ชั้นสมสัณฐานและการระบายสีจุดยอดสำหรับกราฟ $C_G(a,b)$

นายวรเวทย์ ลีลาอภิรดี

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ISOMORPHISM CLASSES AND VERTEX COLORING FOR GRAPHS $C_G(a,b)$

Mr. Worrawate Leela-apiradee

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By	Mr. Worrawate Leela-apiradee
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Thesis Advisor	Assistant Professor Yotsanan Meemark, Ph.D.

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……………………………………………….. Dean of the Faculty of Science
(Professor Supot Hannongbua, Dr.rer.nat.)

THESIS COMMITTEE

……………………………………………….. Chairman
(Assistant Professor Chariya Uiyyasathian, Ph.D.)

……………………………………………….. Thesis Advisor
(Assistant Professor Yotsanan Meemark, Ph.D.)

……………………………………………….. Examiner
(Assistant Professor Tuangrat Chaichana, Ph.D.)

……………………………………………….. External Examiner
(Associate Professor Nittiya Pabhapote, Ph.D.)

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ให้ G เป็นกรุปจำกัดสลับที่ ซึ่ง $G \neq \{0\}$ และ $a,b \in G \setminus \{0\}$ ให้ $C_G(a,b)$ เป็นกราฟที่มีเซตของจุดยอด คือ G และเซตของเส้นเชื่อม คือ

$$E = \{\{x,x+a\},\{x,x+b\},\{x,x-a\},\{x,x-b\} : x \in G\}$$

ในวิทยานิพนธ์ฉบับนี้ เราใช้สมบัติของกรุปจำกัดสลับที่ในการทดสอบการสมสัณฐานบนกราฟ $C_G(a,b)$ ที่ได้นิยามไว้ข้างต้น เราศึกษาชั้นสมสัณฐานของกราฟดังกล่าว ทำให้งานของเราเป็นกรณีทั่วไปของนิโคโลโซและไพโทรเปาลิ ซึ่งผลที่ได้คล้ายคลึงกัน เมื่อ G เป็นกรุปวัฏจักร นอกจากนี้ เรายังศึกษาขั้นตอนวิธีและได้วิธีการระบายสีจุดยอดที่ชัดแจ้งสำหรับกราฟ $C_G(a,b)$ โดยที่จุดยอดประชิดกันใช้สีต่างกัน และมีจำนวนสีที่ใช้น้อยที่สุด

ภาควิชา คณิตศาสตร์และวิทยาการคอมพิวเตอร์ ลายมือชื่อนิสิต ....

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Let G be a finite abelian group such that $G \neq \{0\}$ and $a,b \in G \setminus \{0\}$. Let $C_G(a,b)$ be the graph whose vertex set is G and the edge set is given by

$$E = \{\{x,x+a\},\{x,x+b\},\{x,x-a\},\{x,x-b\} : x \in G\}.$$

In this thesis, we use the properties of finite abelian group to derive isomorphism testing on the graph $C_G(a,b)$ defined above. We study classes of isomorphic graphs. This work generalizes Nicoloso and Pietropaoli's paper, which obtain analogous results when G is a cyclic group. In addition, we study the algorithms to give an explicit assignment of colors to the vertices of graph $C_G(a,b)$ such that adjacent vertices receive different colors and the number of colors is minimized.

Department Mathematics and Computer Science Student's Signature ....

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CONTENTS

CHAPTER I
THE GRAPH $C_G(a,b)$

1.1 Introduction

Let G be a finite abelian group such that $G \neq \{0\}$ and $a, b \in G \setminus \{0\}$ with $o(b) \leq o(a)$. Let $C_G(a,b)$ be the undirected graph whose vertex set is G and the edge set is given by

$$E = \{\{x, x+a\}, \{x, x+b\}, \{x, x-a\}, \{x, x-b\} : x \in G\}.$$

We shall assume that $a \neq \pm b$, otherwise $C_G(a,b)$ degenerates into $C_G(a) = C_G(b)$, where $C_G(a)$ is the graph whose vertex set is G and the edge set is given by $E = \{\{x, x+a\}, \{x, x-a\} : x \in G\}$. A connected component of the graph $C_G(a)$ is called an *a-cycle*.

A graph is *k-regular* if all its vertices have the same degree k. Under the above conditions, we can classify $C_G(a,b)$ into three types of regular graph as follows.

Theorem 1.1.1. *Let $a, b \in G \setminus \{0\}$ such that $a \neq \pm b$ and $o(b) \leq o(a)$.*

(1) $C_G(a,b)$ is 2-regular if and only if a and b are elements of order two.

(2) $C_G(a,b)$ is 3-regular if and only if b is an element of order two but a is not.

(3) $C_G(a,b)$ is 4-regular if and only if a and b are not elements of order two.

Proof. (1) Assume that $C_G(a,b)$ is 2-regular. Let x be a vertex of $C_G(a,b)$. Since $a \neq \pm b$, $x + a \neq x \pm b$, so $x + a = x - a$ and $x + b = x - b$. Thus, $2a = 0$ and $2b = 0$. Since $a, b \neq 0$, $o(a) = 2 = o(b)$. Conversely, suppose that a and b are elements of order two. Then $2a = 0$ and $2b = 0$, so $x+a = x-a$ and $x+b = x-b$

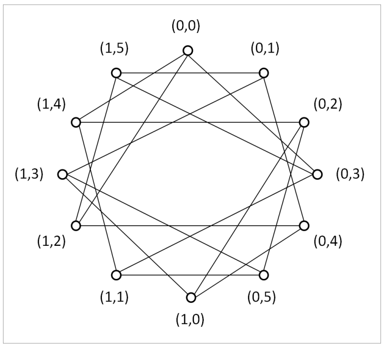
for all $x \in G$. Thus, the edge set of $C_G(a, b)$ is $\{\{x, x+a\}, \{x, x+b\} : x \in G\}$. Since $a \neq b$, the vertices $C_G(a, b)$ is 2-regular.

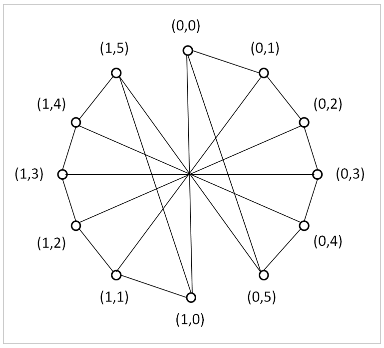
(2) Assume that $C_G(a, b)$ is 3-regular. Let x be a vertex of $C_G(a, b)$. Since $a \neq \pm b$, $x + a \neq x \pm b$, so either $x + a = x - a$ or $x + b = x - b$ Thus, either $2a = 0$ or $2b = 0$. Since $a, b \neq 0$ and $o(b) \leq o(a)$, $o(b) = 2 < o(a)$. Conversely, suppose that b is an element of order two but a is not. Then $2b = 0$ and $2a \neq 0$, so $x+b = x-b$ and $x+a \neq x-a$ for all $x \in G$. Thus, the edge set of $C_G(a, b)$ is $\{\{x, x+a\}, \{x, x-a\}, \{x, x+b\} : x \in G\}$. Since $a \neq \pm b$, $C_G(a, b)$ is 3-regular.

(3) Assume that $C_G(a, b)$ is 4-regular. Let x be a vertex of $C_G(a, b)$. Then $\{x, x+a\}, \{x, x-a\}, \{x, x+b\}, \{x, x-b\}$ are distinct. So $x+a \neq x-a$ and $x+b \neq x-b$, which implies $2a \neq 0$ and $2b \neq 0$. Hence, $o(a) \neq 2$ and $o(b) \neq 2$. Conversely, suppose that a and b are not elements of order two. Then $2a \neq 0$ and $2b \neq 0$, so $x + a \neq x - a$ and $x + b \neq x - b$ for all $x \in G$. Since $a \neq \pm b$, $x + a \neq x \pm b$ for all $x \in G$. Thus, $\{x, x + a\}, \{x, x - a\}, \{x, x + b\}, \{x, x - b\}$ are distinct. Hence, $C_G(a, b)$ is 4-regular.

This completes the proof. □

Example 1.1.2. Since $o((1, 2)) = 6 \neq 2, o((0, 3)) = 2$ and $o((0, 1)) = 6 \neq 2, o((1, 0)) = 2$ in $\mathbb{Z}_2 \times \mathbb{Z}_6$, $C_{\mathbb{Z}_2\times\mathbb{Z}_6}((1, 2), (0, 3))$ and $C_{\mathbb{Z}_2\times\mathbb{Z}_6}((0, 1), (1, 0))$ are 3-regular graphs as respectively shown below.





Remark. *When $G = \langle g \rangle$ is cyclic, G has a unique element of order 2, so $C_G(a, b)$ is not 2-regular. Then it is either 3-regular or 4-regular.*

Next, we give a condition for $C_G(a, b)$ to be connected.

Theorem 1.1.3. *The graph $C_G(a,b)$ is connected if and only if the group G is generated by a and b.*

Proof. Assume that $C_G(a,b)$ is connected. Let $y \in G$ and $y \neq 0$. Then there is a path between vertices 0 and y, so $y = ka + lb$ for some $k, l \in \mathbb{Z}$. Thus, $y \in \langle a, b\rangle$. Hence, $G = \langle a, b\rangle$. Conversely, suppose that $G = \langle a, b\rangle$. Let x and y be two distinct vertices in $C_G(a,b)$. Then $y - x \in G = \langle a, b\rangle$. Thus, $y - x = ka + lb$ for some $k, l \in \mathbb{Z}$. This means that there is a path between x and y. Hence, $C_G(a,b)$ is connected. □

Corollary 1.1.4. *Let $G = \langle g\rangle$ be a cyclic group. The graph $C_G(a,b)$ is connected if and only if $g = ka + lb$ for some $k, l \in \mathbb{Z}$.*

Proof. It directly follows from Theorem 1.1.3 because $G = \langle g\rangle = \langle a, b\rangle$ is equivalent to $g = ka + lb$ for some $k, l \in \mathbb{Z}$. □

Two graphs (V,E) and (V',E') are said to be *isomorphic*, denoted by $(V,E) \simeq (V',E')$, if there exists a bijection $f : V \to V'$ such that $\{x,y\} \in E$ if and only if $\{f(x), f(y)\} \in E'$ for all $x, y \in V$. Note that $C_G(a,b)$ and $C_G(b,a)$ are trivially isomorphic. Moreover, since the edge sets of the graphs $C_G(a,b)$, $C_G(-a,b)$, $C_G(a,-b)$, $C_G(-a,-b)$ are the same sets, they are also isomorphic.

We can generalize Theorem 1.1.3 as follows.

Theorem 1.1.5. *If $H = \langle a, b\rangle$, then the graph $C_G(a,b)$ has $[G : H] = |G|/|H|$ connected components, each of which is isomorphic to $C_H(a,b)$.*

Proof. Let $x \in G$ and let $x + C_H(a,b)$ be the translation graph whose vertex set is $x + H$ and edge set is $\{\{x+h, x+h+a\}, \{x+h, x+h+b\}, \{x+h, x+h-a\}, \{x+h, x+h-b\} : h \in H\}$. Clearly, $x + C_H(a,b)$ is isomorphic to $C_H(a,b)$. By Theorem 1.1.3, $C_H(a,b)$ is connected, so $x + C_H(a,b)$ is a connected component of $C_G(a,b)$ for all $x \in G$. Since $\bigcup_{x\in G}\big(x + C_H(a,b)\big) = C_G(a,b)$ and $|\{x + C_H(a,b) : x \in G\}| = |\{x + H : x \in G\}| = [G : H] = |G|/|H|$, we have $C_G(a,b)$ has $|G|/|H|$ connected components and each component is isomorphic to $C_H(a,b)$. □

Remark. *If* $G = \mathbb{Z}_n$ *is a cyclic group of order* $n \geq 2$*, then* $H = \langle a, b\rangle = \langle \gcd(a, b)\rangle$*, so* $|H| = \frac{n}{\gcd(n,\gcd(a,b))} = \frac{n}{\gcd(n,a,b)}$ *and* $C_G(a, b)$ *has* $\gcd(n, a, b)$ *connected components.*

Furthermore, for a cyclic group G, Nicoloso and Pietropaoli [2] studied the isomorphism testing problem for connected circulant graphs $C_G(a, b)$ and derived a necessary and sufficient condition to test whether two circulant graphs $C_G(a, b)$ and $C_G(a', b')$ are isomorphic. They proposed an elementary method to solve isomorphism testing, which is purely combinatorial and new for the problem.

In addition, properties of the classes of mutually isomorphic graphs were analyzed. Later, they studied vertex coloring for connected circulant graphs $C_G(a, b)$ in [3]. They provided an algorithm to find an assignment of colors to the vertices of circulant graph $C_G(a, b)$ such that adjacent vertices receive different colors and the number of colors is minimized. The vertex coloring of connected graph $C_G(a, b)$ is based on the representative matrix of $C_G(a, b)$.

In this work, we let G be any finite abelian group and use their properties to define the representative matrix of $C_H(a, b)$ and derive isomorphism testing on the graph $C_G(a, b)$ defined above. We study classes of isomorphic graphs. This generalizes Nicoloso and Pietropaoli's paper [2]. In addition, we shall study the algorithms in [3] to give an explicit assignment of colors to the vertices of graph $C_G(a, b)$ such that adjacent vertices receive different colors and the number of colors is minimized.

The thesis is organized as follows. In the next section, we represent our graph $C_G(a, b)$ as the matrix $M_G(a, b)$ and study its properties including a-cycles, b-cycles, column jumps and block jumps. Isomorphism criteria are studied in Chapter II. The final chapter gives some results on chromatic numbers and explicit vertex coloring schemes from the algorithms presented in [3].

1.2 Cycles and matrices

In the previous section, we learn that each connected components of $C_G(a,b)$ is isomorphic to $C_H(a,b)$ where $H = \langle a,b\rangle$. Now we start with the definition of representative matrix of $C_H(a,b)$ denote by $M_G(a,b)$, which will be used to prove the isomorphism testing as our main theorem in the next chapter. The representative matrix $M_G(a,b)$ for the graph $C_H(a,b)$ can be defined as the following table.

0	a	$2a$	$\dots$	$(o(a)-1)a$
b	$b+a$	$b+2a$	$\dots$	$b+(o(a)-1)a$
$2b$	$2b+a$	$2b+2a$	$\dots$	$2b+(o(a)-1)a$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$(o(b+\langle a\rangle)-1)b$	$(o(b+\langle a\rangle)-1)b+a$	$(o(b+\langle a\rangle)-1)b+2a$	$\dots$	$(o(b+\langle a\rangle)-1)b+(o(a)-1)a$

Lemma 1.2.1. *Let $a,b \in G \setminus \{0\}$ with $a \neq \pm b$ and $H = \langle a,b\rangle$. Then*

$$H/\langle a\rangle = \{\langle a\rangle, b+\langle a\rangle, 2b+\langle a\rangle, \dots, (o(b+\langle a\rangle)-1)b+\langle a\rangle\} = \langle b+\langle a\rangle\rangle.$$

In particular, $o(b+\langle a\rangle) = \frac{|H|}{o(a)}$.

