tilde{\odot} \right\rangle$ be a vague ideal of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ and $\left\{ \left\langle J_i, \tilde{\oplus}, \tilde{\odot} \right\rangle \mid i \in \Omega \right\}$ be the collection of all vague irreducible ideals of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ containing $\left\langle A, \tilde{\oplus}, \tilde{\odot} \right\rangle$. Obviously, $A \subseteq \bigcap J_i$. For the reverse inclusion, let $a \notin A$. By Lemma 2.1.6, there exists a vague irreducible ideal J of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ containing $\left\langle A, \tilde{\oplus}, \tilde{\odot} \right\rangle$ but not containing a. Thus $a \notin \bigcap J_i$. Hence $\left\langle A, \tilde{\oplus}, \tilde{\odot} \right\rangle = \left\langle \bigcap J_i, \tilde{\oplus}, \tilde{\odot} \right\rangle$. $\square$

Proposition 2.1.8. *Every maximal vague ideal of* $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ *is a vague irreducible ideal of* $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$.

Proof. Let $\langle M, \tilde{\circ}, \tilde{\bullet} \rangle$ be maximal vague ideal of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$. Suppose that $\langle M, \tilde{\circ}, \tilde{\bullet} \rangle$ is not a vague irreducible ideal of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$, i.e., $\langle M, \tilde{\circ}, \tilde{\bullet} \rangle = \langle A, \tilde{\circ}, \tilde{\bullet} \rangle \cap \langle B, \tilde{\circ}, \tilde{\bullet} \rangle$ where $\langle A, \tilde{\circ}, \tilde{\bullet} \rangle$ and $\langle B, \tilde{\circ}, \tilde{\bullet} \rangle$ are vague ideals of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ properly containing $\langle M, \tilde{\circ}, \tilde{\bullet} \rangle$. Consider the following sequence

$$\langle M, \tilde{\circ}, \tilde{\bullet} \rangle \subsetneqq \langle A, \tilde{\circ}, \tilde{\bullet} \rangle \subseteq \langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle \text{ and } \langle M, \tilde{\circ}, \tilde{\bullet} \rangle \subsetneqq \langle B, \tilde{\circ}, \tilde{\bullet} \rangle \subseteq \langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle .$$

By the maximality of $\langle M, \tilde{\circ}, \tilde{\bullet} \rangle$, we have $\langle A, \tilde{\circ}, \tilde{\bullet} \rangle = \langle B, \tilde{\circ}, \tilde{\bullet} \rangle = \langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$. Therefore, $M = A \cap B = \mathcal{H}$ so that a contradiction occurs. Hence $\langle M, \tilde{\circ}, \tilde{\bullet} \rangle$ is a vague irreducible ideal of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$. $\square$

The converse of Proposition 2.1.8 is not true as we can see from the next example.

Example 2.1.9. Consider the non-commutative ring $\langle \mathbb{M}_n(\mathbb{Z}_{12}), +, \cdot \rangle$ where $+$ and $\cdot$ are the usual addition and usual multiplication of matrices. Let $\langle \mathbb{M}_n(\mathbb{Z}_{12}), \tilde{\circ}, \tilde{\bullet} \rangle$ be a vague ring where $\tilde{\circ}$ and $\tilde{\bullet}$ are any vague binary operations of which $+$ and $\cdot$ are the ordinary descriptions, respectively. Since there is a one-to-one correspondence between the ideals of the ring $\mathbb{M}_n(\mathbb{Z}_{12})$ and the ideals of the ring $\mathbb{Z}_{12}$, evidently, $\mathbb{Z}_{12}$ has 6 ideals namely:

$$\{\bar{0}\}, \{\bar{0}, \bar{6}\}, \{\bar{0}, \bar{4}, \bar{8}\}, \{\bar{0}, \bar{3}, \bar{6}, \bar{9}\}, \{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \bar{10}\}, \mathbb{Z}_{12}.$$

Consider the vague ideal $\left\langle \mathbb{M}_n(\{\bar{0}, \bar{4}, \bar{8}\}), \tilde{\circ}, \tilde{\bullet} \right\rangle$ of $\langle \mathbb{M}_n(\mathbb{Z}_{12}), \tilde{\circ}, \tilde{\bullet} \rangle$. Write

$$\left\langle \mathbb{M}_n(\{\bar{0}, \bar{4}, \bar{8}\}), \tilde{\circ}, \tilde{\bullet} \right\rangle = \langle A, \tilde{\circ}, \tilde{\bullet} \rangle \cap \langle B, \tilde{\circ}, \tilde{\bullet} \rangle$$

for some vague ideals $\langle A, \tilde{\circ}, \tilde{\bullet} \rangle$ and $\langle B, \tilde{\circ}, \tilde{\bullet} \rangle$ of $\langle \mathbb{M}_n(\mathbb{Z}_{12}), \tilde{\circ}, \tilde{\bullet} \rangle$. From the above observation, clearly, $A = \mathbb{M}_n\big(\{\bar{0}, \bar{4}, \bar{8}\}\big)$ or $B = \mathbb{M}_n\left(\{\bar{0}, \bar{4}, \bar{8}\}\right)$.

Therefore $\left\langle \mathbb{M}_n(\{\bar{0}, \bar{4}, \bar{8}\}), \tilde{\circ}, \tilde{\bullet} \right\rangle$ is a vague irreducible ideal of $\langle \mathbb{M}_n(\mathbb{Z}_{12}), \tilde{\circ}, \tilde{\bullet} \rangle$ but $\left\langle \mathbb{M}_n(\{\bar{0}, \bar{4}, \bar{8}\}), \tilde{\circ}, \tilde{\bullet} \right\rangle$ is not a maximal vague ideal since

$$\left\langle \mathbb{M}_n(\{\bar{0}, \bar{4}, \bar{8}\}), \tilde{\circ}, \tilde{\bullet} \right\rangle \subsetneqq \langle \mathbb{M}_n(\{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \bar{10}\}), \tilde{\circ}, \tilde{\bullet} \rangle \subseteq \langle \mathbb{M}_n(\mathbb{Z}_{12}), \tilde{\circ}, \tilde{\bullet} \rangle.$$

Hence $\left\langle \mathbb{M}_n(\{\bar{0}, \bar{4}, \bar{8}\}), \tilde{\circ}, \tilde{\bullet} \right\rangle$ is a vague irreducible ideal which is not a maximal vague ideal of $\langle \mathbb{M}_n(\mathbb{Z}_{12}), \tilde{\circ}, \tilde{\bullet} \rangle$ as desired.

2.2 A Vague Ring without Vague Prime Ideals

In this section, we give an example of the vague ring of matrices having none of vague prime ideals. In the following, we give an example of classical rings possesses three prime ideals but all of the corresponding vague rings does not have any vague prime ideals.

Example 2.2.1. Consider the ring $\langle \mathbb{M}_n(\mathbb{Z}_6), +, \cdot \rangle$ and the vague ring $\langle \mathbb{M}_n(\mathbb{Z}_6), \tilde{\circ}, \tilde{\bullet} \rangle$. Obviously, there are only three proper ideals of the ring $\langle \mathbb{M}_n(\mathbb{Z}_6), +, \cdot \rangle$, say

$$\mathcal{I}_1 = \left\{ \left(\begin{smallmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{smallmatrix} \right) \right\}, \ \mathcal{I}_2 = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \mid \ a, b, c, d \in \{\bar{0}, \bar{3}\} \right\}, \ \mathcal{I}_3 = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \mid \ a, b, c, d \in \{\bar{0}, \bar{2}, \bar{4}\} \right\}.$$

Moreover, all of them are prime ideals of $\langle \mathbb{M}_n(\mathbb{Z}_6), +, \cdot \rangle$.

Proposition 1.5.8 shows that all possibilities of vague prime ideals of the vague ring $\langle \mathbb{M}_n(\mathbb{Z}_6), \tilde{\circ}, \tilde{\bullet} \rangle$ are $\langle \mathcal{I}_1, \tilde{\circ}, \tilde{\bullet} \rangle$, $\langle \mathcal{I}_2, \tilde{\circ}, \tilde{\bullet} \rangle$ or $\langle \mathcal{I}_3, \tilde{\circ}, \tilde{\bullet} \rangle$.
Note that

$$\mu_{\tilde{\bullet}} \left(\left(\begin{smallmatrix} \bar{1} & \bar{1} \\ \bar{1} & \bar{1} \end{smallmatrix} \right), \left(\begin{smallmatrix} \bar{5} & \bar{1} \\ \bar{1} & \bar{5} \end{smallmatrix} \right), \left(\begin{smallmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{smallmatrix} \right) \right) = 1,$$

but $\left(\begin{smallmatrix} \bar{1} & \bar{1} \\ \bar{1} & \bar{1} \end{smallmatrix} \right), \left(\begin{smallmatrix} \bar{5} & \bar{1} \\ \bar{1} & \bar{5} \end{smallmatrix} \right) \notin \mathcal{I}_1 \cup \mathcal{I}_2 \cup \mathcal{I}_3$ and $\left(\begin{smallmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{smallmatrix} \right) \in \mathcal{I}_1 \cap \mathcal{I}_2 \cap \mathcal{I}_3$. Therefore $\langle \mathcal{I}_1, \tilde{\circ}, \tilde{\bullet} \rangle$, $\langle \mathcal{I}_2, \tilde{\circ}, \tilde{\bullet} \rangle$ and $\langle \mathcal{I}_3, \tilde{\circ}, \tilde{\bullet} \rangle$ are not vague prime ideals of $\langle \mathbb{M}_n(\mathbb{Z}_6), \tilde{\circ}, \tilde{\bullet} \rangle$.

