CHAPTER II

SKEW SEMIFIELDS AND SKEW RATIO SEMIRINGS OF RIGHT [LEFT] QUOTIENTS OF SEMIRINGS

In this chapter, we shall generalize the concept of the semifield and the ratio semiring of quotients of a commutative semiring to the skew semifield and the skew ratio semiring of right [left] quotients of a semiring which gives P. Sinutoke's construction when the semiring is commutative.

<u>Definition 2.1</u>. Let S be a semiring with a multiplicative zero 0 such that $|S| > 1$. Then a skew semifield K is said to be a <u>skew semifield of right [left] quotients of S</u> iff there exists a monomorphism $i : S \rightarrow K$ such that for all $x \in K$ there exist $a \in S$, $b \in S \setminus \{0\}$ such that $x = i(a)i(b)^{-1}$ $[x = i(b)^{-1}i(a)]$. A monomorphism i satisfying the above property is said to be a <u>right [left] quotient embedding</u> of S into K. Note that it is easily proved that for a right [left] quotient embedding $i(a) = 0$ iff $a = 0$.

<u>Example 2.2</u>. Let $S = \{\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} / x,z \in \mathbb{Z}^+ \text{ and } y \in \mathbb{Z}\} \cup \{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\}$ and $K = \{\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} / x,z \in \mathbb{Q}^+ \text{ and } y \in \mathbb{Q}\} \cup \{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\}$. Then S and K with the usual addition and multiplication are a semiring with multiplicative zero and a skew semifield, respectively. To show that K is a skew semifield of right quotients of S, let $X \in K$. If $X = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ then

let $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $B \in S\setminus\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\}$ so $X = AB^{-1}$. Suppose that $X = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Then $x = \frac{p}{q}$, $y = \frac{m}{n}$ and $z = \frac{u}{v}$ where $p,q,n,u,v \in \mathbb{Z}^+$ and $m \in \mathbb{Z}$. Let $A = \begin{bmatrix} p & mv + p \\ 0 & un \end{bmatrix}$ and $B = \begin{bmatrix} q & q \\ 0 & vn \end{bmatrix}$. Then $A,B \in S\setminus\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\}$ and $AB^{-1} = \begin{bmatrix} p & mv + p \\ 0 & un \end{bmatrix}\begin{bmatrix} 1/q & -1/vn \\ 0 & 1/vn \end{bmatrix} = X$. Hence K is a skew semifield of right quotients of S.

<u>Remark 2.3.</u> In this chapter, we shall prove some theorems for skew semifields of right quotients of a semiring S. The theorems are true for skew semifields of left quotients of S and the proofs are similar so we shall not give the proofs for skew semifields of left quotients.

<u>Theorem 2.4.</u> Let S be a semiring with a multiplicative zero 0 such that $|S| > 1$. Then a skew semifield of right [left] quotients of S exists iff

(i) S is multiplicatively cancellative

and (ii) $(S,\cdot)$ satisfies the right [left] Ore condition.

<u>Proof.</u> Assume that (i) and (ii) hold. Consider $S \times (S\setminus\{0\})$. Let $(a,b),(c,d) \in S \times (S\setminus\{0\})$. Define a relation $\sim$ on $S \times (S\setminus\{0\})$ by $(a,b) \sim (c,d)$ iff there exist $x,y \in S\setminus\{0\}$ such that $ax = cy$ and $bx = dy$. Clearly $\sim$ is reflexive and symmetric. Let $(a,b),(c,d),(e,f) \in S \times (S\setminus\{0\})$ be such that $(a,b) \sim (c,d)$ and $(c,d) \sim (e,f)$. Then there exist $x,y,z,w \in S\setminus\{0\}$ such that $ax = cy$, $bx = dy$, $cz = ew$ and $dz = fw$. Since $y,z \in S\setminus\{0\}$, there exist $u,v \in S\setminus\{0\}$ such that $yu = zv$. Let $p = xu$ and $q = wv$. Then $p,q \in S\setminus\{0\}$. Since $ap = axu = cyu = czv = ewv = eq$

and $bp = bxu = dyu = dzv = fwv = fq$, $(a,b) \sim (e,f)$, so $\sim$ is transitive. Hence $\sim$ is an equivalence relation on $S \times (S\setminus\{0\})$. Let $K = \frac{S \times (S\setminus\{0\})}{\sim}$.

Let $\alpha,\beta \in K$. Define $\cdot$ on K in the following way : Choose $(a,b) \in \alpha$ and $(c,d) \in \beta$. Since $b \in S\setminus\{0\}$ and $c \in S$, there exist $x \in S$ and $y \in S\setminus\{0\}$ such that $bx = cy$. Define $\alpha \cdot \beta = [(ax,dy)]$. We must show that $\cdot$ is well-defined. We shall show this in three steps.

1) We shall show that $\cdot$ is independent of the choice of x,y. Suppose that there exist $x' \in S$ and $y' \in S\setminus\{0\}$ such that $bx' = cy'$. We must show that $(ax,dy) \sim (ax',dy')$. Since $y,y' \in S\setminus\{0\}$, there exist $z,w \in S\setminus\{0\}$ such that $yz = y'w$. Since $bxz = cyz = cy'w = bx'w$ and $b \neq 0$, $xz = x'w$. Thus $axz = ax'w$ and $dyz = dy'w$. Hence $(ax,dy) \sim (ax',dy')$.

2) Fix (a,b). Suppose that $(c,d) \sim (c',d')$. Since $b \in S\setminus\{0\}$ and $c' \in S$, there exist $z \in S$ and $w \in S\setminus\{0\}$ such that $bz = c'w$. We must show that $(ax,dy) \sim (az,d'w)$. Since $(c,d) \sim (c',d')$, there exist $u,v \in S\setminus\{0\}$ such that $cu = c'v$ and $du = d'v$. Since $dy,d'w \in S\setminus\{0\}$, there exist $p,q \in S\setminus\{0\}$ such that $dyp = d'wq$. Since $wq,v \in S\setminus\{0\}$, there exist $g,h \in S\setminus\{0\}$ such that $wqg = vh$. Since $duh = d'vh = d'wqg = dypg$ and $d \neq 0$, $uh = ypg$. Since $bxpg = cypg = cuh = c'vh = c'wqg = bzqg$ and $b,g \neq 0$, $xp = zq$. Thus $axp = azq$ and $dyp = d'wq$. Hence $(ax,dy) \sim (az,d'w)$.

3) Fix (c,d). Suppose that $(a,b) \sim (a',b')$. Since $b' \in S\setminus\{0\}$ and $c \in S$, there exist $z \in S$ and $w \in S\setminus\{0\}$ such that $b'z = cw$. We must show that $(ax,dy) \sim (a'z,dw)$. Since $(a,b) \sim (a',b')$, there exist $u,v \in S\setminus\{0\}$ such that $au = a'v$ and $bu = b'v$. Since $y,w \in S\setminus\{0\}$, there exist $p,q \in S\setminus\{0\}$ such that $yp = wq$. Since $zq \in S$ and $v \in S\setminus\{0\}$, there exist $g \in S\setminus\{0\}$ and $h \in S$ such that $zqg = vh$. Since $buh = b'vh = b'zqg = cwqg = cypg = bxpg$ and $b \neq 0$, $uh = xpg$. Since $axpg = auh = a'vh = a'zqg$

and $g \neq 0$, $axp = a'zq$. Hence $(ax,dy) \sim (a'z,dw)$. Therefore $\cdot$ is well-defined.

To show that $\cdot$ is associative, let $\alpha,\beta,\gamma \in K$. Choose $(a,b) \in \alpha$, $(c,d) \in \beta$ and $(e,f) \in \gamma$. There exist $x \in S$ and $y \in S\setminus\{0\}$ such that $bx = cy$, so $\alpha\beta = [(ax,dy)]$. There exist $z \in S$ and $w \in S\setminus\{0\}$ such that $dyz = ew$, so $(\alpha\beta)\gamma = [(axz,fw)]$. Since $dyz = ew, \beta\gamma = [(cyz,fw)]$. There exist $p \in S$ and $q \in S\setminus\{0\}$ such that $bp = cyzq$, so $\alpha(\beta\gamma) = [(ap,fwq)]$. Let $h \in S\setminus\{0\}$ and $g = qh$. Since $bph = cyzqh = bxzg$ and $b \neq 0$, $ph = xzg$. Thus $aph = axzg$ and $fwqh = fwg$, so $(ap,fwq) \sim (axz,fw)$. Hence $\alpha(\beta\gamma) = (\alpha\beta)\gamma$. Therefore $\cdot$ is associative.

Since $(a,a) \sim (b,b)$ for all $a,b \in S\setminus\{0\}$, denote $[(a,a)]$ by 1 where $a \in S\setminus\{0\}$. To show that 1 is a multiplicative identity, let $\alpha \in K$. Choose $(a,b) \in \alpha$. There exist $x \in S\setminus\{0\}$ and $y \in S$ such that $ax = by$, so $1\alpha = [(b,b)][(a,b)] = [(by,bx)]$. Let $z \in S\setminus\{0\}$ and $w = xz$. then $aw = axz = byz$ and $bw = bxz$. Hence $(a,b) \sim (by,bx)$, so $\alpha = 1\alpha$. Also, $\alpha 1 = [(a,b)][(b,b)] = [(ab,bb)] = [(a,b)] = \alpha$. Hence 1 is a multiplicative identity of K.

Since $(0,a) \sim (0,b)$ for all $a,b \in S\setminus\{0\}$, denote $[(0,a)]$ by 0 where $a \in S\setminus\{0\}$. To show that 0 is a multiplicative zero, let $\alpha \in K$. Choose $(a,b) \in \alpha$. Since $b0 = 0b$, $\alpha 0 = [(a,b)][(0,b)] = [(a0,bb)] = [(0,bb)] = 0$. There exist $x \in S\setminus\{0\}$ and $y \in S$ such that $ax = by$, so $0\alpha = [(0,b)][(a,b)] = [(0y,bx)] = [(0,bx)] = 0$. Hence 0 is a multiplicative zero of K.

Let $\alpha \in K\setminus\{0\}$. Choose $(a,b) \in \alpha$. Then $a \neq 0$. Let $\beta = [(b,a)]$. Let $c \in S\setminus\{0\}$ be arbitary. Then $\alpha\beta = [(ac,ac)] = 1 = [(bc,bc)] = \beta\alpha$, so $\beta = \alpha^{-1}$. Hence $(K\setminus\{0\},\cdot)$ is a group.

Let $\alpha,\beta \in K$. Define + on K in the following way : Choose $(a,b) \in \alpha$ and $(c,d) \in \beta$. There exist $x,y \in S\setminus\{0\}$ such that $bx = dy$.

Define $\alpha + \beta = [(ax + cy, bx)]$. We must show that + is well-defined. We shall show this in three steps.