Proof. Clearly, $\langle b+\langle a\rangle\rangle \subset H/\langle a\rangle$. Let $x \in H$. Then $x = ka+lb$ for some $k,l \in \mathbb{Z}$, so $x+\langle a\rangle = ka+lb+\langle a\rangle = lb+\langle a\rangle \in \langle b+\langle a\rangle\rangle$. Hence, $H/\langle a\rangle = \langle b+\langle a\rangle\rangle$. Moreover, $o(b+\langle a\rangle) = |\langle b+\langle a\rangle\rangle| = |H/\langle a\rangle| = \frac{|H|}{o(a)}$. □

From the above matrix, $M_G(a,b)$ has $r = o(b+\langle a\rangle)$ rows and $c = o(a)$ columns. The number of entries of $M_G(a,b)$ is $o(b+\langle a\rangle)o(a) = |H|$. Each row corresponds to a coset in the quotient $H/\langle a\rangle$ and all entries of $M_G(a,b)$ are distinct. In other words, vertices of $C_H(a,b)$ appear exactly once.

Two vertices $x,y \in G$ are said to be *a-adjacent* and $\{x,y\}$ is an *a-edge* if $y-x = \pm a$ and x,y are in an a-cycle if $y-x \in \langle a\rangle$. Notice that two consecutive entries of a row are a-adjacent and the first and the last entries of a same row also are a-adjacent, so that each row of $M_G(a,b)$ corresponds to an a-cycle of $C_H(a,b)$. Thus, $C_H(a,b)$ consists of $o(b+\langle a\rangle)$ a-cycles of length $o(a)$. In addition, two consecutive entries of a column are b-adjacent, that is, their difference is $\pm b$.

However, the first and the last entries of a same column are not necessarily b-adjacent. It depends on the *column-jump* of $M_G(a,b)$ denoted by $\lambda_G(a,b)$.

Lemma 1.2.2. *Let $a, b \in G \setminus \{0\}$ with $a \neq \pm b$. Then there exists a unique number $\lambda_G(a,b) \in \{0, 1, \ldots, o(a)-1\}$, called the column-jump of $M_G(a,b)$, satisfying*

$$rb = \lambda_G(a,b)a. \tag{1.2.1}$$

Proof. Since $b + \langle a \rangle \in G/\langle a \rangle$, we have

$$rb + \langle a \rangle = o(b + \langle a \rangle)b + \langle a \rangle = o(b + \langle a \rangle)(b + \langle a \rangle) = \langle a \rangle,$$

so $rb \in \langle a \rangle$. Hence, there exists a unique $\lambda_G(a,b) \in \{0, 1, \ldots, o(a)-1\}$ such that $rb = \lambda_G(a,b)a$ as desired. □

Some remarks on the column-jump of $M_G(a,b)$ are studied in the next theorem.

Theorem 1.2.3. *Let $a, b \in G \setminus \{0\}$ with $a \neq \pm b$.*

(1) $\lambda_G(-a,b) = 0$ if and only if $\lambda_G(a,b) = 0$.

(2) $\lambda_G(-a,b) = o(a) - \lambda_G(a,b)$ if $\lambda_G(a,b)$ and $\lambda_G(-a,b)$ are nonzero.

(3) $\lambda_G(a,-b) = 0$ if and only if $\lambda_G(a,b) = 0$.

(4) $\lambda_G(a,-b) = o(a) - \lambda_G(a,b)$ if $\lambda_G(a,b)$ and $\lambda_G(a,-b)$ are nonzero.

(5) $\lambda_G(-a,-b) = \lambda_G(a,b)$.

Proof. (1) Assume $\lambda_G(-a,b) = 0$. By Lemma 1.2.2, we have $\lambda_G(a,b)a = o(b + \langle a \rangle)b = o(b + \langle -a \rangle)b = -\lambda_G(-a,b)a = 0$. Since $\lambda_G(a,b) \in \{0, 1, \ldots, o(a)-1\}$, $\lambda_G(a,b) = 0$. A similar argument proves the reverse.

(2) From Lemma 1.2.2, we have $(o(a) - \lambda_G(-a,b))a = o(a)a - \lambda_G(-a,b))a = o(b + \langle -a \rangle)b = o(b + \langle a \rangle)b = \lambda_G(a,b)a$. Then $(\lambda_G(a,b) + \lambda_G(-a,b) - o(a))a = 0$. Since $\lambda_G(a,b), \lambda_G(-a,b) \in \{1, 2, \ldots, o(a)-1\}$, $\lambda_G(-a,b) = o(a) - \lambda_G(a,b)$.

(3) Analogous to the proof of (1).

(4) Analogous to the proof of (2).

(5) If $\lambda_G(a,b) = 0$, then by (3) and (1), we have $\lambda_G(a,b) = 0 \Leftrightarrow \lambda_G(a,-b) = 0 \Leftrightarrow \lambda_G(-a,-b) = 0$, so $\lambda_G(-a,-b) = \lambda_G(a,b)$. Assume that $\lambda_G(a,b) \neq 0$. From (2) and (4), we have

$$\lambda_G(-a,-b) = o(a) - \lambda_G(a,-b) = o(a) - (o(a) - \lambda_G(a,b)) = \lambda_G(a,b).$$

The proof completes. □

Theorem 1.2.4. *Let $a, b \in G$. Then* $\dfrac{c}{o(a+\langle b\rangle)} = \dfrac{o(b)}{r}$.

Proof. If a or $b = 0$, the conclusion is trivial. Assume that $a, b \in G \setminus \{0\}$. From Lemma 1.2.1, we have $r = \dfrac{|H|}{c}$ and $o(a+\langle b\rangle) = \dfrac{|H|}{o(b)}$. Then $rc = |H| = o(a+\langle b\rangle)o(b)$, so $\dfrac{c}{o(a+\langle b\rangle)} = \dfrac{o(b)}{r}$. □

Theorem 1.2.5. *Let $a, b \in G \setminus \{0\}$ with $a \neq \pm b$ and write $\lambda = \lambda_G(a,b) \neq 0$. Then* $\gcd(\lambda, o(a)) = o(a+\langle b\rangle)$.

Proof. From Eq. (1.2.1) and $r = o(b+\langle a\rangle) \mid o(b)$,

$$\frac{o(b)}{r} = \frac{o(b)}{\gcd(r, o(b))} = o(rb) = o(\lambda a) = \frac{o(a)}{\gcd(\lambda, o(a))}.$$

Thus, we have $\gcd(\lambda, o(a)) = o(a) \cdot \frac{r}{o(b)} = o(a+\langle b\rangle)$ by Theorem 1.2.4. □

From the above theorem, $\langle \lambda a\rangle = \{0, \lambda a, 2\lambda a, \ldots, \big(\frac{o(a)}{o(a+\langle b\rangle)} - 1\big)\lambda a\}$. This implies that a b-cycle of $C_H(a,b)$ consists of $h = \frac{o(a)}{o(a+\langle b\rangle)} = \frac{c}{o(a+\langle b\rangle)}$ columns. As a consequence, $M_G(a,b)$ can be partitioned into h equally sized submatrices, the blocks denoted by β_l where $l \in \{0, 1, \ldots, h-1\}$. The block β_l is defined on all the r rows and $o(a+\langle b\rangle)$ consecutive columns from column $lo(a+\langle b\rangle)+1$ to $(l+1)o(a+\langle b\rangle)$.

Since $o(a+\langle b\rangle) \mid \lambda_G(a,b)$, that is, $\lambda_G(a,b)$ is a multiple of $o(a+\langle b\rangle)$. From this, we define the *block-jump* of $M_G(a,b)$ to be

$$\Lambda_G(a,b) = \frac{\lambda_G(a,b)}{o(a+\langle b\rangle)}, \tag{1.2.2}$$

where $\Lambda_G(a,b) \in \{0,1,\dots,h-1\}$. Moreover, we have

Theorem 1.2.6. *Let $a,b \in G \setminus \{0\}$ with $a \neq \pm b$.*

(1) $\Lambda_G(-a,b) = h - \Lambda_G(a,b)$.

(2) $\Lambda_G(a,-b) = h - \Lambda_G(a,b)$.

(3) $\Lambda_G(-a,-b) = \Lambda_G(a,b)$.

Proof. We use the definition of block-jump and Theorem 1.2.3 to prove (1)–(3) as follows.

(1) $\Lambda_G(-a,b) = \frac{\lambda_G(-a,b)}{o(-a+\langle b\rangle)} = \frac{o(a)-\lambda_G(a,b)}{o(a+\langle b\rangle)} = \frac{o(a)}{o(a+\langle b\rangle)} - \frac{\lambda_G(a,b)}{o(a+\langle b\rangle)} = h - \Lambda_G(a,b)$.

(2) $\Lambda_G(a,-b) = \frac{\lambda_G(a,-b)}{o(a+\langle -b\rangle)} = \frac{o(a)-\lambda_G(a,b)}{o(a+\langle b\rangle)} = \frac{o(a)}{o(a+\langle b\rangle)} - \frac{\lambda_G(a,b)}{o(a+\langle b\rangle)} = h - \Lambda_G(a,b)$.

(3) $\Lambda_G(-a,-b) = \frac{\lambda_G(-a,-b)}{o(-a+\langle -b\rangle)} = \frac{\lambda_G(a,b)}{o(a+\langle b\rangle)} = \Lambda_G(a,b)$.

This completes the proof. □

Example 1.2.7. Let $a=(1,2), b=(0,3)$ be in $G=\mathbb{Z}_2\times\mathbb{Z}_6$. Since $\mathbb{Z}_2\times\mathbb{Z}_6 = \langle(1,2),(0,3)\rangle$, the graph $C_{\mathbb{Z}_2\times\mathbb{Z}_6}((1,2),(0,3))$ is connected. We have $r = o((0,3)+\langle(1,2)\rangle) = 2$ and

$$(0,0) = 2(0,3) = r(0,3) = \lambda_G((1,2),(0,3))(1,2),$$

so $\lambda_G((1,2),(0,3)) = 0$, which implies $\Lambda_G((1,2),(0,3)) = \frac{\lambda_G((1,2),(0,3))}{o((1,2)+\langle(0,3)\rangle)} = \frac{0}{6} = 0$. Since $c = o((1,2)) = 6$, $M_G(a,b)$ has $r=2$ rows and $c=6$ columns. The representative matrix $M_G((1,2),(0,3))$ for the graph $C_{\mathbb{Z}_2\times\mathbb{Z}_6}((1,2),(0,3))$ is the following table (the blocks are separated by double lines)

$(0,0)$	$(1,2)$	$(0,4)$	$(1,0)$	$(0,2)$	$(1,4)$
$(0,3)$	$(1,5)$	$(0,1)$	$(1,3)$	$(0,5)$	$(1,1)$

Example 1.2.8. Let $a = (0,1,2), b = (1,1,1)$ be in $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$. Since $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 = \langle (0,1,2),(1,1,1)\rangle$, the graph $C_{\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3}((0,1,2),(1,1,1))$ is connected. We have $r = o((1,1,1) + \langle (0,1,2)\rangle) = 2$ and

$$(0,0,2) = 2(1,1,1) = r(1,1,1) = \lambda_G((0,1,2),(1,1,1))(0,1,2),$$

so $\lambda_G((0,1,2),(1,1,1)) = 4$, which implies $\Lambda_G((0,1,2),(1,1,1)) = \frac{\lambda_G((0,1,2),(1,1,1))}{o((0,1,2)+\langle(1,1,1)\rangle)} = \frac{4}{2} = 2$. Since $c = o((0,1,2)) = 6$, $M_G(a,b)$ has $r = 2$ rows and $c = 6$ columns. The representative matrix $M_G((0,1,2),(1,1,1))$ for the graph $C_{\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3}((0,1,2),(1,1,1))$ is the following table (the blocks are separated by double lines)

$(0,0,0)$	$(0,1,2)$	$(0,0,1)$	$(0,1,0)$	$(0,0,2)$	$(0,1,1)$
$(1,1,1)$	$(1,0,0)$	$(1,1,2)$	$(1,0,1)$	$(1,1,0)$	$(1,0,2)$

CHAPTER II
ISOMORPHISM TESTING

In the previous chapter, we defined the representative matrix $M_G(a,b)$ for the graph $C_H(a,b)$, which has $r = o(b + \langle a\rangle)$ rows and $c = o(a)$ columns. Moreover, we defined the block-jump of $M_G(a,b)$ denoted by $\Lambda_G(a,b)$, which is a constant in $\{0,\ldots,h-1\}$, where $h = \frac{o(a)}{o(a+\langle b\rangle)}$. In this chapter, we study the isomorphism testing problem for the graphs $C_H(a,b)$ and use the properties of $M_G(a,b)$ to derive a necessary and sufficient condition to test whether two graphs $C_G(a,b)$ and $C_G(a',b')$ are isomorphic. We analyze more results when G is cyclic in Section 2.2.

2.1 Isomorphism Theorem

Our main theorem is as follows.

Theorem 2.1.1. *Let $a, a', b, b' \in G \setminus \{0\}$ such that $a \neq \pm b, a' \neq \pm b', o(b) \leq o(a)$ and $o(b') \leq o(a')$. Then $C_H(a,b)$ and $C_{H'}(a',b')$ are isomorphic if and only if either one of the following two conditions holds:*

(1) $r = r'$, $o(b') = o(b) < c = c'$ and $\Lambda_G(a,b) = \pm\Lambda_G(a',b')$;

(2) $r = r'$, $o(b') = o(b) = c = c'$ and either $\Lambda_G(a,b) = \pm\Lambda_G(a',b')$ or $\Lambda_G(a,b) = \pm\Lambda_G(b',a')$,

where $H = \langle a,b\rangle, H' = \langle a',b'\rangle, r = o(b+\langle a\rangle), r' = o(b'+\langle a'\rangle), c = o(a), c' = o(a'), \Lambda_G(a,b) = \frac{\lambda_G(a,b)}{o(a+\langle b\rangle)}$ and $\Lambda_G(a',b') = \frac{\lambda_G(a',b')}{o(a'+\langle b'\rangle)}$, where $\lambda_G(a,b)$ and $\lambda_G(a',b')$ are the column-jump of $M_G(a,b)$ and $M_G(a',b')$ respectively.

Proof. **Case 1.** $r = r'$, $o(b') = o(b) < c = c'$ and $\Lambda_G(a,b) = \pm\Lambda_G(a',b')$. By Theorem 1.2.4, $o(a+\langle b\rangle) = \frac{r}{o(b)} \cdot c = \frac{r'}{o(b')} \cdot c' = o(a'+\langle b'\rangle)$ and observe that $h = \frac{c}{o(a+\langle b\rangle)} = \frac{c'}{o(a'+\langle b'\rangle)} = h'$. Then $M_G(a,b)$ and $M_G(a',b')$ have the same number of rows and columns and the same size of the blocks.

1.1 $\Lambda_G(a,b) = \Lambda_G(a',b')$. Then $\lambda_G(a',b') = \Lambda_G(a',b')o(a' + \langle b' \rangle) = \Lambda_G(a,b)o(a + \langle b \rangle) = \lambda_G(a,b)$. Let $\lambda = \lambda_G(a,b) = \lambda_G(a',b')$. The representative matrices $M_G(a,b)$ and $M_G(a',b')$ are shown below.