Hence the vague ring $\langle \mathbb{M}_n(\mathbb{Z}_6), \tilde{\circ}, \tilde{\bullet} \rangle$ does not contain any vague prime ideals no matter what vague binary operations $\tilde{\circ}$ and $\tilde{\bullet}$ are.

In fact, the ring $\mathbb{Z}_6$ in Example 2.2.1 can be replaced by any ring with identity.

Theorem 2.2.2. *Let R be any ring with identity. Then $\langle \mathbb{M}_n(R), \tilde{\circ}, \tilde{\bullet} \rangle$ is a vague ring without vague prime ideals where $\tilde{\circ}$ and $\tilde{\bullet}$ are vague binary operations induced by the usual addition $+$ and the usual multiplication $\cdot$ of matrices, respectively.*

Proof. Recall the fact that there is a one - to - one correspondence between the ideals of the ring $\mathbb{M}_n(R)$ and the ideals of the ring R via the map $J \longmapsto \mathbb{M}_n(J)$

where J is an ideal of R and

$$\mathbb{M}_n(J) = \left\{[a_{ij}] \,|a_{ij} \in J \text{ for all } i,j \in \{1,2,\ldots,n\}\right\}.$$

Let J be a proper ideal of R,

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ -1 & -1 & \cdots & -1 \end{pmatrix}.$$

Then $A, B \notin \mathbb{M}_n(J)$ since $1 \notin J$. Moreover, $AB = \underline{0} \in J$, i.e., $\mu_{\tilde{\bullet}}(A, B, \underline{0}) = 1$. Thus $\langle \mathbb{M}_n(J), \tilde{\circ}, \tilde{\bullet} \rangle$ is not a vague prime ideal of $\langle \mathbb{M}_n(R), \tilde{\circ}, \tilde{\bullet} \rangle$.

This shows that $\langle \mathbb{M}_n(R), \tilde{\circ}, \tilde{\bullet} \rangle$ is a vague ring without vague prime ideals. $\square$

From the previous theorem, we conclude that $\langle \mathbb{M}_n(\mathbb{Z}_m), \tilde{\circ}, \tilde{\bullet} \rangle$ is a vague ring without vague prime ideals for any $m, n \in \mathbb{N}$.

2.3 Vague Prime Right Ideals and Vague Prime Left Ideals

In this section, we define a vague prime right ideal, a vague prime left ideal and define specific sets induced from a vague prime ideal, a vague prime right ideal and a vague prime left ideal. We also give some elementary attributes. In order to define a vague prime right ideal and a vague prime left ideal, we first give definitions of a vague right ideal and a vague left ideal.

Definition 2.3.1. Let $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ be a vague ring and $\left\langle A, \tilde{\oplus}, \tilde{\odot} \right\rangle \overset{v.r.}{\leq} \langle \mathcal{H}, \tilde{\circ}.\tilde{\bullet} \rangle$. If for all $a \in A$ and $h, t \in \mathcal{H}$, $\mu_{\tilde{\circ}}(a, h, t) = 1 \Rightarrow t \in A$ holds, then $\left\langle A, \tilde{\oplus}, \tilde{\odot} \right\rangle$ is said to be a ***vague right ideal*** of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$, denoted by $\left\langle A, \tilde{\oplus}, \tilde{\odot} \right\rangle \overset{v.r.i.}{\leq} \langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$.

Analogously, if for all $a \in A$ and $h, s \in \mathcal{H}$, $\mu_{\tilde{\circ}}(h, a, s) = 1 \Rightarrow s \in A$ holds, then $\langle A, \tilde{\oplus}, \tilde{\odot} \rangle$ is said to be a ***vague left ideal*** of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$, denoted by $\langle A, \tilde{\oplus}, \tilde{\odot} \rangle \overset{v.l.i.}{\leq} \langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$.

Now, we define a vague prime right ideal, a vague prime left ideal and study their elementary properties .

Definition 2.3.2. Let $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ be a vague ring, A be a nonempty proper crisp subset of $\mathcal{H}$ and $\langle A, \tilde{\oplus}, \tilde{\odot} \rangle \overset{v.r.i}{\leq} \langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$. Then $\langle A, \tilde{\oplus}, \tilde{\odot} \rangle$ is said to be a ***vague prime right ideal*** of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ if

$$\mu_{\tilde{\bullet}}(x, y, z) < 1 \quad \text{for each } z \in A \text{ and for each } x, y \in \mathcal{H} \setminus A.$$

Definition 2.3.3. Let $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ be a vague ring, A be a nonempty proper crisp subset of $\mathcal{H}$ and $\langle A, \tilde{\oplus}, \tilde{\odot} \rangle \overset{v.l.i}{\leq} \langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$. Then $\langle A, \tilde{\oplus}, \tilde{\odot} \rangle$ is said to be a ***vague prime left ideal*** of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ if

$$\mu_{\tilde{\bullet}}(x, y, z) < 1 \quad \text{for each } z \in A \text{ and for each } x, y \in \mathcal{H} \setminus A.$$

Definition 2.3.2, Definition 2.3.3 and the analogous proof of Theorem 2.2.2 yields the following proposition.

Proposition 2.3.4. *Let R be any ring with identity. Then $\langle \mathbb{M}_n(R), \tilde{\circ}, \tilde{\bullet} \rangle$ is a vague ring without any vague prime right ideals and vague prime left ideals where $\tilde{\circ}$ and $\tilde{\bullet}$ are vague binary operations induced by the usual addition $+$ and the usual multiplication $\cdot$ of matrices, respectively.*

Definition 2.3.5. Let $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ be a vague ring, $a \in \mathcal{H}$ and P, I and J be a vague prime ideal, a vague prime right ideal and a vague prime left ideal of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$, respectively. Define

$$\begin{aligned}(P:a) &= \big\{x\in\mathcal{H} \mid \forall b\in\mathcal{H}\forall c\in\mathcal{H}\,[\mu_{\tilde{\bullet}}(a,x,b)=1\Rightarrow b\in P\\ &\qquad \text{and } \mu_{\tilde{\bullet}}(x,a,c)=1\Rightarrow c\in P]\big\},\\ (I:a)_r &= \big\{x\in\mathcal{H} \mid \forall b\in\mathcal{H}\,[\mu_{\tilde{\bullet}}(a,x,b)=1\Rightarrow b\in I]\big\},\\ (J:a)_l &= \big\{x\in\mathcal{H} \mid \forall c\in\mathcal{H}\,[\mu_{\tilde{\bullet}}(x,a,c)=1\Rightarrow c\in J]\big\}.\end{aligned}$$

By observing the above definition, if P is a vague prime ideal, then

$$(P:a)\subseteq (P:a)_r\cap (P:a)_l\,.$$

It is easy to see that for a vague prime right ideal I of a vague ring $\langle\mathcal{H},\tilde{\circ},\tilde{\bullet}\rangle$, if $a\in I$, then $(I:a)_r=\mathcal{H}$, i.e., $(I:a)_r$ is not a proper vague ideal of $\langle\mathcal{H},\tilde{\circ},\tilde{\bullet}\rangle$ so that $(I:a)_r$ is not a vague prime right ideal. Similarly, if $a\in J$, where J is a vague prime left ideal of a vague ring, then $(J:a)_l$ is not a vague prime left ideal.

Let $\langle\mathcal{H},\tilde{\circ},\tilde{\bullet}\rangle$ be a vague ring. Then $\langle\mathcal{H},\circ,\bullet\rangle$ is a ring by Proposition 1.4.3. From now on, we denote $x\circ y$ and $x\bullet y$ by $x+y$ and xy, respectively, for any $x,y\in\mathcal{H}$. Moreover, the inverse of $x\in\mathcal{H}$ under $\circ$ is written by $-x$.