1) We shall show that + is independent of the choice of x, y. Suppose that there exist $x', y' \in S\setminus\{0\}$ such that $bx' = dy'$. We must show that $(ax + cy, bx) \sim (ax' + cy', bx')$. There exist $z, w \in S\setminus\{0\}$ such that $xz = x'w$. Since $dyz = bxz = bx'w = dy'w$ and $d \neq 0, yz = y'w$. Thus $(ax + cy)z = axz + cyz = ax'w + cy'w = (ax' + cy')w$ and $bxz = b'xw$. Hence $(ax + cy, bx) \sim (ax' + cy', bx')$.

2) Fix (a,b). Suppose that $(c,d) \sim (c',d')$. There exist $z, w \in S\setminus\{0\}$ such that $bz = d'w$. We must show that $(ax + cy, bx) \sim (az + c'w, bz)$. Since $(c,d) \sim (c',d')$, there exist $u, v \in S\setminus\{0\}$ such that $cu = c'v$ and $du = d'v$. There exist $p, q \in S\setminus\{0\}$ such that $xp = zq$. There exist $g, h \in S\setminus\{0\}$ such that $vg = wqh$. Since $dug = d'vg = d'wqh = bzqh = bxph = dyph$ and $d \neq 0$, $ug = yph$. Since $cyph = cug = c'vg = c'wqh$ and $h \neq 0$, $cyp = c'wq$. Thus $(ax + cy)p = axp + cyp = azq + c'wq = (az + c'w)q$ and $bxp = bzq$. Hence $(ax + cy, bx) \sim (az + c'w, bz)$.

3) Fix (c,d). Suppose that $(a,b) \sim (a',b')$. There exist $z, w \in S\setminus\{0\}$ such that $b'z = dw$. We must show that $(ax + cy, bx) \sim (a'z + cw, b'z)$. Since $(a,b) \sim (a',b')$, there exist $u, v \in S\setminus\{0\}$ such that $au = a'v$ and $bu = b'v$. There exist $p, q \in S\setminus\{0\}$ such that $bxp = b'zq$. Since $dyp = bxp = b'zq = dwq$ and $d \neq 0$, $yp = wq$. There exist $g, h \in S\setminus\{0\}$ such that $vg = zqh$. Since $bug = b'vg = b'zqh = bxph$ and $b \neq 0$, $ug = xph$. Since $axph = aug = a'vg = a'zqh$ and $h \neq 0$, $axp = a'zq$. Thus $(ax + cy)p = axp + cyp = a'zq + cwq = (a'z + cw)q$ and $bxp = b'zq$. Hence $(ax + cy, bx) \sim (a'z + cw, b'z)$. Therefore + is well-defined.

To show that + is associative, let $\alpha, \beta, \gamma \in K$. Choose $(a,b) \in \alpha$,

$(c,d) \in \beta$ and $(e,f) \in \gamma$. There exist $x,y \in S\setminus\{0\}$ such that $bx = dy$, so $\alpha + \beta = [(ax + cy,bx)]$. There exist $z,w \in S\setminus\{0\}$ such that $bxz = fw$, so $(\alpha + \beta) + \gamma = [((ax + cy)z + ew,bxz)]$. There exist $u,v \in S\setminus\{0\}$ such that $du = fv$, so $\beta + \gamma = [(cu + ev,du)]$. There exist $p,q \in S\setminus\{0\}$ such that $bp = duq$, so $\alpha + (\beta + \gamma) = [(ap + (cu + ev)q,bp)]$. We must show that $((ax + cy)z + ew,bxz) \sim (ap + (cu + ev)q,bp)$. There exist $g,h \in S\setminus\{0\}$ such that $xzg = ph$. Since $dyzg = bxzg = bph = duqh$ and $d \neq 0$, $yzg = uqh$. Since $fwg = bxzg = duqh = fvqh$ and $f \neq 0$, $wg = vqh$. Thus $((ax + cy)z + ew)g = axzg + cyzg + ewg = aph + cuqh + evqh = (ap + (cu + ev)q) h$ and $bxzg = bph$, so $((ax + cy)z + ew,bxz) \sim (ap + (cu + ev)q,bp)$. Hence $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$. Therefore + is associative.

We shall show that multiplication distributes over addition. Let $\alpha,\beta,\gamma \in K$. Choose $(a,b) \in \alpha$, $(c,d) \in \beta$ and $(e,f) \in \gamma$. First, we shall show that $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$. There exist $x,y \in S\setminus\{0\}$ such that $dx = fy$, so $\beta + \gamma = [(cx + ey,dx)]$. There exist $z \in S$ and $w \in S\setminus\{0\}$ such that $bz = (cx + ey)w$, so $\alpha(\beta + \gamma) = [(az,dxw)]$. There exist $u \in S$ and $v \in S\setminus\{0\}$ such that $bu = cv$, so $\alpha\beta = [(au,dv)]$. There exist $p \in S$ and $q \in S\setminus\{0\}$ such that $bp = eq$, so $\alpha\gamma = [(ap,fq)]$. There exist $g,h \in S\setminus\{0\}$ such that $dvg = fqh$, so $\alpha\beta + \alpha\gamma = [(aug + aph,dvg)]$. We must show that $(az,dxw) \sim (aug + aph,dvg)$. There exist $m,n \in S\setminus\{0\}$ such that $xwm = vgn$. Since $fywm = dxwm = dvgn = fqhn$ and $f \neq 0$, $ywm = qhn$. Since $bzm = (cx + ey)wm = cxwm + eywm = cvgn + eqhn = bugn + bphn = b(ug + ph)n$ and $b \neq 0$, $zm = (ug + ph)n$. Thus $azm = (aug + aph)n$ and $dxwm = dvgn$, so $(az,dxw) \sim (aug + aph,dvg)$. Hence $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$.

Next, we shall show that $(\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma$. There exist $x,y \in S\setminus\{0\}$ such that $bx = dy$, so $\alpha + \beta = [(ax + cy,bx)]$.

There exist $z \in S$ and $w \in S\setminus\{0\}$ such that $bxz = ew$, so $(\alpha + \beta)\gamma = [((ax + cy)z, fw)]$. Since $bxz = ew$, $\alpha\gamma = [(axz, fw)]$. There exist $p \in S$ and $q \in S\setminus\{0\}$ such that $dp = eq$, so $\beta\gamma = [(cp, fq)]$. There exist $g, h \in S\setminus\{0\}$ such that $wg = qh$, so $\alpha\gamma + \beta\gamma = [(axzp + cph, fwg)]$. We must show that $((ax + cy)z, fw) \sim (axzg + cph, fwg)$. Let $n \in S\setminus\{0\}$ and $m = gn$. Since $dyzm = bxzm = ewm = ewgn = eqhn = dphn$ and $d \neq 0$, $yzm = phn$. Thus $(ax + cy)zm = axzm + cyzm = axzgn + cphn = (axzg + cph)n$ and $fwm = fwgn$, so $((ax + cy)z, fw) \sim (axzg + cph, fwg)$. Hence $(\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma$. Therefore $(K, +, \cdot)$ is a skew semifield.

Fix $c \in S\setminus\{0\}$. Define $i : S \to K$ by $i(x) = [(xc, c)]$ for all $x \in S$. Note that $i(x) = 0$ iff $x = 0$. To show that i is a homomorphism, let $a, b \in S$. Then there exist $x \in S$ and $y \in S\setminus\{0\}$ such that $cx = bcy$, so $i(a)i(b) = [(ac, c)][(bc, c)] = [(acx, cy)]$. Let $z \in S\setminus\{0\}$ and $w = yz$. Since $abcw = abcyz = acxz$ and $cw = cyz$, $(abc, c) \sim (acx, cy)$, hence $i(ab) = i(a)i(b)$. Also, $i(a) + i(b) = [(acc + bcc, cc)] = [((a + b)c, c)] = i(a + b)$. Hence i is a homomorphism. Let $a, b \in S$ be such that $i(a) = i(b)$. Then $[(ac, c)] = [(bc, c)]$, so there exist $x, y \in S\setminus\{0\}$ such that $acx = bcy$ and $cx = cy$, hence $a = b$. Thus i is a monomorphism. Let $\alpha \in K$. Choose $(a, b) \in \alpha$. Then $\alpha = [(a, b)] = [(acc, bcc)] = [(ac, c)][(c, bc)] = i(a)i(b)^{-1}$. Therefore $(K, +, \cdot)$ is a skew semifield of right quotients of S.

Conversely, assume that a skew semifield of right quotients of S exists. Let i be a right quotient embedding of S into K. To show that S is multiplicatively cancellative, let $a, b, c \in S$ be such that $ab = ac$ and $a \neq 0$. Then $i(a)i(b) = i(a)i(c)$, so $i(b) = i(a)^{-1}i(a)i(b) = i(a)^{-1}i(a)i(c) = i(c)$, hence $b = c$. To show that $(S, \cdot)$ satisfies the right Ore condition, let $a, b \in S\setminus\{0\}$. Then $i(a)^{-1}i(b) \in K$, so there exist $x \in S$ and $y \in S\setminus\{0\}$ such that $i(a)^{-1}i(b) = i(x)i(y)^{-1}$. Thus

$i(a)i(x) = i(b)i(y)$, hence $ax = by$.. Since $ax = by \neq 0$ and $a \neq 0$, $x \neq 0$, hence we have the theorem.

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Remark 2.5. Let S be a semiring having K as a skew semifield of right [left] quotients. In the proof of Theorem 2.4 we can see that

1) if S is additively commutative then K is additively commutative.

2) if S is multiplicatively commutative then K is multiplicatively commutative.

3) if S is commutative then the construction of K is the same as the construction of the semifield of quotients of S given by P. Sinutoke in [1].

4) In [4] it was shown that if 0 is a multiplicative zero of a skew semifield K then either 0 is a left or right additive identity of K and either 0 is a left or right additive zero of K. Clearly, 0 is an additive identity of K iff 0 is an additive identity of S and 0 is an additive zero of K iff 0 is an additive zero of S. So we get as a corollary of the preceding theorem that if S is a semiring with a multiplicative zero 0 satisfying both the multiplicative cancellativity condition and the right [left] Ore condition then 0 must be either a left or right additive identity of S and either a left or right additive zero of S.

Remark 2.6. Let S be a semiring having K as a skew semifield of right [left] quotients, $i : S \rightarrow K$ a right [left] quotient embedding and $x \in K$. If $x = i(a)i(b)^{-1} = i(c)i(d)^{-1}$ where $a,c \in S$ and $b,d \in S\setminus\{0\}$ then there exist $p,q \in S\setminus\{0\}$ such that $ap = cq$ and $bp = dq$.

