ตัวแบบสโทแคสติกสำหรับราคาทองคำและการประยุกต์ใช้เพื่อหาราคายุติธรรมของอนุพันธ์
ทองคำ

นาย นราธิป อิสรานุสรณ์

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต
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STOCHASTIC MODELING FOR GOLD PRICES AND ITS APPLICATION TO GOLD DERIVATIVE PRICING

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A Thesis Submitted in Partial Fulfillment of the Requirements
for the Degree of Master of Science Program in Computational Science
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นราธิป อิสรานุสรณ์ : ตัวแบบสโทแคสติกสำหรับราคาทองคำและการประยุกต์ใช้เพื่อหาราคายุติธรรมของอนุพันธ์ทองคำ. (STOCHASTIC MODELING FOR GOLD PRICES AND ITS APPLICATION TO GOLD DERIVATIVE PRICING) **อ. ที่ปรึกษาวิทยานิพนธ์หลัก : อ. ดร.คำรณ เมฆฉาย, อ. ที่ปรึกษาวิทยานิพนธ์ร่วม : อ. ดร. กิตติพัฒน์ วอง, อ. ดร. เสน่ห์ รุจิวรรณ, 42 หน้า**

ในวิทยานิพนธ์นี้ เราพัฒนาตัวแบบสโทแคสติกหนึ่งปัจจัย (one-factor) ที่อธิบายพฤติกรรมของราคาทองคำ โดยเรากำหนดราคาทองคำให้สอดคล้องกับสมการ extended Geometric Brownian Motion และอธิบายฤดูกาลของราคาทองคำ โดยพจน์ drift ประกอบด้วยอัตราผลตอบแทนความสะดวกที่กำหนดให้สอดคล้องกับสมการอนุพันธ์เชิงสามัญ นอกจากนี้เราได้หาสมการราคายุติธรรมของทองคำล่วงหน้าและออปชันที่มีทองคำเป็นสินค้าอ้างอิง ภายใต้สมมติฐานที่ว่า ไม่มีโอกาสค้ากำไรโดยไม่มีความเสี่ยงเกิดขึ้นในตลาด (no-arbitrage opportunity)

ภาควิชา.............คณิตศาสตร์............... ลายมือชื่อนิสิต...

สาขาวิชา...........วิทยาการคณนา........... ลายมือชื่อ อ.ที่ปรึกษาวิทยานิพนธ์หลัก.............

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In this thesis, we developed a one-factor model of stochastic behavior of gold prices. The gold prices are assumed to follow an extended Geometric Brownian Motion with a time-varying drift which describes seasonal variation in gold prices. The drift includes instantaneous convenience yields which follow an ordinary differential equation. Moreover, we derive closed-form solutions for no-arbitrage prices of gold futures and European gold options under the no-arbitrage assumptions.

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CHAPTER I
INTRODUCTION

Gold has been a valuable metal throughout the ages because of its versatility. It can be used in many applications. One of its primary uses is as jewelry and adornment. It is even used in aerospace, dentistry and electronics. People can also use gold as the standard value for the money of each country. Furthermore, gold is extremely important to the development of industry and technology of each country because it is considered as a liquid asset and has low fluctuations. Therefore it is used as an assurance when issuing bank notes. We can see that in many countries they have gold for a guaranteed source of investment funds. Moreover, gold investment is a good way for getting high return in the long run. Nowadays, the investment in commodity derivatives markets which have gold as an underlying asset is receiving widespread attention. Gold derivatives markets have witnessed a tremendous growth in recent years. In order to provide closed-form solutions for gold derivatives such as futures and options, one can treat gold spot prices as a random walk. In other words, gold spot prices are assumed to follow a stochastic differential equation (SDE). Using the SDE theory, the closed-form solutions can be obtained by solving the associated partial differential equations.

By considering gold as one of those commodities, one can choose a model proposed by Brennan-Schwartz (1985) [1] to describe the dynamics of gold spot prices. To follow the model, gold spot prices are assumed to follow a Geometric Brownian Motion (GBM) and the convenience yield, which is an important factor influencing commodity prices, is described in the same way as a dividend yield. Nevertheless, this specification is inappropriate because it does not take into account the mean-reversion property of commodity prices. Schwartz (1997) [2] introduced variation of this model in which the convenience yield is mean reverting and intervenes in the commodity price dynamics. Besides the mean reversion property of commodity prices, the other main empirical characteristic that makes commodities noticeable difference from stocks, bonds and other financial assets is seasonality in prices. Many commodities, such as agricultural commodities or natural gas, exhibit seasonality in prices, due to harvest cycles in the case of agricultural commodities and change consumptions in the case of natural gas. In addition, gold also have seasonal variation in prices as well (see Chapter 3).

In this thesis, we developed a one-factor model of stochastic behavior gold prices, which is an extension of Schwartz model 1 [2] in the following form.

The Model

$$dS_t = (r - \delta(t))S_t dt + \sigma S_t dW_t, \quad S_0 = s_0 > 0, \tag{1.1}$$

$$\frac{d\delta(t)}{dt} = \kappa(\alpha(t) - \delta(t)), \quad \delta(0) = \delta_0, \tag{1.2}$$

$$\alpha(t) = \alpha_0 + \alpha_1 \sin(2\pi(t - t_\alpha)), \tag{1.3}$$

The first factor is the gold price S_t which follows an extended Geometric Brownian Motion with a time-varying drift which describes seasonal variation in gold prices. The drift includes an instantaneous convenience yield $\delta(t)$ which follows an ordinary differential equation (ODE). The function $\alpha(t)$ represents seasonal variation in convenience yields.

The No-Arbitrage Assumptions

1. The market is arbitrage-free, that is for any portfolio $\varphi = (\varphi_i),\ V_\varphi(0) = 0$ and $V_\varphi(T) \geq 0,\ \mathrm{P}-$a.s. for all time $T > 0$ imply $V_\varphi(T) = 0,\ \mathrm{P}-$a.s., where $V_\varphi(t) \equiv V_\varphi(t, S_t, \varphi_t)$ denotes the value of the portfolio φ at time t and P denotes an original probability measures. Namely, if a portfolio requires a null investment and is riskless (there is no possible loss at the time horizon T), then its terminal value at time T has to be zero.
2. The market participants are subject to no transaction costs when they trade.
3. The market participants are subject to the same or no tax rate on all net trading profits.
4. The market participants can borrow/lend money at the same risk-free rate of interest.

The aim of this thesis is twofold.

(I) To derive a closed-form solution for no-arbitrage gold futures prices (fair-prices).

(II) To derive a closed-form solution for no-arbitrage European options for gold.

The remaining of this dissertation is organized as follows.

In Chapter 2, we gave some background knowledge to develop a one-factor model of stochastic behavior of gold prices. The gold prices are assumed to follow an extended Geometric Brownian Motion with a time-varying drift which describes seasonal variation in gold prices. The drift includes instantaneous convenience yields which follow an ordinary differential equation. In addition, we also provide the solutions and simulating of instantaneous convenience yield and gold prices process.

In Chapter 3, we gave some basic concepts for both futures and European options contract. Moreover, we summarized the necessary theorems relating to obtain the no-arbitrage gold futures price and European options written on a gold contract. Finally, we illustrated the evolutions of the gold futures and the European gold option prices.

In Chapter 4, we concluded the result of the thesis.

CHAPTER II
STOCHASTIC MODELING FOR GOLD PRICES

In this Chapter, we provided some background knowledge to develop a one-factor model of stochastic behavior of gold prices. The gold prices are assumed to follow an extended Geometric Brownian Motion with a time-varying drift which describes seasonal variation in gold prices. The drift includes instantaneous convenience yields which follow an ordinary differential equation. In addition, we also provided the solutions and simulating of instantaneous convenience yield and gold prices process.

2.1 Mean reversion property

Mean reversion is one of the main mathematical methodology. That is sometimes used for investments such as stock, rice, gold, etc, and it can also be applied to some other processes. In general terms, the idea of high and low prices are temporary, when the current price is less than the average price, the stock is considered attractive for purchase, with the expectation that the price will rise. Similarly, when the current price is above the average price, the price is expected to fall. In other words, deviations from the average price fluctuates around their long-run mean. It means that they conduce to mean-revert to a level which may be viewed as a marginal cost of production.

2.2 Instantaneous convenience yields

Brennan and Schwartz's article [7] fixes the definition of the convenience yield that the convenience yield is referred to as the flow of services that accrues to an owner of the physical commodity but not to the owner of a contract for future delivery of the commodity. Recognizing the time lost and the costs incurred in transporting a commodity from one location to another, the convenience yield may be thought of as the value of being able to profit from temporary local shortages of the commodity through ownership of the physical commodity. The profit may arise either from local price variations or from the ability to maintain a production process as a result of ownership of an inventory of raw material.

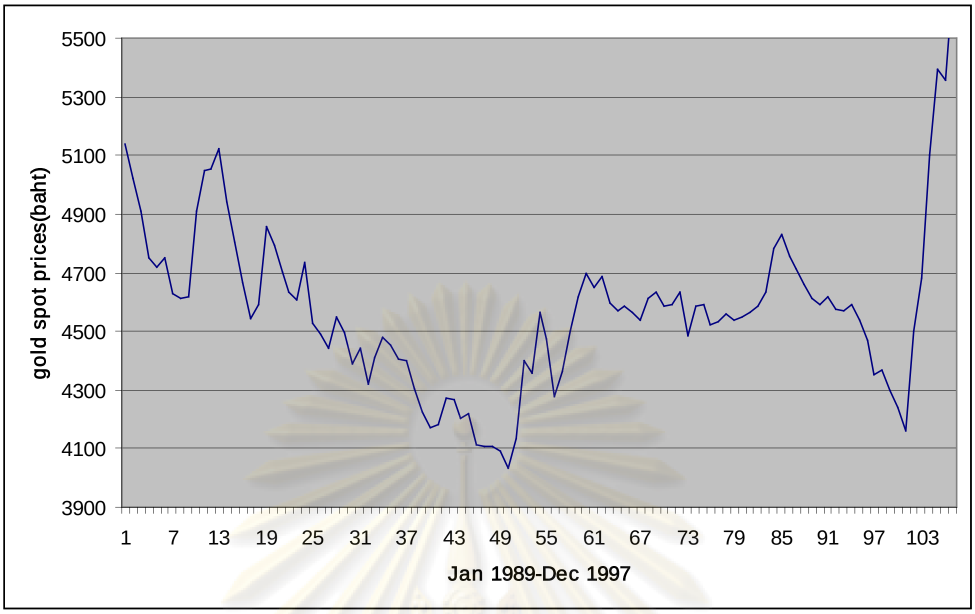
2.3 An equivalent martingale measure $\mathbb{Q}$

In financial mathematics, an equivalent martingale measure $\mathbb{Q}$ (a risk-neutral measure) is one kind of probability measure that results when one assumes that the current value of all financial assets is equal to the expected value of the future payoff of the assets discounted at the risk-free rate. This knowledge is used in the pricing of derivatives.