Next, we show that $(P:a)\,,(I:a)_r$ and $(J:a)_l$ are a vague ideal, a vague right ideal and a vague left ideal of a vague ring, respectively.

Lemma 2.3.6. *Let $\langle\mathcal{H},\tilde{\circ},\tilde{\bullet}\rangle$ be a vague ring and $\langle P,\tilde{\oplus},\tilde{\odot}\rangle$ [$\langle I,\tilde{\oplus},\tilde{\odot}\rangle$, $\langle J,\tilde{\oplus},\tilde{\odot}\rangle$] be a vague prime ideal [vague prime right ideal, vague prime left ideal] of $\langle\mathcal{H},\tilde{\circ},\tilde{\bullet}\rangle$. Then $\langle (P:a)\,,\tilde{\oplus},\tilde{\odot}\rangle$ [$\langle (I:a)_r\,,\tilde{\oplus},\tilde{\odot}\rangle$, $\langle (J:a)_l\,,\tilde{\oplus},\tilde{\odot}\rangle$] is a vague ideal [vague right ideal, vague left ideal] of $\langle\mathcal{H},\tilde{\circ},\tilde{\bullet}\rangle$ for all $a\in\langle\mathcal{H},\tilde{\circ},\tilde{\bullet}\rangle$.*

Proof. Evidently, if $a\in P$, then $\langle (P:a)\,,\tilde{\oplus},\tilde{\odot}\rangle$ is a vague ideal. Assume that $a\notin P$. First, we show that $\langle (P:a)\,,\tilde{\oplus}\rangle\overset{v.s.}{\leq}\langle\mathcal{H},\tilde{\circ}\rangle$. Let $x\in(P:a)$ and $b,c\in\mathcal{H}$. Assume that $\mu_{\tilde{\bullet}}(a,x,b)=1$ and $\mu_{\tilde{\bullet}}(x,a,c)=1$. By Theorem 1.3.4, we have

$b = ax$ and $c = xa$. Since $x \in (P : a)$, it follows that $ax, xa \in P$. Since $\langle P, \tilde{\oplus} \rangle \overset{v.s.}{\leq} \langle \mathcal{H}, \tilde{\circ} \rangle$, we obtain $-ax, -xa \in P$. Likewise, $-x \in (P : a)$, by Theorem 1.3.4 and $\mu_{\tilde{\bullet}}(a, -x, -ax) = 1 = \mu_{\tilde{\bullet}}(-x, a, -xa)$. By Proposition 1.3.9 we have $\langle (P : a), \tilde{\oplus} \rangle \overset{v.s.}{\leq} \langle \mathcal{H}, \tilde{\circ} \rangle$ as desired.

Moreover, let $h \in \mathcal{H}$. Since P is a vague ideal, $ax, xa \in P$ and $\mu_{\tilde{\bullet}}(ax, h, axh) = 1 = \mu_{\tilde{\bullet}}(h, xa, hxa)$, it follows that $axh, hxa \in P$. Similarly, we have $xh, hx \in (P : a)$. Hence $\langle (P : a), \tilde{\oplus}, \tilde{\odot} \rangle$ is a vague ideal by Proposition 1.5.4.

Analogously, $\langle (I : a)_r, \tilde{\oplus}, \tilde{\odot} \rangle$ [$\langle (J : a)_l, \tilde{\oplus}, \tilde{\odot} \rangle$] is a vague right [left] ideal of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$. $\square$

Theorem 2.3.7. *Let $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ be a vague ring, $a \in \mathcal{H}$, $\langle P, \tilde{\oplus}, \tilde{\odot} \rangle$ be a vague prime ideal of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$. Then*

$$\langle (P : a), \tilde{\oplus}, \tilde{\odot} \rangle \text{ is a vague prime ideal } \Leftrightarrow a \notin P.$$

Proof. First, observe that if $a \in P$, then $(P : a) = \mathcal{H}$. Therefore $\langle (P : a), \tilde{\oplus}, \tilde{\odot} \rangle$ is not a vague prime ideal. Conversely, assume that $a \notin P$. Suppose that $\langle (P : a), \tilde{\oplus}, \tilde{\odot} \rangle$ is not a vague prime ideal, i.e., there exist $x, y \notin (P : a)$ and $z \in (P : a)$ such that $\mu_{\tilde{\bullet}}(x, y, z) = 1$. Since $\mu_{\tilde{\bullet}}(a, x, ax) = 1$ and $\mu_{\tilde{\bullet}}(x, a, xa) = 1$, we obtain that $ax \notin P$ or $xa \notin P$. Similarly, $ay \notin P$ or $ya \notin P$. Since $\mu_{\tilde{\bullet}}(x, y, z) = 1$, it follows that $xy = z \in (P : a)$. Consequently, $axy \in P$ since $\mu_{\tilde{\bullet}}(a, xy, axy) = 1$. We consider the following 4 cases.

1. $ax \notin P$ and $ay \notin P$.
   Observe that $\mu_{\tilde{\bullet}}(ax, y, axy) = 1, ax \notin P$ and $axy \in P$. Thus $y \in P$. Since P is a vague ideal and $\mu_{\tilde{\bullet}}(a, y, ay) = 1$, we have $ay \in P$ leading to a contradiction.

2. $ax \notin P$ and $ya \notin P$.
   Similar to case 1, we yield a contradiction.

3. $xa \notin P$ and $ay \notin P$.

   First, we show that $ax \notin P$. Suppose not, i.e., $ax \in P$. Since $\mu_{\tilde{\bullet}}(ax, a, axa) = 1$ and P is a vague ideal, $axa \in P$. But since $\mu_{\tilde{\bullet}}(a, xa, axa) = 1$ and $xa \notin P$, it follows that $axa \notin P$, a contradiction. Thus $ax \notin P$. Hence another contradiction occurs so we obtain that $ax \notin P$ and $ay \notin P$.

4. $xa \notin P$ and $ya \notin P$.

   Similar to case 3, using the result of Case 2 yields a contradiction.

From the above observations, $\left\langle (P : a), \tilde{\oplus}, \tilde{\odot} \right\rangle$ is a vague prime ideal as desired.

□

For a vague ring $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ and $a \in \mathcal{H}$, we show that if $\left\langle I, \tilde{\oplus}, \tilde{\odot} \right\rangle$ and $\left\langle J, \tilde{\oplus}, \tilde{\odot} \right\rangle$ are vague prime right ideal and a vague prime left ideal of $\mathcal{H}$, respectively, then $(I : a)_r \cap (J : a)_l$ need not be vague prime ideals.

Example 2.3.8. Consider the vague ring $\langle \mathbb{Z}, \tilde{\circ}, \tilde{\bullet} \rangle$. We see that $\langle 3\mathbb{Z}, \tilde{\circ}, \tilde{\bullet} \rangle$ and $\langle 5\mathbb{Z}, \tilde{\circ}, \tilde{\bullet} \rangle$ are a vague prime right ideal and a vague prime left ideal of $\langle \mathbb{Z}, \tilde{\circ}, \tilde{\bullet} \rangle$, respectively. Evidently, $(3\mathbb{Z} : 7) = 3\mathbb{Z}$ while $(5\mathbb{Z} : 7) = 5\mathbb{Z}$. Therefore

$$(3\mathbb{Z} : 7) \cap (5\mathbb{Z} : 7) = 3\mathbb{Z} \cap 5\mathbb{Z} = 15\mathbb{Z}$$

which is not a vague prime ideal by Theorem 1.5.8.

Theorem 2.3.9. *Let $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ be a non-commutative vague ring with identity and $\left\langle I, \tilde{\oplus}, \tilde{\odot} \right\rangle$ be any vague prime right ideal of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$. If*

$$\begin{aligned} Q = \bigcap_{a \in \mathcal{H}} (I : a)_r &= \bigcap_{a \in \mathcal{H} \setminus I} (I : a)_r \\ &= \left\{ x \in \mathcal{H} \mid \forall a \in \mathcal{H} \setminus I \forall b \in \mathcal{H} \left[\mu_{\tilde{\bullet}}(a, x, b) = 1 \Rightarrow b \in I \right] \right\}, \end{aligned}$$

then $\left\langle Q, \tilde{\oplus}, \tilde{\odot} \right\rangle$ is a vague ideal of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ contained in I.

In particular, $\langle Q, \tilde{\circ}, \tilde{\bullet} \rangle$ is the largest vague ideal of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ contained in I.