0	$\ldots$	$\ldots$	$\ldots$	$(\lambda-1)a$	λa	$\ldots$	$(c-1)a$
$\vdots$							$\vdots$
$(r-1)b$	$\ldots$	$(r-1)b+(c-\lambda-1)a$	$(r-1)b+(c-\lambda)a$	$\ldots$	$\ldots$	$\ldots$	$(r-1)b+(c-1)a$

0	$\ldots$	$\ldots$	$\ldots$	$(\lambda-1)a'$	$\lambda a'$	$\ldots$	$(c-1)a'$
$\vdots$							$\vdots$
$(r-1)b'$	$\ldots$	$(r-1)b'+(c-\lambda-1)a'$	$(r-1)b'+(c-\lambda)a'$	$\ldots$	$\ldots$	$\ldots$	$(r-1)b'+(c-1)a'$

We see that $\{\{(r-1)b+ja, (\lambda+j)a\} : 0 \leq j \leq c-\lambda-1\}$ and $\{\{(r-1)b+(c-\lambda+j)a, ja\} : 0 \leq j \leq \lambda-1\}$ contain boundary b-edges connecting an entry of the last row with an entry of the first row of $M_G(a,b)$. While, $\{\{(r-1)b' + ja', (\lambda+j)a'\} : 0 \leq j \leq c-\lambda-1\}$ and $\{\{(r-1)b' + (c-\lambda+j)a', ja'\} : 0 \leq j \leq \lambda-1\}$ contain boundary b'-edges connecting an entry of the last row with an entry of the first row of $M_G(a',b')$. We define a bijection $f : H \to H'$ by $f(ib+ja) = ib' + ja'$ where $i \in \{0,1,\ldots,r-1\}$ and $j \in \{0,1,\ldots,c-1\}$. Then f is a bijection preserving the adjacency condition, so $C_H(a,b)$ and $C_{H'}(a',b')$ are isomorphic.

1.2 $\Lambda_G(a,b) = -\Lambda_G(a',b')$. Then $\lambda_G(a',b') = \Lambda_G(a',b')o(a' + \langle b'\rangle) = (h - \Lambda_G(a,b))o(a + \langle b\rangle) = c - \lambda_G(a,b)$. Let $\lambda = \lambda_G(a,b)$. The representative matrices $M_G(a,b)$ and $M_G(a',b')$ are shown below.

0	$\dots$	$(\lambda-1)a$	λa	$\dots$	$(c-\lambda-1)a$	$(c-\lambda)a$	$\dots$	$(c-1)a$
$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$
$(r-1)b$	$\dots$	$(r-1)b+(\lambda-1)a$	$(r-1)b+\lambda a$	$\dots$	$(r-1)b+(c-\lambda-1)a$	$(r-1)b+(c-\lambda)a$	$\dots$	$(r-1)b+(c-1)a$

0	$\dots$	$(\lambda-1)a'$	$\lambda a'$	$\dots$	$(c-\lambda-1)a'$	$(c-\lambda)a'$	$\dots$	$(c-1)a'$
$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$
$(r-1)b'$	$\dots$	$(r-1)b'+(\lambda-1)a'$	$(r-1)b'+\lambda a'$	$\dots$	$(r-1)b'+(c-\lambda-1)a'$	$(r-1)b'+(c-\lambda)a'$	$\dots$	$(r-1)b'+(c-1)a'$

We see that $\{\{(r-1)b+ja, (\lambda+j)a\} : 0 \le j \le c-\lambda-1\}$ and $\{\{(r-1)b+(c-\lambda+j)a, ja\} : 0 \le j \le \lambda-1\}$ contain boundary b-edges connecting an entry of the last row with an entry of the first row of $M_G(a,b)$. While, $\{\{(r-1)b'+ja', (c-\lambda+j)a'\} : 0 \le j \le \lambda-1\}$ and $\{\{(r-1)b' + (\lambda+j)a', ja'\} : 0 \le j \le c-\lambda-1\}$ contain boundary b'-edges connecting an entry of the last row with an entry of the first row of $M_G(a',b')$. We define a bijection $f : H \to H'$ by $f(ib+ja) = ib' + (c-j-1)a'$ where $i \in \{0,1,\dots,r-1\}$ and $j \in \{0,1,\dots,c-1\}$. Then f is a bijection preserving the adjacency condition, so $C_H(a,b)$ and $C_{H'}(a',b')$ are isomorphic.

Case 2. *$r = r'$, $o(b') = o(b) = c = c'$ and either $\Lambda_G(a,b) = \pm\Lambda_G(a',b')$ or $\Lambda_G(a,b) = \pm\Lambda_G(b',a')$.* Clearly, if $\Lambda_G(a,b) = \pm\Lambda_G(a',b')$, then $C_H(a,b)$ and $C_{H'}(a',b')$ are isomorphic. Suppose $\Lambda_G(a,b) = \pm\Lambda_G(b',a')$. Then we can apply Case 1 by swapping a' and b', we have $C_H(a,b) \simeq C_{H'}(b',a')$. From trivially isomorphic, $C_H(a,b)$ and $C_{H'}(a',b')$ are isomorphic as desired.

Conversely, assume that $C_H(a,b)$ and $C_{H'}(a',b')$ are isomorphic. Then the representative matrices $M_G(a,b)$ and $M_G(a',b')$ have the same number of rows and columns and the same size of the blocks, which implies $r = r', c = c'$ and $o(a + \langle b\rangle) = o(a' + \langle b'\rangle)$. So $o(b) = \frac{rc}{o(a+\langle b\rangle)} = \frac{r'c'}{o(a'+\langle b'\rangle)} = o(b')$. Since we have assumed that $o(b) \leq c$, we clearly obtain in the following two conditions.

(1) $r = r'$, $o(b') = o(b) < c = c'$ and $\Lambda_G(a,b) = \pm\Lambda_G(a',b')$;

(2) $r = r'$, $o(b') = o(b) = c = c'$ and either $\Lambda_G(a,b) = \pm\Lambda_G(a',b')$ or $\Lambda_G(a,b) = \pm\Lambda_G(b',a')$,

as desired. □

Lemma 2.1.2. *Let $a, a', b, b' \in G \setminus \{0\}$ such that $a \neq \pm b$ and $a' \neq \pm b'$.*

(1) If $r = r', o(b) = o(b')$ and $c = c'$, then $rb' = \pm\lambda_G(a,b)a'$ if and only if $\Lambda_G(a,b) = \pm\Lambda_G(a',b')$.

(2) If $r = r'$ and $o(b') = o(b) = c = c'$, then $ra' = \pm\lambda_G(a,b)b'$ if and only if $\Lambda_G(a,b) = \pm\Lambda_G(b',a')$.

Here, $r = o(b + \langle a\rangle), r' = o(b' + \langle a'\rangle), c = o(a), c' = o(a'), \Lambda_G(a,b) = \frac{\lambda_G(a,b)}{o(a+\langle b\rangle)}$ and $\Lambda_G(a',b') = \frac{\lambda_G(a',b')}{o(a'+\langle b'\rangle)}$, where $\lambda_G(a,b)$ and $\lambda_G(a',b')$ are the column-jump of $M_G(a,b)$ and $M_G(a',b')$ respectively.

Proof. (1) Let $r = r', o(b) = o(b')$ and $c = c'$. By Theorem 1.2.4, we have

$$o(a + \langle b\rangle) = \frac{r}{o(b)} \cdot c = \frac{r'}{o(b')} \cdot c' = o(a' + \langle b'\rangle).$$

Assume that $rb' = \pm\lambda_G(a,b)a'$. From

$$\lambda_G(a,b)a' = \pm rb' = \pm r'b' = \pm\lambda_G(a',b')a',$$

so $(\lambda_G(a,b) \pm \lambda_G(a',b'))a' = 0$. Then $c' = o(a')$ divides $\lambda_G(a,b) \pm \lambda_G(a',b')$, which implies $\lambda_G(a,b) \pm \lambda_G(a',b') = 0$. By Eq. (1.2.2), $\Lambda_G(a,b) = \frac{\lambda_G(a,b)}{o(a+\langle b\rangle)} = \pm\frac{\lambda_G(a',b')}{o(a'+\langle b'\rangle)} = \pm\Lambda_G(a',b')$. Conversely, assume that $\Lambda_G(a,b) = \pm\Lambda_G(a',b')$. By Eq. (1.2.2), we have $\lambda_G(a,b) = \Lambda_G(a,b)o(a+\langle b\rangle) = \pm\Lambda_G(a',b')o(a'+\langle b'\rangle) = \pm\lambda_G(a',b')$, so

$$rb' = r'b' = \lambda_G(a',b')a' = \pm\lambda_G(a,b)a'$$

as desired.

(2) We can apply (1) by swapping a' and b'.

Hence, we have the lemma. □

Corollary 2.1.3. *Let $a, a', b, b' \in G \setminus \{0\}$ such that $a \neq \pm b, a' \neq \pm b', o(b) \leq o(a)$ and $o(b') \leq o(a')$. Then $C_G(a,b)$ and $C_G(a',b')$ are isomorphic if and only if either one of the following two conditions holds:*

(1) $r = r'$, $o(b') = o(b) < c = c'$ and $rb' = \pm\lambda_G(a,b)a'$;

(2) $r = r'$, $o(b') = o(b) = c = c'$ and either $rb' = \pm\lambda_G(a,b)a'$ or $ra' = \pm\lambda_G(a,b)b'$,

where $r = o(b+\langle a\rangle), r' = o(b'+\langle a'\rangle), c = o(a), c' = o(a')$ and $\lambda_G(a,b)$ is the column-jump of $M_G(a,b)$.

Proof. Let $H = \langle a,b\rangle$ and $H' = \langle a',b'\rangle$. Assume $C_H(a,b)$ and $C_{H'}(a',b')$ are isomorphic. Then $|H| = |H'|$, which implies $C_G(a,b)$ and $C_G(a',b')$ have the same number of connected components. Moreover, each connected components of $C_G(a,b)$ is isomorphic to $C_H(a,b)$ and each connected components of $C_G(a',b')$ is isomorphic to $C_{H'}(a',b')$ by Theorem 1.1.5. So $C_G(a,b)$ and $C_G(a',b')$ are isomorphic. Clearly, if $C_G(a,b)$ and $C_G(a',b')$ are isomorphic, then $C_H(a,b)$ and $C_{H'}(a',b')$ are isomorphic. Hence, this corollary follows from Theorem 2.1.1 and Lemma 2.1.2. □

We give examples to demonstrate the above corollary.

Example 2.1.4. Let $a = (1,0), a' = (0,1), b = (0,2), b' = (2,0)$ be in $G = \mathbb{Z}_4 \times \mathbb{Z}_4$. Then $o(b') = o((2,0)) = 2 = o((0,2)) = o(b) < c = o((1,0)) = 4 = o((0,1)) = c'$ and $r = o((0,2) + \langle(1,0)\rangle) = 2 = o((2,0) + \langle(0,1)\rangle) = r'$. From Lemma 1.2.2, since

$$(0,0) = 2(0,2) = rb = \lambda_G(a,b)a = \lambda_G(a,b)(1,0)$$

for some $\lambda_G(a,b) \in \{0,1,2,3 = c-1\}$, $\lambda_G(a,b) = 0$. Thus,

$$rb' = 2(2,0) = (0,0) = 0(0,1) = \lambda_G(a,b)a'.$$

By Corollary 2.1.3, $C_{\mathbb{Z}_4 \times \mathbb{Z}_4}((1,0),(0,2))$ is isomorphic to $C_{\mathbb{Z}_4 \times \mathbb{Z}_4}((0,1),(2,0))$.

Example 2.1.5. Let $a = (1,0), a' = (1,1), b = (0,1), b' = (2,0)$ be in $G = \mathbb{Z}_4 \times \mathbb{Z}_4$. Since $o(b') = o((2,0)) = 2 \neq 4 = o((0,1)) = o(b)$, $C_{\mathbb{Z}_4 \times \mathbb{Z}_4}((1,0),(0,1))$ and $C_{\mathbb{Z}_4 \times \mathbb{Z}_4}((1,1),(2,0))$ are not isomorphic by Corollary 2.1.3.

We quote two results on finite abelian groups as follows.

Theorem 2.1.6. *[1] Let G be a finite abelian group. Then there exist integers $n_1, \ldots, n_t > 1$ such that*

$$G \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_t}.$$

Theorem 2.1.7. *[1] Let $G_1, G_2, \ldots, G_t$ be finite abelian groups and $(a_1, a_2, \ldots, a_t) \in \prod_{i=1}^{t} G_i$. Then*

$$o((a_1, a_2, \ldots, a_t)) = \operatorname{lcm}(o(a_1), o(a_2), \ldots, o(a_t)),$$

where $o(a_i)$ denotes order of a_i in G_i for all $i \in \{1, 2, \ldots, t\}$.

The next corollary gives an easier way to compute the order of elements.

Corollary 2.1.8. *Let $a = (a_1, a_2, \ldots, a_t), b = (b_1, b_2, \ldots, b_t) \in \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_t}$ where $n_1, \ldots, n_t > 1$. Then*

(1) $o(a) = \text{lcm}\Big(\frac{n_1}{\gcd(n_1,a_1)}, \frac{n_2}{\gcd(n_2,a_2)}, \ldots, \frac{n_t}{\gcd(n_t,a_t)}\Big)$.

(2) $|H| = \dfrac{o(a)\cdot o(b)}{|\langle a\rangle \cap \langle b\rangle|}$, *where* $H = \langle a, b\rangle$.

(3) $o(b + \langle a\rangle) = \dfrac{|H|}{o(a)}$.

Example 2.1.9. Let $a = (1, 2), a' = (0, 1), b = (0, 3), b' = (1, 0)$ be in $G = \mathbb{Z}_2 \times \mathbb{Z}_6$. By Example 1.2.7, we have $r = o((0, 3) + \langle(1, 2)\rangle) = 2$ and $\lambda_G((1, 2), (0, 3)) = 0$, so

$$rb' = 2(1, 0) = (0, 0) = 0(0, 1) = \lambda_G(a, b)a'.$$

From

$$\begin{aligned}
c = o(a) = o((1,2)) &= \text{lcm}\Big(\tfrac{2}{\gcd(2,1)}, \tfrac{6}{\gcd(6,2)}\Big) = \text{lcm}(2,3) = 6,\\
c' = o(a') = o((0,1)) &= \text{lcm}\Big(\tfrac{2}{\gcd(2,0)}, \tfrac{6}{\gcd(6,1)}\Big) = \text{lcm}(1,6) = 6,\\
o(b) = o((0,3)) &= \text{lcm}\Big(\tfrac{2}{\gcd(2,0)}, \tfrac{6}{\gcd(6,3)}\Big) = \text{lcm}(1,2) = 2,\\
o(b') = o((1,0)) &= \text{lcm}\Big(\tfrac{2}{\gcd(2,1)}, \tfrac{6}{\gcd(6,0)}\Big) = \text{lcm}(2,1) = 2
\end{aligned}$$

and $\langle a'\rangle \cap \langle b'\rangle = \langle(0, 1)\rangle \cap \langle(1, 0)\rangle = \{(0, 0)\}$, we have $o(b') = o(b) < c = c'$ and $|H'| = \frac{o(a')\cdot o(b')}{|\langle a'\rangle\cap\langle b'\rangle|} = \frac{6(2)}{1} = 12$, which imply $r' = o(b' + \langle a'\rangle) = \frac{|H'|}{o(a')} = \frac{12}{6} = 2 = r$. By Corollary 2.1.3, $C_{\mathbb{Z}_2\times\mathbb{Z}_6}((1, 2), (0, 3))$ is isomorphic to $C_{\mathbb{Z}_2\times\mathbb{Z}_6}((0, 1), (1, 0))$.