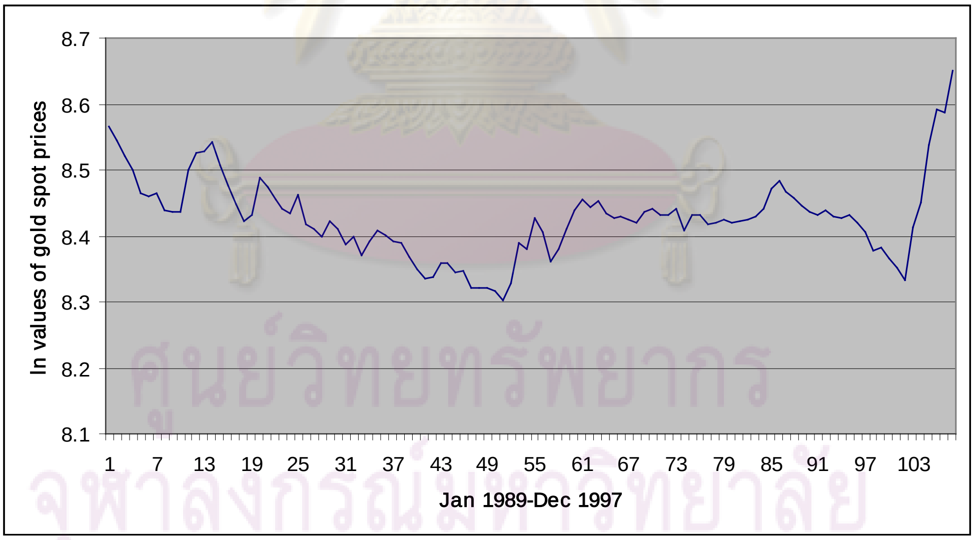
2.4 Seasonality in gold prices

Besides the mean reversion property of commodity prices, the other main empirical characteristic that makes commodities noticeable different from stocks, bonds and other financial assets, is seasonality in prices. Many commodities, such as agricultural commodities or natural gas, exhibit seasonality in prices, due to harvest cycles in the case of agricultural commodities and change consumptions as a result of weather patterns in the case of natural gas. In addition, gold also have seasonal variation in prices as well (Figure 2.1). We use the gold spot prices data [6] during January 1989 to December 1997 to observe the tendency of gold spot prices as shown in Figure 2.1.

In Figure 2.1, we see the season variation of gold prices, namely, the gold spot prices vary up and down annually, up in the beginning or end of the years and down in the middles. This motivated the idea that a model of gold spot prices should take into account a seasonal variation which behave like a wave.



(a)



(b)

Figure 2.1: (a) Monthly averages of gold spot prices over nine years

(b) Natural logarithm of data in (a)

2.5 Stochastic model for gold spot prices

Stochastic modeling for commodity prices significantly plays a role in pricing commodity derivatives for both futures and options, under the no-arbitrage assumptions. The literature review on stochastic models of commodity spot prices can be found in Lautier [7]. One can also derive the no-arbitrage prices (fair-prices) of futures or options in closed-form solutions or numerical solutions [5]. We presented here a stochastic model of gold spot prices which is developed in this research as an enlargement of the model proposed by Schwartz [2]. We typically assumed that the gold spot prices S_t is random and the instantaneous convenience yield $\delta(t)$ is deterministic under an equivalent martingale measure $\mathbb{Q}$.

We developed a one-factor model of stochastic behavior gold prices, which is an extension of Schwartz model 1 [2],

$$dS_t = \kappa(r - \ln S_t)S_t dt + \sigma S_t dW_t, \qquad (S)$$

where ,

$(S_t(\omega))_{t\in[0,T]}$ is the commodity prices process at time t, r is a risk-free interest rate, $\ln S_t$ is instantaneous convenience yield at time t, κ is the speed of adjustment of the commodity prices, σ is the volatility of commodity prices, $(W_t(\omega))_{t\in[0,T]}$ is a-one dimensional standard Brownian motion.

The main empirical characteristic that makes commodities noticeable difference from stocks, bonds and other financial assets is seasonality in prices. In case of we know seasonality in prices of commodities, we could precisely predict the commodities spot prices. For this reason, we examined the data of gold spot prices from January 1989 - December 1997 (see Figure 2.1) and modeled seasonality in gold prices defined as $\alpha(t)$ in (2.3). We also modified, Schwartz model 1 (S), the term instantaneous convenience yields $\ln S_t$ to $\delta(t)$ satisfying the term as motivated by Figure 2.1; instantaneous convenience yields $\ln S_t$ for Schwartz model (S) also has seasonality. For instantaneous convenience yields $\delta(t)$, similar to gold spot prices, also has seasonality and to have mean reversion property that the solution converges to long run mean α_0, we use an ordinary differential equation (2.2) which the solution is in the form (2.4). Thus, (2.2) is the equation for $\delta(t)$ that reflects the seasonality and mean reversion property. Finally, the stochastic model for gold prices is obtained in (2.1) which the seasonality is entered in the term of $\delta(t)$.

By the given motivation, we presented our stochastic model for gold spot prices as follows,

$$dS_t = (r - \delta(t))S_t dt + \sigma S_t dW_t, \quad S_0 = s_0 > 0, \tag{2.1}$$

$$\frac{d\delta(t)}{dt} = \kappa(\alpha(t) - \delta(t)), \quad \delta(0) = \delta_0, \tag{2.2}$$

$$\alpha(t) = \alpha_0 + \alpha_1 \sin(2\pi(t - t_\alpha)), \tag{2.3}$$

where $(S_t(\omega))_{t\in[0,T]}$ is the gold prices process at time t, r is a risk-free interest rate, $\delta(t)$ is instantaneous convenience yield at time t which is assumed to follow the ordinary differential equation (2) and has mean-reversion property, S_0 and δ_0 is an initial condition. $(W_t(\omega))_{t\in[0,T]}$ is a-one dimensional standard Brownian motion under a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ with a filtration $(\mathcal{F}_t)_{t\geq 0}$ and equivalent martingale measure $\mathbb{Q}$, κ is the speed of adjustment of the gold prices, α_0 is a long run mean, σ is the volatility of gold prices, and $\alpha(t)$ represents seasonal variation. The parameters α_1 and t_α denote the annual seasonality parameter representing the amplitude of seasonality and the seasonality centering parameter representing the starting point of seasonality in each year, respectively.

Due to the parameter t_α, the function cosine is already included in the term of seasonality in the model (2.3) because $\sin(t) = \cos(t - \pi/2)$. Figure 2.2 shows the example of plots of t_α. We fixed $\alpha_0 = 2$, $t \in [0, 2]$, $\alpha_1 = 1$, but varying t_α. The values of $\alpha(t)$ in this figure do not start at the same points.

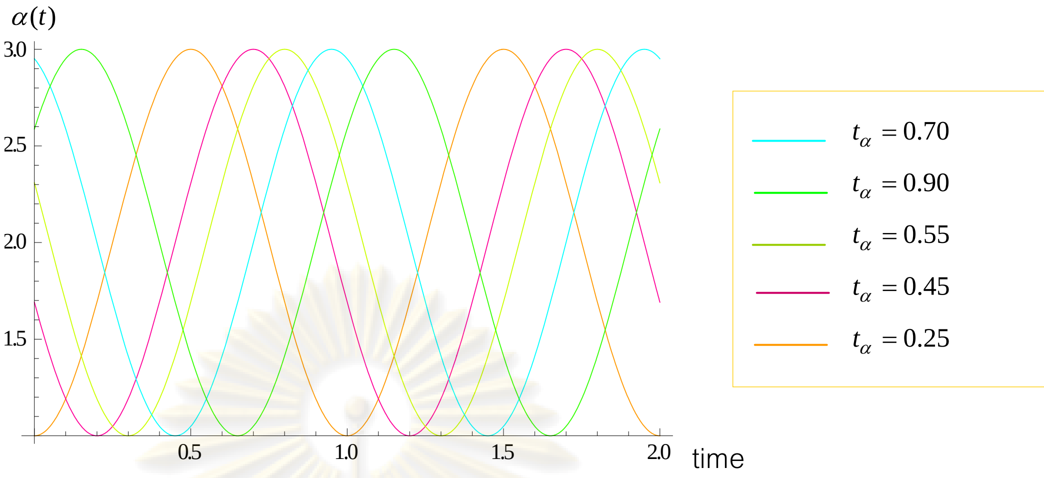


Figure 2.2: Seasonal variation in convenience yields $\alpha(t)$

2.6 The solution for instantaneous convenience yields

Let t be in the interval $[0,\infty)$ and $\delta(t)$ be the instantaneous convenience yields satisfying (2.2). Then, the closed-form solution of the instantaneous convenience yield is

$$\delta(t)=\alpha_0+C(t)+(\delta_0-\alpha_0-C(0))\exp(-\kappa t), \tag{2.4}$$

where

$$C(t)=\frac{-2\alpha_1\kappa\pi\cos(2\pi(t-t_\alpha))+\alpha_1\kappa^2\sin(2\pi(t-t_\alpha))}{\kappa^2+4\pi^2}. \tag{2.5}$$

Examples of plots for the instantaneous convenience yields

In Figure 2.3, we plotted the graph for the various values of κ to show the difference magnitude of speeds and given the other variables are fixed. This shows the speed of approaching the long-run mean

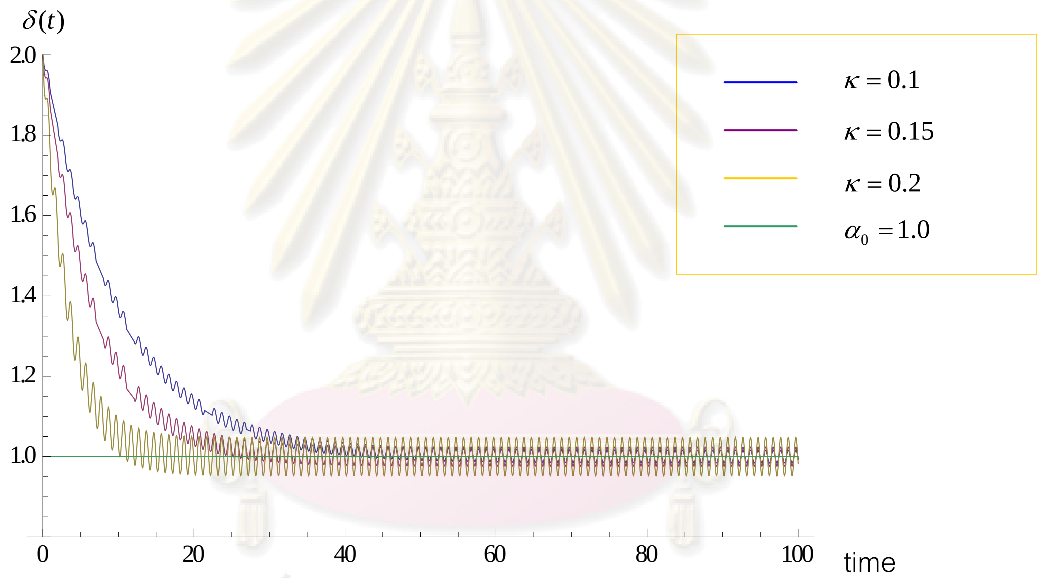


Figure 2.3: Graph of the instantaneous convenience yields for $\kappa = 0.1$, $\kappa = 0.15$, $\kappa = 2.0$ with fixed parameters $(\alpha_0 = 1, \delta_0 = 2, \alpha_1 = 1, t_\alpha = 0.2)$.

In Figure 2.4, we plotted the graphs with various values of δ_0 for the different initial values plotted with the other variables.