Proof. First, note that if $a \in I$, then $(I : a)_r = \mathcal{H}$. Therefore

$$\bigcap_{a \in \mathcal{H}} (I : a)_r = \bigcap_{a \in \mathcal{H} \smallsetminus I} (I : a)_r .$$

By Lemma 2.3.6 and Proposition 1.5.5, we obtain that Q is a vague right ideal of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$. Thus, it suffices to show that Q is a vague left ideal. Let $x \in Q, a \in \mathcal{H} \smallsetminus I$ and $b, y \in \mathcal{H}$. Evidently, $ay \in \mathcal{H}$. Assume that $\mu_{\tilde{\bullet}}(ay, x, b) = 1$. By theorem 1.3.4, we have $b = ayx$. Since $x \in Q$, it follows that $ayx \in I$. Since a is arbitary, $yx \in Q$. Therefore, Q is a vague ideal of $\mathcal{H}$ as desired. Since $e_{\tilde{\bullet}} \in \mathcal{H}, x \in I$ for all $x \in Q$. This yields $Q \subseteq I$. Hence $\left\langle Q, \tilde{\oplus}, \tilde{\odot} \right\rangle$ is a vague ideal of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ contained in I as desired.

For the last assertion, let $\langle K, \tilde{\circ}, \tilde{\bullet} \rangle$ be any vague ideal of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ such that $K \subseteq I$. We show that $\langle K, \tilde{\circ}, \tilde{\bullet} \rangle \subseteq \langle Q, \tilde{\circ}, \tilde{\bullet} \rangle$. Let $x \in K$ and $a \in \mathcal{H} \smallsetminus I$. Since $\mu_{\tilde{\bullet}}(a, x, ax) = 1$, we obtain $ax \in K \subseteq I$. Consequently, $x \in Q$. Hence $\langle Q, \tilde{\circ}, \tilde{\bullet} \rangle$ is the largest vague ideal of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ contained in I. $\square$

From the previous theorem, the non-commutativity is neccessary because if $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ is a commutative vague ring, then $\bigcap_{a \in \mathcal{H}} (I : a)_r = I$.

Note that it is not neccessary true that the intersection of vague right ideals is a vague ideal as seen in Theorem 2.3.9. The following corollary is proved analogously.

Corollary 2.3.10. *Let $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ be a non-commutative vague ring with identity and $\left\langle J, \tilde{\oplus}, \tilde{\odot} \right\rangle$ be any vague prime left ideal of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$. If*

$$\begin{aligned} K = \bigcap_{a \in \mathcal{H}} (J : a)_l &= \bigcap_{a \in \mathcal{H} \smallsetminus J} (J : a)_l \\ &= \left\{ x \in \mathcal{H} \,|\, \forall a \in \mathcal{H} \smallsetminus J \forall b \in \mathcal{H} \left[\mu_{\tilde{\bullet}}(x, a, b) = 1 \Rightarrow b \in J \right] \right\}, \end{aligned}$$

then $\left\langle K, \tilde{\oplus}, \tilde{\odot} \right\rangle$ is a vague ideal of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ contained in J.

In particular, $\langle K, \tilde{\circ}, \tilde{\bullet} \rangle$ is the largest vague ideal of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ contained in J.

Finally, Corollary 2.3.11 is the immediate result of Theorem 2.3.9 and Corollary 2.3.10.

Corollary 2.3.11. *Let* $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ *be a non-commutative vague ring with identity and* $\left\langle P, \tilde{\oplus}, \tilde{\odot} \right\rangle$ *be any vague prime ideal of* $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$*. If*

$$M = \bigcap_{a \in \mathcal{H}} (P : a) = \bigcap_{a \in \mathcal{H} \setminus P} (P : a),$$

then $\left\langle M, \tilde{\oplus}, \tilde{\odot} \right\rangle$ *is a vague ideal of* $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ *contained in* P*.*

In particular, $\langle M, \tilde{\circ}, \tilde{\bullet} \rangle$ *is the largest vague ideal of* $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ *contained in* P*.*

In classical ring, we have a generalization of prime ideal which we called a primary ideal, similar to vague ring we define a vague primary ideal in the next chapter.

CHAPTER III

Vague Primary Ideals

In rings, ideals are crucial notion and primary ideals are directly closed to prime ideals. This inspired us to define a vague primary ideal of a vague ring which is similar to a primary ideal of a ring. We see that every vague prime ideal of a vague ring is a vague primary ideal. Consequently, we are interested in studying some properties of vague prime ideals and vague primary ideals which are parallel to those of prime ideals and primary ideals of classical rings.

In classical sense, Noetherian ring is a well known ring which is beneficial to investigate some properties of vague primary ideals. This led us to define a vague Noetherian ring. We also define some crucial vague ideals such as vague irreducible ideal, vague semiprime ideal in order to explore their related properties. Eventually, we give some sufficient condition which vague prime ideals and vague primary ideals are coincide.

3.1 Vague Primary Ideals

In this section, we define a vague primary ideal and a vague irreducible ideal. We also find out some elementary properties and their relations. In order to define a vague primary ideal, we need to recall the definition of a primary ideal of a classical ring as follows.

Definition 3.1.1. Let $\langle \mathcal{H}, \circ, \bullet \rangle$ be a commutative ring and P be a proper ideal of $\mathcal{H}$. Then an ideal P of $\mathcal{H}$ is said to be a ***primary ideal*** if for each $x, y \in \mathcal{H}$,

$$xy \in P \Rightarrow x \in P \text{ or } y^n \in P \text{ for some } n \in \mathbb{N}.$$

In the following, we define a vague primary ideal of a commutative vague ring $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$.

Definition 3.1.2. Let $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ be a commutative vague ring, A be a nonempty proper crisp subset of $\mathcal{H}$ and $\langle A, \tilde{\oplus}, \tilde{\odot} \rangle \overset{v.i}{\leq} \langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$. Then $\langle A, \tilde{\oplus}, \tilde{\odot} \rangle$ is said to be a ***vague primary ideal*** of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ if

$$\forall x, y \notin P \, \exists n \in \mathbb{N} \, \forall z \in P, \left[y^n \notin P \Rightarrow \mu_{\tilde{\bullet}} (x, y^n, z) < 1 \right].$$

Proposition 3.1.3. *Every vague prime ideal is a vague primary ideal.*

Proof. By choosing $n = 1$, the result holds. $\square$

The converse of Proposition 3.1.3 is not true shown in the following example.

Example 3.1.4. Let p be a prime number. Consider $(p^3)_i$ as a principal ideal of $\langle \mathbb{Z}, \circ, \bullet \rangle$ containing p^3. We show that $\langle (p^3)_i, \tilde{\circ}, \tilde{\bullet} \rangle$ is a vague primary ideal which is not a vague prime ideal where $\tilde{\circ}$ and $\tilde{\bullet}$ are any vague binary operations of which $\circ$ and $\bullet$ are the ordinary descriptions, respectively.

First, observe that $p, p^2 \notin (p^3)_i$ but

$$p \bullet p^2 = p^3 \in \left(p^3\right)_i.$$

Thus $\langle (p^3)_i, \tilde{\circ}, \tilde{\bullet} \rangle$ is not a prime ideal of $\langle \mathbb{Z}, \circ, \bullet \rangle$. By Proposition 1.5.8, we have $\langle (p^3)_i, \tilde{\circ}, \tilde{\bullet} \rangle$ is not a vague prime ideal of $\langle \mathbb{Z}, \tilde{\circ}, \tilde{\bullet} \rangle$ as desired.

Next, suppose that $\langle (p^3)_i, \tilde{\circ}, \tilde{\bullet} \rangle$ is not a vague primary ideal, i.e.,

$$\exists x, y \notin \left(p^3\right)_i, \forall n \in \mathbb{N}, \exists z \in \left(p^3\right)_i \text{ such that } y^n \notin \left(p^3\right)_i \text{ and } \mu_{\tilde{\bullet}} (x, y^n, z) = 1.$$

Thus $xy^n = z \in (p^3)_i$ for all $n \in \mathbb{N}$. Therefore, we have $xy \in (p^3)_i$, i.e., $p^3 | xy$. Consider 3 cases as follows:

1. ($p^2 | x$ and $p | y$) or ($p^2 | y$ and $p | x$)

- If ($p^2|x$ and $p|y$), then $p^3|y^3$. Thus $y^3 \in (p^3)_i$ which is a contradiction.
- If ($p^2|y$ and $p|x$), then $p^3|y^2$. Thus $y^2 \in (p^3)_i$ which is a contradiction.

2. $p^3|x$ or $p^3|y$

- If $p^3|x$, then $x \in (p^3)_i$ which is a contradiction.
- If $p^3|y$, then $y \in (p^3)_i$ which is a contradiction.

Hence $\left\langle (p^3)_i, \tilde{\circ}, \tilde{\bullet} \right\rangle$ is a vague primary ideal as desired.