Proof. Assume that $x = i(a)i(b)^{-1} = i(c)i(d)^{-1}$ where $a,c \in S$ and $b,d \in S\setminus\{0\}$. Then there exist $p,q \in S\setminus\{0\}$ such that $bp = dq$. Thus

$i(a)i(p)i(q)^{-1}i(d)^{-1} = i(a)i(b)^{-1} = i(c)i(d)^{-1}$, so $i(ap) = i(cq)$. Hence $ap = cq$.

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Corollary 2.7. Let S be a semiring having K as a skew semifield of right [left] quotients, $i : S \to K$ a right [left] quotient embedding, L a skew semifield and $f : S \to L$ a homomorphism such that $f(x) = 0$ iff $x = 0$. Then there exists a unique homomorphism $g : K \to L$ such that $g \circ i = f$. Furthermore, if f is a monomorphism then g is a monomorphism.

Proof. Define $g : K \to L$ in the following way : Let $x \in K$. Then $x = i(a)i(b)^{-1}$ where $a,b \in S$ and $b \neq 0$. Define $g(x) = f(a)f(b)^{-1}$.

We must show that g is well-defined. Suppose that there exist $a' \in S$ and $b' \in S\setminus\{0\}$ such that $x = i(a')i(b')^{-1}$. By Remark 2.6, there exist $p,q \in S\setminus\{0\}$ such that $ap = a'q$ and $bp = b'q$. Then $f(a)f(b)^{-1} = f(a')f(q)f(p)^{-1}f(p)f(q)^{-1}f(b')^{-1} = f(a')f(b')^{-1}$. Hence g is well-defined.

To show that $g \circ i = f$, let $a \in S$. If $a = 0$ then $i(a) = 0$ and $g(i(a)) = 0 = f(a)$ so we are done. Assume that $a \neq 0$. Then $g \circ i(a) = g(i(a)) = g(i(aa)i(a)^{-1}) = f(aa)f(a)^{-1} = f(a)$, so $g \circ i = f$.

To show that g is a homomorphism, let $x,y \in K$. Then $x = i(a)i(b)^{-1}$ and $y = i(c)i(d)^{-1}$ where $a,c \in S$ and $b,d \in S\setminus\{0\}$. There exist $m \in S$ and $n \in S\setminus\{0\}$ such that $bm = cn$. Thus $g(xy) = g(i(a)i(b)^{-1}i(c)i(d)^{-1}) = g(i(a)i(m)i(n)^{-1}i(d)^{-1}) = g(i(am)i(dn)^{-1}) = f(am)f(dn)^{-1} = f(a)f(m)f(n)^{-1}f(d)^{-1} = f(a)f(b)^{-1}f(c)f(d)^{-1} = g(x)g(y)$.
There exist $p,q \in S\setminus\{0\}$ such that $bp = dq$. Thus $g(x + y) = g(i(a)i(b)^{-1} + i(c)i(d)^{-1}) = g(i(a)i(p)i(bp)^{-1} + i(c)i(q)i(dq)^{-1}) = g(i(ap + cq)i(bp)^{-1}) = f(ap + cq)f(bp)^{-1} = f(ap)f(bp)^{-1} + f(cq)f(dq)^{-1} =$

$f(a)f(b)^{-1} + f(c)f(d)^{-1} = g(x) + g(y)$. Hence g is a homomorphism.

To show that g is unique, suppose that there exists a homomorphism $h : K \to L$ such that $h \circ i = f$. Let $x \in K$. Then $x = i(a)i(b)^{-1}$ where $a,b \in S$ and $b \neq 0$. Then $g(x) = f(a)f(b)^{-1} = (h \circ i(a))(h \circ i(b))^{-1} = h(i(a)i(b)^{-1}) = h(x)$, so $g = h$. Hence g is unique.

Suppose that f is an injection. To show that g is an injection, let $x,y \in K$ be such that $g(x) = g(y)$. There exist $a,c \in S$ and $b,d \in S\backslash\{0\}$ such that $x = i(a)i(b)^{-1}$ and $y = i(c)i(d)^{-1}$. There exist $p,q \in S\backslash\{0\}$ such that $bp = dq$. Then $f(a)f(p)f(q)^{-1}f(d)^{-1} = f(a)f(b)^{-1} = g(x) = g(y) = f(c)f(d)^{-1}$, thus $f(ap) = f(cq)$, so $ap = cq$ because f is an injection. Hence $x = i(a)i(b)^{-1} = i(c)i(q)i(p)^{-1}i(p)i(q)^{-1}i(d)^{-1} = i(c)i(d)^{-1} = y$. Therefore g is an injection.

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<u>Corollary 2.8</u>. Let S be a semiring having K as a skew semifield of right [left] quotients. If L is a skew semifield and L contiains an isomorphic copy of S then L contains an isomorphic copy of K.

<u>Corollary 2.9</u>. If S is a semiring having K and K' as skew semifields of right or left quotients then $K \cong K'$.

<u>Proof</u>. There exist right or left quotient embedding $i : S \to K$ and $j : S \to K'$. By Corollary 2.7, there exists a unique monomorphism $f : K \to K'$ and $g : K' \to K$ such that $f \circ i = j$ and $g \circ j = i$. Then $(g \circ f) \circ i = i$. By Corollary 2.7 and $Id_K \circ i = i$ we get that $g \circ f = Id_K$. Similarly, $f \circ g = Id_{K'}$. Thus $f = g^{-1}$, hence f is an isomorphism.

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Remark 2.10. Let S be a semiring having K as a skew semifield of right [left] quotients. If S is additively cancellative then K is additively commutative and additively cancellative.

Proof. Let i be a right quotient embedding of S into K. Assume that S is additively cancellative. To show that K is additively commutative, let $\alpha,\beta \in K$. Then $\alpha = i(x)i(y)^{-1}$ and $\beta = i(z)i(w)^{-1}$ where $x,z \in S$ and $y,w \in S\setminus\{0\}$. There exist $a,b \in S\setminus\{0\}$ such that $ya = wb$. Then $\alpha + \beta = i(xa + zb)i(ya)^{-1} = i(zb + xa)i(wb)^{-1} = \beta + \alpha$. To show that K is additively cancellative, let $\alpha,\beta,\gamma \in K$ be such that $\alpha + \beta = \alpha + \gamma$. There exist $x,z,u \in S$ and $y,w,v \in S\setminus\{0\}$ such that $\alpha = i(x)i(y)^{-1}$, $\beta = i(z)i(w)^{-1}$ and $\gamma = i(u)i(v)^{-1}$. There exist $a,b,c,d \in S\setminus\{0\}$ such that $ya = wb$ and $yc = vd$. Then $i(xa + zb)i(ya)^{-1} = \alpha + \beta = \alpha + \gamma = i(xc + ud)i(yc)^{-1}$. Hence there exist $e,f \in S\setminus\{0\}$ such that $yae = ycf$ and $(xa + zb)e = (xc + ud)f$. Thus $wbe = yae = ycf = vdf$ and $ae = cf$. Since $xae + zbe = xcf + udf$ and S is additively cancellative, $zbe = udf$. So $\beta = i(zbe)i(wbe)^{-1} = i(udf)i(vdf)^{-1} = \gamma$. Hence we have the remark.

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Corollary 2.11. Let S be a semiring of order > 1 with a multiplicative zero which is additively cancellative and which satisfies properties (i),(ii) of Theorem 2.4. Then S is additively commutative.

Corollary 2.12. Let R be a skew ring such that $|R| > 1$. Then a skew semifield K of right [left] quotients of R exists iff

(i) R has no left zero divisors and no right zero disvisors

and (ii) $(R,\cdot)$ satisfies the right [left] Ore condition.

Furthermore, K is a skew field.

Proof. We need only prove that K is a skew field. We must show that an additive inverse of x belongs to K for all $x \in K$. Let $x \in K$. Then $x = i(a)i(b)^{-1}$ where $a \in R$, $b \in R\setminus\{0\}$ and i is a right quotient embedding of R into K. Let $y = i(-a)i(b)^{-1}$. Then $y \in K$ and $x + y = i(a - a)i(b)^{-1} = 0 = i(-a + a)i(b)^{-1} = y + x$. Thus y is an additive inverse of x and hence K is a skew field.

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Remark 2.13. In this case, we shall call K the skew field of right [left] quotients of R.

Remark 2.14.

1) Let R be a skew ring of order > 1 satisfying properties (i) and (ii) of Corollary 2.12. Then R is a ring.

2) Let R be a ring having K as a skew field of right [left] quotients. If R is commutative then K is a field of quotients of R.

We shall now give an example of a skew field of right quotients of a noncommutative ring.

Example 2.15. Let K be a field and $\sigma : K \to K$ an automorphism of K such that $\sigma \neq Id_K$. Let $(a_i)_{i\in\mathbf{Z}_o^+}$ denote an infinite sequence in K whose i^{th} term is a_i.

Let $K[[X]] = \{(a_i)_{i\in\mathbf{Z}_o^+} | a_i \in K \text{ for all } i \in \mathbf{Z}_o^+\}$.

Denote $(a_i)_{i\in\mathbf{Z}_o^+} \in K[[X]]$ by $\sum_{i=0}^{\infty} a_iX^i$. Let $f = \sum_{i=0}^{\infty} a_iX^i$, $g = \sum_{i=0}^{\infty} b_iX^i$ and $h = \sum_{i=0}^{\infty} c_iX^i \in K[[X]]$. Define

$$f + g = \sum_{i=0}^{\infty} (a_i + b_i)X^i$$

and $\quad f\cdot g = \sum_{\ell=0}^{\infty} d_\ell X^\ell$ where $d_\ell = \sum_{i+j=\ell} a_i\sigma^i(b_j)$.