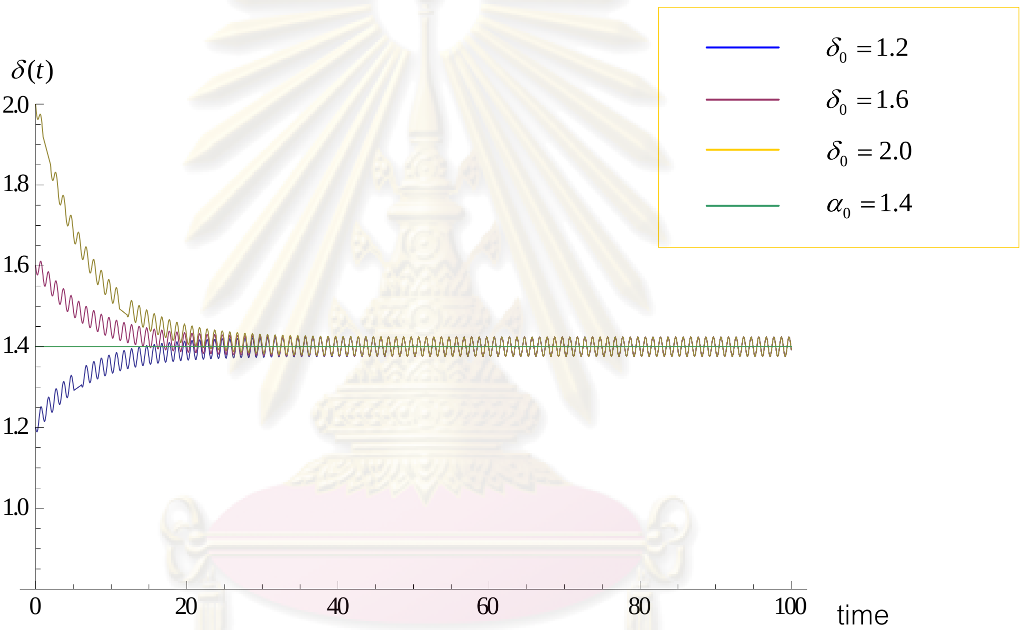


Figure 2.4: Graph of the instantaneous convenience yields for $\delta_0 = 1.2$ $\delta_0 = 1.6$ $\delta_0 = 2.0$ with fixed parameters $(\kappa = 0.15, \alpha_0 = 1.4, \alpha_1 = 1, t_\alpha = 0.2)$.

In Figure 2.5, we plotted the graph for the various values of α_0 to show the different long-run means plotted with the other variables.

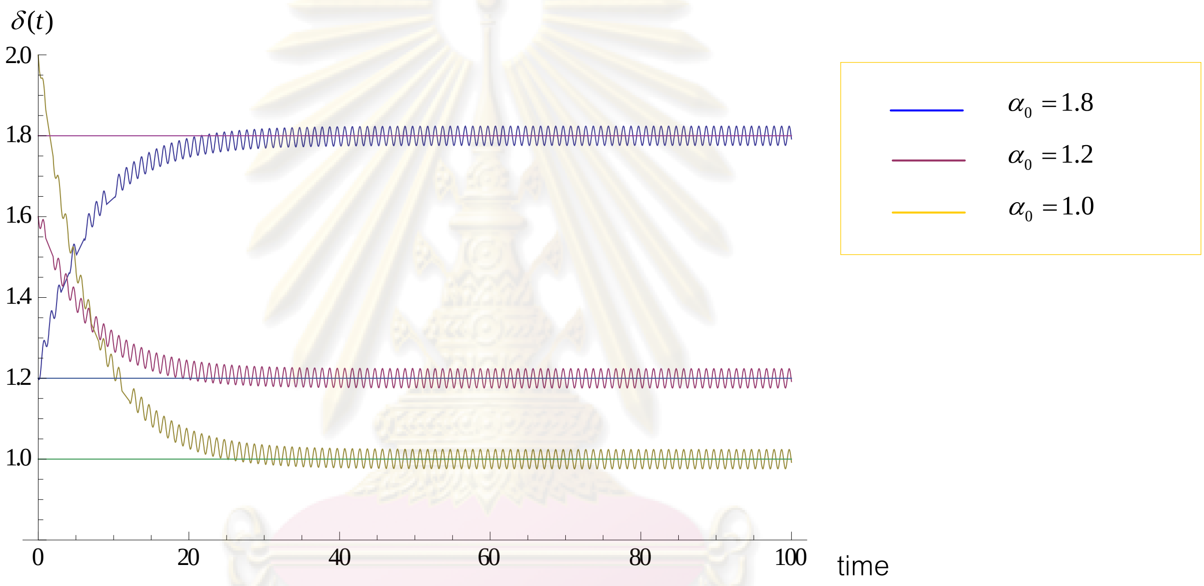


Figure 2.5: Graph of the instantaneous convenience yields for $\alpha_0 = 1.8$, $\alpha_0 = 1.2, \alpha_0 = 1$ with fixed parameters $(\kappa = 0.15, \delta_0 = 1.4, \alpha_1 = 1, t_\alpha = 0.2)$.

The instantaneous convenience yields $\delta(t)$ depend on 5 parameters $\alpha_0,\ \alpha_1,$ $\kappa,\ \delta_0,\ t_\alpha,$ where each parameter has its own meaning as illustrated in Figures 2.2-2.5. One can actually estimate the values of these parameters by using some method such as maximum likelihood method, given that the data of the gold spot prices are known.

2.7 The solution of the SDE

The strong solution of the SDE (2.1) [4, p.120] can be expressed as

$$S_t = S_{t_0} \exp\left(\int_{t_0}^{t} (r - \frac{\sigma^2}{2} - \delta(s))ds + \sigma\sqrt{(t-t_0)}Z \right), \tag{2.6}$$

for all $0 \le t_0 \le t \le T$, where Z is the standard normal random variable.

Proposition 2.1. *The gold spot price* S_t (2.6) *is neither negative nor zero for all* $t \ge 0$. *Because its closed-form solution* (2.6) *is exponential function and its initial values is equal to or greater than zero. Moreover, for a fixed* $t_0 \ge 0$, $\ln\left(S_t / S_{t_0}\right)$ *is normal distributed with mean*
$\int_{t_0}^{t} (r - \frac{\sigma^2}{2} - \delta(s))ds$ *and variance* $\sigma^2(t-t_0)$.

Proof.

Obviously, It follows from (2.6) *that* $S_t \ge 0$, *for all* $t \ge 0$.

From (2.6), *we obtain*

$$\ln\left(\frac{S_t}{S_{t_0}}\right) = \ln\exp\left(\int_{t_0}^{t} (r - \frac{\sigma^2}{2} - \delta(s))ds + \sigma\sqrt{(t-t_0)}Z \right),$$

$$\ln\left(\frac{S_t}{S_{t_0}}\right) = \int_{t_0}^{t} (r - \frac{\sigma^2}{2} - \delta(s))ds + \sigma\sqrt{(t-t_0)}Z.$$

Let $\int_{t_0}^{t} (r - \frac{\sigma^2}{2} - \delta(s))ds$ *be* μ *and* $\sigma\sqrt{(t-t_0)}$ *be* σ_1.

Then, we obtain $\ln\left(\frac{S_t}{S_{t_0}}\right) \sim N\left(\mu, \sigma_1^2\right).$ □

Simulation

We use the Euler method [4] to simulate sample paths for the SDE (2.1), Here, we consider the time discretization

$$t = t_0 < t_1 < \ldots < t_n < \ldots < t_N = T,$$

on the time interval $[t, T]$ in which the equidistant case has step size

$$\Delta t = \frac{T - t_0}{N},$$

for some integer N, large enough so that $\Delta t \in (0,1)$.

The Euler approximation satisfies the recursive formula

$$S_{n+1} = S_n + (r - \delta(t))S_n(t_{n+1} - t_n) + \sigma S_n(W_{n+1} - W_n), \qquad (2.7)$$

for $n = 0,1,2,\ldots,N-1$ of the Wiener process $W = \{W_t, t \geq 0\}$ with the initial condition $S_{t_0} = s_0$.

We show the simulation of spot prices (F_t^T) from (2.7) to analyze the tendency of the gold prices by r is 0.07, S_0=15,000 baht, $\alpha_0 = 0.05$, $\delta_0 = 0.5$, $\kappa = 0.2$, $\alpha_1 = 0.1$, $t_a = 0$, $\sigma = 0.01$, $N = 500$, $\Delta t = 1/500$.

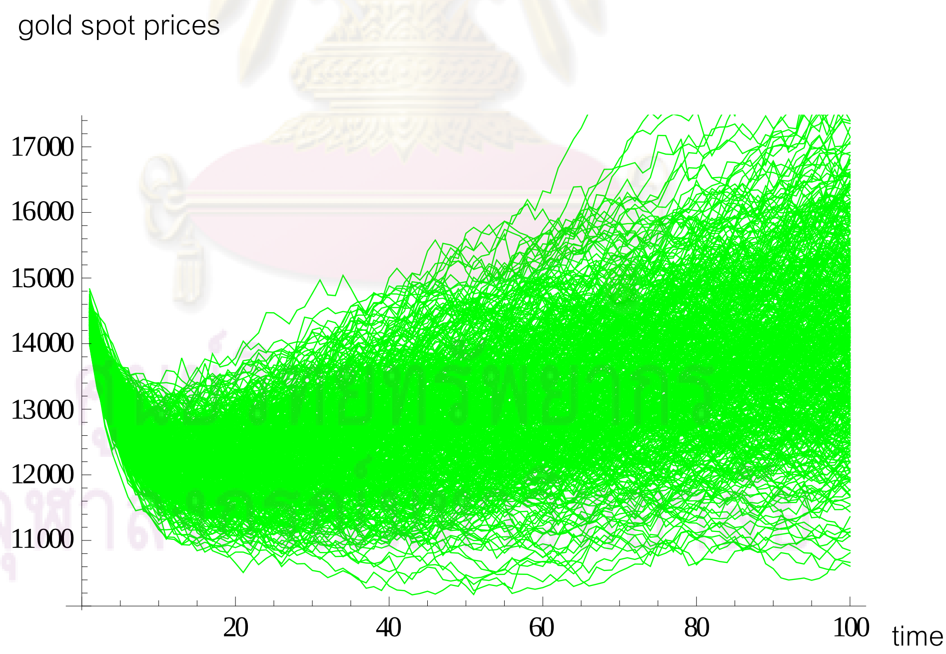
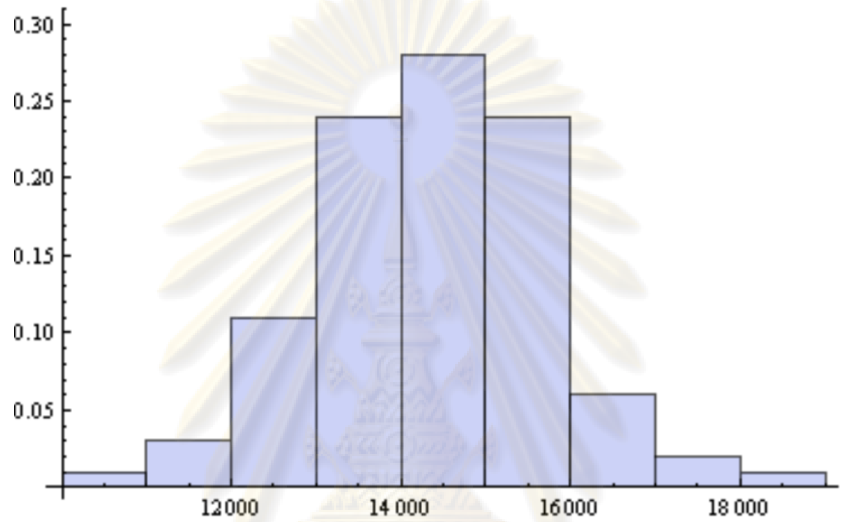


Figure 2.6: The gold future prices sample paths.