Proposition 3.1.5. *If $\left\langle A, \tilde{\oplus}, \tilde{\odot} \right\rangle$ is a vague primary ideal of a vague ring $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$, then $\langle A, \oplus, \odot \rangle$ is a primary ideal of the corresponding ring $\langle \mathcal{H}, \circ, \bullet \rangle$.*

Proof. Let $x, y \in \mathcal{H}$ be such that $x, y^n \notin P$ for all $n \in \mathbb{N}$. But $\mu_{\tilde{\bullet}}(x, y^n, xy^n) = 1$ for all $n \in \mathbb{N}$. Since P is a vague primary ideal, $xy^n \notin P$ for all $n \in \mathbb{N}$. Therefore, $xy \notin P$ as desired. □

Next, we give an example of a vague ideal of a vague ring such that it is not a vague primary ideal.

Example 3.1.6. Let p and q be distinct prime numbers. Consider $(pq)_i$ as a principal ideal of the ring $\langle \mathbb{Z}, \circ, \bullet \rangle$. We show that $\left\langle (pq)_i, \tilde{\circ}, \tilde{\bullet} \right\rangle$ is not a vague primary ideal of any vague ring $\langle \mathbb{Z}, \tilde{\circ}, \tilde{\bullet} \rangle$, i.e.,

$$\exists x, y \notin (pq)_i, \forall n \in \mathbb{N}, \exists z \in (pq)_i, \ y^n \notin (pq)_i \text{ and } \mu_{\tilde{\bullet}}(x, y^n, z) = 1. \qquad (3.1.7)$$

By choosing $x = p$ and $y = q$. Evidently, $y^n = q^n \notin (pq)_i$ for all $n \in \mathbb{N}$. Since $n \geq 1$, we have

$$xy^n = pq^n = (pq)\, q^{n-1} \in (pq)_i$$

for all $n \in \mathbb{N}$. This shows that $xy^n \in (pq)_i$ and clearly $\mu_{\tilde{\bullet}}(x, y^n, xy^n) = 1$ for all $n \in \mathbb{N}$. Thus the equation (3.1.7) holds. Hence $\left\langle (pq)_i, \tilde{\circ}, \tilde{\bullet} \right\rangle$ is not a vague primary ideal of $\langle \mathbb{Z}, \tilde{\circ}, \tilde{\bullet} \rangle$ as desired.

3.2 Vague Semiprime Ideals

In this section, we define a vague semiprime ideal, a vague regular ring and also study their elementary properties. The main theorem of this section shows that if $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ is a commutative vague regular ring, then every vague primary ideal of a vague ring $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ is a maximal vague ideal.

In order to define a vague semiprime ideal, we need to recall the definition of a semiprime ideal of a classical ring.

Definition 3.2.1. Let $\langle \mathcal{H}, \circ, \bullet \rangle$ be a ring and A be a proper ideal of $\mathcal{H}$. Then an ideal A of $\mathcal{H}$ is said to be a ***semiprime ideal*** if for each ideal J of $\mathcal{H}$,

$$J^2 \subseteq A \Rightarrow J \subseteq A$$

where $J^2 = \left\{ \sum_{i=1}^{n} x_i y_i \mid n \in \mathbb{N}, x_i, y_i \in J \right\}$.

Definition 3.2.2. Let $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ be a vague ring, A be a nonempty proper crisp subset of $\mathcal{H}$ and $\left\langle A, \tilde{\oplus}, \tilde{\odot} \right\rangle \overset{v.i}{\leq} \langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$. Then $\left\langle A, \tilde{\oplus}, \tilde{\odot} \right\rangle$ is said to be a ***vague semiprime ideal*** of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ if

$$\mu_{\tilde{\bullet}}(x, x, z) < 1 \text{ for each } z \in A \text{ and for each } x \in \mathcal{H} \setminus A.$$

In a classical ring, every prime ideal is a semiprime ideal. Similar to vague ring, we obtain the analogous result.

Proposition 3.2.3. *Every vague prime ideal is a vague semiprime ideal.*

Proof. By choosing $y = x$ in Proposition 1.5.7, the result holds. $\square$

Proposition 3.2.4. *If $\left\langle A, \tilde{\oplus}, \tilde{\odot} \right\rangle$ is a vague semiprime ideal of a vague ring $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$, then $\langle A, \oplus, \odot \rangle$ is a semiprime ideal of $\langle \mathcal{H}, \circ, \bullet \rangle$.*

Proof. Let M be an ideal of A such that $M \nsubseteq A$. Thus there exists $x \in M$ such that $x \in \mathcal{H} \setminus A$. But $\mu_{\tilde{\bullet}}(x, x, x^2) = 1$, we conclude that $x^2 \notin A$ because $\langle A, \tilde{\oplus}, \tilde{\odot} \rangle$ is a vague semiprime ideal. Since $x^2 \in M^2$, we have $M^2 \nsubseteq A$. Hence $\langle A, \oplus, \odot \rangle$ is a semiprime ideal of $\langle \mathcal{H}, \circ, \bullet \rangle$ as desired. □

In order to define a vague regular element, we need to recall the definition of a regular element of a classical ring.

Definition 3.2.5. Let $\langle \mathcal{H}, \circ, \bullet \rangle$ be a ring. Then an element $a \in \langle \mathcal{H}, \circ, \bullet \rangle$ is said to be a ***regular element*** if $a = axa$ for some $x \in \mathcal{H}$.

A ring $\langle \mathcal{H}, \circ, \bullet \rangle$ is said to be a ***regular ring*** if every element in $\langle \mathcal{H}, \circ, \bullet \rangle$ is a regular element.

Analogously, we give the definition of a vague regular element as follows:

Definition 3.2.6. Let $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ be a vague ring. Then an element $a \in \langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ is said to be a ***vague regular element*** if a is a regular element of $\langle \mathcal{H}, \circ, \bullet \rangle$ where $\circ$ and $\bullet$ are the ordinary descriptions of $\tilde{\circ}$ and $\tilde{\bullet}$, respectively.

A vague ring $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ is said to be a ***vague regular ring*** if every element in $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ is a vague regular element.

Proposition 3.2.7. *If $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ is a vague regular ring, then $\langle \mathcal{H}, \circ, \bullet \rangle$ is a regular ring where $\circ$ and $\bullet$ are the ordinary descriptions of $\tilde{\circ}$ and $\tilde{\bullet}$, respectively.*

Proof. This follows immediately from Definition 3.2.6. □

Theorem 3.2.8. *If $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ is a commutative vague regular ring, then every vague ideal $\langle I, \tilde{\oplus}, \tilde{\odot} \rangle$ of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ is a vague semiprime ideal of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$.*

Proof. Let $\langle I, \tilde{\oplus}, \tilde{\odot} \rangle$ be a vague ideal of a commutative vague regular ring $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$. Suppose that $\langle I, \tilde{\oplus}, \tilde{\odot} \rangle$ is not a vague semiprime ideal of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$, i.e., there exist $x \notin I$ and $z \in I$ such that $\mu_{\tilde{\odot}}(x, x, z) = 1 = \mu_{\tilde{\bullet}}(x, x, z)$. Thus $x^2 = z \in I$. Since

$\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ is a commutative vague regular ring, by Proposition 1.3.5 and Proposition 3.2.7, we obtain that $\langle \mathcal{H}, \circ, \bullet \rangle$ is a commutative regular ring. Therefore $x = xyx = x^2y$ for some $y \in \mathcal{H}$. Since I is an ideal of $\mathcal{H}$ and $x^2 \in I$, we have $x = x^2y \in Iy \subseteq I$ which is a contradiction. Hence a vague ideal $\left\langle I, \tilde{\oplus}, \tilde{\odot} \right\rangle$ is a vague semiprime ideal of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ as desired. $\square$

Corollary 3.2.9. *If $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ is a commutative vague regular ring, then every vague primary ideal of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ is a vague semiprime ideal of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$.*

To give next lemma, we recall Proposition 1.5.2 that if $\left\langle I, \tilde{\oplus}, \tilde{\odot} \right\rangle \overset{v.i.}{\leq} \langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$, then $\langle I, \oplus, \odot \rangle$ is an ideal of $\langle \mathcal{H}, \circ, \bullet \rangle$. In fact $\oplus$ and $\odot$ are $\circ$ and $\bullet$, respectively. So that we write $\langle I, \circ, \bullet \rangle$ instead of $\langle I, \oplus, \odot \rangle$.