Example 2.1.10. Let $a = (0, 1, 2), a' = (1, 0, 1), b = (1, 1, 1), b' = (0, 1, 1)$ be in $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$. From Example 1.2.8, we have $r = o((1, 1, 1) + \langle(0, 1, 2)\rangle) = 2$ and $\lambda_G((0, 1, 2), (1, 1, 1)) = 4$, so

$$rb' = 2(0, 1, 1) = (0, 0, 2) = -4(1, 0, 1) = -\lambda_G(a, b)a'.$$

From

$$\begin{aligned}
c = o(a) = o((0,1,2)) &= \operatorname{lcm}\left(\tfrac{2}{\gcd(2,0)}, \tfrac{2}{\gcd(2,1)}, \tfrac{3}{\gcd(3,2)}\right) = \operatorname{lcm}(1,2,3) = 6,\\
c' = o(a') = o((1,0,1)) &= \operatorname{lcm}\left(\tfrac{2}{\gcd(2,1)}, \tfrac{2}{\gcd(2,0)}, \tfrac{3}{\gcd(3,1)}\right) = \operatorname{lcm}(2,1,3) = 6,\\
o(b) = o((1,1,1)) &= \operatorname{lcm}\left(\tfrac{2}{\gcd(2,1)}, \tfrac{2}{\gcd(2,1)}, \tfrac{3}{\gcd(3,1)}\right) = \operatorname{lcm}(2,2,3) = 6,\\
o(b') = o((0,1,1)) &= \operatorname{lcm}\left(\tfrac{2}{\gcd(2,0)}, \tfrac{2}{\gcd(2,1)}, \tfrac{3}{\gcd(3,1)}\right) = \operatorname{lcm}(1,2,3) = 6
\end{aligned}$$

and $\langle a'\rangle \cap \langle b'\rangle = \langle (1,0,1)\rangle \cap \langle (0,1,1)\rangle = \{(0,0,0),(0,0,1),(0,0,2)\}$, we have $o(b') = o(b) = c = c'$ and $|H'| = \frac{o(a')\cdot o(b')}{|\langle a'\rangle \cap \langle b'\rangle|} = \frac{6(6)}{3} = 12$, which imply $r' = o(b' + \langle a'\rangle) = \frac{|H'|}{o(a')} = \frac{12}{6} = 2 = r$. By Corollary 2.1.3, $C_{\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3}((0,1,2),(1,1,1))$ is isomorphic to $C_{\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3}((1,0,1),(0,1,1))$.

Example 2.1.11. Let $a = (6,9), a' = (6,15), b = (12,18), b' = (12,6)$ be in $G = \mathbb{Z}_{36} \times \mathbb{Z}_{36}$. Then

$$\begin{aligned}
c = o(a) = o((6,9)) &= \operatorname{lcm}\left(\tfrac{36}{\gcd(36,6)}, \tfrac{36}{\gcd(36,9)}\right) = \operatorname{lcm}(6,4) = 12,\\
c' = o(a') = o((6,15)) &= \operatorname{lcm}\left(\tfrac{36}{\gcd(36,6)}, \tfrac{36}{\gcd(36,15)}\right) = \operatorname{lcm}(6,12) = 12,\\
o(b) = o((12,18)) &= \operatorname{lcm}\left(\tfrac{36}{\gcd(36,12)}, \tfrac{36}{\gcd(36,18)}\right) = \operatorname{lcm}(3,2) = 6,\\
o(b') = o((12,6)) &= \operatorname{lcm}\left(\tfrac{36}{\gcd(36,12)}, \tfrac{36}{\gcd(36,6)}\right) = \operatorname{lcm}(3,6) = 6,\\
\langle a\rangle \cap \langle b\rangle = \langle (6,9)\rangle \cap \langle (12,18)\rangle &= \{(0,0),(12,18),(24,0),(0,18),(12,0),(24,18)\},\\
\langle a'\rangle \cap \langle b'\rangle = \langle (6,15)\rangle \cap \langle (12,6)\rangle &= \{(0,0),(0,18)\}.
\end{aligned}$$

Since $|H| = \frac{o(a)\cdot o(b)}{|\langle a\rangle \cap \langle b\rangle|} = \frac{12(6)}{6} = 12$ and $|H'| = \frac{o(a')\cdot o(b')}{|\langle a'\rangle \cap \langle b'\rangle|} = \frac{12(6)}{2} = 36$, $r = o(b + \langle a\rangle) = \frac{|H|}{o(a)} = \frac{12}{12} = 1 \neq 3 = \frac{36}{12} = \frac{|H'|}{o(a')} = o(b' + \langle a'\rangle) = r'$. By Corollary 2.1.3, $C_{\mathbb{Z}_{36}\times\mathbb{Z}_{36}}((6,9),(12,18))$ is not isomorphic to $C_{\mathbb{Z}_{36}\times\mathbb{Z}_{36}}((6,15),(12,6))$.

2.2 $C_G(a,b)$ when G is cyclic

When G is a cyclic group, the next proposition gives the explicit form of elements in G which have the same order. The proof is immediate.

Proposition 2.2.1. *Let $a, a' \in G$. Then $o(a) = o(a')$ if and only if $a' = ka$ for some $1 \leq k \leq o(a)$ and $\gcd(k, o(a)) = 1$.*

Theorem 2.2.2. *Let G be a cyclic group such that $a, a', b, b' \in G \setminus \{0\}$ with $a \neq \pm b, a' \neq \pm b', o(b) \leq o(a)$ and $o(b') \leq o(a')$. If $o(a) = o(a')$ and $o(b) = o(b')$, then $C_G(a, b)$ is connected if and only if $C_G(a', b')$ is connected.*

Proof. Assume $C_G(a, b)$ is connected. Then $G = \langle a, b \rangle$ by Corollary 1.1.3. By the above proposition, since $o(a) = o(a')$ and $o(b) = o(b')$, we have $a' = ka$ and $b' = lb$ for some $1 \leq k \leq o(a), \gcd(k, o(a)) = 1, 1 \leq l \leq o(b)$ and $\gcd(l, o(b)) = 1$. Then

$$\langle a', b' \rangle = \langle ka, lb \rangle = \langle a, b \rangle = G,$$

so $C_G(a', b')$ is connected. By symmetry, we have the theorem. $\square$

Corollary 2.2.3. *Let G be a cyclic group such that $a, a', b, b' \in G \setminus \{0\}$ with $a \neq \pm b, a' \neq \pm b', o(b) \leq o(a)$ and $o(b') \leq o(a')$. Then $C_G(a, b)$ and $C_G(a', b')$ are isomorphic if and only if either one of the following two conditions holds:*

(1) $o(b') = o(b) < c = c'$ and $rb' = \pm\lambda_G(a, b)a'$;

(2) $o(b') = o(b) = c = c'$ and either $rb' = \pm\lambda_G(a, b)a'$ or $ra' = \pm\lambda_G(a, b)b'$,

where $r = o(b + \langle a \rangle), c = o(a), c' = o(a')$ and $\lambda_G(a, b)$ is the column-jump of $M_G(a, b)$.

Proof. Assume $o(a) = c = c' = o(a')$ and $o(b) = o(b')$. From Corollary 2.1.3, it suffices to show that $r = r'$. By Proposition 2.2.1, since G is cyclic, $a' = ka$ and $b' = lb$ for some $\gcd(k, o(a)) = 1$ and $\gcd(l, o(b)) = 1$. Then $H' = \langle a', b' \rangle = \langle ka, lb \rangle = \langle a, b \rangle = H$. From Lemma 1.2.1, we have $r = \frac{|H|}{c} = \frac{|H'|}{c'} = r'$ as desired. $\square$

Proposition 2.2.4. *Let G be a cyclic group of order n such that $a, b \in G$. Then*

(1) $o(a) = \dfrac{n}{\gcd(n, a)}$.

(2) $o(b+\langle a\rangle) = \dfrac{\gcd(n,a)}{\gcd(n,a,b)}$.

(3) $\dfrac{o(a)}{o(a+\langle b\rangle)} = \dfrac{n\gcd(n,a,b)}{\gcd(n,a)\gcd(n,b)}$.

Proof. (1) comes from Corollary 2.1.8 (1). (2) is obtained from Corollary 2.1.8 (3) and remark after Theorem 1.1.5. (3) can be proved by (1) and (2). □

Example 2.2.5. Let $a = 3, a' = 21, b = 5, b' = 55$ be in $G = \mathbb{Z}_{60}$. Then $o(b') = o(55) = \frac{60}{\gcd(60,55)} = 12 = \frac{60}{\gcd(60,5)} = o(5) = o(b) < c = o(3) = \frac{60}{\gcd(60,3)} = 20 = \frac{60}{\gcd(60,21)} = o(21) = c'$ and $r = o(5+\langle 3\rangle) = \frac{\gcd(60,3)}{\gcd(60,3,5)} = 3$. From Lemma 1.2.2, since

$$15 = 3(5) = rb = \lambda_G(a,b)a = \lambda_G(a,b)3$$

for some $\lambda_G(a,b) \in \{0,1,\ldots,19 = c-1\}$, $\lambda_G(a,b) = 5$. Thus,

$$rb' = 3(55) = 165 = 45 = 105 = 5(21) = \lambda_G(a,b)a'.$$

This shows that $C_{\mathbb{Z}_{60}}(3,5)$ is isomorphic to $C_{\mathbb{Z}_{60}}(21,55)$.

Example 2.2.6. Let $a = a' = 2, b = 9, b' = 15$ be in $G = \mathbb{Z}_{42}$. Then $o(b') = o(15) = \frac{42}{\gcd(42,15)} = 14 = \frac{42}{\gcd(42,9)} = o(9) = o(b) < c = c' = o(2) = \frac{42}{\gcd(42,2)} = 21$ and $r = o(9+\langle 2\rangle) = \frac{\gcd(42,2)}{\gcd(42,2,9)} = 2$. From Lemma 1.2.2, since

$$18 = 2(9) = rb = \lambda_G(a,b)a = \lambda_G(a,b)2$$

for some $\lambda_G(a,b) \in \{0,1,\ldots,20 = c-1\}$, $\lambda_G(a,b) = 9$. Thus,

$$rb' = 2(15) = 30 \neq \pm 18 = 9(2) = \pm\lambda_G(a,b)a'.$$

This shows that $C_{\mathbb{Z}_{42}}(2,15)$ is not isomorphic to $C_{\mathbb{Z}_{42}}(2,9)$.

Lemma 2.2.7. *Let G be a cyclic group such that $a, a', b, b' \in G \smallsetminus \{0\}$ with $a \neq \pm b$ and $a' \neq \pm b'$.*

(1) $o(b') = o(b) < c = c'$ and $rb' = \pm\lambda_G(a,b)a'$ if and only if $o(b) < c$ and

$$(a',b') \in \{(ka,lb) : 1 \le k \le o(a), \gcd(k,o(a)) = 1,$$
$$1 \le l \le o(b), \gcd(l,o(b)) = 1 \textit{ and } k \equiv \pm l \mod h\}.$$

(2) $o(b') = o(b) = c = c'$ *and either* $rb' = \pm\lambda_G(a,b)a'$ *or* $ra' = \pm\lambda_G(a,b)b'$ *if and only if* $o(b) = c$ *and either*

$$(a',b') \in \{(ka,lb) : 1 \le k \le o(a), \gcd(k,o(a)) = 1,$$
$$1 \le l \le o(b), \gcd(l,o(b)) = 1 \textit{ and } k \equiv \pm l \mod h\} \textit{ or}$$
$$(a',b') \in \{(l'b,k'a) : 1 \le k' \le o(a), \gcd(k',o(a)) = 1,$$
$$1 \le l' \le o(b), \gcd(l',o(b)) = 1 \textit{ and } k' \equiv \pm l' \mod h\}.$$

Here, $h = \frac{o(b)}{o(b+\langle a\rangle)}, r = o(b+\langle a\rangle), c = o(a), c' = o(a')$ *and* $\lambda_G(a,b)$ *is the column-jump of* $M_G(a,b)$.

Proof. (1) Assume $o(b) = o(b') < o(a) = c = c' = o(a')$ and $rb' = \pm\lambda_G(a,b)a'$. From Proposition 2.2.1, we have $a' = ka$ and $b' = lb$ where $1 \le k \le o(a), \gcd(k,o(a)) = 1, 1 \le l \le o(b)$ and $\gcd(l,o(b)) = 1$. Then

$$rlb = rb' = \pm\lambda_G(a,b)a' = \pm\lambda_G(a,b)ka = \pm rkb.$$

So $(k \pm l)rb = 0$, which implies $o(b)$ divides $(k \pm l)r = (k \pm l)o(b+\langle a\rangle)$. Since $o(b)(b+\langle a\rangle) = o(b)b + \langle a\rangle = \langle a\rangle$, $o(b+\langle a\rangle) \mid o(b)$. Thus, $h = \frac{o(b)}{o(b+\langle a\rangle)}$ divides $k \pm l$, hence $k \equiv \pm l \mod h$.

Conversely, assume that $o(b) < c$ and $(a',b') \in \{(ka,lb) : 1 \le k \le o(a), \gcd(k,o(a)) = 1, 1 \le l \le o(b), \gcd(l,o(b)) = 1$ and $k \equiv \pm l \mod h\}$. By Proposition 2.2.1, $o(b) = o(b') < o(a) = c = c' = o(a')$. Since $k \equiv \pm l \mod h = \frac{o(b)}{o(b+\langle a\rangle)}$, $o(b)$ divides $(k \pm l)o(b+\langle a\rangle) = (k \pm l)r$. That is, $krb \pm lrb = (k \pm l)rb = 0$ implies $rlb = \pm rkb$. From Lemma 1.2.2, we have

$$rb' = rlb = \pm rkb = \pm\lambda_G(a,b)ka = \pm\lambda_G(a,b)a'.$$

(2) Assume $o(b') = o(b) = c = o(a) = o(a') = c'$ and either $rb' = \pm\lambda_G(a,b))a'$ or $ra' = \pm\lambda_G(a,b)b'$. By Proposition 2.2.1, $a' = l'b$ and $b' = k'a$ where

$1 \leq k' \leq o(a), \gcd(k', o(a)) = 1, 1 \leq l' \leq o(b)$ and $\gcd(l', o(b)) = 1$. Clearly, if $rb' = \pm\lambda_G(a,b)a'$, then $(a', b') \in \{(ka, lb) : 1 \leq k \leq o(a), \gcd(k, o(a)) = 1, 1 \leq l \leq o(b), \gcd(l, o(b)) = 1$ and $k \equiv \pm l \mod h\}$. Suppose $ra' = \pm\lambda_G(a,b)b'$. From

$$rl'b = ra' = \pm\lambda_G(a,b)b' = \pm\lambda_G(a,b)k'a = \pm rk'b,$$

so $k' \equiv \pm l' \mod h$.

On the contrary, suppose that $o(b) = c$ and either $(a', b') \in \{(ka, lb) : 1 \leq k \leq o(a), \gcd(k, o(a)) = 1, 1 \leq l \leq o(b), \gcd(l, o(b)) = 1$ and $k \equiv \pm l \mod h\}$ or $(a', b') \in \{(l'b, k'a) : 1 \leq k' \leq o(a), \gcd(k', o(a)) = 1, 1 \leq l' \leq o(b), \gcd(l', o(b)) = 1$ and $k' \equiv \pm l' \mod h\}$. By Proposition 2.2.1, $o(b') = o(b) = c = o(a) = o(a') = c'$. Clearly, if $(a', b') \in \{(ka, lb) : 1 \leq k \leq o(a), \gcd(k, o(a)) = 1, 1 \leq l \leq o(b), \gcd(l, o(b)) = 1$ and $k \equiv \pm l \mod h\}$, then $rb' = \pm\lambda_G(a,b)a'$. Suppose $(a', b') \in \{(l'b, k'a) : 1 \leq k' \leq o(a), \gcd(k', o(a)) = 1, 1 \leq l' \leq o(b), \gcd(l', o(b)) = 1$ and $k' \equiv \pm l' \mod h\}$. Then

$$ra' = rl'b = \pm rk'b = \pm\lambda_G(a,b)k'a = \pm\lambda_G(a,b)b'.$$

This completes the proof. □

Next, we use Corollary 2.2.3 and Lemma 2.2.7 to derive classes of isomorphic graphs. The results have necessary and sufficient conditions which not depends on $\lambda_G(a,b)$ in the next theorem. It is equivalent to Theorem 5.2 of [2] but our presentation is simpler.