They start at 15,000 baht and end in the terminal time with various prices where $T = 1$ means that we divided 1 duration of time to 100 sub-durations.

probabilities



gold spot prices

Figure 2.7: Distribution of the gold spot prices

We captured the prices in the terminal time of every sample paths in Figure 2.6 and the histogram in Figure 2.7. We obtained that the gold spot prices in the future will higher or lower than the initial price (in this case is 15,000 baht) at the various probabilities.

CHAPTER III

PRICING OF GOLD FUTURES AND EUROPEAN GOLD OPTIONS

In this Chapter, we explained some basic concepts about futures and options. Moreover, we summarized the necessary theorems relating to obtain the no-arbitrage gold futures price and European options written on a gold contract. Finally, we illustrate the evolutions of the gold futures and the European gold option prices.

3.1 A futures contract

A futures contract of an underlying asset is a derivative security giving one has an obligation to a specified transaction of the underlying asset, at a certain future time for a certain price. The certain future time is known as the expiration date or maturity of the contract and the certain price is known as the futures price. The expiration date of the contract and the futures price are written when the contract is entered by two parties. After that the futures price is known as the delivery price. The futures contract is traded on an exchange. The exchange specifies certain standardized features of the contract and provides a mechanism that gives the two parties a guarantee that the contract will be honored. Such the exchange is known as the futures market.

Example of trading a futures contract

Suppose that the current spot price of gold is 1,000 baht. Party A wants to buy 1 share of gold. Party A have to make a contract by taking the position “Long” with party B who want to sell 1 share in the exchange market by taking position “Short”. Suppose that the price rises to 1,100 baht in later time. Then, party A can make a profit by selling 1 share to someone in the futures market (by taking the position “Short”). These processes will be done in the futures market by a broker. Then, he will benefit (1,100-1,000)x1=100 baht and party B will lost 100 baht. On the other hand, if the price falls to 900 baht, party A can limit his loss by selling 1 share from someone in the futures market. Party B will benefit 100 baht. On the other hand, party A want to sell the share he can take the short position. If the price falls to 900 at later time, he can offset the position by taking the long position to receive the profit 100 baht. Meanwhile, the price rises to 1,100 baht, he will lose the 100 baht as well.

If the parties do not want to offset (taking the position opposite in the first) their positions, when they reach the maturity time, the party who takes the short position must prepare the commodity to sell to the parties who take the long position.

3.2 Arbitrageurs

In economics and finance, arbitrage is a transaction of taking advantage of price different between two or more markets. It also makes of a gain through trading without committing any money and without taking a list of loosing money (arbitrage transaction).

Arbitrageurs are the third important group of the participants in futures or option market who can do transactions by locking in a risk profit by simultaneously entering in transactions in the markets. In the simplest example, any goods sold in one market should sell for the same price in another market. Trader may find that the price of commodity is lower in someplace than in cities and they purchase the goods and transport it to another place to sell at a higher price. This type of price arbitrage is the most common, but this simple example ignores the cost of transport, storage, risk, and other factors.

3.3 Valuation of gold derivatives

In order to price futures and option of a commodity, gold in this case, we need the following theorem available in [5].

Theorem 3.1. (No- Arbitrage futures prices)

Under the no-arbitrage assumptions in a futures market, the no-arbitrage price or fair-price of gold futures at a current time t *can be represented by the expected value of its discounted payoff function at the maturity time* T *under a risk-neutral probability measure denoted by* $F^T(t,S_t)$*, satisfying,*

$$F^T(t,S_t) = E_{\mathbb{Q}}[S_T \mid \mathcal{F}_t]\,, \tag{3.1}$$

where the expectation is taken under the equivalent martingale measure $\mathbb{Q}$ *conditioned on* $\mathcal{F}_t$ *.*

The relation (3.1) implies that F^T solves the partial differential equation [5]:

$$-\frac{\partial F^T}{\partial t} = (r-\delta(t))S\frac{\partial F^T}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F^T}{\partial S^2}, \tag{3.2}$$

subject to a terminal condition

$$F^T(T,S) = S, \text{ for all } S \geq 0\,. \tag{3.3}$$

We now solve the PDE (3.2) subject to (3.3) to obtained closed-form solutions for no-arbitrage gold futures price.

Theorem 3.2. (Determination of gold futures prices)

For given and fixed maturity date T *, no-arbitrage gold futures prices to the PDE* (3.2) *can be expressed as*

$$F^T(t,S_t) = S_t e^{A(T-t)} \tag{3.4}$$

where

$$A(\tau) = r\tau - \int_0^{\tau} \delta(T-s)ds, \tag{3.5}$$

for all $\tau \geq 0$ *.*

proof

To avoid confusion about the notations, we omit writing the subscript t of S_t and write $F^T \equiv F^T(t,S_t)$.

Let $\tau = T-t$, and we calculate

$$\frac{\partial F^T}{\partial t} = -\left(F^T A'(\tau)\right),\ \frac{\partial F^T}{\partial S} = \frac{F^T}{S},\ \frac{\partial^2 F^T}{\partial S^2} = 0,$$

where $' = \frac{d}{d\tau}.$

Replacing the above partial derivatives into (3.2) , we then obtain the following ODE,

$$A'(\tau) = (r-\delta(T-\tau)) \tag{3.6}$$

and the terminal condition implies that

$$A(0) = 0\,. \tag{3.7}$$

Then, we solve (3.6) subject to (3.7) to obtain (3.5) . □

The solution of $\int_0^{\tau} \delta(T-s)ds$ is in the form of $B+C+D+E-F+G+H$,

where $B = \frac{-((\alpha_0 e^{-\kappa T}\ (-1\ +\ e^{\kappa\tau}))}{\kappa}$, $C = \frac{((\delta_0 e^{-\kappa T}(-1+e^{\kappa\tau}))}{\kappa}$, $D = \alpha_0 \tau$,

$$E = \frac{(2\alpha_1 e^{-\kappa T}(-1+e^{\kappa\tau})\pi cos[2\pi t_a]}{\kappa^2+4\pi^2},\quad F = \frac{(\alpha_1 \kappa^2\ cos[2\pi(-T+t_a)]-cos[2\pi(-T+t_a+\tau)]}{2\pi(\kappa^2+4\pi^2)},$$

$$G = \frac{((\alpha_1 e^{-\kappa T}(-1+e^{\kappa\tau})\ \kappa sin[2\pi t_a]}{\kappa^2+4\pi^2},\quad H = \frac{((\alpha_1 \kappa sin[2\pi(-T+t_a)]-sin[2\pi(-T+t_a+\tau)]}{\kappa^2+4\pi^2}.$$

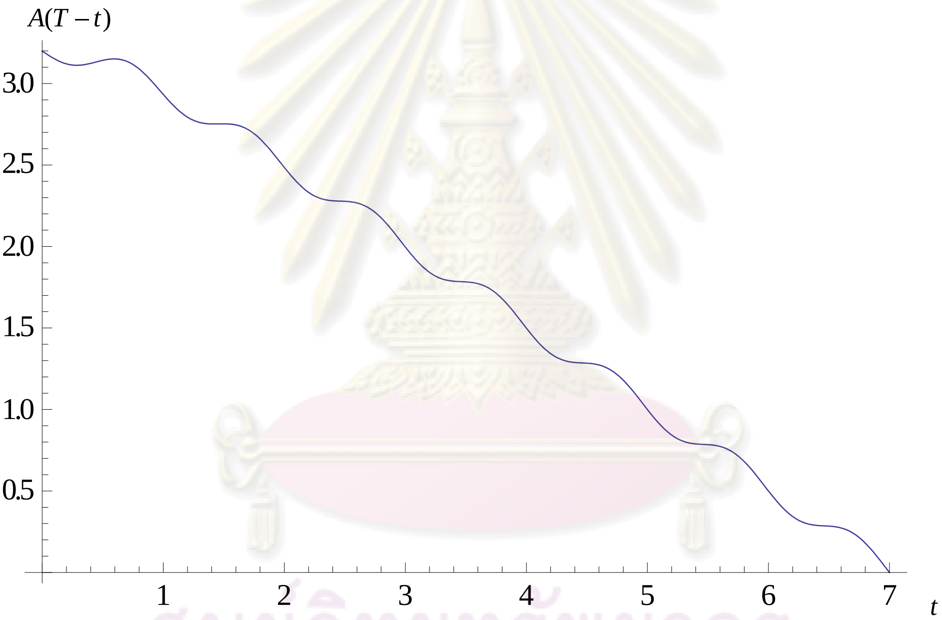


Figure 3.1: graph of $A(T-t)$, where $T=7$ and $t \in [0,7]$

We fixed T and we obtain the values of $A(T-t)$ have nonnegative values from day $t=0$ (the day that the futures contract initiated) until the maturity date $t=T$. This implies, on day t, $A(\tau)>0$ then the future price is higher than spot price and vice versa.

We ended this Chapter by showing the evolution of the futures prices obtained from the closed-form solution (3.4) with the parameters in various cases.

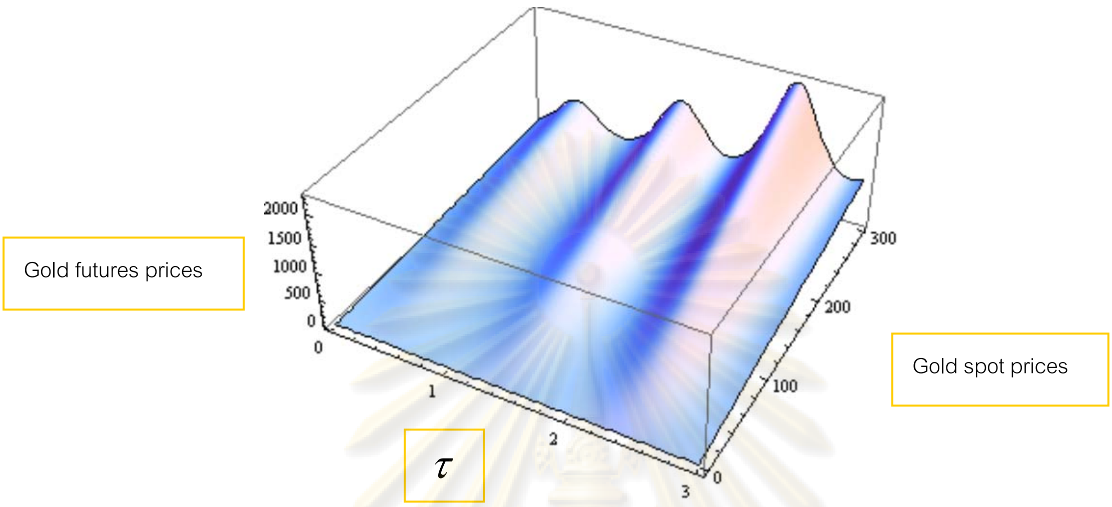


Figure 3.2 (a): Evolution of the futures prices with the parameters

$\alpha_0 = 1,\ \alpha_1 = 5, r = 1.5,\ \kappa=5,\ \delta_0 = 0.7, \sigma = 1,\ T = 3,\ t_\alpha = 4.$

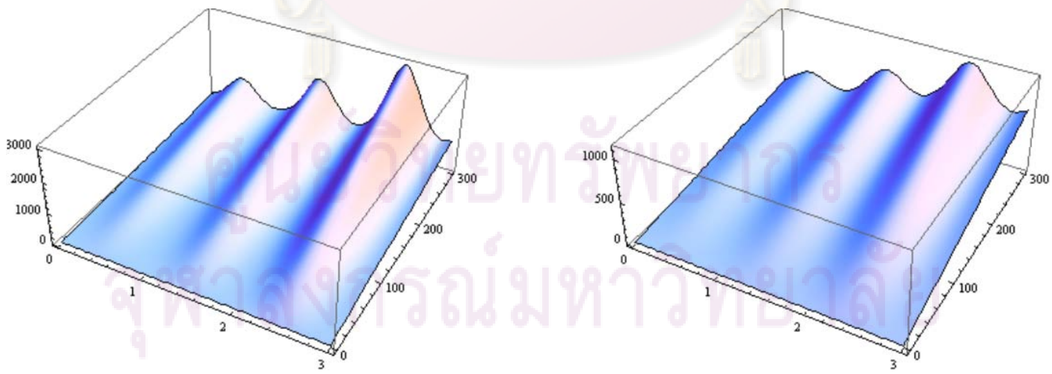


Figure 3.2 (b): Evolution of the futures prices with the same parameters as Figure 3.2 (a) excepted κ=8 (left) and κ=8 (right).