Lemma 3.2.10. *Let $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ be a commutative vague regular ring and $\left\langle I, \tilde{\oplus}, \tilde{\odot} \right\rangle$ be a vague ideal of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$. If $a \notin I$, then $a^n \notin I$ for all $n \in \mathbb{N}$.*

Proof. Let $n \in \mathbb{N}$. First, observe that if $n = 1$, then we done. Assume that $n > 1$. Note that $\langle \mathcal{H}, \circ, \bullet \rangle$ is a commutative regular ring and $\langle I, \circ, \bullet \rangle$ is an ideal of $\langle \mathcal{H}, \circ, \bullet \rangle$. Let $a \notin I$. From the commutativity and regularity of $\mathcal{H}$,

$$a = axa = a^2x = a\,(ax) = (axa)\,ax = a^3x^2$$

for some $x \in \mathcal{H}$. By the same process as above, we have $a = a^n y$ for some $y \in \mathcal{H}$. Evidently, if $a^n \in I$, then $a \in I$ since I is an ideal of $\mathcal{H}$ which is a contradiction. Consequently, $a^n \notin I$. The result holds. $\square$

Recall that for a vague ideal $\left\langle I, \tilde{\oplus}, \tilde{\odot} \right\rangle$ of a vague ring $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ where $\mu_{\tilde{\oplus}} \neq \mu_{\tilde{\circ}}$ and $\mu_{\tilde{\odot}} \neq \mu_{\tilde{\bullet}}$, we obtain that $\left\langle I, \tilde{\oplus}, \tilde{\odot} \right\rangle$ cannot be a maximal vague ideal since

$$\left\langle I, \tilde{\oplus}, \tilde{\odot} \right\rangle \subsetneq \langle I, \tilde{\circ}, \tilde{\bullet} \rangle \subseteq \langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle .$$

Theorem 3.2.11. *Let $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ be a commutative vague regular ring. Then every vague primary ideal $\langle P, \tilde{\circ}, \tilde{\bullet} \rangle$ of a vague ring $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ is a maximal vague ideal of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$.*

Proof. Let $\langle P, \tilde{\circ}, \tilde{\bullet} \rangle$ be a vague primary ideal of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$. Proposition 1.3.5 and Proposition 3.1.5 provide that $\langle \mathcal{H}, \circ, \bullet \rangle$ is a commutative regular ring and $\langle P, \circ, \bullet \rangle$ is a primary ideal of $\langle \mathcal{H}, \circ, \bullet \rangle$. Let $\langle J, \circ, \bullet \rangle$ be an ideal of $\langle \mathcal{H}, \circ, \bullet \rangle$ such that

$$\langle P, \circ, \bullet \rangle \subsetneqq \langle J, \circ, \bullet \rangle \subseteq \langle \mathcal{H}, \circ, \bullet \rangle .$$

Thus there exists $p \in J$ such that $p \notin P$. Since $p \in J \subseteq \mathcal{H}$ and $\mathcal{H}$ is a regular ring, $p = p^2 k$ for some $k \in \mathcal{H}$. By the distributive laws, we have

$$p\,(1 - pk) = 0 \in P. \tag{3.2.12}$$

But $p \notin P$ and P is an ideal of $\mathcal{H}$, by Lemma 3.2.10, we have $p^n \notin P$ for all $n \in \mathbb{N}$. Since $\langle P, \circ, \bullet \rangle$ is a primary ideal of $\langle \mathcal{H}, \circ, \bullet \rangle$ and (3.2.12), we obtain that

$$(1 - pk) \in P \subseteq J.$$

Since $p \in J$ and J is an ideal of $\mathcal{H}$, this implies that $pk \in J$. Thus $1 = (1 - pk) + pk \in J$. Hence $\langle J, \circ, \bullet \rangle = \langle \mathcal{H}, \circ, \bullet \rangle$. Therefore, $\langle P, \tilde{\circ}, \tilde{\bullet} \rangle$ is a maximal vague ideal of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ as desired. □

3.3 Related Relations

In this section, we define a new vague structures called a vague ideal quotient and a vague Notherian ring. The main theorem of this section shows that if $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ is a commutative vague Notherian ring with identity and $\tilde{\oplus}$, $\tilde{\odot}$ are vague binary operations on $\mathcal{H}$ such that for each $a, b, c \in \mathcal{H}$,

$$\mu_{\tilde{\oplus}}\,(a,b,c) \le \mu_{\tilde{\circ}}\,(a,b,c) \text{ and } \mu_{\tilde{\odot}}\,(a,b,c) \le \mu_{\tilde{\bullet}}\,(a,b,c)\,,$$

then every vague irreducible ideal $\left\langle I, \tilde{\oplus}, \tilde{\odot} \right\rangle$ in $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ is primary.

Finally, we summarise related relations of the following vague ideals: maximal vague ideals, vague irreducible ideals, vague prime ideals, vague primary ideals and vague semiprime ideals. Therefore, at the end, we obtain that the condition for vague prime ideals and vague primary ideals to be coincide.

Let $\left\langle A,\tilde{\oplus},\tilde{\odot}\right\rangle$ and $\left\langle B,\tilde{\boxplus},\tilde{\boxdot}\right\rangle$ be vague ideals of a vague ring $\langle\mathcal{H},\tilde{\circ},\tilde{\bullet}\rangle$. We define

$$\mathcal{V}\left(A:B\right)=\left\{h\in\mathcal{H}\mid\forall b\in B\,\forall c\in\mathcal{H}\;\left[\mu_{\tilde{\bullet}}\left(h,b,c\right)=1\Rightarrow c\in A\right]\right\}.$$

We observe that $\mathcal{V}\left(A:B\right)\neq\phi$ because $A\subseteq\mathcal{V}\left(A:B\right).$

Lemma 3.3.1. *Let $\langle\mathcal{H},\tilde{\circ},\tilde{\bullet}\rangle$ be a vague ring, $\tilde{\heartsuit}$ and $\tilde{\spadesuit}$ be vague binary operations such that for each $a,b,c\in\mathcal{H}$,*

$$\mu_{\tilde{\heartsuit}}\left(a,b,c\right)\leq\mu_{\tilde{\circ}}\left(a,b,c\right)\text{ and }\mu_{\tilde{\spadesuit}}\left(a,b,c\right)\leq\mu_{\tilde{\bullet}}\left(a,b,c\right).$$

Then $\left\langle\mathcal{V}\left(A:B\right),\tilde{\heartsuit},\tilde{\spadesuit}\right\rangle$ is a vague ideal of $\langle\mathcal{H},\tilde{\circ},\tilde{\bullet}\rangle$.

Proof. First, we show that $\left\langle\mathcal{V}\left(A:B\right),\tilde{\heartsuit},\tilde{\spadesuit}\right\rangle\overset{v.s.}{\leq}\langle\mathcal{H},\tilde{\circ}\rangle$. Let $b\in B$, $c\in\mathcal{H}$ and $x\in\mathcal{V}\left(A:B\right)$. Since $\mu_{\tilde{\bullet}}\left(x,b,xb\right)=1$, we have $xb\in A$. Thus $-xb\in A$ because $\left\langle A,\tilde{\oplus},\tilde{\odot}\right\rangle$ is a vague ideal of $\langle\mathcal{H},\tilde{\circ},\tilde{\bullet}\rangle$. Now, we have $\mu_{\tilde{\bullet}}\left(-x,b,-xb\right)=1$ and $-xb\in A$. Thus $-x\in\mathcal{V}\left(A:B\right).$ Because of the choices of $\tilde{\heartsuit}$, $\tilde{\spadesuit}$ and Proposition 1.3.9, we have $\left\langle\mathcal{V}\left(A:B\right),\tilde{\heartsuit},\tilde{\spadesuit}\right\rangle\overset{v.s.}{\leq}\langle\mathcal{H},\tilde{\circ}\rangle$. Similarly to above, we have $\mu_{\tilde{\spadesuit}}\left(a,b,c\right)\leq\mu_{\tilde{\bullet}}\left(a,b,c\right)$ for each $a,b,c\in\mathcal{V}\left(A:B\right).$

Finally, let $a\in\mathcal{V}\left(A:B\right)$, $b\in B$ and $h,t\in\mathcal{H}$ be such that $\mu_{\tilde{\bullet}}\left(a,h,t\right)=1$, i.e., $ah=t$. Since $\mu_{\tilde{\bullet}}\left(h,b,tb\right)=1$ and B is a vague ideal of $\langle\mathcal{H},\tilde{\circ},\tilde{\bullet}\rangle$, we obtain that $hb\in B$. For each $c\in\mathcal{H}$, assume that $\mu_{\tilde{\bullet}}\left(t,b,c\right)=1$, i.e., $c=tb$. We show that $c\in A$. Since $a\in\mathcal{V}\left(A:B\right)$ and $\mu_{\tilde{\bullet}}\left(a,hb,ahb\right)=1$, this implies that $c=tb=ahb\in A$ as desired. Analogously, we can show that if $\mu_{\tilde{\bullet}}\left(h,a,t\right)=1$ then $t\in\mathcal{V}\left(A:B\right).$ Hence by Proposition 1.5.4, we obtain that $\left\langle\mathcal{V}\left(A:B\right),\tilde{\heartsuit},\tilde{\spadesuit}\right\rangle$ is a vague ideal of

$\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ as desired. □

$\left\langle \mathcal{V}(A:B), \tilde{\heartsuit}, \tilde{\spadesuit} \right\rangle$ in the above lemma is called a vague ideal quotient.