Then $(K[[X]],+)$ is an abelian group. To show that $(K[[X]],+,\cdot)$ is a ring, note that

$$(fg)h = (\sum_{p=0}^{\infty}(\sum_{i+j=p} a_i\sigma^i(b_j))X^p)(\sum_{\ell=0}^{\infty} c_\ell X^\ell)$$

$$= \sum_{q=0}^{\infty}(\sum_{p+\ell=q}(\sum_{i+j=p} a_i\sigma^i(b_j))\sigma^p(c_\ell))X^q$$

$$= \sum_{q=0}^{\infty}(\sum_{i+j+\ell=q} a_i\sigma^i(b_j)\sigma^{i+j}(c_\ell))X^q$$

and $$f(gh) = (\sum_{i=0}^{\infty} a_iX^i)(\sum_{p=0}^{\infty}(\sum_{j+\ell=p} b_j\sigma^j(c_\ell))X^p)$$

$$= \sum_{q=0}^{\infty}(\sum_{i+p=q} a_i\sigma^i(\sum_{j+\ell=p} b_j\sigma^j(c_\ell)))X^q$$

$$= \sum_{q=0}^{\infty}(\sum_{i+p=q} a_i(\sum_{j+\ell=p}\sigma^i(b_j)\sigma^{i+j}(c_\ell)))X^q$$

$$= \sum_{q=0}^{\infty}(\sum_{i+j+\ell=q} a_i\sigma^i(b_j)\sigma^{i+j}(c_\ell))X^q,$$

so $\quad (fg)h = f(gh)$, thus $\cdot$ is associative. Also,

$$(f+g)h = (\sum_{i=0}^{\infty}(a_i+b_i)X^i)(\sum_{j=0}^{\infty} c_jX^j)$$

$$= \sum_{\ell=0}^{\infty}(\sum_{i+j=\ell}(a_i+b_i)\sigma^i(c_j))X^\ell$$

$$= \sum_{\ell=0}^{\infty}(\sum_{i+j=\ell} a_i\sigma^i(c_j))X^\ell + \sum_{\ell=0}^{\infty}(\sum_{i+j=\ell} b_i\sigma^i(c_j))X^\ell$$

$$= fh + gh$$

and $$f(g+h) = (\sum_{i=0}^{\infty} a_iX^i)(\sum_{j=0}^{\infty}(b_j+c_j)X^j)$$

$$= \sum_{\ell=0}^{\infty}(\sum_{i+j=\ell} a_i\sigma^i(b_j+c_j))X^\ell$$

$$= \sum_{\ell=0}^{\infty} \Big(\sum_{i+j=\ell} a_i\sigma^i(b_j)\Big)X^\ell + \sum_{\ell=0}^{\infty} \Big(\sum_{i+j=\ell} a_i\sigma^i(c_j)\Big)X^\ell$$

$$= fg + fh,$$

so multiplication distributes over addition. To show that $\cdot$ is noncommutative, note that since $\sigma \neq Id_K$, there exists a $d \in K$ such that $\sigma(d) \neq d$. Then $(X)(dX) = \sigma(d)X^2$ and $(dX)(X) = dX^2$, so $(X)(dX) \neq (dX)(X)$. Hence $(K[[X]],+,\cdot)$ is a noncommutative ring.

Let $(a_i)_{i\in\mathbb{Z}}$ denote an infinite sequence in K whose i^{th} term is a_i. Let $K((X)) = \{(a_i)_{i\in\mathbb{Z}} | a_i \in K$ for all $i \in \mathbb{Z}$ and the number of $i \in \mathbb{Z}^-$ such that $a_i \neq 0$ is finite$\}$. Denote $(a_i)_{i\in\mathbb{Z}} \in K((X))$ by $\sum_{i=-\infty}^{\infty} a_iX^i$.

Let $f = \sum_{i=-\infty}^{\infty} a_iX^i$ and $g = \sum_{i=-\infty}^{\infty} b_iX^i \in K((X))$. Define

$$f + g = \sum_{i=-\infty}^{\infty} (a_i + b_i)X^i$$

and $$f \cdot g = \sum_{\ell=-\infty}^{\infty} c_\ell X^\ell \text{ where } c_\ell = \sum_{i+j=\ell} a_i\sigma^i(b_j).$$

Then $(K((X)),+)$ is an abelian group. To show that $\cdot$ is well-defined, let $n,m \in \mathbb{Z}$ be such that $a_p = b_q = 0$ for all $p < n$ and $q < m$. Let $k = n + m$. Let $r \in \mathbb{Z}$ be such that $r < k$. Then $c_r = \sum_{i+j=r} a_i\sigma^i(b_j)$. Let $i,j \in \mathbb{Z}$ be such that $i + j = r$.

<u>Case 1</u> $i < n$. Thus $a_i = 0$, so $a_i\sigma^i(b_j) = 0$.

<u>Case 2</u> $n \leqslant i$. Thus $j = r - i < k - i \leqslant n + m - n = m$, so $b_j = 0$, hence $a_i\sigma^i(b_j) = 0$.

Therefore $c_r = 0$. Hence $\cdot$ is well-defined. A proof similar to the one just given shows that $\cdot$ is associative and distributive. Since $K[[X]]$ is a subring of $K((X))$, $K((X))$ is noncommutative. Let 1 be the multiplicative identity of K. Then 1 is the multiplicative identity of $K((X))$ if we identify 1 with $\sum_{i=-\infty}^{\infty} a_iX^i$ where $a_o = 1$ and $a_i = 0$ for

$i \neq 0$. Let $f \in K((X))\setminus\{0\}$. Then there exists an $n \in \mathbb{Z}$ such that $a_m = 0$ for all $m < n$ and $a_n \neq 0$, so $f = a_nX^n + a_{n+1}X^{n+1} + \ldots$. Thus

$$f^{-1} = \frac{1}{a_nX^n + a_{n+1}X^{n+1} + \ldots} = \frac{1}{a_nX^n\,(1 + \frac{a_{n+1}}{a_n}X + \ldots)} =$$

$\frac{1}{a_n}X^{-n}(1 - A + A^2 - A^3 + \ldots)$, where $A = \frac{a_{n+1}}{a_n}X + \frac{a_{n+2}}{a_n}X^2 + \ldots$, so $f^{-1} \in K((X))\setminus\{0\}$. Hence $(K((X)),+,\cdot)$ is a skew field and $K[[X]] \subseteq K((X))$. Let $f \in K((X))$. We must show that there exist $g \in K[[X]]$ and $h \in K[[X]]\setminus\{0\}$ such that $f = gh^{-1}$. If $f \in K[[X]]$ then let $g = f$ and $h = 1$. Suppose that $f = \sum_{i=-\infty}^{\infty} z_iX^i \in K((X))\setminus K[[X]]$. Let $m \in \mathbb{Z}^-$ be such that $z_i = 0$ for all $i < m$ and $z_m \neq 0$. Let $h = \sum_{i=0}^{\infty} b_iX^i \in K[[X]]\setminus\{0\}$ be such that $b_{-m} \neq 0$ and $b_i = 0$ for all $i < -m$. Then h^{-1} exists in $K((X))$ and $h^{-1} = \sum_{i=-\infty}^{\infty} c_iX^i$ where $c_m \neq 0$ and $c_i = 0$ for all $i < m$. Let $g = \sum_{i=0}^{\infty} a_iX^i \in K[[X]]$ where $a_o = \frac{z_m}{c_m}$ and

$a_i = \frac{1}{\sigma^i(c_m)}(z_{m+i} - a_oc_{m+i} - a_1\sigma(c_{m+i-1}) - \ldots - a_{i-1}\sigma^{i-1}(c_{m+1}))$ for all $i > 0$. Then $g = \sum_{i=-\infty}^{\infty} a_iX^i \in K((X))$ where $a_i = 0$ for all $i < 0$. Let $gh^{-1} = \sum_{\ell=-\infty}^{\infty} d_\ell X^\ell$ where $d_\ell = \sum_{i+j=\ell} a_i\sigma^i(c_j)$. To show that $f = gh^{-1}$, consider terms of degree $\ell < m$. Let $i,j \in \mathbb{Z}$ be such that $i + j = \ell$.

<u>Case 1</u> $i < 0$. Thus $a_i = 0$, so $a_i\sigma^i(c_j) = 0$.

<u>Case 2</u> $0 \leqslant i$. Thus $j = \ell - i < m$, so $c_j = 0$, hence $a_i\sigma^i(c_j) = 0$.

Then $d_\ell = 0 = z_\ell$. Consider the term of degree m. Thus

$$d_m = \sum_{i\leqslant -1} a_i\sigma^i(c_{m-i}) + a_oc_m + \sum_{i\geqslant 1} a_i\sigma^i(c_{m-i})$$

$$= a_oc_m = \frac{z_m}{c_m}c_m = z_m.$$

Consider terms of degree $m + n$ where $n > 1$. Thus

$$d_{m+n} = \sum_{i\leq -1} a_i\sigma^i(c_{m+n-i}) + a_o c_{m+n} + a_1\sigma(c_{m+n-1}) + \ldots + a_n\sigma^n(c_m) + \sum_{i>n} a_i\sigma^i(c_{m+n-i})$$

$$= a_o c_{m+n} + a_1\sigma(c_{m+n-1}) + \ldots + a_n\sigma^n(c_m)$$

$$= a_o c_{m+n} + a_1\sigma(c_{m+n-1}) + \ldots + a_{n-1}\sigma^{n-1}(c_{m+1}) + \frac{1}{\sigma^n(c_m)}\left[z_{m+n} - a_o c_{m+n} - a_1\sigma(c_{m+n-1}) - \ldots - a_{n-1}\sigma^{n-1}(c_{m+1})\right]\sigma^n(c_m)$$

$$= z_{m+n}.$$

Hence $f = gh^{-1}$. Therefore $(K((X)),+,\cdot)$ is a skew field of right quotients of $K[[X]]$.

Corollary 2.16. Let R be a ring having K as a skew field of right [left] quotients, $i : R \to K$ a right quotient embedding, L a skew field and $f : R \to L$ a monomorphism. Then there exists a monomorphism $g : K \to L$ such that $g \circ i = f$.

Corollary 2.17. Let R be a ring having K as a skew field of right [left] quotients. If L is a skew field and L contains an isomorphic copy of R then L contains an isomorphic copy of K.

Corollary 2.18. If R is a ring having K and K' as skew fields of right or left quotients then $K \cong K'$.

Definition 2.19. Let S be a semiring without a multiplicative zero. Then a skew ratio semiring D is said to be a skew ratio semiring of right [left] quotients of S iff there exists a monomorphism $i : S \to D$ such that for all $x \in D$ there exist $a,b \in S$ such that $x = i(a)i(b)^{-1}[x = i(b)^{-1}i(a)]$. A monomorphism i satisfying the above property is said to be a right [left] quotient embedding of S into D.

Example 2.20. Let $S = \{\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} / x,z \in \mathbb{Z}^+ \text{ and } y \in \mathbb{Z}\}$ and $D = \{\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} / x,z \in \mathbb{Q}^+ \text{ and } y \in \mathbb{Q}\}$. Then S and D with the usual addition and multiplication are a semiring and its skew ratio semiring of right quotients, respectively.

Remark 2.21. In this chapter, we shall prove some theorems for skew ratio semirings of right quotients of a semiring S. The theorems are true for skew ratio semirings of left quotients of S and the proofs are similar so we shall not give the proofs for skew ratio semirings of left quotients.

The construction of a skew ratio semiring of right [left] quotients of a semiring S is the same as the construction of a skew semifield of right [left] quotients of S, so we have the following theorem.

Theorem 2.22. Let S be a semiring without a multiplicative zero. Then a skew ratio semiring of right [left] quotients of S exists iff

(i) S is multiplicatively cancellative

and (ii) $(S,\cdot)$ satisfies the right [left] Ore condition.