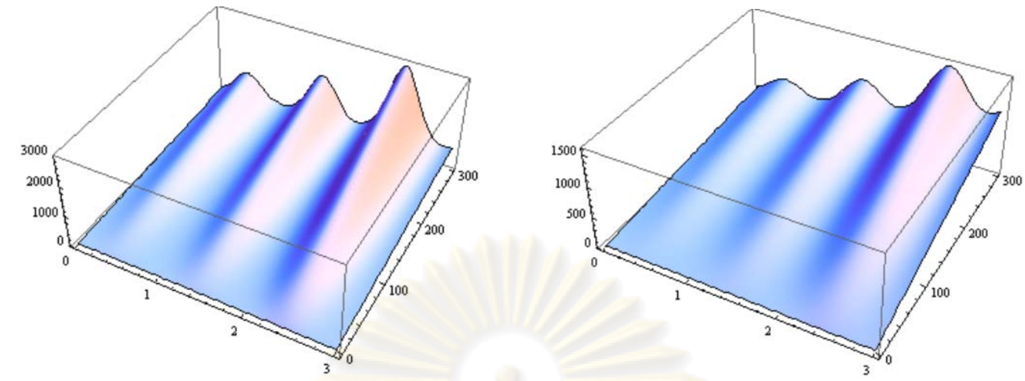


Figure 3.2 (c): Evolution of the futures prices with the same parameters as Figure 3.2 (a) excepted α_1=7 (left) and α_1=3 (right).

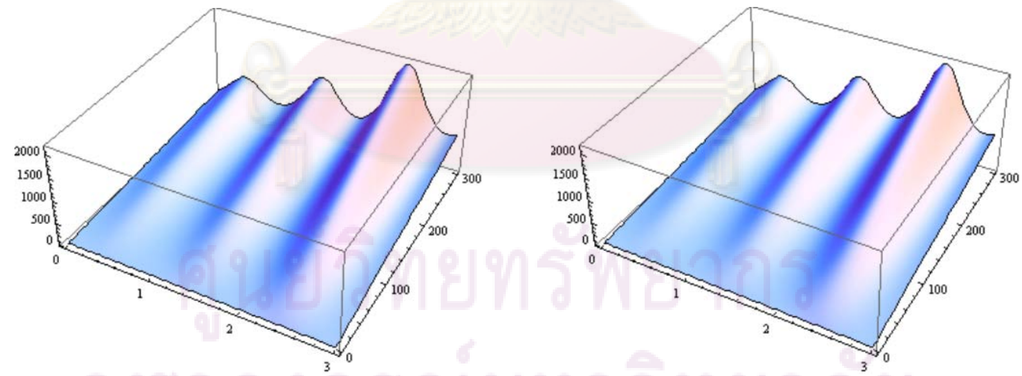


Figure 3.2 (d): Evolution of the futures prices with the same parameters as Figure 3.2 (a) excepted σ=2 (left) and σ=0.5 (right).

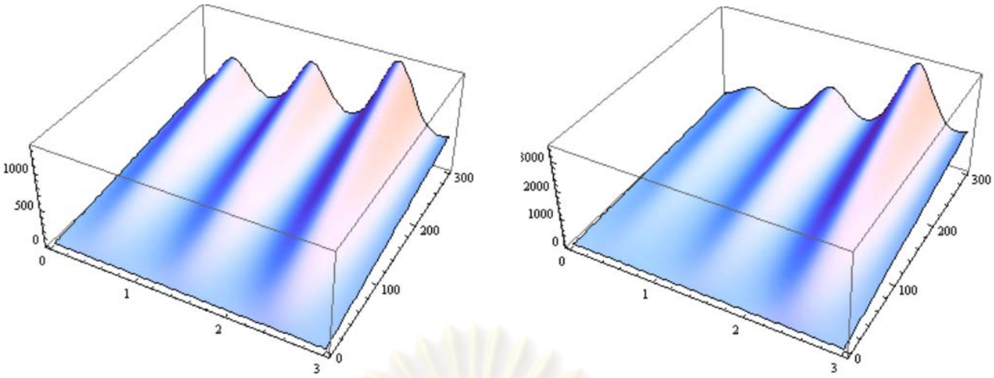


Figure 3.2 (e): Evolution of the futures prices with the same parameters as Figure 3.2 (a) excepted α_0=1.2 (left) and α_0=0.8 (right).

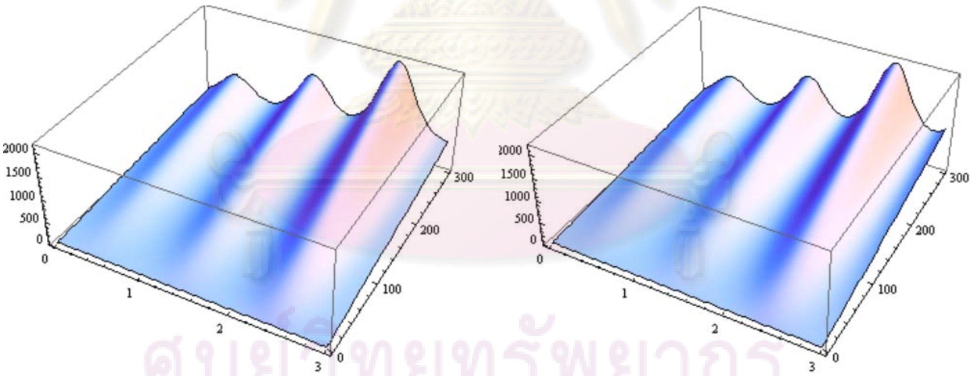


Figure 3.2 (f): Evolution of the futures prices with the same parameters as Figure 3.2 (a) excepted δ_0=2.0 (left) and δ_0=0.1 (right).

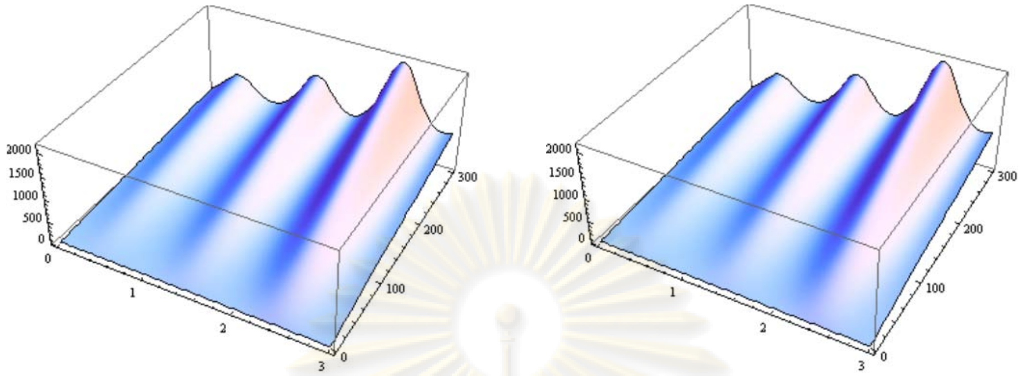


Figure 3.2 (g): Evolution of the futures prices with the same parameters as Figure 3.2 (a) excepted $t_\alpha = 8$ (left) and $t_\alpha = 2$ (right).

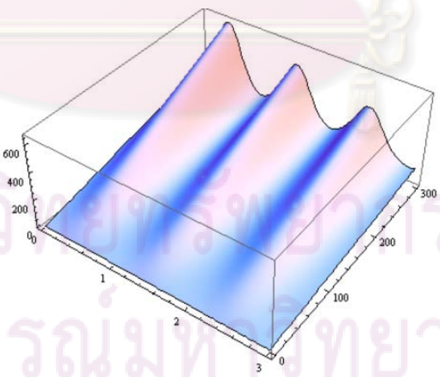


Figure 3.2 (h): Evolution of the futures prices with the same parameters as Figure 3.2 (a) excepted $r = 0.8$.

Futures prices will increase or decrease when the time passes and they are also up to the parameters $\kappa, \alpha_1, \sigma, \alpha_0, \delta_0, t_\alpha, r$. Moreover, it will be nonnegative because the solution (3.4) is exponential and it has the nonnegative initial condition. When we substitute any cases of the parameter values to evaluate the futures prices, they make the results in various cases. One thing that makes futures prices increase or decrease is the difference between the term of $r\tau$ and $\int_0^\tau \delta(T-s)ds$. If $r\tau > \int_0^\tau \delta(T-s)ds,$ we obtained $A(\tau)$ be positive and it makes futures prices be higher than spot prices (Figures 3.2 (a) – (h)). If $r\tau < \int_0^\tau \delta(T-s)ds,$ we obtained $A(\tau)$ be negative and it makes futures prices be lower than spot prices (Figure 3.2 (h)).

As seen in Figures 3.2 (a) - (h), we obtained these figures by varying parameters, Moreover, the tendency of gold futures prices for increasing or decreasing depends on the condition of $r\tau$ and $\int_0^\tau \delta(T-s)ds$. We also obtained from Figures 3.2 (as a result from (3.4)) that the gold futures prices behave like a wave showing the seasonality of variation in gold prices, which is somehow similar to the seasonality of the gold spot prices shown in Figures 2. There is the term of instantaneous convenience yields consists of the seasonality in gold prices term, then we obtained that the gold futures prices depend on seasonality in gold as well.

3.4 An option contract

An options contract is a derivative security giving one the right but not obligation to make a specified transaction of the underlying asset at a future date at a certain price. The futures date is known as the expiration date or maturity of the option and the specified price is known as the exercise price or strike price. Call options give one the right to buy. Put options give one the right to sell. European options give one the right to exercise the options only on expiration date. If the underlying asset is referred to a futures contract then the European options are known as the European Option.

3.5 Payoff of European options

Let consider the simple situation of a party who buys a European call option with a strike price of $\$K \ \ (K \geq 0)$ 1 share of a certain stock. Suppose that if the stock price on the maturity date S_t is more than the strike price $\$K$, the call option will be exercised. Then, the party can buy a share for $\$K$ and sell the same share for S_t to realize a gain of $S_T - K$ for a share. (ignoring any transaction costs) $\$K$ If the stock price (S_T) is less than the strike price (K), the party does not use his right to exercise at the price $\$K$ because if he would be interested in buying that share, he could get it cheaper in the market. Thus, in this case, the gain from holding the call option is zero. Putting the two cases together, we see that holding a European call option leads to a payoff max $(0, S_T - K)$ at the maturity date T. In the other hand, the party who buys a European put option with a strike price of $\$K \ \ (K \geq 0)$ 1 share of a certain stock. Suppose that if the current stock price on the maturity date (S_T) is less than the strike price (K), the put option will be exercised. Then, the party can buy a share for S_T per share and under the right of the put option, he can sell the same share for $\$K$ to realize a gain of $K - S_T$ per share. Thus, in this case, the gain from holding the put option is zero. Putting the two cases together, we see that holding a European put option leads to a payoff max $(0, K - S_T)$ at the maturity date T.