Definition 3.3.2. Let $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ be a vague ring. Let $\left\langle A, \tilde{\oplus}, \tilde{\odot} \right\rangle$ and $\left\langle B, \tilde{\boxplus}, \tilde{\boxdot} \right\rangle$ be vague ideals of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$. We define a ***vague ideal quotient*** $\mathcal{V}(A:B)$ of A and B as

$$\mathcal{V}(A:B) = \left\{ h \in \mathcal{H} \mid \forall b \in B \, \forall c \in \mathcal{H} \ \left[\mu_{\tilde{\bullet}}(h,b,c) = 1 \Rightarrow c \in A \right] \right\}.$$

Definition 3.3.3. Let $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ be a vague ring. Then $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ is said to be a ***vague Notherian ring*** if every increasing chain of vague ideals

$$\left\langle J_1, \tilde{\oplus}, \tilde{\odot} \right\rangle \subseteq \left\langle J_2, \tilde{\oplus}, \tilde{\odot} \right\rangle \subseteq \ldots$$

eventually stops, that is $\left\langle J_k, \tilde{\oplus}, \tilde{\odot} \right\rangle = \left\langle J_{k+1}, \tilde{\oplus}, \tilde{\odot} \right\rangle = \ldots$ for some k.

Next Lemma shows that if $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ is a vague Notherian ring, then $\langle \mathcal{H}, \circ, \bullet \rangle$ is a Notherian ring.

Lemma 3.3.4. *If $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ is a vague Notherian ring, then $\langle \mathcal{H}, \circ, \bullet \rangle$ is a Notherian ring where $\tilde{\circ}$ and $\tilde{\bullet}$ are any vague binary operations of which $\circ$ and $\bullet$ are the ordinary descriptions, respectively.*

Proof. Since the assumption of a vague Noetherian ring holds for all vague binary operations, by choosing

$$\mu_{\tilde{\circ}}(a,b,c) = \begin{cases} 1, & \text{if } a \circ b = c, \\ 0, & \text{if } a \circ b \neq c, \end{cases}$$

and

$$\mu_{\tilde{\bullet}}(a,b,c) = \begin{cases} 1, & \text{if } a \bullet b = c, \\ 0, & \text{if } a \bullet b \neq c, \end{cases}$$

the result holds. □

Note that if A and B are nonempty crisp sets, $\tilde{\oplus}, \tilde{\odot}, \tilde{\circ}, \tilde{\bullet}$ are any vague binary operations, then $\langle A, \tilde{\oplus}, \tilde{\odot} \rangle \subseteq \langle B, \tilde{\circ}, \tilde{\bullet} \rangle$ if and only if

1. A is a crisp subset of B,
2. $\mu_{\tilde{\oplus}}(a, b, c) \leq \mu_{\tilde{\circ}}(a, b, c)$ for all $a, b, c \in A$, and
3. $\mu_{\tilde{\odot}}(a, b, c) \leq \mu_{\tilde{\bullet}}(a, b, c)$ for all $a, b, c \in A$.

Next, we give an example of a vague Noetherian ring.

Example 3.3.5. Consider a vague ring $\langle \mathbb{Z}, \tilde{\circ}, \tilde{\bullet} \rangle$ where $\tilde{\circ}$ and $\tilde{\bullet}$ are any vague binary operations of which the usual addition $+$ and the usual multiplication $\cdot$ on $\mathbb{Z}$ are the ordinary descriptions, respectively. See that if $\langle I, \tilde{\oplus}, \tilde{\odot} \rangle$ is a vague ideal of $\langle \mathbb{Z}, \tilde{\circ}, \tilde{\bullet} \rangle$, then $\langle I, \tilde{\oplus}, \tilde{\odot} \rangle = \langle (m)_i, \tilde{\oplus}, \tilde{\odot} \rangle$ for some $m \in \mathbb{Z}$. From the above note, it is easy to show that

$$\langle (m)_i, \tilde{\oplus}, \tilde{\odot} \rangle \subseteq \langle (n)_i, \tilde{\oplus}, \tilde{\odot} \rangle \Leftrightarrow n|m.$$

We show that $\langle \mathbb{Z}, \tilde{\circ}, \tilde{\bullet} \rangle$ is a vague Noetherian ring. Let

$$\langle J_1, \tilde{\oplus}, \tilde{\odot} \rangle \subseteq \cdots \subseteq \langle J_{k-1}, \tilde{\oplus}, \tilde{\odot} \rangle \subseteq \langle J_k, \tilde{\oplus}, \tilde{\odot} \rangle \subseteq \langle J_{k+1}, \tilde{\oplus}, \tilde{\odot} \rangle \subseteq \ldots$$

be an ascending chain of vague ideals of $\langle \mathbb{Z}, \tilde{\circ}, \tilde{\bullet} \rangle$. Since for each $h \in \mathbb{N}$, $J_h = (n_h)_i$ for some $n_h \in \mathbb{Z}$, we obtain the following ascending chain of vague ideals

$$\langle (n_1)_i, \tilde{\oplus}, \tilde{\odot} \rangle \subseteq \cdots \subseteq \langle (n_{k-1})_i, \tilde{\oplus}, \tilde{\odot} \rangle \subseteq \langle (n_k)_i, \tilde{\oplus}, \tilde{\odot} \rangle \subseteq \langle (n_{k+1})_i, \tilde{\oplus}, \tilde{\odot} \rangle \subseteq \ldots.$$

If the chain did not terminate, n_2 would have been a proper divisor of n_1 and n_3 a proper divisor of n_2, etc. In particular,

$$|n_1| > \cdots > |n_{k-1}| > |n_k| > |n_{k+1}| > \ldots$$

which is impossible since there can only be finitely many positive integers strictly less than n_1. Consequently, $\langle \mathbb{Z}, \tilde{\circ}, \tilde{\bullet} \rangle$ is a vague Noetherian ring.

Lemma 3.3.6. *Let $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ be a vague ring, $\langle I, \tilde{\oplus}, \tilde{\odot} \rangle$, $\langle A, \tilde{\oplus}, \tilde{\odot} \rangle$ and $\langle B, \tilde{\oplus}, \tilde{\odot} \rangle$ be vague ideals of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$. If $\langle A, \tilde{\oplus}, \tilde{\odot} \rangle \subseteq \langle B, \tilde{\oplus}, \tilde{\odot} \rangle$, then*

$$\langle \mathcal{V}(I : B), \tilde{\oplus}, \tilde{\odot} \rangle \subseteq \langle \mathcal{V}(I : A), \tilde{\oplus}, \tilde{\odot} \rangle .$$

Proof. Let $x \in \langle \mathcal{V}(I : B), \tilde{\oplus}, \tilde{\odot} \rangle, a \in A$ and $c \in \mathcal{H}$ be such that $\mu_{\tilde{\bullet}}(x, a, c) = 1$. Then $c = xa$. Since $\mu_{\tilde{\bullet}}(x, a, xa) = 1$ and $a \in A \subseteq B$, we obtain that $c = xa \in I$. Therefore, $x \in \langle \mathcal{V}(I : A), \tilde{\oplus}, \tilde{\odot} \rangle$. Hence $\langle \mathcal{V}(I : B), \tilde{\oplus}, \tilde{\odot} \rangle \subseteq \langle \mathcal{V}(I : A), \tilde{\oplus}, \tilde{\odot} \rangle$ as desired. □

Next theorem and corollary show that every vague ideal in a commutative vague Notherian ring with identity is decomposition (means that any vague ideal $\langle A, \tilde{\circ}, \tilde{\bullet} \rangle$ of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ is the intersection of the primary vague ideal of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$).

Theorem 3.3.7. *Let $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ be a commutative vague Notherian ring with identity. If $\tilde{\oplus}$ and $\tilde{\odot}$ are vague binary operations on $\mathcal{H}$ such that for each $a, b, c \in \mathcal{H}$,*

$$\mu_{\tilde{\oplus}}(a, b, c) \leq \mu_{\tilde{\circ}}(a, b, c) \text{ and } \mu_{\tilde{\odot}}(a, b, c) \leq \mu_{\tilde{\bullet}}(a, b, c),$$

then every vague irreducible ideal $\langle I, \tilde{\oplus}, \tilde{\odot} \rangle$ in $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ is primary.