Remark 2.23. Let S be a semiring having D as a skew ratio semiring of right [left] quotients. Then

1) if S is additively commutative the D is additively commutative.

2) if S is multiplicatively commutative then D is multiplicatively commutative.

3) if S is commutative then the construction of D is the same as the construction of the ratio semiring of quotients of S given by P.Sinutoke in [1].

4) if S is additively cancellative then D is additively commutative and additively cancellative.

Corollary 2.24. Let S be a semiring having D as a skew ratio semiring of right [left] quotients, $i : S \to D$ a right [left] quotient embedding, E a skew ratio semiring and $f : S \to E$ a homomorphism. Then there exists a unique homomorphism $g : D \to E$ such that $g \circ i = f$. Furthermore, if f is a monomorphism then g is a monomorphism.

Corollary 2.25. Let S be a semiring having D as a skew ratio semiring of right [left] quotients. If E is a skew ratio semiring and E contains an isomorphic copy of S then E contains an isomorphic copy of D.

Corollary 2.26. If S is a semiring having D and D' as skew ratio semirings of right or left quotients then $D \cong D'$.

Remark 2.27. Let S be a semiring having D as a skew ratio semiring of right [left] quotients. Then S is strongly multiplicatively cancellative iff D is precise.

Proof. Let $i : S \to D$ be a right quotient embedding.

Assume that S is strongly multiplicatively cancellative. To show that D is precise, let $\alpha, \beta \in D$ be such that $1 + \alpha\beta = \alpha + \beta$.

There exist $x,y,z,w \in S$ such that $\alpha = i(x)i(y)^{-1}$ and $\beta = i(z)i(w)^{-1}$. There exist $a,b,c,d \in S$ such that $ya = zb$ and $yc = wd$. Then $\alpha\beta = i(xa)i(wb)^{-1}$ and $\alpha + \beta = i(xc + zd)i(wd)^{-1}$. Thus $i(wb + xa)i(wb)^{-1} = 1 + \alpha\beta = \alpha + \beta = i(xc + zd)i(wd)^{-1}$. Hence there exist $e,f \in S$ such that $(wb + xa)e = (xc + zd)f$ and $wbe = wdf$, so $be = df$. Since $xcf + yae = xcf + zbe = xcf + zdf = wbe + xae = wdf + xae = ycf + xae$ and S is S.M.C., $x = y$ or $ae = cf$. If $x = y$ then $\alpha = 1$. Suppose that $ae = cf$. Then $zbe = yae = ycf = wdf$, thus $z = w$, hence $\beta = 1$. Similarly, if $1 + \alpha\beta = \beta + \alpha$ then $\alpha = 1$ or $\beta = 1$.

Conversely, assume that D is precise. To show that S is strongly multiplicatively cancellative it suffices to show that D is strongly multiplicatively cancellative. Let $\alpha,\beta,\gamma,\delta \in D$ be such that $\alpha\gamma + \beta\delta = \alpha\delta + \beta\gamma$. Then $1 + \alpha^{-1}\beta\delta\gamma^{-1} = \delta\gamma^{-1} + \alpha^{-1}\beta$, thus $\alpha^{-1}\beta = 1$ or $\delta\gamma^{-1} = 1$, so $\alpha = \beta$ or $\gamma = \delta$. Similarly, if $\alpha\gamma + \beta\delta = \beta\gamma + \alpha\delta$ then $\alpha = \beta$ or $\gamma = \delta$. Thus D is strongly multiplicatively cancellative and hence S is also.

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Theorem 2.28. Let $(R,\leqslant)$ be a partially ordered ring having K as a skew field of right [left] quotients such that for all $x,y \in R$ with $x > 0$ and $y > 0$ there exist $a,b \in R$ with $a > 0$ and $b > 0$ such that $xa = yb$ and $i : R \to K$ a right [left] quotient embedding. Then there exists a partial order $\leqslant^*$ on K such that $(K,\leqslant^*)$ is a partially ordered skew field and i is an increasing map iff $\leqslant$ is multiplicatively regular. Furthermore, if $\leqslant$ is total then $\leqslant^*$ is total.

Proof. Assume that $\leqslant$ is multiplicatively regular. Let $E = \{\alpha \in K | \alpha = i(x)i(y)^{-1}$ for some $x,y \in R$ such that $x \geqslant 0$ and $y > 0\}$. Define $\alpha \leqslant^* \beta$ iff $\beta - \alpha \in E$ for all $\alpha,\beta \in K$.

From now on, in the proof of this theorem, we shall denote α and $\beta \in K$ by $i(x)i(y)^{-1}$ and $i(z)i(w)^{-1}$, respectively, where $x,z \in R$ and $y,w \in R\setminus\{0\}$.

We must show that $\leqslant^*$ is a partial order on K. Clearly $\leqslant^*$ is reflexive. Let $\alpha,\beta \in K$ be such that $\alpha \leqslant^* \beta$ and $\beta \leqslant^* \alpha$. Then $\beta - \alpha$ and $\alpha - \beta \in E$. Let $\beta - \alpha = i(a)i(b)^{-1}$ and $\alpha - \beta = i(c)i(d)^{-1}$ where $a,c \geqslant 0$ and $b,d > 0$ in R. Since $i(c)i(d)^{-1} = i(-a)i(b)^{-1}$, by Remark 2.6 and the hypothesis, there exist $e,f > 0$ in R such that $-ae = cf$ and $be = df$. Since $-ae \geqslant 0$ and $\leqslant$ is M.R., $-a \geqslant 0$, so $a \leqslant 0$. Then $a = 0$ because $a \geqslant 0$ and $a \leqslant 0$. Thus $\alpha = \beta$. Hence $\leqslant^*$ is anti-symmetric. Claim that $\alpha + \beta \in E$ for all $\alpha,\beta \in E$. To prove this, let $\alpha,\beta \in E$. Then $x,z \geqslant 0$ and $y,w > 0$. There exist $a,b > 0$ in R such that $ya = wb$. Thus $(xa + zb) \geqslant 0$ and $ya > 0$, so $\alpha + \beta = i(xa + zb)i(ya)^{-1} \in E$, hence we have the claim. Let $\alpha,\beta,\gamma \in K$ be such that $\alpha \leqslant^* \beta$ and $\beta \leqslant^* \gamma$. Then $\beta - \alpha$ and $\gamma - \beta \in E$. Thus $\gamma - \alpha = (\gamma - \beta) + (\beta - \alpha) \in E$, so $\alpha \leqslant^* \gamma$. Hence $\leqslant^*$ is transitive. Therefore $\leqslant^*$ is a partial order on K.

Let $\alpha,\beta,\gamma \in K$ be such that $\alpha \leqslant^* \beta$. Then $(\beta + \gamma) - (\alpha + \gamma) = \beta - \alpha \in E$, so $(\alpha + \gamma) \leqslant^* (\beta + \gamma)$. Claim that $\alpha\beta \in E$ for all $\alpha,\beta \in E$. To prove this, let $\alpha,\beta \in E$. Then $x,z \geqslant 0$ and $y,w > 0$. There exist $a \geqslant 0$ and $b > 0$ in R such that $ya = zb$. Thus $xa \geqslant 0$ and $wb > 0$, so $\alpha\beta = i(xa)i(wb)^{-1} \in E$. Hence we have the claim. Let $\alpha,\beta \in K$ be such that $\alpha \leqslant^* \beta$. Then $\beta - \alpha \in E$. Let $\gamma \in E$. Then $\beta\gamma - \alpha\gamma = (\beta - \alpha)\gamma \in E$ and $\gamma\beta - \gamma\alpha = \gamma(\beta - \alpha) \in E$, so $\alpha\gamma \leqslant^* \beta\gamma$ and $\gamma\alpha \leqslant^* \gamma\beta$. Hence $(K,\leqslant^*)$ is a partially ordered skew field.

To show that i is an increasing map, let $a,b,c \in R$ be such that $a < b$ and $c > 0$. Then $(b - a)c > 0$, so $i(b) - i(a) = i((b - a)c)i(c)^{-1} \in E$. Hence $i(a) <^* i(b)$. Claim that if $\alpha = i(p)i(q)^{-1}$ where $p,q \in R$ such that $q > 0$ and $\alpha \in E$ then $p \geqslant 0$. Since $\alpha \in E$, $x \geqslant 0$ and $y > 0$.

There exist $g,h > 0$ in R such that $xg = ph$ and $yg = qh$, so $ph \geqslant 0$. Thus $p \geqslant 0$ because $\leqslant$ is M.R.. Hence we have the claim. Let $a,b,c \in R$ be such that $i(a) <^* i(b)$ and $c > 0$. Then $i((b - a)c)i(c)^{-1} = i(b) - i(a) \in E$, so $(b - a)c \geqslant 0$, hence $b - a > 0$, thus $a < b$. Hence i is an increasing map.

Conversely, assume that there exists a partial order $\leqslant^*$ on K such that $(K,\leqslant^*)$ is a partially ordered skew field and i is an increasing map. Let $a,b,c \in R$ be such that $ac \leqslant bc$ and $c > 0$. Then $i(a)i(c) \leqslant^* i(b)i(c)$, so $0 \leqslant^* (i(b) - i(a))i(c)$ and $0 <^* i(c)$. Thus $0 \leqslant^* i(b) - i(a)$, so $i(a) \leqslant^* i(b)$. Hence $a \leqslant b$. Similarly, if $ca \leqslant cb$ and $c > 0$ then $a \leqslant b$. Therefore $\leqslant$ is multiplicatively regular.

Furthermore, assume that $\leqslant$ is total. Let $\alpha,\beta \in K$. There exist $a,b \in R\setminus\{0\}$ such that $ya = wb$. Then $((zb - xa) \leqslant 0$ or $(zb - xa) \geqslant 0)$ and $(ya < 0$ or $ya > 0)$.

<u>Case 1</u> $(zb - xa) \leqslant 0$ and $ya < 0$. Thus $(xa - zb) \geqslant 0$ and $-ya > 0$, so $\beta - \alpha = i(zb - xa)i(ya)^{-1} = i(xa - zb)i(-ya)^{-1} \in E$. Hence $\alpha \leqslant^* \beta$.

<u>Case 2</u> $(zb - xa) \leqslant 0$ and $ya > 0$. Thus $(xa - zb) \geqslant 0$, so $\alpha - \beta = i(xa - zb)i(ya)^{-1} \in E$. Hence $\beta \leqslant^* \alpha$.

<u>Case 3</u> $(zb - xa) \geqslant 0$ and $ya < 0$. Thus $-ya > 0$, so $\alpha - \beta = i(xa - zb)i(ya)^{-1} = i(zb - xa)i(-ya)^{-1} \in E$. Hence $\beta \leqslant^* \alpha$.