Example of call options

Party A buy a call option contract to obtain the right to buy 1 share of gold. The spot price is equal to 1,000 baht and it may increase more that 1,000 baht in the later time. Party A must pay a premium by 100 baht (100 baht per share) to guarantee to buy 1 share of gold by the contract price so that party A can buy it which is not too high at that time. Suppose the contract price (strike price) is 1,050 baht. If the price rises to 1,200 baht, party A can exercise the call option by buying 1 share of gold for $1\times 1{,}050=1{,}050$ baht and sell the shares in the exchange market at $1\times 1{,}200=1{,}200$ baht. Then, party A will profit $1{,}200-1{,}050-100=50$ baht per share. However, if the price less than 1,050 baht, party A need not to exercise the call option, then, party A will lose the total only the premium 100 baht.

Example of put options

Party A buy a put option contract to obtain the right to sell 1 share of gold. The spot price is equal to 1,000 baht and it may decrease less than 1,000 baht in the later time. Party A must pay a premium by 100 baht (100 baht per share) to guarantee to sell 1 share of gold by the contract price so that party A can sell it which is not too low at that time. Suppose the contract price is 950 baht. If the price drops to 800 baht, party A can exercise the put option by selling 1 share of gold for $1\times 950=950$ baht and buy the shares in the exchange market at $1\times 800=800$ baht. Then, party A will profit $950-800-100=50$ baht per share. However, if the price more than 950 baht, party A need not to exercise the put option, then, party A will lose the total only the premium 100 baht.

3.6 Put-call parity

The relationship between the price of a call and the price of a put for an option with the same characteristics (strike price, expiration date, underlying) called put-call parity. It is used in arbitrage theory. If different portfolios comprised of calls and puts have the same value at expiration, it is implied that they will have the same value leading up to the expiration point. Thus, the values of the portfolios move in lock step. Portfolio price equality is calculated as $C + Ke^{-rT} = P + S_0$ (put-call parity equation), where C is the market value of the call, Ke^{-rT} is the present value of the strike price, P is the market value of the put, and S_0 is the market value of the underlying security. If the two sides of the equation are not equal, arbitrage profit could be gained by investing in the less expensive portfolio. Analysis of the parity relationship assumes that other factors, such as a dividend, are not taken into account.

3.7 European gold options pricing

In this section, we consider European options written on gold spot prices. We first consider a European call option. Let T and K be respectively, a maturity date and strike price of the call option. From S. Rujivan [5], under the no-arbitrage assumptions, the call option price must equal to the present value of the expected payoff of the call option under the equivalent martingale measure $\mathbb{Q}$,

$$C(T,t,S_t,K) = e^{-r(T-t)} E_{\mathbb{Q}}[\max(0, S_T - K) \mid \mathcal{F}_t],$$

Theorem 3.3. (No-arbitrage prices of European call options for gold)

The no- arbitrage European gold call options prices with strike price K is

$$C(T,t,S_t,K) = S_t e^{-\int_t^T \delta(s)ds} \Phi(d_1) - Ke^{-r(T-t)} \Phi(d_2) \tag{3.19}$$

where

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + r(T-t) - \int_t^T \delta(s)ds + \frac{\sigma^2 (T-t)}{2}}{\sigma\sqrt{T-t}}, \tag{3.20}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}, \tag{3.21}$$

where Φ is the standard normal distribution function.

Proof.

We omit writing the martingale measure $\mathbb{Q}$ and the filtration $\mathcal{F}_t$ in this proof to avoid confusion about notations. Then, we obtain

$$\begin{aligned} C(T,t,S_t,K) &= e^{-r(T-t)}E[\max(0,S_T - K)], \\ &= e^{-r(T-t)}E[I(S_T - K)], \\ &= e^{-r(T-t)}E[IS_T] - Ke^{-r(T-t)}E[I], \end{aligned} \tag{3.22}$$

Where I is the indicator random variable for the event that the option finishes in the money, that is,

$$I = \begin{cases} 1 & if \ S_T > K, \\ 0 & if \ S_T \le K. \end{cases} \tag{3.23}$$

Next, we will show that

$$I = \begin{cases} 1 & if \ Z > \sigma\sqrt{T-t} - d_1, \\ 0 & otherwise. \end{cases} \tag{3.24}$$

where d_1 is given in (3.20).

Using (2.6) and (3.23), we obtain

$$\begin{aligned} S_T > K &\leftrightarrow S_t e^{\int_t^T (r - \frac{\sigma^2}{2} - \delta(s))ds + \sigma\sqrt{(T-t)}Z} > K \\ &\leftrightarrow \int_t^T (r - \frac{\sigma^2}{2} - \delta(s))ds + \sigma\sqrt{(T-t)}Z > \ln\left(\frac{K}{S_t}\right) \\ &\leftrightarrow Z > \frac{\ln\left(\frac{K}{S_t}\right) - r(T-t) + \int_t^T \delta(s)ds + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} \\ &\leftrightarrow Z > \sigma\sqrt{T-t} - d_1, \end{aligned}$$

and we also obtain (3.24). From (3.24), we have

$$\begin{aligned} E[I] &= P(S_T > K) \\ &= P(Z > \sigma\sqrt{T-t} - d_1) \\ &= P(Z < d_1 - \sigma\sqrt{T-t}) \\ &= \Phi(d_1 - \sigma\sqrt{T-t}) \\ &= \Phi(d_2), \end{aligned} \tag{3.25}$$

Using (2.6) and (3.24) with $a=\sigma\sqrt{T-t}-d_1$

We then obtain,

$$
\begin{aligned}
&e^{-r(T-t)}E[IS_T] \\
&= e^{-r(T-t)}\int_a^\infty (S_t e^{r(T-t)-\int_t^T \delta(s)ds-\frac{1}{2}\sigma^2(T-t)+\sigma\sqrt{T-t}y})\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy, \\
&= S_t e^{-\int_t^T \delta(s)ds}\frac{1}{\sqrt{2\pi}}\int_a^\infty e^{-\frac{1}{2}(y^2-2\sigma\sqrt{T-t}y+\sigma^2(T-t))}dy, \\
&= S_t e^{-\int_t^T \delta(s)ds}\frac{1}{\sqrt{2\pi}}\int_a^\infty e^{\frac{-(y-\sigma\sqrt{T-t})^2}{2}}dy, \\
&= S_t e^{-\int_t^T \delta(s)ds}\frac{1}{\sqrt{2\pi}}\int_{-d_1}^\infty e^{\frac{-x^2}{2}}dx, \qquad (let\ x=y-\sigma\sqrt{T-t}) \\
&= S_t e^{-\int_t^T \delta(s)ds}P(Z>-d_1), \\
&= S_t e^{-\int_t^T \delta(s)ds}P(Z<d_1), \\
&= S_t e^{-\int_t^T \delta(s)ds}\Phi(d_1). \qquad (3.26)
\end{aligned}
$$

Replacing (3.25) and (3.26) into (3.22) we then reach (3.19) □

Next, we consider European put options written on gold spot prices. From [5], under the no-arbitrage assumptions, the put option price must equal to the present value of the expected payoff of the call option under the equivalent martingale measure $\mathbb{Q}$,

$$P(T,t,S_t,K)=e^{-r(T-t)}E_{\mathbb{Q}}[\max(0,K-S_T)\,|\,\mathcal{F}_t],$$

Theorem 3.4. (No- arbitrage prices of European put options for gold)

The no- arbitrage European gold put options prices with strike price K is

$$P(T,t,S_t,K)=Ke^{-r(T-t)}\Phi(-d_2)-S_t e^{-\int_t^T \delta(s)ds}\Phi(-d_1). \qquad (3.27)$$

where d_1 and d_1 satisfy, respectively, (3.20) *and* (3.21)

Proof.

$$
\begin{aligned}
P(T,t,S_t,K) &= e^{-r(T-t)}E_{\mathbb{Q}}[\max(0,K-S_T)\,|\,\mathcal{F}_t], \\
&= e^{-r(T-t)}E[I(K-S_T)], \\
&= Ke^{-r(T-t)}E[I]-e^{-r(T-t)}E[IS_T], \qquad (3.28)
\end{aligned}
$$

Where I is the indicator random variable for the event that the option finishes in the money, that is,

$$I = \begin{cases} 1 & if\ S_T < K, \\ 0 & if\ S_T \geq K. \end{cases} \tag{3.29}$$

Next, we will show that

$$I = \begin{cases} 1 & if\ Z < \sigma\sqrt{T-t} - d_1, \\ 0 & otherwise. \end{cases} \tag{3.30}$$

where d_1 is given in (3.20).

Using (2.6) and (3.29), we obtain

$$\begin{aligned} S_T < K &\leftrightarrow S_t e^{\int_t^T (r - \frac{\sigma^2}{2} - \delta(s))ds + \sigma\sqrt{(T-t)}Z} < K \\ &\leftrightarrow \int_t^T (r - \frac{\sigma^2}{2} - \delta(s))ds + \sigma\sqrt{(T-t)}Z < \ln\left(\frac{K}{S_t}\right) \\ &\leftrightarrow Z < \frac{\ln\left(\frac{K}{S_t}\right) - r(T-t) + \int_t^T \delta(s)ds + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} \\ &\leftrightarrow Z < \sigma\sqrt{T-t} - d_1, \end{aligned}$$

and we also obtain (3.30). From (3.30), we have

$$\begin{aligned} E[I] &= P(S_T < K) \\ &= P(Z < \sigma\sqrt{T-t} - d_1) \\ &= P(Z < -(d_1 - \sigma\sqrt{T-t})) \\ &= \Phi(-(d_1 - \sigma\sqrt{T-t})) \\ &= \Phi(-d_2), \end{aligned} \tag{3.31}$$

where Φ is the standard normal distribution function.

Using (2.6) and (3.30) with $a = \sigma\sqrt{T-t} - d_1$

We then obtain,

$$\begin{aligned} &e^{-r(T-t)}E[IS_T] \\ &= e^{-r(T-t)}\int_{-\infty}^{a} (S_t e^{r(T-t) - \int_t^T \delta(s)ds - \frac{1}{2}\sigma^2(T-t) + \sigma\sqrt{T\text{-}t}y}) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy, \\ &= S_t e^{-\int_t^T \delta(s)ds} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-\frac{1}{2}(y^2 - 2\sigma\sqrt{T\text{-}t}y + \sigma^2(T-t))} dy, \\ &= S_t e^{-\int_t^T \delta(s)ds} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{\frac{-(y - \sigma\sqrt{T\text{-}t})^2}{2}} dy, \\ &= S_t e^{-\int_t^T \delta(s)ds} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-d_1} e^{\frac{-x^2}{2}} dx, \qquad (let\ x = y - \sigma\sqrt{T\text{-}t}) \end{aligned}$$

$$= S_t e^{-\int_t^T \delta(s)ds} P(Z < -d_1),$$

$$= S_t e^{-\int_t^T \delta(s)ds} \Phi(-d_1). \tag{3.32}$$

Replacing (3.31) and (3.32) into (3.28) we then reach (3.27) □

Corollary 3.5. (Put-Call Parity)

From theorem 3.3. and 3.4., we have the put call parity in the following form,

$$C - P = S_t e^{-\int_t^T \delta(s)ds} - Ke^{-r(T-t)}.$$

We end this chapter by showing the evolution of the options prices obtained from the closed-form solution of call prices (3.19) and put prices (3.27) with the parameters follow in various cases.