Theorem 2.2.8. *Let G be a cyclic group such that $a, a', b, b' \in G \setminus \{0\}$ with $a \neq \pm b, a' \neq \pm b', o(b) \leq o(a)$ and $o(b') \leq o(a')$. Then $C_G(a,b)$ and $C_G(a', b')$ are isomorphic if and only if either one of the following two conditions holds:*

(1) $o(b) < c$ and

$$(a', b') \in \{(ka, lb) : 1 \leq k \leq o(a), \gcd(k, o(a)) = 1, \quad 1 \leq l \leq o(b), \gcd(l, o(b)) = 1 \text{ and } k \equiv \pm l \mod h\};$$

(2) $o(b) = c$ *and either*

$$\begin{aligned}(a', b') \in \{(ka, lb) :& 1 \leq k \leq o(a), \gcd(k, o(a)) = 1,\\ & 1 \leq l \leq o(b), \gcd(l, o(b)) = 1 \text{ and } k \equiv \pm l \mod h\} \text{ or}\\ (a', b') \in \{(l'b, k'a) :& \ 1 \leq k' \leq o(a), \gcd(k', o(a)) = 1,\\ & 1 \leq l' \leq o(b), \gcd(l', o(b)) = 1 \text{ and } k' \equiv \pm l' \mod h\},\end{aligned}$$

where $c = o(a)$ *and* $h = \frac{o(b)}{o(b+\langle a \rangle)}$.

Example 2.2.9. $C_{\mathbb{Z}_{42}}(a', b')$ is isomorphic to $C_{\mathbb{Z}_{42}}(2, 9)$ if and only if $(a', b') \in \{(2k, 9l) : 1 \leq k \leq 21, \gcd(k, 21) = 1, 1 \leq l \leq 14, \gcd(l, 14) = 1$ and $k \equiv \pm l \mod h\}$, where $h = \frac{o(b)}{r} = \frac{14}{2} = 7$. Then $k \in \{1, 2, 4, 5, 8, 10, 11, 13, 16, 17, 19, 20\}$ and $l \in \{1, 3, 5, 9, 11, 13\}$. Since

$$\begin{aligned}\{(k, l) : k \equiv \pm l \mod 7\} = \{&(1, 1), (1, 13), (2, 5), (2, 9), (4, 3), (4, 11), (5, 5), (5, 9)\\ &(8, 1), (8, 13), (10, 3), (10, 11), (11, 3), (11, 11), (13, 1),\\ &(13, 13), (16, 5), (16, 9), (17, 3), (17, 11), (19, 5), (19, 9),\\ &(20, 1), (20, 13)\},\end{aligned}$$

we have

$$\begin{aligned}(a', b') \in \{&(2, 9), (2, 33), (4, 3), (4, 39), (8, 27), (8, 15), (10, 3),\\ &(10, 39), (16, 9), (16, 33), (20, 27), (20, 15), (22, 27),\\ &(22, 15), (26, 9), (26, 33), (32, 3), (32, 39), (34, 27),\\ &(34, 15), (38, 3), (38, 39), (40, 9), (40, 33)\}.\end{aligned}$$

CHAPTER III

VERTEX COLORING

In this chapter, we give some results on chromatic numbers of $C_G(a, b)$ and explicit vertex coloring schemes from the algorithms presented in [3]. The conclusion is recorded in tables in Section 3.4.

3.1 Elementary results

Let G be a finite abelian group such that $G \neq \{0\}$ and $a, b \in G \setminus \{0\}$ with $a \neq \pm b$ and $o(b) \leq o(a)$. In this chapter, we shall study the algorithms in [3] to give an explicit assignment of colors to the vertices of graph $C_G(a, b)$ such that adjacent vertices receive different colors and the number of colors is minimized. Since each connected componets of $C_G(a, b)$ is isomorphic to $C_H(a, b)$ where $H = \langle a, b \rangle$, we just consider the assignment on the vertices of graph $C_H(a, b)$ by using the representative matrix $M_G(a, b)$. Recall that $M_G(a, b)$ is defined on $r = o(b + \langle a \rangle)$ rows and $c = o(a)$ columns and it can be partitioned into $h = \frac{c}{o(a+\langle b \rangle)}$ blocks, each block is equally sized $r \times o(a + \langle b \rangle)$ submatrices and let λ denotes $\lambda_G(a, b) \in \{0, 1, \ldots, c - 1\}$ such that $rb = \lambda a$.

A *k-coloring* of a graph G is an assignment of k colors to the vertices of G. It is *feasible* if adjacent vertices receive different colors. A graph G is *k-colorable* if it has a feasible k-coloring. The *chromatic number* $\chi(G)$ is the smallest k such that G is k-colorable.

The representative matrix $M_G(a, b)$ for the graph $C_H(a, b)$ is as follows

0	$\ldots$	$\ldots$	$\ldots$	$(\lambda - 1)a$	λa	$\ldots$	$(c - 1)a$
$\vdots$ $\vdots$							
$(r - 1)b$	$\ldots$	$(r - 1)b + (c - \lambda - 1)a$	$(r - 1)b + (c - \lambda)a$	$\ldots$	$\ldots$	$\ldots$	$(r - 1)b + (c - 1)a$

As for the a-edges of $M_G(a,b)$, we distinguish into two types: $A_0 = \{\{ib + ja, ib + (j+1)a\} : 0 \leq i \leq r-1 \text{ and } 0 \leq j \leq c-2\}$, which contains ordinary a-edges connecting consecutive entries in the same row and $A_1 = \{\{ib, ib+(c-1)a\} : 0 \leq i \leq r-1\}$, which contains boundary a-edges connecting an entry of the last column with an entry of the first column in the same row. As for the b-edges, we distinguish into three types: $B_0 = \{\{ib+ja, (i+1)b+ja\} : 0 \leq i \leq r-2 \text{ and } 0 \leq j \leq c-1\}$, which contains ordinary b-edges connecting consecutive entries in the same column and $B_1 = \{\{(r-1)b + ja, (\lambda + j)a\} : 0 \leq j \leq c - \lambda - 1\}$ and $B_2 = \{\{(r-1)b + (c-\lambda+j)a, ja\} : 0 \leq j \leq \lambda - 1\}$. We can use the fact that $rb = \lambda a$ to prove that B_1 and B_2 contain boundary b-edges connecting an entry of the last row with an entry of the first row of $M_G(a,b)$. Throughout the whole chapter, we denote by B, W, R, G the colors black, white, red and green respectively.

The coloring some entries of $M_G(a,b)$ according to the *BW-schema*, we mean the assignment $BW : H \to \{B, W\}$ such that

$$BW(ib+ja) = \begin{cases} B, & \text{if } i+j \text{ is even;} \\ W, & \text{if } i+j \text{ is odd,} \end{cases}$$

for all $i \in \{0, 1, \ldots, r-1\}$ and $j \in \{0, 1, \ldots, c-1\}$, while the *WB-schema*, we mean the assignment $WB : H \to \{B, W\}$ such that

$$WB(ib+ja) = \begin{cases} W, & \text{if } i+j \text{ is even;} \\ B, & \text{if } i+j \text{ is odd,} \end{cases}$$

for all $i \in \{0, 1, \ldots, r-1\}$ and $j \in \{0, 1, \ldots, c-1\}$.

Next, we shall describe the vertex coloring algorithm for the graph $C_H(a,b)$. These colorings are usually given as elements of the free monoid generated by

$\{B, W, R, G\}$, for instance,

$$B(WB)^k WB = \begin{cases} B\underbrace{WB\ldots WB}_{k\ copies\ of\ WB}WB, & \text{if } k \geq 1; \\ BWB, & \text{if } k = 0, \end{cases}$$

and we write B for $B(WB)^{-1}WB$. In addition, when we write these elements in a column, we shall mean we color vertically.

Let the *chessboard coloring* of $M_G(a, b)$ denoted by C^2, be a 2-coloring of $M_G(a, b)$ for which we color all its entries according to the BW-schema. The *block chessboard coloring*, denoted by BC^2, is a 2-coloring for which we color each block $\beta_0, \ldots, \beta_{h-1}$ of $M_G(a, b)$ like a chessboard, in such a way that the upper left corner of each block has always the same color (B or W does not matter).

When we assign C^2 on $M_G(a, b)$, we just consider coloring feasibility for three sets of boundary edges, namely, A_1, B_1 and B_2.

A criterian for 2-colorable is given in the next theorem.

Theorem 3.1.1. *Let $a, b \in G \setminus \{0\}$ and $H = \langle a, b\rangle$. Then $\chi(C_H(a, b)) = 2$ if and only if $r + \lambda$ and c are both even.*

Proof. Assume $r + \lambda$ and c are both even.

Case 1. *r and λ are even.* Then $c - \lambda$ is even. When we assign color to all entries of $M_G(a, b)$ according to the C^2, we have

B	$(WB)^{\frac{\lambda}{2}-1}$	W	B	$(WB)^{\frac{c-\lambda}{2}-1}$	W
$(WB)^{\frac{r}{2}-1}$					$(BW)^{\frac{r}{2}-1}$
W	$(BW)^{\frac{c-\lambda}{2}-1}$	B	W	$(BW)^{\frac{\lambda}{2}-1}$	B

We see that A_1, B_1 and B_2 are feasible. The coloring completes.

Case 2. *r and λ are odd.* Then $c - \lambda$ is odd. When we assign color to all entries of $M_G(a, b)$ according to the C^2, we have

B	$(WB)^{\frac{\lambda-3}{2}}W$	B	W	$(BW)^{\frac{c-\lambda-3}{2}}B$	W
$(WB)^{\frac{r-3}{2}}W$					$(BW)^{\frac{r-3}{2}}B$
B	$(WB)^{\frac{c-\lambda-3}{2}}W$	B	W	$(BW)^{\frac{\lambda-3}{2}}B$	W

We see that A_1, B_1 and B_2 are feasible. The coloring completes.

So C^2 is a vertex coloring for $r+\lambda$ and c are both even. Hence $\chi(C_H(a,b)) = 2$.

Conversely, assume that $\chi(C_H(a,b)) = 2$. Then $C_H(a,b)$ is bipartite. Recall that a graph is bipartite if and only if there is no odd cycle. So $\{(r-1)b, (r-2)b, \ldots, b, 0, a, 2a, \ldots, \lambda a\}$ and $\{0, a, 2a, \ldots, (c-1)a\}$ are even cycles. Hence $r+\lambda$ and c are both even. $\square$

We give a example to demonstrate the above theorem.

Example 3.1.2. Consider the graph $C_{\mathbb{Z}_2\times\mathbb{Z}_6}((1,2),(0,3))$ in Example 1.2.7.

$(0,0)$	$(1,2)$	$(0,4)$	$(1,0)$	$(0,2)$	$(1,4)$
$(0,3)$	$(1,5)$	$(0,1)$	$(1,3)$	$(0,5)$	$(1,1)$

Since $r+\lambda = 2+0 = 2$ and $c = 6$ are even, by Case 1, we have the vertex coloring for $C_{\mathbb{Z}_2\times\mathbb{Z}_6}((1,2),(0,3))$ in the following table.

B	W	B	W	B	W
W	B	W	B	W	B

Lemma 3.1.3. *Let $a, b \in G \smallsetminus \{0\}$ with $b \in \langle a\rangle$. Then there exists $2 \le \lambda \le \frac{c}{2}$ such that $C_H(a,b) \simeq C_H(a,\lambda a)$ where $H = \langle a\rangle$.*

Proof. Assume $b \in \langle a\rangle$. Then $r = o(b+\langle a\rangle) = 1$ and $H = \langle a,b\rangle = \langle a\rangle$. By Lemma 1.2.2, since $b \neq 0$ and $a \neq \pm b$, we have $b = \lambda a$ for some $2 \le \lambda \le c-2$. Thus $C_H(a,b) \simeq C_H(a,\lambda a)$. By trivially isomorphic, $C_H(a,\lambda a) \simeq C_H(a,-\lambda a) \simeq C_H(a,(c-\lambda)a)$. If $2 \mid c$, then $C_H(a,b) \simeq C_H(a,\lambda a)$ for some $2 \le \lambda \le \frac{c}{2}$, otherwise $C_H(a,b) \simeq C_H(a,\lambda a)$ for some $2 \le \lambda \le \frac{c-1}{2}$. Hence $C_H(a,b) \simeq C_H(a,\lambda a)$ for some $2 \le \lambda \le \frac{c}{2}$. $\square$

From the above lemma, we now focus on $2 \le \lambda \le \frac{c}{2}$ for $C_H(a,b)$ with $b = \lambda a$.

Lemma 3.1.4. *If c is odd, then $C_H(a,2a) \simeq C_H(a,(\frac{c-1}{2})a)$.*

Proof. Assume that c is odd. Since $c - 2(\frac{c-1}{2}) = 1$, $\gcd(\frac{c-1}{2}, c) = 1$. Thus,

$$o(2a) = \frac{o(a)}{\gcd(2, o(a))} = \frac{c}{\gcd(2,c)} = \frac{c}{1} = c$$

and

$$o\Big((\frac{c-1}{2})a\Big) = \frac{o(a)}{\gcd(\frac{c-1}{2}, o(a))} = \frac{c}{\gcd(\frac{c-1}{2}, c)} = \frac{c}{1} = c.$$

Clearly, $r = o(2a + \langle a\rangle) = 1 = o\Big((\frac{c-1}{2})a + \langle a\rangle\Big) = r'$. From Lemma 1.2.2, $\lambda_G(a, 2a) = 2$. Since $o(2a) = o((\frac{c-1}{2})a) = o(a) = c$ and $ra = -\lambda_G(a, 2a)(\frac{c-1}{2})a$, we have $C_H(a, 2a) \simeq C_H(a, (\frac{c-1}{2})a)$ by Corollary 2.1.3. □

Lemma 3.1.5 (Theorem 4.1 of [3]). *Let G be a cyclic group such that $a, b \in G \setminus \{0\}$ and $a \neq \pm b$. Consider the graph $C_H(a, b)$ such that either $(a \notin \langle b\rangle \wedge b \notin \langle a\rangle)$ or $(a \in \langle b\rangle \vee b \in \langle a\rangle)$. Then*

$$\chi(C_H(a,b)) = \begin{cases} 2, & \text{if } |H| \text{ even } a, b \text{ odd;} \\ 5, & \text{if } |H| = 5; \\ 4, & \text{if } |H| = 13, a = 1, b = 5 \text{ or } |H| \neq 5, 3 \nmid |H|, a = 1, b \in \{2, \frac{|H|-1}{2}\}; \\ 3, & \text{otherwise.} \end{cases}$$

Lemma 3.1.6. *Let G be a cyclic group such that $a, b \in G \setminus \{0\}$ with $a \neq \pm b$ and $o(b) \leq o(a)$. Consider the graph $C_H(a, b)$ such that either $b \notin \langle a\rangle$ or $b = \lambda a$ with $2 \leq \lambda \leq \frac{c}{2}$. Then*

$$\chi(C_H(a,b)) = \begin{cases} 2, & \text{if } r + \lambda \text{ and } c \text{ are even;} \\ 5, & \text{if } r = 1 \text{ and } c = 5; \\ 4, & \text{if } r = 1, c = 13, \lambda = 5 \text{ or } r = 1, c \neq 5, 3 \nmid c, \lambda \in \{2, \frac{c-1}{2}\}; \\ 3, & \text{otherwise.} \end{cases}$$

Proof. Since $r = o(b + \langle a\rangle) \leq o(b)$ and $o(b) \leq o(a) = c$, we have $r \leq c$. The lemma obtains directly from Lemma 3.1.5 and the fact that $|H| = rc$ and $r \leq c$. □

We shall apply the above lemmas in the following sections.