<u>Case 4</u> $(zb - xa) \geqslant 0$ and $ya > 0$. Thus $\beta - \alpha = i(zb - xa)i(ya)^{-1} \in E$, so $\alpha \leqslant^* \beta$. Hence $\alpha \leqslant^* \beta$ or $\beta \leqslant^* \alpha$. Therefore $\leqslant^*$ is total.

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<u>Corollary 2.29</u>. Let $(R,\leqslant)$ be a partially ordered ring having K as a field of quotients and $i : R \to K$ a quotient embedding. Then there exists a partial order $\leqslant^*$ on K such that $(K,\leqslant^*)$ is a partially ordered field and i is an increasing map iff $\leqslant$ is multiplicatively regular. Furthermore, if $\leqslant$ is total then $\leqslant^*$ is total.

<u>Theorem 2.30</u>. Let $(S,\leqslant)$ be a partially ordered semiring having D as a skew ratio semiring of right [left] quotients and $i : S \to D$ a right [left] quotient embedding. Then there exists a unique partial order $\leqslant^*$ on D such that $(D,\leqslant^*)$ is a partially ordered skew ratio semiring and i is an increasing map iff $\leqslant$ is multiplicatively regular. Furthermore, if $\leqslant$ is total then $\leqslant^*$ is total.

<u>Proof</u>. Assume that $\leqslant$ is multiplicatively regular. Let $E = \{\alpha \in D \mid \alpha = i(x)i(y)^{-1}$ for some $x,y \in S$ such that $y \leqslant x\}$. It follows from the fact that $\leqslant$ is multiplicatively regular that if $\alpha = i(p)i(q)^{-1}$ where $p,q \in S$ and $\alpha \in E$ then $q \leqslant p$. Define a relation $\leqslant^*$ on D by $\alpha \leqslant^* \beta$ iff $\beta\alpha^{-1} \in E$ for all $\alpha,\beta \in D$.

From now on, in the proof of this theorem, we shall denote α,β and $\gamma \in D$ by $i(x)i(y)^{-1}, i(z)i(w)^{-1}$ and $i(u)i(v)^{-1}$, respectively, where $x,y,z,w,u,v \in S$.

We must show that $\leqslant^*$ is a partial order on D. Clearly $\leqslant^*$ is reflexive. Let $\alpha,\beta \in D$ be such that $\alpha \leqslant^* \beta$ and $\beta \leqslant^* \alpha$. Let $\beta\alpha^{-1} = i(p)i(q)^{-1}$ and $\alpha\beta^{-1} = i(m)i(n)^{-1}$ where $p,q,m,n \in S$. Thus $q \leqslant p$, $n \leqslant m$ and $i(p)i(q)^{-1} = i(n)i(m)^{-1}$. There exist $a,b \in S$ such that $pa = nb$ and $qa = mb$. Suppose that $q < p$. Then $mb = qa < pa = nb$, so $m < n$ because $\leqslant$ is M.R, we have a contradiction. Thus $p = q$, so $\beta\alpha^{-1} = 1$. Hence $\alpha = \beta$, so $\leqslant^*$ is anti-symmetric. Claim that if $\alpha,\beta \in E$ then $\alpha\beta \in E$. To prove this, let $\alpha,\beta \in E$. Then $y \leqslant x$ and $w \leqslant z$. There exist $a,b \in S$ such that $ya = zb$. Since $wb \leqslant zb = ya \leqslant xa$, $\alpha\beta = i(xa)i(wb)^{-1} \in E$, so we have the claim. Let α,β and $\gamma \in D$ be such that $\alpha \leqslant^* \beta$ and $\beta \leqslant^* \gamma$. So $\beta\alpha^{-1}, \gamma\beta^{-1} \in E$. Thus $\gamma\alpha^{-1} = \gamma\beta^{-1}\beta\alpha^{-1} \in E$, so $\alpha \leqslant^* \gamma$. Hence $\leqslant^*$ is transitive. Therefore $\leqslant^*$ is a partial order on D.

Let α,β and $\gamma \in D$ be such that $\alpha \leqslant^* \beta$. We must show that $(\alpha + \gamma) \leqslant^* (\beta + \gamma)$ and $(\gamma + \alpha) \leqslant^* (\gamma + \beta)$. There exist $a,b \in S$ such that $ya = vb$, so $\alpha + \gamma = i(xa + ub)i(ya)^{-1}$. There exist $c,d \in S$ such that $wc = vd$, so $\beta + \gamma = i(zc + ud)i(wc)^{-1}$. There exist $e,f \in S$ such that $wce = yaf$, so $(\beta + \gamma)(\alpha + \gamma)^{-1} = i((zc + ud)e)i((xa + ub)f)^{-1}$. We must show that $(\beta + \gamma)(\alpha + \gamma)^{-1} \in E$. Since $vbf = yaf = wce = vde$, $bf = de$. Since $\alpha \leqslant^* \beta$ and $\beta\alpha^{-1} = i(zce)i(xaf)^{-1}$, $(xa + ub)f = (xaf + ubf) \leqslant (zce + ude) = (zc + ud)e$, so $(\beta + \gamma)(\alpha + \gamma)^{-1} \in E$. Hence $(\alpha + \gamma) \leqslant^* (\beta + \gamma)$. Similarly, $(\gamma + \alpha) \leqslant^* (\gamma + \beta)$.

Claim that $\beta\alpha\beta^{-1} \in E$ for all $\alpha \in E$, $\beta \in D$. To prove this, let $\alpha \in E$ and $\beta \in D$. Then there exist $a,b \in S$ such that $wa = xb$, so $\beta\alpha = i(za)i(yb)^{-1}$. There exist $c,d \in S$ such that $ybc = wd$, so $\beta\alpha\beta^{-1} = i(zac)i(zd)^{-1}$. It follows from $\alpha \in E$ that $wd = ybc \leqslant xbc = wac$, so $d \leqslant ac$. Thus $zd \leqslant zac$, hence $\beta\alpha\beta^{-1} \in E$, so we have the claim.

Let α,β and $\gamma \in D$ be such that $\alpha \leqslant^* \beta$. Then $\beta\alpha^{-1} \in E$. We must show that $\alpha\gamma \leqslant^* \beta\gamma$ and $\gamma\alpha \leqslant^* \gamma\beta$. Since $(\beta\gamma)(\alpha\gamma)^{-1} = \beta\alpha^{-1} \in E$, $\alpha\gamma \leqslant^* \beta\gamma$. Since $(\gamma\beta)(\gamma\alpha)^{-1} = \gamma\beta\alpha^{-1}\gamma^{-1} \in E$, $\gamma\alpha \leqslant^* \gamma\beta$. Hence $(D,\leqslant^*)$ is a partially ordered skew ratio semiring

To show that i is an increasing map, let $a,b \in S$. Then $a \leqslant b$ iff $i(b)i(a)^{-1} \in E$ and $i(a) \leqslant^* i(b)$ iff $i(b)i(a)^{-1} \in E$, so $a \leqslant b$ iff $i(a) \leqslant^* i(b)$. Hence $a < b$ iff $i(a) <^* i(b)$.

To show that $\leqslant^*$ is unique, suppose that there exists a partial order $\leqslant^{**}$ on D such that $(D,\leqslant^{**})$ is a partially ordered skew ratio semiring and i is an increasing map. Let $\alpha,\beta \in D$ be such that $\alpha \leqslant^* \beta$. There exist $a,b \in S$ such that $ya = wb$, so $\beta\alpha^{-1} = i(zb)i(xa)^{-1} \in E$. Thus $xa \leqslant zb$, so $i(xa) \leqslant^{**} i(zb)$. Then $\alpha = i(xa)i(ya)^{-1} \leqslant^{**} i(zb)i(wb)^{-1} = \beta$. Suppose that $\alpha \leqslant^{**} \beta$. Then $i(xa) = i(x)i(y)^{-1}i(ya) \leqslant^{**} i(z)i(w)^{-1}i(wb) = i(zb)$, so $xa \leqslant zb$. Thus $\beta\alpha^{-1} = i(zb)i(xa)^{-1} \in E$, so $\alpha \leqslant^* \beta$. Hence $<^* = <^{**}$.

Conversely, assume that there exists a unique partial order $\leqslant^*$ on D such that $(D,\leqslant^*)$ is a partially ordered skew ratio semiring and i is an increasing map. Let $x,y,z \in S$ be such that $xz \leqslant yz$. Then $i(xz) \leqslant^* i(yz)$, thus $i(x) = i(xz)i(z)^{-1} \leqslant^* i(yz)i(z)^{-1} = i(y)$ which implies that $x \leqslant y$. Similarly, if $zx \leqslant zy$ then $x \leqslant y$. Therefore $\leqslant$ is multiplicatively regular.

Furthermore, assume that $\leqslant$ is total. To show that $\leqslant^*$ is total, let $\alpha,\beta \in D$. Then there exist $a,b \in S$ such that $ya = wb$. Since $\leqslant$ is total, $xa \leqslant zb$ or $zb \leqslant xa$, hence $\beta\alpha^{-1} = i(zb)i(xa)^{-1} \in E$ or $\alpha\beta^{-1} = i(xa)i(zb)^{-1} \in E$. Thus $\alpha \leqslant^* \beta$ or $\beta \leqslant^* \alpha$, so $\leqslant^*$ is total.

#

<u>Corollary 2.31.</u> Let $(S,\leqslant)$ be a partially ordered semiring having D as a ratio semiring of quotients and $i : S \to D$ a quotient embedding. Then there exists a unique partial order $\leqslant^*$ on D such that $(D,\leqslant^*)$ is a partially ordered ratio semiring and i is an increasing map iff $\leqslant$ is multiplicatively regular. Furthermore, if $\leqslant$ is total then $\leqslant^*$ is total.

<u>Theorem 2.32.</u> Let S be a semiring having D as a skew ratio semiring of right [left] quotients, $i : S \to D$ a right [left] quotient embedding, A the lower semilattice of multiplicatively regular partial orders $\leqslant$ on S such that $(S,\leqslant)$ is a partially ordered semiring and B the lower semilattice of partial orders $\leqslant^*$ on D such that $(D,\leqslant^*)$ is a partially ordered skew ratio semiring and i is an increasing map. Then there exists an order isomorphism between A and B.

<u>Proof.</u> Define a map $f : A \to B$ in the following way : Let $\leqslant \in A$. Then Theorem 2.30 determines a unique $\leqslant^* \in B$. Define $f(\leqslant) = \leqslant^*$.