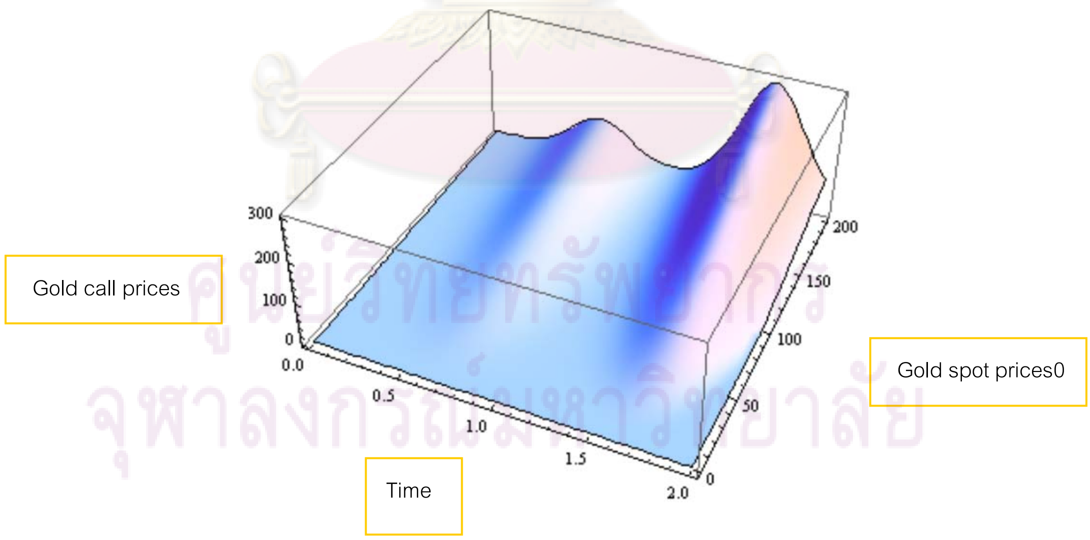


Figure 3.3 (a): Evolution of the gold call option prices with parameters

$\alpha_0 = 1,\ \alpha_1 = 6,\ r = 1,\ \kappa = 5,\ \delta_0 = 0.7, \sigma = 1, t_\alpha = 4,\ K = 100,\ T = 2$

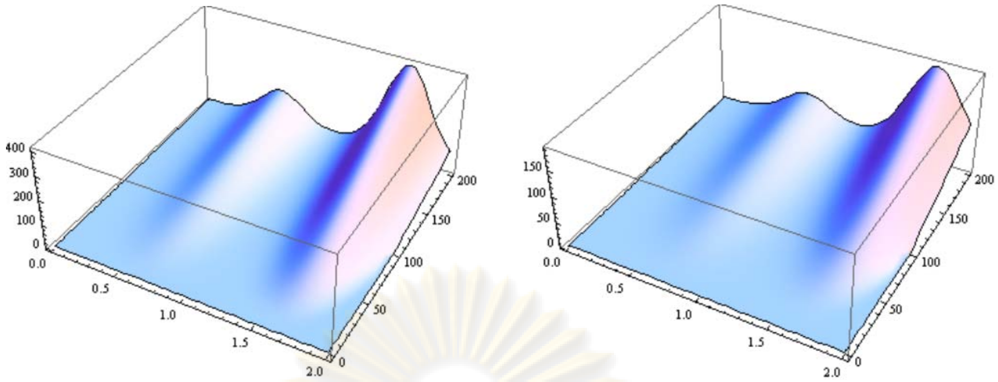


Figure 3.3 (b): Evolution of the gold call option prices with the same parameters as Figure 3.3 (a) excepted κ=7 (left) and κ=3 (right).

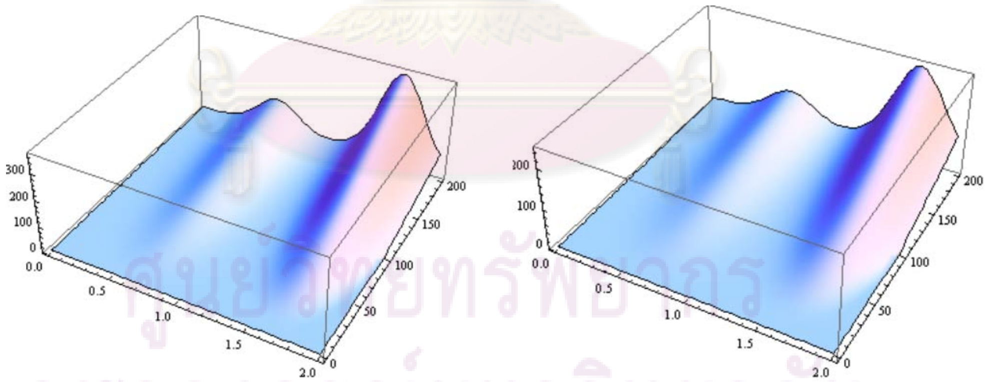


Figure 3.3 (c): Evolution of the gold call option prices with the same parameters as Figure 3.3 (a) excepted α_1=7 (left) and α_1=5 (right).

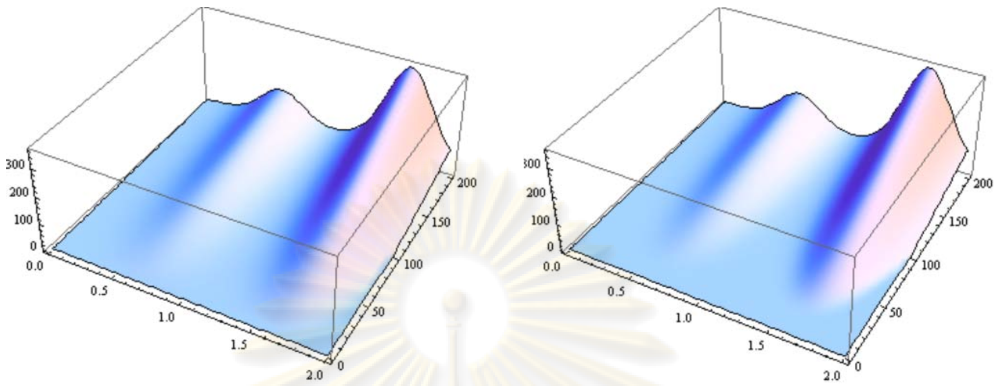


Figure 3.3 (d): Evolution of the gold call option prices with the same parameters as Figure 3.3 (a) excepted σ=2 (left) and σ=0.5 (right).

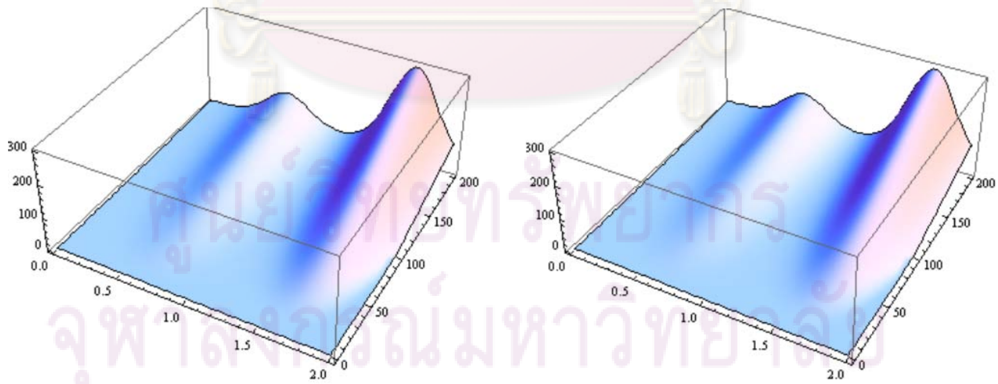


Figure 3.3 (e): Evolution of the gold call option prices with the same parameters as Figure 3.3 (a) excepted δ_0=2 (left) and δ_0=0.1 (right).

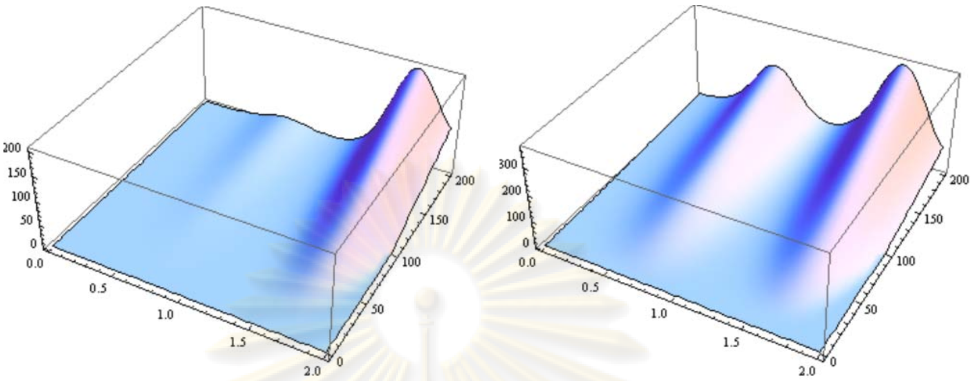


Figure 3.3 (f): Evolution of the gold call option prices with the same parameters as Figure 3.3 (a) excepted α_0=2 (left) and α_0=0.5 (right).

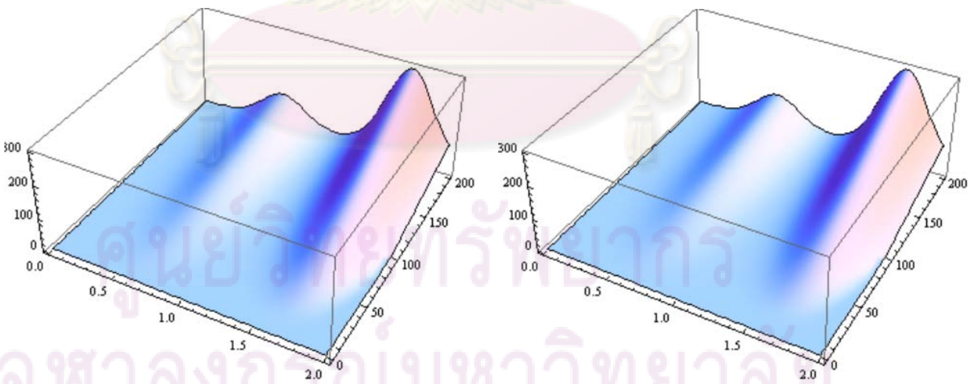


Figure 3.3 (g): Evolution of the gold call option prices with the same parameters as Figure 3.3 (a) excepted t_α=8. (left) and t_α=2 (right).