Proof. Let $\langle I, \tilde{\oplus}, \tilde{\odot} \rangle$ be a vague irreducible ideal in $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$. Suppose that

$$\exists x, y \notin I, \forall k \in \mathbb{N}, \exists z_k \in I, y^k \notin I \text{ and } \mu_{\tilde{\bullet}}(x, y^k, z_k) = 1,$$

i.e., $xy^k \in I$ for all $k \in \mathbb{N}$. For each $t \in \mathbb{N}$, define $J_t = \mathcal{V}(I : (y^t)_i)$, the vague ideal quotient of vague ideal I and $(y^t)_i$. For each $n \geq 2$, since

$$\cdots \subseteq \langle (y^n)_i, \tilde{\oplus}, \tilde{\odot} \rangle \subseteq \cdots \subseteq \langle (y^2)_i, \tilde{\oplus}, \tilde{\odot} \rangle \subseteq \langle (y)_i, \tilde{\oplus}, \tilde{\odot} \rangle,$$

by Lemma 3.3.6, evidently, we have

$$\left\langle J_1, \tilde{\oplus}, \tilde{\odot} \right\rangle \subseteq \left\langle J_2, \tilde{\oplus}, \tilde{\odot} \right\rangle \subseteq \cdots \subseteq \left\langle J_n, \tilde{\oplus}, \tilde{\odot} \right\rangle \subseteq \cdots .$$

Since $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ is a vague Notherian ring,

$$\left\langle J, \tilde{\oplus}, \tilde{\odot} \right\rangle := \left\langle J_n, \tilde{\oplus}, \tilde{\odot} \right\rangle = \left\langle J_m, \tilde{\oplus}, \tilde{\odot} \right\rangle \text{ for all } m \geq n.$$

Next, define $\left\langle K, \tilde{\oplus}, \tilde{\odot} \right\rangle = \left\langle I + (y^n)_i, \tilde{\oplus}, \tilde{\odot} \right\rangle$, the sum of ideals I and $(y^n)_i$. By assumption and Lemma 2.1.1, we have $\left\langle K, \tilde{\oplus}, \tilde{\odot} \right\rangle$ is a vague ideal of $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$. We show that $\left\langle I, \tilde{\oplus}, \tilde{\odot} \right\rangle = \left\langle J \cap K, \tilde{\oplus}, \tilde{\odot} \right\rangle$. Obviously, $\left\langle I, \tilde{\oplus}, \tilde{\odot} \right\rangle \subseteq \left\langle J, \tilde{\oplus}, \tilde{\odot} \right\rangle$ and $\left\langle I, \tilde{\oplus}, \tilde{\odot} \right\rangle \subseteq \left\langle K, \tilde{\oplus}, \tilde{\odot} \right\rangle$. Now, let $r \in \left\langle J \cap K, \tilde{\oplus}, \tilde{\odot} \right\rangle$. Then $r = s + ty^n$ where $s \in I$ and $t \in \mathcal{H}$. Since $\left\langle J, \tilde{\oplus}, \tilde{\odot} \right\rangle := \left\langle J_n, \tilde{\oplus}, \tilde{\odot} \right\rangle = \left\langle \mathcal{V}\left(I : (y^n)_i\right), \tilde{\oplus}, \tilde{\odot} \right\rangle$, $y^n \in (y^n)_i$, and $\mu_{\tilde{\bullet}}\left(r, y^n, ry^n\right) = 1$, we obtain that $ry^n \in \left\langle I, \tilde{\oplus}, \tilde{\odot} \right\rangle$. Therefore, $ty^{2n} = ry^n - sy^n \in I$, i.e., $t \in \left\langle \mathcal{V}\left(I : (y^{2n})_i\right), \tilde{\oplus}, \tilde{\odot} \right\rangle = \left\langle \mathcal{V}\left(I : (y^n)_i\right), \tilde{\oplus}, \tilde{\odot} \right\rangle$. This yields $r = s + ty^n \in \left\langle I, \tilde{\oplus}, \tilde{\odot} \right\rangle$. Therefore $\left\langle I, \tilde{\oplus}, \tilde{\odot} \right\rangle = \left\langle J \cap K, \tilde{\oplus}, \tilde{\odot} \right\rangle$. Since $\left\langle I, \tilde{\oplus}, \tilde{\odot} \right\rangle$ is a vague irreducible ideal, $\left\langle I, \tilde{\oplus}, \tilde{\odot} \right\rangle = \left\langle J, \tilde{\oplus}, \tilde{\odot} \right\rangle$ or $\left\langle I, \tilde{\oplus}, \tilde{\odot} \right\rangle = \left\langle K, \tilde{\oplus}, \tilde{\odot} \right\rangle$, we analyze the following two cases :

- If $\left\langle I, \tilde{\oplus}, \tilde{\odot} \right\rangle = \left\langle J, \tilde{\oplus}, \tilde{\odot} \right\rangle = \left\langle \mathcal{V}\left(I : (y^n)_i\right), \tilde{\oplus}, \tilde{\odot} \right\rangle$. By assumption $xy^n \in \left\langle I, \tilde{\oplus}, \tilde{\odot} \right\rangle$, this implies $x \in \left\langle \mathcal{V}\left(I : (y^n)_i\right), \tilde{\oplus}, \tilde{\odot} \right\rangle = \left\langle I, \tilde{\oplus}, \tilde{\odot} \right\rangle$ which is a contradiction.

- If $\left\langle I, \tilde{\oplus}, \tilde{\odot} \right\rangle = \left\langle K, \tilde{\oplus}, \tilde{\odot} \right\rangle = \left\langle (y^n)_i + I, \tilde{\oplus}, \tilde{\odot} \right\rangle$, then $y^n \in \left\langle I, \tilde{\oplus}, \tilde{\odot} \right\rangle$ which is a contradiction.

Hence $\left\langle I, \tilde{\oplus}, \tilde{\odot} \right\rangle$ is a vague primary ideal. □

From Lemma 2.1.1, Theorem 2.1.7 and Theorem 3.3.7, we can conclude the following corollary.

Corollary 3.3.8. *Let $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ be a commutative vague Notherian ring with identity.*

Then any vague ideal $\langle A, \tilde{\circ}, \tilde{\bullet} \rangle$ *of* $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ *is the intersection of all primary vague ideals of* $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ *.*

Theorem 2.1.2, Proposition 3.1.3, Theorem 3.3.7, Proposition 2.1.8 and Theorem 3.2.11 imply the following corollary.

Corollary 3.3.9. *If* $\langle \mathcal{H}, \tilde{\circ}, \tilde{\bullet} \rangle$ *be a commutative regular Noethrerian vague ring with identity, then a maximal vague ideal, a vague irreducible ideal, a vague prime ideal and a vague primary ideal under vague binary operations* $\tilde{\circ}$ *and* $\tilde{\bullet}$ *are coincide.*

Note that, in fact, the vague binary operations $\tilde{\circ}$ and $\tilde{\bullet}$ in Corollary 3.3.9 can be replaced by any vague binary operations $\tilde{\oplus}$ and $\tilde{\odot}$ on $\mathcal{H}$ such that for each $a, b, c \in \mathcal{H}$,

$$\mu_{\tilde{\oplus}}(a, b, c) \leq \mu_{\tilde{\circ}}(a, b, c) \text{ and } \mu_{\tilde{\odot}}(a, b, c) \leq \mu_{\tilde{\bullet}}(a, b, c) .$$

Eventually, we can conclude the related relations of the following vague ideals: maximal vague ideals, vague irreducible ideals, vague prime ideals, vague primary ideals and vague semiprime ideals with vague binary operations $\tilde{\circ}$ and $\tilde{\bullet}$ in the following diagram.

$$\begin{tikzcd}
\text{Maximal Vague Ideals} \arrow[r, "\text{(Proposition 2.1.8)}"] \arrow[d, "\substack{\text{+ comm vague}\\ \text{ring with 1}\\ \text{(Theorem 2.1.2)}}"'] & \text{Vague Irreducible Ideals} \arrow[d, "\substack{\text{+ comm vague}\\ \text{Notherian ring with 1}\\ \text{(Theorem 3.3.7)}}"] \\
\text{Vague Prime Ideals} \arrow[r, "\text{+commutative}", "\text{(Proposition 3.1.3)}"'] \arrow[d, "\text{(Proposition 3.2.3)}"'] & \text{Vague Primary Ideals} \arrow[ul, "\substack{\text{+ comm vague}\\ \text{regular ring}\\ \text{(Theorem 3.2.11)}}"'] \arrow[dl, squiggly, "\substack{\text{+ comm vague}\\ \text{regular ring}\\ \text{(Corollary 3.2.9)}}"] \\
\text{Vague Semiprime Ideal} &
\end{tikzcd}$$

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