To show that f is an injection, let $\leqslant_1, \leqslant_2 \varepsilon A$ be such that $f(\leqslant_1) = f(\leqslant_2)$. Then $\leqslant_1^* = \leqslant_2^*$. Let $x,y \varepsilon S$. Then $x \leqslant_1 y$ iff $i(x) \leqslant_1^* i(y)$ and $x \leqslant_2 y$ iff $i(x) \leqslant_2^* i(y)$, so $x \leqslant_1 y$ iff $x \leqslant_2 y$. Hence $\leqslant_1 = \leqslant_2$. To show that f is a surjection, let $\leqslant' \varepsilon B$. Define a relation $\leqslant$ on S by $x \leqslant y$ iff $i(x) \leqslant' i(y)$ for all $x,y \varepsilon S$. Clearly $\leqslant \varepsilon A$. By the uniqueness, $\leqslant' = \leqslant^*$, so $f(\leqslant) = \leqslant'$. Hence f is a bijection.

Let $\leqslant_1, \leqslant_2 \varepsilon A$ be such that $\leqslant_1 \subseteq \leqslant_2$. To show that $f(\leqslant_1) \subseteq f(\leqslant_2)$, let $\alpha,\beta \varepsilon D$ be such that $\alpha \leqslant_1^* \beta$. Then $\alpha = i(x)i(y)^{-1}$ and $\beta = i(z)i(w)^{-1}$ where $x,y,z,w \varepsilon S$. There exist $a,b \varepsilon S$ such that $wa = yb$. Thus $xb \leqslant_1 za$, so $xb \leqslant_2 za$, hence $\alpha \leqslant_2^* \beta$. Therefore $f(\leqslant_1) = \leqslant_1^* \subseteq \leqslant_2^* = f(\leqslant_2)$.

Let $\leqslant_1^*, \leqslant_2^* \varepsilon B$ be such that $\leqslant_1^* \subseteq \leqslant_2^*$. To show that $f^{-1}(\leqslant_1^*) \subseteq f^{-1}(\leqslant_2^*)$, let $x,y \varepsilon S$ be such that $x \leqslant_1 y$. Then $i(x) \leqslant_1^* i(y)$, so $i(x) \leqslant_2^* i(y)$, hence $x \leqslant_2 y$. Therefore $f^{-1}(\leqslant_1^*) = \leqslant_1 \subseteq \leqslant_2 = f^{-1}(\leqslant_2^*)$. Hence f is an order isomorphism.

#

Corollary 2.33. Let $(S,\leqslant)$ be a semiring having D as a ratio semiring of quotients, $i : S \to D$ a quotient embedding, A the lower semilattice of multiplicatively regular partial orders $\leqslant$ on S such that $(S,\leqslant)$ is a partially ordered semiring and B the lower semilattice of partial orders $\leqslant^*$ on D such that $(D,\leqslant^*)$ is a partially ordered ratio semiring and i is an increasing map. Then there exists an order isomorphism between A and B.

Theorem 2.34. Let S be a semiring having K as a skew semifield of right [left] quotients, $i : S \to K$ a right [left] quotient embedding and ρ a congruence on S. Then there exists a unique congruence ρ^* on K such that ($i(x)\ \rho^*\ i(y)$ iff $x\ \rho\ y$ for all $x,y \varepsilon S$) iff ρ is multiplicatively regular.

Proof. Assume that ρ is multiplicatively regular. Define a relation ρ^* on K in the following way : Let $\alpha,\beta \in K$. Then $\alpha = i(x)i(y)^{-1}$ and $\beta = i(z)i(w)^{-1}$ where $x,z \in S$ and $y,w \in S\setminus\{0\}$. Define $\alpha\ \rho^*\ \beta$ iff there exist $a,b \in S\setminus\{0\}$ such that $xa\ \rho\ zb$ and $ya\ \rho\ wb$.

To show that ρ^* is well-defined, suppose that there exist $x',z' \in S$ and $y',w' \in S\setminus\{0\}$such that $\alpha = i(x')i(y')^{-1}$ and $\beta = i(z')i(w')^{-1}$. There exist $p,q,m,n \in S\setminus\{0\}$ such that $xp = x'q$, $yp = y'q$, $zm = z'n$ and $wm = w'n$. There exist $c,d,e,f \in S\setminus\{0\}$ such that $pc = ad$ and $bde = mf$. Thus $x'qce = xpce = xade$ and $z'nf = zmf = zbde$. Since $xa\ \rho\ zb$, $xade\ \rho\ zbde$, so $x'qce\ \rho\ z'nf$. Similarly, $y'qce\ \rho\ w'nf$. Hence ρ^* is well-defined.

From now on, in the proof of this theorem, we shall denote α,β and $\gamma \in K$ by $i(x)i(y)^{-1}$, $i(z)i(w)^{-1}$and $i(u)i(v)^{-1}$, respectively, where $x,z,u \in S$ and $y,w,v \in S\setminus\{0\}$.

We must show that ρ^* is an equivalence relation on K. Clearly ρ^* is reflexive and symmetric. Let α,β and $\gamma \in K$ be such that $\alpha\ \rho^*\ \beta$ and $\beta\ \rho^*\ \gamma$. Then there exist $a,b,c,d \in S\setminus\{0\}$ be such that $xa\ \rho\ zb$, $ya\ \rho\ wb$, $zc\ \rho\ ud$ and $wc\ \rho\ vd$. There exist $e,f \in S\setminus\{0\}$ such that $be = cf$. Since $xae\ \rho\ zbe$ and $zcf\ \rho\ udf$, $xae\ \rho\ udf$. Similarly, $yae\ \rho\ vdf$. Thus $\alpha\ \rho^*\ \gamma$, so ρ^* is transitive. Hence ρ^* is an equivalence relation on K.

To show that ρ^* is a congruence on K, let α,β and $\gamma \in K$ be such that $\alpha\ \rho^*\ \beta$. Then there exist $a,b \in S\setminus\{0\}$ such that $xa\ \rho\ zb$ and $ya\ \rho\ wb$. First, we shall show that $(\alpha + \gamma)\ \rho^*\ (\beta + \gamma)$ and $(\gamma + \alpha)\ \rho^*\ (\gamma + \beta)$. There exist $c,d \in S\setminus\{0\}$ such that $yc = vd$, so $\alpha + \gamma = i(xc + ud)i(yc)^{-1}$. There exist $e,f \in S\setminus\{0\}$ such that $we = vf$, so $\beta + \gamma = i(ze + uf)i(we)^{-1}$. There exist $g,h \in S\setminus\{0\}$ such that $ag = ch$. There exist $p,q \in S\setminus\{0\}$ such that $bgp = eq$. Thus $xchp = xagp$,

$zeq = zbgp$ and $xagp \, \rho \, zbgp$, so $xchp \, \rho \, zeq$. Similarly, $ychp \, \rho \, weq$. Since $vdhp = ychp$ and $vfq = weq$, $vdhp \, \rho \, vfq$. Since ρ is M.R., $dhp \, \rho \, fq$, so $udhp \, \rho \, ufq$. Then $(xc + ud)hp = (xchp + udhp) \, \rho$ $(zeq + ufq) = (ze + uf)q$ and $ychp \, \rho \, weq$, so $(\alpha + \gamma) \, \rho^* \, (\beta + \gamma)$. Similarly, $(\gamma + \alpha) \, \rho^* \, (\gamma + \beta)$.

Next, we shall show that $\alpha\gamma \, \rho^* \, \beta\gamma$. This is clear if $\gamma = 0$. Suppose that $\gamma \neq 0$, so $u \neq 0$. There exist $c,d,e,f \in S\backslash\{0\}$ such that $yc = ud$ and $we = uf$, so $\alpha\gamma = i(xc)i(vd)^{-1}$ and $\beta\gamma = i(ze)i(vf)^{-1}$. There exist $g,h,p,q \in S\backslash\{0\}$ such that $cg = ah$ and $bhp = eq$. Since $xcgp = xahp$, $zeq = zbhp$ and $xahp \, \rho \, zbhp$, $xcgp \, \rho \, zeq$. Since $udgp = ycgp = yahp$, $ufq = weq = wbhp$ and $yahp \, \rho \, wbhp$, $udgp \, \rho \, ufq$. Since ρ is M.R., $dgp \, \rho \, fq$, so $vdgp \, \rho \, vfq$. Hence $\alpha\gamma \, \rho^* \, \beta\gamma$. Similarly, $\gamma\alpha \, \rho^* \, \gamma\beta$. Thus ρ^* is a congruence on K.

Let $a,b \in S$ be arbitary and $c \in S\backslash\{0\}$. Suppose that $i(a) \, \rho^* \, i(b)$. Then $i(ac)i(c)^{-1} \, \rho^* \, i(bc)i(c)^{-1}$. Thus there exist $p,q \in S\backslash\{0\}$ such that $acp \, \rho \, bcq$ and $cp \, \rho \, cq$. There exist $m,n \in S\backslash\{0\}$ such that $pm = qn$, so $cpm \, \rho \, cpn$, hence $m \, \rho \, n$. Then $acpm \, \rho \, bcqn$, so $a \, \rho \, b$. Suppose that $a \, \rho \, b$. Then $acc \, \rho \, ycc$ and $cc \, \rho \, cc$, so $i(ac)i(c)^{-1} \, \rho^* \, i(bc)i(c)^{-1}$, hence $i(a) \, \rho^* \, i(b)$. Therefore $i(a) \, \rho^* \, i(b)$ iff $a \, \rho \, b$ for all $a,b \in S$.

To show that ρ^* is unique, suppose that there exists a congruence ρ^{**} on K such that $i(x) \, \rho^{**} \, i(y)$ iff $x \, \rho \, y$ for all $x,y \in S$. Let $\alpha,\beta \in K$ be arbitary. Suppose that $\alpha \, \rho^* \, \beta$. Then there exist $a,b \in S\backslash\{0\}$ such that $xa \, \rho \, zb$ and $ya \, \rho \, wb$. Thus $i(xa) \, \rho^{**} \, i(zb)$ and $i(ya) \, \rho^{**} \, i(wb)$, so $i(ya)^{-1} = i(ya)^{-1}i(wb)i(wb)^{-1} \, \rho^{**} i(ya)^{-1}i(ya)i(wb)^{-1} = i(wb)^{-1}$. Hence $\alpha = i(xa)i(ya)^{-1} \, \rho^{**} \, i(zb)i(wb)^{-1} = \beta$. Suppose that $\alpha \, \rho^{**} \, \beta$. There exist $c,d \in S\backslash\{0\}$ such that $yc = wd$, so $i(yc) = i(wd)$. Since $i(xc) = i(x)i(y)^{-1}i(yc)$, $i(zd) = i(z)i(w)^{-1}i(wd)$ and $\alpha i(yc) \, \rho^{**} \, \beta i(wd)$,

$i(xc)\ \rho^{**}\ i(zd)$, so $xc\ \rho\ zd$. Hence $\alpha\ \rho^{*}\ \beta$. Hence $\rho^{*} = \rho^{**}$.