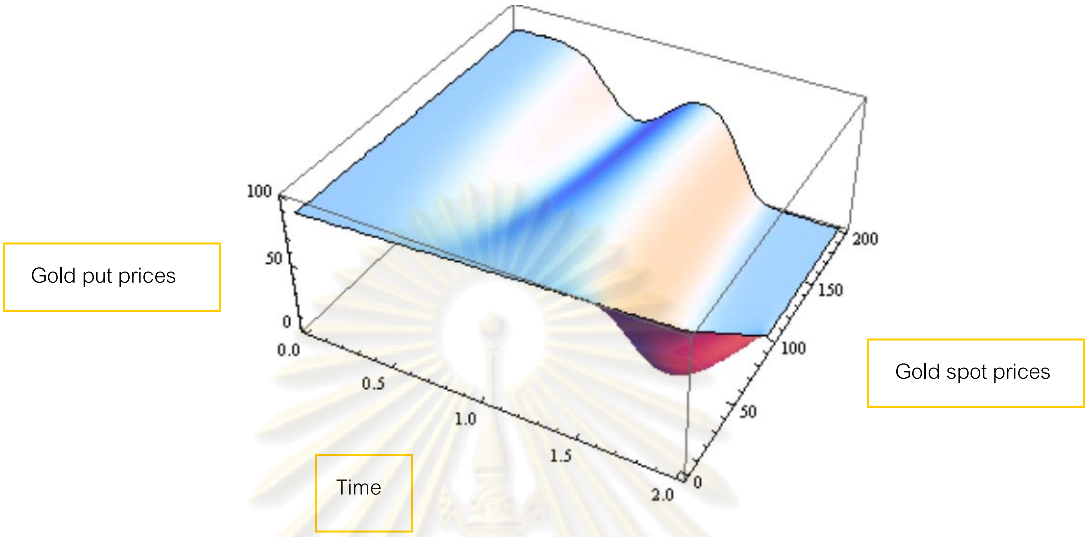


Figure 3.4 (a): Evolution of the gold put option prices with parameters

$\alpha_0 = 1,\ \alpha_1 = 25,\ r = 0.07,\ \kappa{=}1.7,\ \delta_0 = 0.7, \sigma = 1, t_\alpha = -4,\ K = 100,\ T = 2$

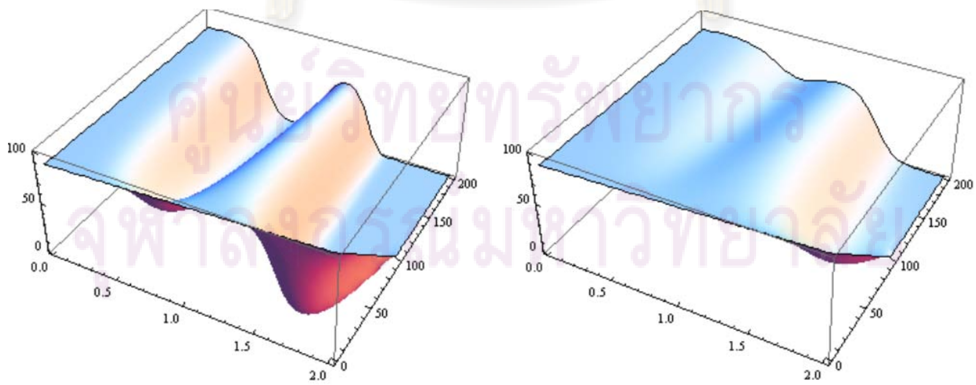


Figure 3.4 (b): Evolution of the gold put option prices with the same parameters as Figure 3.4 (a) excepted κ=3 (left) and κ=1 (right).

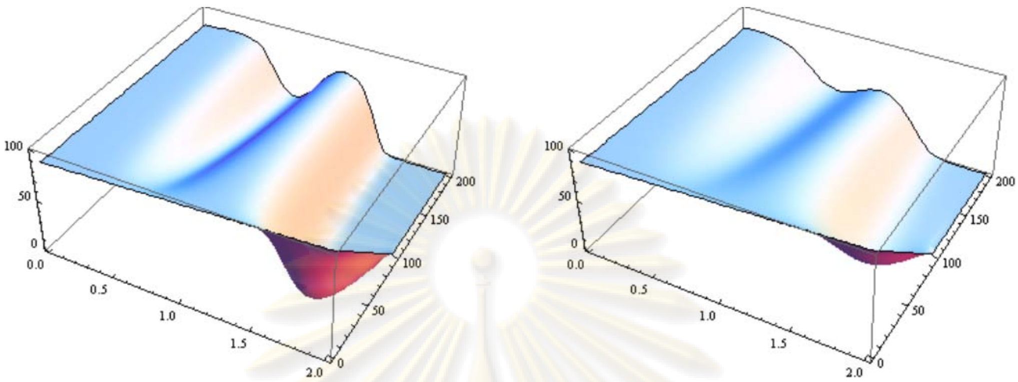


Figure 3.4 (c): Evolution of the gold put option prices with the same parameters as Figure 3.4 (a) excepted α_1=35 (left) and α_1=15 (right).

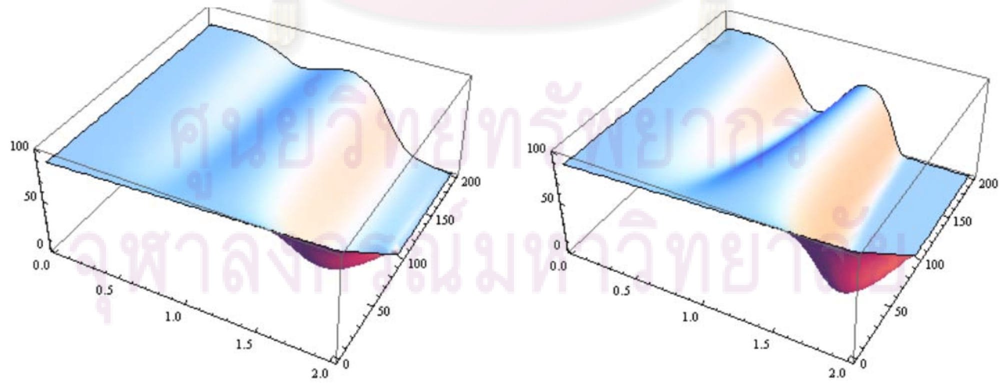


Figure 3.4 (d): Evolution of the gold put option prices with the same parameters as Figure 3.4 (a) excepted σ=2 (left) and σ=0.5 (right).

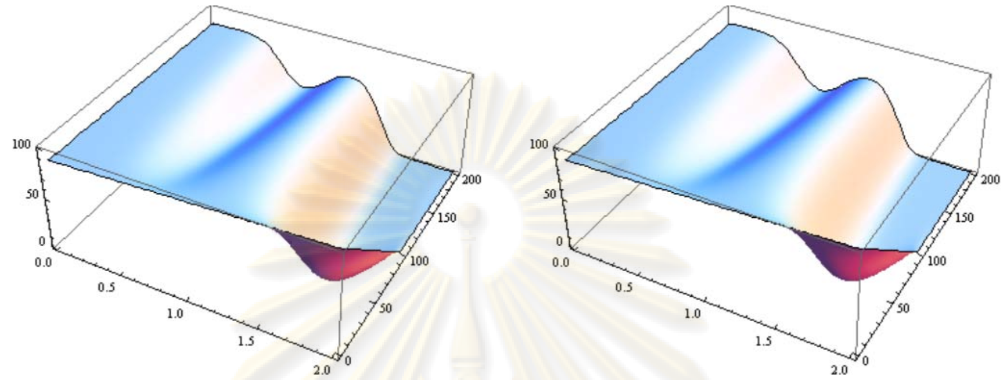


Figure 3.4 (e): Evolution of the gold put option prices with the same parameters as Figure 3.4 (a) excepted δ_0=2 (left) and δ_0=0.1 (right).

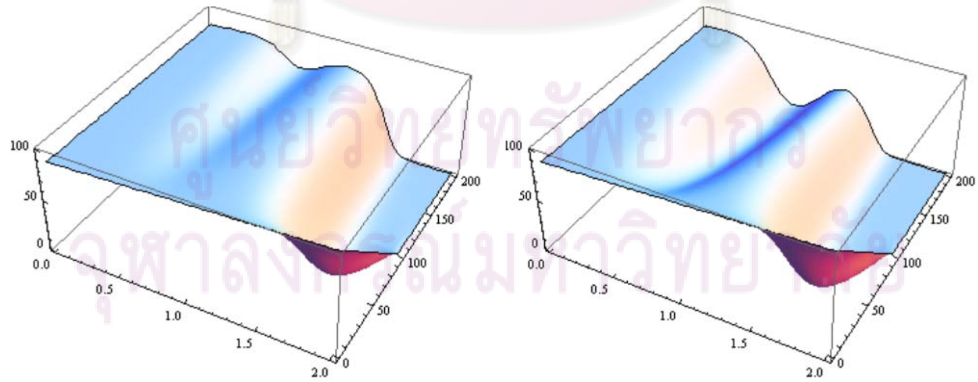


Figure 3.4 (f): Evolution of the gold put option prices with the same parameters as Figure 3.4 (a) excepted α_0=2 (left) and α_0=0.5 (right).

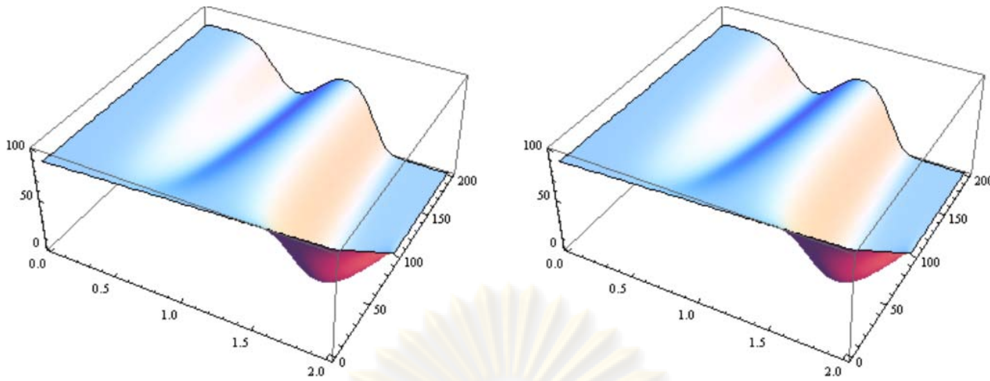


Figure 3.4 (g): Evolution of the gold put option prices with the same parameters as Figure 3.4 (a) excepted t_α=4 (left) and $t_\alpha = -8$ (right).

Options prices will increase or decrease when the time passes and they are also up to the parameters $\kappa, \alpha_1, \sigma, \alpha_0, \delta_0, t_\alpha, r$ and property of call or put options. Moreover, it will be nonnegative because of their terminal conditions (payoff) (Section 3.5). When we substitute any cases of the parameter values to evaluate the options prices, they make the results in various cases. One thing that has an effect on the shape of the evolution of options prices is payoff of options. Payoff of call option is in the form $\max(0, S_T - K)$ and put option is $\max(0, K - S_T)$. In case of call option shown as Figures 3.3 (a) – (g), if the spot prices is greater than strike price ($K = 100$) in the future and the traders want to buy this underlying asset be equal to strike price, they have to buy call option in high price meanwhile the expiration date has not reached yet. In case of put option shown as Figures 3.4 (a) – (g), if the spot prices is lower than strike price ($K = 100$) in the future and the traders want to sell this underlying asset be equal to strike price, they have to buy put option in high price meanwhile the expiration date has not reached yet.

As seen in Figures 3.3 (a) – (g), we obtained that the gold options prices behave like a wave showing the seasonality of variation in gold prices. This is a result from

(3.19), (3.27) each equation has the term of instantaneous convenience yields consists of the seasonality in gold prices term, then we obtained that the gold options prices depend on seasonality in gold as well.

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CHAPTER IV

CONCLUSIONS

In this thesis, we have introduced a one-factor model for gold prices which is an extension of the model proposed by Schwartz [2]. The gold prices are assumed to follow an extended Geometric Brownian Motion with a time-varying drift which describes seasonal variation in gold prices. The drift includes instantaneous convenience yields which follow an ordinary differential equation. Moreover, we derive closed-form solutions for no-arbitrage prices of gold futures and European gold options under the no-arbitrage assumptions. In addition, we obtain that both the gold futures prices and the gold European options prices depend on the seasonality in gold prices. Moreover, one can use our model to predict gold prices in the future if the model parameters are estimated using historical data of gold spot prices. Finally, the challenging thing about the next work is to price American gold options.

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