3.2 The case $b \in \langle a \rangle$

We study the vertex coloring for $C_H(a, b)$ when $b \in \langle a \rangle$ in this section.

Theorem 3.2.1. *Let $a, b \in G \setminus \{0\}$ and $a \neq \pm b$ be such that $b = \lambda a$ with $2 \leq \lambda \leq \frac{c}{2}$. If λ is even or c is odd, then*

$$\chi(C_H(a,b)) = \begin{cases} 5, & \text{if } c = 5; \\ 4, & \text{if } c = 13, \lambda = 5 \text{ or } c \neq 5, 3 \nmid c, \lambda \in \{2, \frac{c-1}{2}\}; \\ 3, & \text{otherwise.} \end{cases}$$

Proof. Assume λ is even or c is odd. By Theorem 3.1.1, since $r = o(b + \langle a \rangle) = o(\lambda a + \langle a \rangle) = 1$, $\chi(C_H(a,b)) > 2$. Then $C_H(a,b) \simeq C_H(a, \lambda a)$ where $H = \langle a, b \rangle = \langle a \rangle$ is a cyclic group. We obtain the chromatic number of $C_H(a,b)$ from Lemma 3.1.6. □

Let $S(a, \lambda a)$ be the pseudo-matrix of the graph $C_H(a, \lambda a)$, which will be used to assign the vertex coloring for $C_H(a, \lambda a)$. It can be defined as the following pseudo-matrix.

0	a	$\ldots$	$(l-1)a$	la	$\ldots$	$(\lambda-1)a$
λa	$(\lambda+1)a$	$\ldots$	$(\lambda+l-1)a$	$(\lambda+l)a$	$\ldots$	$(2\lambda-1)a$
$2\lambda a$	$(2\lambda+1)a$	$\ldots$	$(2\lambda+l-1)a$	$(2\lambda+l)a$	$\ldots$	$(3\lambda-1)a$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$(q-1)\lambda a$	$((q-1)\lambda+1)a$	$\ldots$	$((q-1)\lambda+l-1)a$	$((q-1)\lambda+l)a$	$\ldots$	$(q\lambda-1)a$
$q\lambda a$	$(q\lambda+1)a$	$\ldots$	$(q\lambda+l-1)a$			

From the above pseudo-matrix, $S(a, \lambda a)$ has $q+1$ rows where $q \geq 1$ and λ columns, the number of its entries is $c = q\lambda + l$ where $1 \leq l \leq \lambda$. The last row of $S(a, \lambda a)$ is full if $l = \lambda$, otherwise it contains $l = c - q\lambda < \lambda$ entries.

0	$\ldots$	$\ldots$	$\ldots$	$(\lambda-l-1)a$	$(\lambda-l)a$	$\ldots$	$(\lambda-1)a$
λa							$(2\lambda-1)a$
$\vdots$							$\vdots$
$(q-1)\lambda a$	$\ldots$	$\ldots$	$((q-1)\lambda+l)a$	$\ldots$	$\ldots$	$\ldots$	$(q\lambda-1)a$
$q\lambda a$	$\ldots$	$(q\lambda+l-1)a$					

We distinguish the a-edges of $S(a, \lambda a)$ into three types: $A_0 = \{\{(i\lambda+j)a, (i\lambda+(j+1))a\} : 0 \leq i \leq q \text{ and } 0 \leq j \leq \lambda - 2\}$, which contains ordinary a-edges connecting consecutive entries in the same row while $A_1 = \{\{0, (q\lambda + l - 1)a\}\}$ and $A_2 = \{\{i\lambda a, (i\lambda - 1)a\} : 1 \leq i \leq q\}$, which contain boundary a-edges. On the other hand, we distinguish the b-edges of $S(a, \lambda a)$ into three types: $B_0 = \{\{(i\lambda + j)a, ((i + 1)\lambda + j)a\} : 0 \leq i \leq q - 1 \text{ and } 0 \leq j \leq \lambda - 1\}$, which contains ordinary b-edges connecting consecutive entries in the same column while $B_1 = \{\{(q\lambda+(l+j-\lambda))a, ja\} : \lambda-l \leq j \leq \lambda-1\}$ and $B_2 = \{\{((q-1)\lambda+(l+j))a, ja\} : 0 \leq j \leq \lambda - l - 1\}$, which contain boundary b-edges.

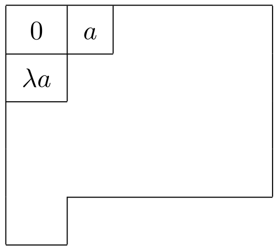
The coloring some entries of $S(a, \lambda a)$ according to the *BW-schema*, we mean the assignment $BW : \langle a \rangle \to \{B, W\}$ such that

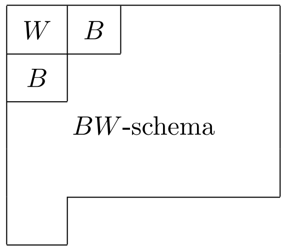
$$BW((i\lambda + j)a) = \begin{cases} B, & \text{if } i + j \text{ is even;} \\ W, & \text{if } i + j \text{ is odd,} \end{cases}$$

for all $i \in \{0, 1, \ldots, q\}$ and $j \in \{0, 1, \ldots, \lambda - 1\}$, while the *WB-schema*, we mean the assignment $WB : \langle a \rangle \to \{B, W\}$ such that

$$WB((i\lambda + j)a) = \begin{cases} W, & \text{if } i + j \text{ is even;} \\ B, & \text{if } i + j \text{ is odd,} \end{cases}$$

for all $i \in \{0, 1, \ldots, q\}$ and $j \in \{0, 1, \ldots, \lambda - 1\}$. Let the *chessboard coloring* of $S(a, \lambda a)$, denoted by C^2, be a 2-coloring of $S(a, \lambda a)$ for which we color all its entries according to the BW-schema. The *corner complemented chessboard coloring*, denoted by C^4, is a 2-coloring such that color entries $0, a, \lambda a$ of $S(a, \lambda a)$ according to the WB-schema and color the remaining entries according to the BW-schema.





In addition, the coloring some entries of $S(a, \lambda a)$ according to the *BWR-schema*, we mean the assignment $BWR : \langle a \rangle \to \{B, W, R\}$ such that

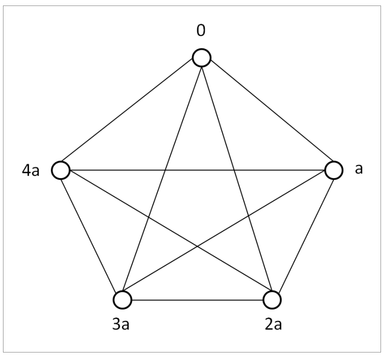
$$BWR(ja) = \begin{cases} B, & \text{if } j \equiv 0 \mod 3; \\ W, & \text{if } j \equiv 1 \mod 3; \\ R, & \text{if } j \equiv 2 \mod 3, \end{cases}$$

for all $j \in \{0, 1, \ldots, c-1\}$. The *BWRG-schema* is the assignment $BWRG : \langle a \rangle \to \{B, W, R, G\}$ such that

$$BWRG(ja) = \begin{cases} B, & \text{if } j \equiv 0 \mod 4; \\ W, & \text{if } j \equiv 1 \mod 4; \\ R, & \text{if } j \equiv 2 \mod 4; \\ G, & \text{if } j \equiv 3 \mod 4, \end{cases}$$

for all $j \in \{0, 1, \ldots, c-1\}$.

Case 1. $c = 5$. Since $2 \le \lambda \le \frac{c}{2}$, $\lambda = 2$, so we have only one graph $C_H(a, 2a)$, which isomorphic to K_5, the complete graph on five vertices. The coloring of its vertices consists of assigning a different color to each vertex.



Case 2. $c = 13$ *and* $\lambda = 5$. Since $c = 2\lambda + 3$, $S(a, \lambda a)$ has $q + 1 = 3$ rows and $\lambda = 5$ columns. The last row contains $l = 3$ entries.

0	a	$2a$	$3a$	$4a$
$5a$	$6a$	$7a$	$8a$	$9a$
$10a$	$11a$	$12a$		

Assign color according to the BWR-schema to entries $0, a, \ldots, 11a$ as shown.

B	W	R	B	W
R	B	W	R	B
W	R			

Since $\{11a, 12a\} \in A_0, \{12a, 0\} \in A_1$ and $\{7a, 12a\} \in B_0$, we assign the cell $12a$ in different color from R, B and W. Pick the color G for $12a$.

B	W	R	B	W
R	B	W	R	B
W	R	G		

Case 3. $c \neq 5$ *and* $\lambda \in \{2, \frac{c-1}{2}\}$. By Theorem 3.2.1, $\chi(C_H(a, \lambda a)) = 4$ if $3 \nmid c$ and $\chi(C_H(a, \lambda a)) = 3$ if $3 \mid c$. From Lemma 3.1.4, we have the fact that $C_H(a, 2a) \simeq C_H(a, (\frac{c-1}{2})a)$ when c is odd. It suffices to verify the coloring algorithm for $C_H(a, 2a)$ only.

3.1 Assume that $3 \mid c$. Then $c = 6k - 3$ or $6k$ for some $k \in \mathbb{N}$.

3.1.1 $c = 6k - 3$. Since $c = (3k - 2)\lambda + 1$, $S(a, 2a)$ has $q + 1 = 3k - 1$ rows and $\lambda = 2$ columns. The last row contains $l = 1$ entry.

0	a
$2a$	$3a$
$4a$	$5a$
$\vdots$	$\vdots$
$(6k-6)a$	$(6k-5)a$
$(6k-4)a$	

Assign color to all entries of $S(a, 2a)$ according to the BWR-schema. Since $6k - 6 \equiv 0 \mod 3$, $BWR((6k - 6)a) = B$.

B	W
R	B
W	R
$(BRW)^{k-2}$	$(WBR)^{k-2}$
B	W
R	

3.1.2 $c = 6k$. Since $c = (3k-1)\lambda + 2$, then $S(a, 2a)$ has $q + 1 = 3k$ rows and $\lambda = 2$ columns. Since $l = \lambda = 2$, the last row of $S(a, 2a)$ is full.

0	a
$2a$	$3a$
$4a$	$5a$
$\vdots$	$\vdots$
$(6k-4)a$	$(6k-3)a$
$(6k-2)a$	$(6k-1)a$

Assign color to all entries of $S(a, 2a)$ according to the BWR-schema. Since $6k - 4 \equiv 2 \mod 3$, $BWR((6k-4)a) = R$ as follows.

B	W
R	B
W	R
$(BRW)^{k-2}B$	$(WBR)^{k-2}W$
R	B
W	R

3.2 Assume that $3 \nmid c$ and $c \equiv 0$ or $3 \mod 4$. Then

$$c = \begin{cases} 6k-2 & \text{if } k \text{ is odd;} \\ 6k-4, 6k-5, 6k-1 & \text{if } k \text{ is even.} \end{cases}$$

3.2.1 $c = 6k - 1$ and k is even. Since $c = (3k-1)\lambda + 1$, $S(a, 2a)$ has $q + 1 = 3k$ rows and $\lambda = 2$ columns. The last row contains $l = 1$ entry.

0	a
$2a$	$3a$
$4a$	$5a$
$6a$	$7a$
$\vdots$	$\vdots$
$(6k-4)a$	$(6k-3)a$
$(6k-2)a$	

Assign color to all entries of $S(a,2a)$ according to the $BWRG$-schema. Since k is even, $k = 2n$ for some $n \in \mathbb{N}$. Then $6k-4 \equiv 12n-4 \equiv 0 \mod 4$, so $BWRG((6k-4)a) = B$.

B	W
R	G
B	W
R	G
$(BR)^{\frac{3k}{2}-3}$	$(WG)^{\frac{3k}{2}-3}$
B	W
R	

The same argument proves the case $c = 6k-5$ and k is even.

3.2.2 $6k-2$ and k is odd. Since $c = (3k-2)\lambda+2$, $S(a,2a)$ has $q+1 = 3k-1$ rows and $\lambda = 2$ columns. Since $l = \lambda = 2$, the last row of $S(a,2a)$ is full.

0	a
$2a$	$3a$
$4a$	$5a$
$6a$	$7a$
$\vdots$	$\vdots$
$(6k-6)a$	$(6k-5)a$
$(6k-4)a$	$(6k-3)a$

Assign color to all entries of $S(a,2a)$ according to the $BWRG$-schema. Since k is odd, $k = 2n+1$ for some $n \in \mathbb{N}$. Then $6k-6 \equiv 12n \equiv 0 \mod 4$, so $BWRG((6k-6)a) = B$.

B	W
R	G
B	W
R	G
$(BR)^{\frac{3k-7}{2}}$	$(WG)^{\frac{3k-7}{2}}$
B	W
R	G

Similarly, we have the case $c = 6k - 4$ and k is even.

3.3 Assume that $3 \nmid c$ and $c \equiv 1 \mod 4$. Then $c = 6k - 5$ or $6k - 1$ where k is odd. Consider the first case $c = 6k - 1$. Since $c = (3k - 1)\lambda + 1$, $S(a, 2a)$ has $q + 1 = 3k$ rows and $\lambda = 2$ columns. The last row contains $l = 1$ entry.

0	a
$2a$	$3a$
$4a$	$5a$
$6a$	$7a$
$\vdots$	$\vdots$
$(6k-8)a$	$(6k-7)a$
$(6k-6)a$	$(6k-5)a$
$(6k-4)a$	$(6k-3)a$
$(6k-2)a$	

Assign color according to the $BWRG$-schema to entries $0, a, \ldots, (c - 6)a = (6k - 7)a$. Since k is odd, $k = 2n + 1$ for some $n \in \mathbb{N}$. Then $6k - 8 \equiv 12n - 2 \equiv 2 \mod 4$, so $BWRG((6k - 8)a) = R$.

B	W
R	G
B	W
R	G
$(BR)^{\frac{3k-9}{2}}B$	$(WG)^{\frac{3k-9}{2}}W$
R	G

Since $\{(6k-2)a, 0\} \in A_1$ and $\{(6k-2)a, a\} \in B_1$, we assign the cell $(6k-2)a$ in different color from B and W. Pick the color R for $(6k-2)a$.

B	W
R	G
B	W
R	G
$(BR)^{\frac{3k-9}{2}}B$	$(WG)^{\frac{3k-9}{2}}W$
R	G
R	

Since $\{(6k-3)a, (6k-2)a\} \in A_0$ and $\{(6k-3)a, 0\} \in B_2$, we assign the cell $(6k-3)a$ in different color from R and B. We can assign color W to $(6k-3)a$.