Conversely, assume that there exists a unique congruence ρ^{*} on K such that $i(x)\ \rho^{*}\ i(y)$ iff $x\ \rho\ y$ for all $x,y \in S$. To show that ρ is multiplicatively regular, let $x,y,z \in S$ be such that $xz\ \rho\ yz$ and $z \neq 0$. Then $i(xz)\ \rho^{*}\ i(yz)$, so $i(x) = i(xz)i(z)^{-1}\ \rho^{*}\ i(yz)i(z)^{-1} = i(y)$, thus $x\ \rho\ y$. Similarly, if $zx\ \rho\ zy$ and $z \neq 0$ then $x\ \rho\ y$. Hence we have the theorem.

#

Corollary 2.35. Let S be a semiring having K as a semifield of quotients, $i : S \to K$ a quotient embedding and ρ a congruence on S. Then there exists a unique congruence ρ^{*} on K such that ($i(x)\ \rho^{*}\ i(y)$ iff $x\ \rho\ y$ for all $x,y \in S$) iff ρ is multiplicatively regular.

Corollary 2.36. Let R be a ring having K as a skew field of right [left] quotients. Then R has only two multiplicatively regular congruences (since a skew field has only two ideals).

Corollary 2.37. Let R be a ring having K as a field of quotients. Then R has only two multiplicatively regular congruences.

Theorem 2.38. Let S be a semiring having K as a skew semifield of right [left] quotients, $i : S \to K$ a right [left] quotient embedding, A the lattice of multiplicatively regular congruences on S and B the lattice of congruences on K. Then there exists an order isomorphism between A and B.

Proof. Define a map $f : A \to B$ in the following way : Let $\rho \in A$. Then Theorem 2.34 determines a unique $\rho^{*} \in B$. Define $f(\rho) = \rho^{*}$.

Let $\rho_1, \rho_2 \in A$ be such that $\rho_1 \subseteq \rho_2$. Let $\alpha, \beta \in K$ be such that $\alpha\ \rho_1^*\ \beta$. Then $\alpha = i(x)i(y)^{-1}$ and $\beta = i(z)i(w)^{-1}$ where $x,z \in S$ and $y,w \in S\setminus\{0\}$ and there exist $a,b \in S\setminus\{0\}$ such that $xa\ \rho_1\ zb$ and $ya\ \rho_1\ wb$. Thus $xa\ \rho_2\ zb$ and $ya\ \rho_2\ wb$, hence $\alpha\ \rho_2^*\ \beta$. Therefore $f(\rho_1) = \rho_1^* \subseteq \rho_2^* = f(\rho_2)$. A proof similar to the one given in Theorem 2.31 shows that the remainder of this theorem is true.

#

Corollary 2.39. Let S be a semiring having K as a semifield of quotients, $i : S \to K$ a quotient embedding, A the lattice of multiplicatively regular congruences on S and B the lattice of congruences on K. Then there exists an order isomorphism between A and B.

Theorem 2.40. Let S be a semiring having D as a skew ratio semiring of right [left] quotients, $i : S \to D$ a right [left] quotient embedding and ρ a congruence on S. Then there exists a unique congruence ρ^* on D such that ($i(x)\ \rho^*\ i(y)$ iff $x\ \rho\ y$ for all $x,y \in S$) iff ρ is multiplicatively regular.

Proof. The proof of this theorem is similar to the proof of Theorem 2.34.

#

Corollary 2.41. Let S be a semiring having D as a ratio semiring of quotients, $i : S \to D$ a quotient embedding and ρ a congruence on S. Then there exists a unique congruence ρ^* on D such that ($i(x)\ \rho^*\ i(y)$ iff $x\ \rho\ y$ for all $x,y \in S$) iff ρ is multiplicatively regular.

Theorem 2.42. Let S be a semiring having D as a skew ratio semiring of right [left] quotients, $i : S \to D$ a right [left] quotient embedding, A the lattice of multiplicatively regular congruences on S and B the lattice of congruences on D. Then there exists an order isomorphism between A and B.

Proof. The proof of this theorem is similar to the proof of Theorem 2.38. #

Corollary 2.43. Let S be a semiring having D as a ratio semiring of quotients, $i : S \to D$ a quotient embedding, A the lattice of multiplicatively regular congruences on S and B the lattice of congruences on D. Then there exists an order isomorphism between A and B.

The skew semifield K in Example 2.2 is not a skew field and the skew semifield K[[X]] in Example 2.15 is a skew field. This shows that the skew semifield K of right [left] quotients of a semiring may or may not be a skew field. We shall now give a necessary and sufficient condition on a semiring which quarantees that the skew semifield K of right [left] quotients is a skew field.

In [2] the concept of extensive was defined for commutative semirings with a multiplicative zero. We shall extend this concept to the noncommutative case.

Definition 2.44. Let S be a semiring with multiplicative zero 0. Then S is said to be right [left] extensive iff for all $x \in S$ there exist $a \in S, b \in S\setminus\{0\}$ such that $xb + a = 0$ [$a + bx = 0$].

Example 2.45. $\mathbb{Z}$ with the usual addition and multiplication is a right and left extensive semiring.

Theorem 2.46. Let S be an additively commutative semiring with a multiplicative zero 0 which is also an additive identity and K a skew semifield of right [left] quotients of S. Then K is a skew field iff S is right [left] extensive.

Proof. Let $i : S \to K$ be a right quotient embedding.

Assume that S is right extensive. To show that K is a skew field, it suffices to show that an additive inverse of α belongs to K for all $\alpha \in K$. Let $\alpha \in K$. Then $\alpha = i(x)i(y)^{-1}$ where $x \in S$ and $y \in S\setminus\{0\}$. Since S is right extensive, there exist $a \in S$ and $b \in S\setminus\{0\}$ such that $xb + a = 0$. Since $i(y)^{-1}$ and $i(b)^{-1}$ exist in K,
$\alpha + i(a)i(yb)^{-1} = i(xb)i(yb)^{-1} + i(a)i(yb)^{-1} = i(xb + a)i(yb)^{-1} = i(0)i(yb)^{-1} = 0$, hence $i(a)i(yb)^{-1}$ is an additive identity of α in K.

Conversely, assume that K is a skew field. To show that S is right extensive, let $x \in S$. Then $i(x) \in K$. There exist $a \in S$ and $b \in S\setminus\{0\}$ such that $i(x) + i(a)i(b)^{-1} = 0$. Thus
$i(xb + a) = (i(x) + i(a)i(b)^{-1})i(b) = i(0)$, so $xb + a = 0$.

#

Remark 2.47. In the case that K is a skew field we will call K a skew field of right [left] quotients of S.

Corollary 2.48. Let S be an additively commutative semiring of order > 1 with a multiplicative zero which is also an additive identity. Then a skew field of right [left] quotients of S exists iff

(i) S is multiplicatively cancellative,

(ii) $(S,\cdot)$ satisfies the right [left] Ore condition

and (iii) S is right [left] extensive.

Proof. It follows from Theorem 2.4 and Theorem 2.46.

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Corollary 2.49. Let S be an additively commutative semiring of order > 1 with a multiplicative zero which is also an additive identity satisfying properties (i) - (iii) of Corollary 2.48. Then S is additively cancellative.

Theorem 2.50. Every skew ratio semiring can be embedded into a skew semifield such that the multiplicative zero is an additive identity [zero].

Proof. Let D be a skew ratio semiring. Let 0 be a symbol not representing any element of D. Extend + and $\cdot$ from D to $D \cup \{0\}$ by $x \cdot 0 = 0 \cdot x = 0$ and $x + 0 = 0 + x = x$ $[x+0 = 0+x = 0]$ for all $x \in D \cup \{0\}$. It can be easily shown that $(D \cup \{0\},+,\cdot)$ is a skew semifield. Define $f : D \to D \cup \{0\}$ by $f(x) = x$ for all $x \in D$. Then f is a monomorphism. Hence we have the theorem.

#

Corollary 2.51. Let S be a multiplicatively cancellative semiring without a multiplicative zero such that $(S,\cdot)$ satisfies the right [left] Ore condition. Then S can be embedded into a skew semifield such that the multiplicative zero is an additive identity [zero].

Proof. It follows from Theorem 2.22 and Theorem 2.50.

#

Lemma 2.52. Let D be an additively cancellative skew ratio semiring and $|D| > 1$. Then $x + y \neq x$ and $y + x \neq x$ for all $x,y \in D$.

Proof. Without loss of generality, suppose that there exist $x,y \in D$ such that $x + y = x$. Then $x + y + x = x + x$, so $y + x = x$. Let $z \in D$ be arbitary. Then $z + y + x = z + x$, so $z + y = z$. Let

Let $w \in D$ be arbitary. Then $yw + yw = (y + y)w = yw = yw + y$, so $yw = y$. Similarly, $wy = y$ for all $w \in D$. Hence y is a multiplicative zero of D, a contradiction. #

Theorem 2.53. Let D be an additively cancellative skew ratio semiring. Then the skew semifield $D \cup \{0\}$ such that 0 is an additive identity is also additively cancellative.

Proof. Let $x,y,z \in D \cup \{0\}$ be such that $x + y = x + z$. We must show that $y = z$. If $x,y,z \in D$ then $y = z$. If one of x,y,z is 0 then we will consider the following cases :

Case 1 $x = 0$. Then $y = z$.

Case 2 $y = 0$ or $z = 0$. Without loss of generality, suppose that $y = 0$. Then $x = x + z$. By Lemma 2.52, $z = 0$.

Similarly, if $y + x = z + x$ then $y = z$ for all $x,y,z \in D \cup \{0\}$. Hence $D \cup \{0\}$ is additively cancellative. #

Proposition 2.54. Let K be an additively commutative skew semifield such that the multiplicative zero 0 is an additive identity. If there exists an $x \in K\setminus\{0\}$ such that x has an additive inverse, then every element in K has an additive inverse and K is a skew field.

Proof. Let $y \in K$. We must show that y has an additive inverse. If $y = 0$ then we are done because $0 + 0 = 0$. So assume that $y \neq 0$. Let z be an additive inverse of x. Thus $x + z = 0$, so $y + yx^{-1}z = yx^{-1}(x + z) = 0$. Hence $yx^{-1}z$ is an additive inverse of y. #

Proposition 2.55. Let K be an additively commutative skew semifield such that the multiplicative zero is an additive identity. If K is not a skew field then $(K\setminus\{0\},+,\cdot)$ is a skew ratio semiring.

Proof. It suffices to show that $x + y \in K\setminus\{0\}$ for all $x,y \in K\setminus\{0\}$. Let $x,y \in K\setminus\{0\}$. Suppose that $x + y = 0$. Then x is a nonzero element which has an additive inverse. By Proposition 2.54, K is a skew field, a contradiction. Hence $x + y \in K\setminus\{0\}$. Thus $(K\setminus\{0\},+,\cdot)$ is a skew ratio semiring. #