B	W
R	G
B	W
R	G
$(BR)^{\frac{3k-9}{2}}B$	$(WG)^{\frac{3k-9}{2}}W$
R	G
	W
R	

For the last three entries, we assign colors W, R, G to entries $(6k-6)a, (6k-5)a, (6k-4)a$ respectively. This leads to the following table.

B	W
R	G
B	W
R	G
$(BR)^{\frac{3k-9}{2}}B$	$(WG)^{\frac{3k-9}{2}}W$
R	G
W	R
G	W
R	

A similar argument proves the case $c = 6k-5$ and k is odd.

3.4 Assume that $3 \nmid c$ and $c \equiv 2 \mod 4$. Then

$$c = \begin{cases} 6k-4, & \text{if } k \text{ is odd;} \\ 6k-2, & \text{if } k \text{ is even.} \end{cases}$$

Consider the first case $c = 6k-2$ where k is even. Since $c = (3k-2)\lambda+2$, $S(a, 2a)$ has $q+1 = 3k-1$ rows and $\lambda = 2$ columns. Since $l = \lambda = 2$, the last row of $S(a, 2a)$ is full.

0	a
$2a$	$3a$
$4a$	$5a$
$6a$	$7a$
$\vdots$	$\vdots$
$(6k-6)a$	$(6k-5)a$
$(6k-4)a$	$(6k-3)a$

Assign color according to the $BWRG$-schema to entries $0, a, \ldots, (c-3)a = (6k-5)a$. Since k is even, $k = 2n$ for some $n \in \mathbb{N}$. Then $6k-6 \equiv 12n-6 \equiv 2 \mod 4$, so $BWRG((6k-6)a) = R$.

B	W
R	G
B	W
R	G
$(BR)^{\frac{3k-7}{2}}B$	$(WG)^{\frac{3k-7}{2}}W$
R	G

Since $\{(6k-5)a, (6k-4)a\} \in A_0, \{(6k-6)a, (6k-4)a\} \in B_0$ and $\{(6k-4)a, 0\} \in B_1$, we assign the cell $(6k-4)a$ in different color from G, R and B. Pick the color W for $(6k-4)a$.

B	W
R	G
B	W
R	G
$(BR)^{\frac{3k-7}{2}}B$	$(WG)^{\frac{3k-7}{2}}W$
R	G
W	

Since $\{(6k-4)a, (6k-3)a\} \in A_0, \{(6k-5)a, (6k-3)a\} \in B_0, \{(6k-3)a, 0\} \in A_1$ and $\{(6k-3)a, a\} \in B_2$, we assign the cell $(6k-3)a$ in different color from W, G and B. Pick the color R for $(6k-3)a$. This leads to the following table.

B	W
R	G
B	W
R	G
$(BR)^{\frac{3k-7}{2}}B$	$(WG)^{\frac{3k-7}{2}}W$
R	G
W	R

Similarly, we have the case $c = 6k-4$ and k is odd.

Case 4. $2 < \lambda \leq \frac{c}{2}$ *and* $\lambda \neq \frac{c-1}{2}$, *except* $c = 13$ *and* $\lambda = 5$. The vertex coloring algorithm for this case, we use 2-phases method. In the first phase we start by suitably coloring $S(a, \lambda a)$ with only colors B and W, such as C^2, SbC^2 and C^4, which SbC^2 shall be refered after Subcase 4.4. Certainly, since λ is even or c is odd, the bipartite graphs are excluded from this case, that is, any 2-coloring has to be infeasible. In order to remove infeasibilities we proceed with the second phase, where we suitably modify into R the color of one entry of each infeasible edge, which we shall consider from boundary edges A_1, A_2, B_1 and B_2 of $S(a, \lambda a)$ respectively, which these coloring is not make a and b-edges become infeasible. Thus, we have the approach for $C_H(a, \lambda a)$'s with $2 < \lambda \leq \frac{c}{2}$ as folows.

0	$\ldots$	$(\lambda-l-1)a$	$(\lambda-l)a$	$\ldots$	$(\lambda-1)a$
λa					$\vdots$
$\vdots$			$((q-1)\lambda+l)a$	$\ldots$	$(q\lambda-1)a$
$q\lambda a$	$\ldots$	$(q\lambda+l-1)a$			

4.1 Assume that q, λ and l are even. Then $\lambda - l$ is even.

Phase 1: Assign color to all entries of $S(a, \lambda a)$ according to the C^2.

B	$(WB)^{\frac{\lambda-l}{2}-1}$	W	B	$(WB)^{\frac{l}{2}-1}$	W
W					$(BW)^{\frac{q}{2}-1}$
$(BW)^{\frac{q}{2}-1}$			W	$(BW)^{\frac{\lambda-l}{2}-1}$	B
B	$(WB)^{\frac{l}{2}-1}$	W			

We see that A_1 is feasible.

Phase 2: Since A_2 is infeasible, we can modify by changing B to R in its left column and W to R in its right column.

B	$(WB)^{\frac{\lambda-l}{2}-1}$	W	B	$(WB)^{\frac{l}{2}-1}$	R
W					$(BR)^{\frac{q}{2}-1}$
$(RW)^{\frac{q}{2}-1}$			W	$(BW)^{\frac{\lambda-l}{2}-1}$	B
R	$(WB)^{\frac{l}{2}-1}$	W			

Since B_1 is infeasible and $\{q\lambda a, (q\lambda+1)a\} \in A_0$ where $q\lambda a$ has color R, we can modify by changing B to R in its bottom row and W to R in its top row.

B	$(WB)^{\frac{\lambda-l}{2}-1}$	W	B	$(RB)^{\frac{l}{2}-1}$	R
W					$(BR)^{\frac{q}{2}-1}$
$(RW)^{\frac{q}{2}-1}$			W	$(BW)^{\frac{\lambda-l}{2}-1}$	B
R	$(WR)^{\frac{l}{2}-1}$	W			

We see that B_2 is feasible. The coloring completes.

4.2 Assume that q is odd and λ, l are even. Then $\lambda - l$ is even.

Phase 1: Assign color to all entries of $S(a, \lambda a)$ according to the C^2.

B	$(WB)^{\frac{\lambda-l}{2}-1}$	W	B	$(WB)^{\frac{l}{2}-1}$	W
W					$(BW)^{\frac{q-3}{2}}B$
$(BW)^{\frac{q-3}{2}}B$					
				B $(WB)^{\frac{\lambda-l}{2}-1}$	W
W	$(BW)^{\frac{l}{2}-1}$		B		

Phase 2: Since A_1 is infeasible, we modify the cell 0 into R.

R	$(WB)^{\frac{\lambda-l}{2}-1}$	W	B	$(WB)^{\frac{l}{2}-1}$	W
W					$(BW)^{\frac{q-3}{2}}B$
$(BW)^{\frac{q-3}{2}}B$					
				B $(WB)^{\frac{\lambda-l}{2}-1}$	W
W	$(BW)^{\frac{l}{2}-1}$		B		

Since A_2 is infeasible and $\{0, \lambda a\} \in B_0$ where 0 has color R, we can modify by changing B to R in its left column and W to R in its right column.

R	$(WB)^{\frac{\lambda-l}{2}-1}$	W	B	$(WB)^{\frac{l}{2}-1}$	R
W					$(BR)^{\frac{q-3}{2}}B$
$(RW)^{\frac{q-3}{2}}R$					
				B $(WB)^{\frac{\lambda-l}{2}-1}$	R
W	$(BW)^{\frac{l}{2}-1}$		B		

We see that B_1 is feasible. Since some B_2 is infeasible and $\{0, a\} \in A_0$ where 0 has color R, we can modify by changing B to R in its top row and W to R in its bottom row.

R	$(WR)^{\frac{\lambda-l}{2}-1}$	W	B	$(WB)^{\frac{l}{2}-1}$	R
W					$(BR)^{\frac{q-3}{2}}B$
$(RW)^{\frac{q-3}{2}}R$					
				B $(RB)^{\frac{\lambda-l}{2}-1}$	R
W	$(BW)^{\frac{l}{2}-1}$		B		

The coloring completes.

4.3 Assume that q is odd and λ is even and l is odd. Then $\lambda - l$ is odd.
Phase 1: Assign color to all entries of $S(a, \lambda a)$ according to the C^2.

<table>
<tr><td>B</td><td>$(WB)^{\frac{\lambda-l-3}{2}}W$</td><td>$B$</td><td>$W$</td><td colspan="3">$(BW)^{\frac{l-3}{2}}B$</td><td>$W$</td></tr>
<tr><td>W</td><td colspan="6" rowspan="2"></td><td rowspan="2">$(BW)^{\frac{q-3}{2}}B$</td></tr>
<tr><td rowspan="2">$(BW)^{\frac{q-3}{2}}B$</td></tr>
<tr><td colspan="4"></td><td>W</td><td>$(BW)^{\frac{\lambda-l-3}{2}}B$</td><td>$W$</td></tr>
<tr><td>W</td><td colspan="3">$(BW)^{\frac{l-3}{2}}B$</td><td>W</td><td colspan="3"></td></tr>
</table>

We see that A_1 is feasible.

Phase 2: Since A_2 is infeasible, we can modify by changing W to R in its left column and B to R in its right column.

<table>
<tr><td>B</td><td>$(WB)^{\frac{\lambda-l-3}{2}}W$</td><td>$B$</td><td>$W$</td><td colspan="3">$(BW)^{\frac{l-3}{2}}B$</td><td>$W$</td></tr>
<tr><td>R</td><td colspan="6" rowspan="2"></td><td rowspan="2">$(RW)^{\frac{q-3}{2}}R$</td></tr>
<tr><td rowspan="2">$(BR)^{\frac{q-3}{2}}B$</td></tr>
<tr><td colspan="4"></td><td>W</td><td>$(BW)^{\frac{\lambda-l-3}{2}}B$</td><td>$W$</td></tr>
<tr><td>R</td><td colspan="3">$(BW)^{\frac{l-3}{2}}B$</td><td>W</td><td colspan="3"></td></tr>
</table>

Since some B_1 is infeasible and $\{q\lambda a, (q\lambda+1)a\} \in A_0$ where $q\lambda a$ has color R, we can modify by changing W to R in its bottom row and B to R in its top row.

<table>
<tr><td>B</td><td>$(WB)^{\frac{\lambda-l-3}{2}}W$</td><td>$B$</td><td>$W$</td><td colspan="3">$(RW)^{\frac{l-3}{2}}R$</td><td>$W$</td></tr>
<tr><td>R</td><td colspan="6" rowspan="2"></td><td rowspan="2">$(RW)^{\frac{q-3}{2}}R$</td></tr>
<tr><td rowspan="2">$(BR)^{\frac{q-3}{2}}B$</td></tr>
<tr><td colspan="4"></td><td>W</td><td>$(BW)^{\frac{\lambda-l-3}{2}}B$</td><td>$W$</td></tr>
<tr><td>R</td><td colspan="3">$(BR)^{\frac{l-3}{2}}B$</td><td>R</td><td colspan="3"></td></tr>
</table>

We see that B_2 is feasible. The coloring completes.

4.4 Assume that q, λ are odd and l is even. Then $\lambda - l$ is odd.

Phase 1: Assign color to all entries of $S(a, \lambda a)$ according to the C^2.

<table>
<tr><td>B</td><td>$(WB)^{\frac{\lambda-l-3}{2}}W$</td><td>$B$</td><td>$W$</td><td colspan="2">$(BW)^{\frac{l}{2}-1}$</td><td>$B$</td></tr>
<tr><td>W</td><td colspan="5" rowspan="2"></td><td rowspan="2">$(WB)^{\frac{q-3}{2}}W$</td></tr>
<tr><td rowspan="2">$(BW)^{\frac{q-3}{2}}B$</td></tr>
<tr><td colspan="3"></td><td>B</td><td>$(WB)^{\frac{\lambda-l-3}{2}}W$</td><td>$B$</td></tr>
<tr><td>W</td><td colspan="2">$(BW)^{\frac{l}{2}-1}$</td><td>B</td><td colspan="3"></td></tr>
</table>

Phase 2: Since A_1 is infeasible, we modify the cell 0 into R.

R	$(WB)^{\frac{\lambda-l-3}{2}}W$	B	W	$(BW)^{\frac{l}{2}-1}$		B
W						$(WB)^{\frac{q-3}{2}}W$
$(BW)^{\frac{q-3}{2}}B$						
				B	$(WB)^{\frac{\lambda-l-3}{2}}W$	B
W	$(BW)^{\frac{l}{2}-1}$		B			

We see that A_2 is feasible. Since B_1 is infeasible and $\{(q\lambda+l-1)a, 0\} \in A_1$ where 0 has color R, we can modify by changing W to R in its bottom row and B to R in its top row.

R	$(WB)^{\frac{\lambda-l-3}{2}}W$	B	W	$(RW)^{\frac{l}{2}-1}$		R
W						$(WB)^{\frac{q-3}{2}}W$
$(BW)^{\frac{q-3}{2}}B$						
				B	$(WB)^{\frac{\lambda-l-3}{2}}W$	B
R	$(BR)^{\frac{l}{2}-1}$		B			

Since some B_2 is infeasible and $\{0, a\} \in A_0$ where 0 has color R, we can modify by changing B to R in its top row and W to R in its bottom row.

R	$(WR)^{\frac{\lambda-l-3}{2}}W$	R	W	$(RW)^{\frac{l}{2}-1}$		R
W						$(WB)^{\frac{q-3}{2}}W$
$(BW)^{\frac{q-3}{2}}B$						
				B	$(RB)^{\frac{\lambda-l-3}{2}}R$	B
R	$(BR)^{\frac{l}{2}-1}$		B			

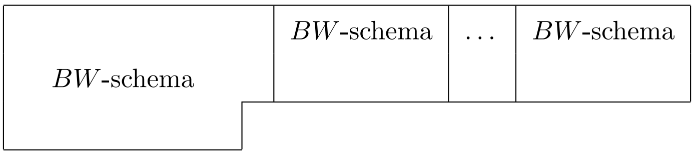
The coloring completes.

In some condition on q, λ and l, the C^2 may not be suitable for coloring of $S(a, \lambda a)$ in Phase 1. We have necessity to introduce the another one namely S-*block chessboard coloring*, denoted SbC^2. It will be useful for vertex coloring in Subcases 4.5.2.2 and 4.6.2. S-block sizes and coloring schema depend on $q = 2$ and some condition on λ, l.

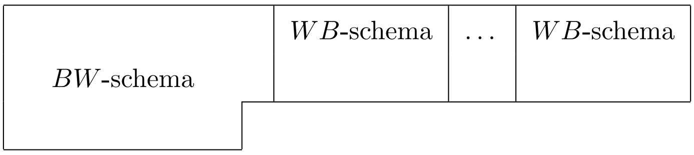
1 If $2 \leq l < \frac{\lambda}{2}$ and l is odd, then color the S-block of the first $l + (\lambda \mod l)$ columns of $S(a, \lambda a)$ according to the BW-schema and partition the remaining columns into S-blocks of consecutive l columns.

$l + (\lambda \mod l)$ columns	l columns	$\ldots$	l columns

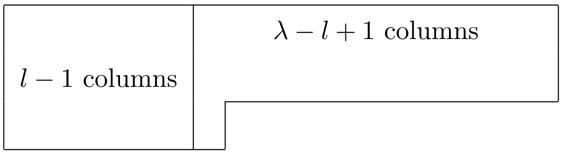
1.1 If ($\lambda \mod l$) is even, then we assign color each S-block according to the BW-schema.

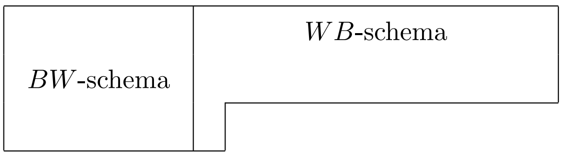


1.2 If ($\lambda \mod l$) is odd, then we assign color each S-block according to the WB-schema.



2 If λ and l are odd and $\frac{\lambda}{2} < l < \lambda$, then color the S-block of the first $l-1$ columns of $S(a, \lambda a)$ according to the BW-schema and color the remaining columns according to the WB-schema.



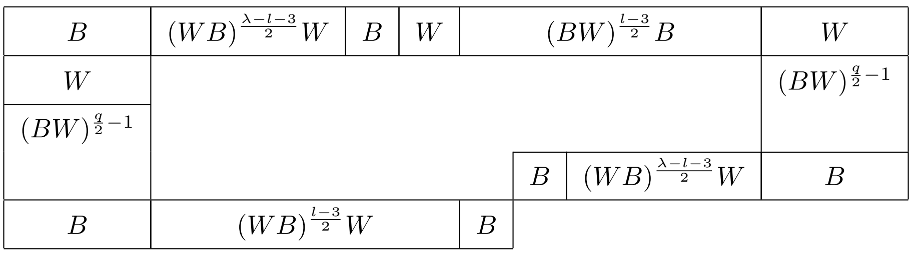


4.5 Assume that q, λ are even and l is odd. Then $\lambda - l$ is odd.

4.5.1 Assume that $q > 2$.

4.5.1.1 $2 < l < \lambda$.

Phase 1: Assign color to all entries of $S(a, \lambda a)$ according to the C^2.



Phase 2: Since A_1 is infeasible, we modify the cell 0 into R.

R	$(WB)^{\frac{\lambda-l-3}{2}}W$	B	W	$(BW)^{\frac{l-3}{2}}B$	W
W					$(BW)^{\frac{q}{2}-1}$
$(BW)^{\frac{q}{2}-1}$			B	$(WB)^{\frac{\lambda-l-3}{2}}W$	B
B	$(WB)^{\frac{l-3}{2}}W$	B			

Since A_2 is infeasible and $\{0, \lambda a\} \in B_0$ where 0 has color R, we can modify by changing B to R in its left column and W to R in its right column.

R	$(WB)^{\frac{\lambda-l-3}{2}}W$	B	W	$(BW)^{\frac{l-3}{2}}B$	R
W					$(BR)^{\frac{q}{2}-1}$
$(RW)^{\frac{q}{2}-1}$			B	$(WB)^{\frac{\lambda-l-3}{2}}W$	B
R	$(WB)^{\frac{l-3}{2}}W$	B			

We see that B_1 is feasible. Since some B_2 is infeasible and $\{0, a\} \in A_0$ where 0 has color R, we can modify by changing B to R in its top row and W to R in its bottom row.

R	$(WR)^{\frac{\lambda-l-3}{2}}W$	R	W	$(BW)^{\frac{l-3}{2}}B$	R
W					$(BR)^{\frac{q}{2}-1}$
$(RW)^{\frac{q}{2}-1}$			B	$(RB)^{\frac{\lambda-l-3}{2}}R$	B
R	$(WB)^{\frac{l-3}{2}}W$	B			

The coloring completes.

4.5.1.2 $l = 1$. Since $W(BW)^{\frac{l-3}{2}}BR = W(BW)^{-1}BR = R$ and $R(WB)^{\frac{l-3}{2}}WB = R(WB)^{-1}WB = R$ and the above complete coloring, we have

R	$(WR)^{\frac{\lambda}{2}-2}W$	R	R
W			$(BR)^{\frac{q}{2}-1}$
$(RW)^{\frac{q}{2}-1}$	B	$(RB)^{\frac{\lambda}{2}-2}R$	B
R			

Since $\{q\lambda a, 0\} \in A_1$ and those entries have the same color R, A_1 is infeasible. This means, we cannot assign color to all entries of $S(a, \lambda a)$ according to the C^2 in Phase 1. So we renew these assigning color according to C^4.

Phase 1: Assign color to all entries of $S(a, \lambda a)$ according to the C^4.

W	B	B	$(WB)^{\frac{\lambda}{2}-2}$	W
B	B			B
B				$(WB)^{\frac{q}{2}-2}W$
$(WB)^{\frac{q}{2}-2}W$				
	B	W	$(BW)^{\frac{\lambda}{2}-2}$	B
B				

We see that A_0 is feasible.

Phase 2: Since some A_2 is infeasible, we can suitably modify by changing B to R in its left column and W to R in its right column.

W	B	B	$(WB)^{\frac{\lambda}{2}-2}$	W
B	B			B
R				$(RB)^{\frac{q}{2}-2}R$
$(WR)^{\frac{q}{2}-2}W$				
	B	W	$(BW)^{\frac{\lambda}{2}-2}$	B
R				

We see that B_1 is feasible. Since some B_2 is infeasible, we can suitably modify by changing W to R in its bottom row and B to R in its top row.

W	B	R	$(WR)^{\frac{\lambda}{2}-2}$	W
B	B			B
R				$(RB)^{\frac{q}{2}-2}R$
$(WR)^{\frac{q}{2}-2}W$				
	B	W	$(BR)^{\frac{\lambda}{2}-2}$	B
R				

Since $\{\lambda a, 2\lambda a\} \in B_0$ where $2\lambda a$ has color R and $\{\lambda a, (\lambda+1)a\} \in A_0$ and those entries have the same color B. Pick the color R for $(\lambda + 1)a$.

W	B	R	$(WR)^{\frac{\lambda}{2}-2}$	W
B	R			B
R				$(RB)^{\frac{q}{2}-2}R$
$(WR)^{\frac{q}{2}-2}W$				
	B	W	$(BR)^{\frac{\lambda}{2}-2}$	B
R				

The coloring completes.

4.5.2 Assume that $q = 2$. Since $\lambda \neq \frac{c-1}{2}$, $2\lambda + l = q\lambda + l = c \neq 2\lambda + 1$, so $l \neq 1$.

4.5.2.1 $\frac{\lambda}{2} \leq l < \lambda$. Since $RW(RW)^{\frac{q}{2}-1}R = RW(RW)^0R = RWR$ and $R(BR)^{\frac{q}{2}-1}B = R(BR)^0B = RB$ with 4.5.1.1, we have

<table>
<tr><td>R</td><td>$(WR)^{\frac{\lambda-l-3}{2}}W$</td><td>$R$</td><td>$W$</td><td colspan="3">$(BW)^{\frac{l-3}{2}}B$</td><td>$R$</td></tr>
<tr><td>W</td><td colspan="3">$(BW)^{\frac{l-3}{2}}B$</td><td>W</td><td>B</td><td>$(RB)^{\frac{\lambda-l-3}{2}}R$</td><td>$B$</td></tr>
<tr><td>R</td><td colspan="3">$(WB)^{\frac{l-3}{2}}W$</td><td>B</td></tr>
</table>

These coloring is feasible for this case.

4.5.2.2 $2 < l < \frac{\lambda}{2}$.

<table>
<tr><td>R</td><td colspan="3">$(WR)^{\frac{\lambda-l-3}{2}}W$</td><td>$R$</td><td>$W$</td><td>$(BW)^{\frac{l-3}{2}}B$</td><td>$R$</td></tr>
<tr><td>W</td><td>$(BW)^{\frac{l-3}{2}}B$</td><td>W</td><td>B</td><td colspan="3">$(RB)^{\frac{\lambda-l-3}{2}}R$</td><td>$B$</td></tr>
<tr><td>R</td><td>$(WB)^{\frac{l-3}{2}}W$</td><td>B</td></tr>
</table>

Since there exists at least one consecutive entries in the same column have color R, the above coloring is infeasible. So we renew these assigning color in Phase 1 according to the SbC^2. Since λ is even and l is odd, we have the following two cases.

<table>
<tr><td>(λ mod l) columns</td><td colspan="2">l columns</td><td>$l^{\lfloor \frac{\lambda}{l} \rfloor - 2}$ columns</td><td>l columns</td></tr>
<tr><td colspan="2">l columns</td><td>(λ mod l) columns</td><td>$l^{\lfloor \frac{\lambda}{l} \rfloor - 2}$ columns</td><td>l columns</td></tr>
<tr><td colspan="2">l columns</td></tr>
</table>

4.5.2.2.1 ($\lambda \mod l$) is even. Then $l + (\lambda \mod l)$ is odd.

Phase 1: Assign color to all entries of $S(a, \lambda a)$ according to the SbC^2.

B	$(WB)^{\frac{\lambda \bmod l}{2}-1}$	W	B	$(WB)^{\frac{l-3}{2}}W$	B	$[B(WB)^{\frac{l-3}{2}}WB]^{\lfloor\frac{\lambda}{l}\rfloor-2}$	B	$(WB)^{\frac{l-3}{2}}W$	B
W	$(BW)^{\frac{l-3}{2}}B$	W	B	$(WB)^{\frac{\lambda \bmod l}{2}-1}$	W	$[W(BW)^{\frac{l-3}{2}}BW]^{\lfloor\frac{\lambda}{l}\rfloor-2}$	W	$(BW)^{\frac{l-3}{2}}B$	W
B	$(WB)^{\frac{l-3}{2}}W$	B							

Phase 2: Since A_1 is infeasible, we modify the cell 0 into R.

R	$(WB)^{\frac{\lambda \bmod l}{2}-1}$	W	B	$(WB)^{\frac{l-3}{2}}W$	B	$[B(WB)^{\frac{l-3}{2}}WB]^{\lfloor\frac{\lambda}{l}\rfloor-2}$	B	$(WB)^{\frac{l-3}{2}}W$	B
W	$(BW)^{\frac{l-3}{2}}B$	W	B	$(WB)^{\frac{\lambda \bmod l}{2}-1}$	W	$[W(BW)^{\frac{l-3}{2}}BW]^{\lfloor\frac{\lambda}{l}\rfloor-2}$	W	$(BW)^{\frac{l-3}{2}}B$	W
B	$(WB)^{\frac{l-3}{2}}W$	B							

We see that A_2 is feasible. Since B_1 is infeasible and $\{0, (q\lambda + l - 1)a\} \in A_1$ where 0 has color R, we can modify by changing W to R in its bottom row and B to R in its top row.

R	$(WB)^{\frac{\lambda \bmod l}{2}-1}$	W	B	$(WB)^{\frac{l-3}{2}}W$	B	$[B(WB)^{\frac{l-3}{2}}WB]^{\lfloor\frac{\lambda}{l}\rfloor-2}$	R	$(WR)^{\frac{l-3}{2}}W$	R
W	$(BW)^{\frac{l-3}{2}}B$	W	B	$(WB)^{\frac{\lambda \bmod l}{2}-1}$	W	$[W(BW)^{\frac{l-3}{2}}BW]^{\lfloor\frac{\lambda}{l}\rfloor-2}$	W	$(BW)^{\frac{l-3}{2}}B$	W
B	$(RB)^{\frac{l-3}{2}}R$	B							

Since some B_2 is infeasible and $\{0, a\} \in A_0$ where 0 has color R, we can modify by changing B to R in its top row and W to R in its bottom row.

R	$(WR)^{\frac{\lambda \bmod l}{2}-1}$	W	B	$(WB)^{\frac{l-3}{2}}W$	B	$[B(WB)^{\frac{l-3}{2}}WB]^{\lfloor\frac{\lambda}{l}\rfloor-2}$	R	$(WR)^{\frac{l-3}{2}}W$	R
W	$(BW)^{\frac{l-3}{2}}B$	W	B	$(RB)^{\frac{\lambda \bmod l}{2}-1}$	R	$[W(BW)^{\frac{l-3}{2}}BW]^{\lfloor\frac{\lambda}{l}\rfloor-2}$	W	$(BW)^{\frac{l-3}{2}}B$	W
B	$(RB)^{\frac{l-3}{2}}R$	B							

Modify the remain infeasible A_0 into R.

R	$(WR)^{\frac{\lambda \bmod l}{2}-1}$	W	B	$(WB)^{\frac{l-3}{2}}W$	B	$[R(WB)^{\frac{l-3}{2}}WB]^{\lfloor\frac{\lambda}{l}\rfloor-2}$	R	$(WR)^{\frac{l-3}{2}}W$	R
W	$(BW)^{\frac{l-3}{2}}B$	W	B	$(RB)^{\frac{\lambda \bmod l}{2}-1}$	R	$[W(BW)^{\frac{l-3}{2}}BR]^{\lfloor\frac{\lambda}{l}\rfloor-2}$	W	$(BW)^{\frac{l-3}{2}}B$	W
B	$(RB)^{\frac{l-3}{2}}R$	B							

The coloring completes.

4.5.2.2.2 $(\lambda \mod l)$ is odd. Then $l + (\lambda \mod l)$ is even.

Phase 1: Assign color to all entries of $S(a, \lambda a)$ according to the SbC^2.

B	$(WB)^{\frac{(\lambda \mod l)-3}{2}}W$	B	W	$(BW)^{\frac{l-3}{2}}B$	W	$[W(BW)^{\frac{l-3}{2}}BW]^{\lfloor\frac{\lambda}{l}\rfloor-2}$	W	$(BW)^{\frac{l-3}{2}}B$	W
W	$(BW)^{\frac{l-3}{2}}B$	W	B	$(WB)^{\frac{(\lambda \mod l)-3}{2}}W$	B	$[B(WB)^{\frac{l-3}{2}}WB]^{\lfloor\frac{\lambda}{l}\rfloor-2}$	B	$(WB)^{\frac{l-3}{2}}W$	B
B	$(WB)^{\frac{l-3}{2}}W$	B							

Phase 2: Since A_1 is infeasible, we modify the cell 0 into R.

R	$(WB)^{\frac{(\lambda \mod l)-3}{2}}W$	B	W	$(BW)^{\frac{l-3}{2}}B$	W	$[W(BW)^{\frac{l-3}{2}}BW]^{\lfloor\frac{\lambda}{l}\rfloor-2}$	W	$(BW)^{\frac{l-3}{2}}B$	W
W	$(BW)^{\frac{l-3}{2}}B$	W	B	$(WB)^{\frac{(\lambda \mod l)-3}{2}}W$	B	$[B(WB)^{\frac{l-3}{2}}WB]^{\lfloor\frac{\lambda}{l}\rfloor-2}$	B	$(WB)^{\frac{l-3}{2}}W$	B
B	$(WB)^{\frac{l-3}{2}}W$	B							

Since A_2 is infeasible and $\{0, \lambda a\} \in B_0$ where 0 has color R, we can modify by changing B to R in its left column and W to R in its right column.

R	$(WB)^{\frac{(\lambda \mod l)-3}{2}}W$	B	W	$(BW)^{\frac{l-3}{2}}B$	W	$[W(BW)^{\frac{l-3}{2}}BW]^{\lfloor\frac{\lambda}{l}\rfloor-2}$	W	$(BW)^{\frac{l-3}{2}}B$	R
W	$(BW)^{\frac{l-3}{2}}B$	W	B	$(WB)^{\frac{(\lambda \mod l)-3}{2}}W$	B	$[B(WB)^{\frac{l-3}{2}}WB]^{\lfloor\frac{\lambda}{l}\rfloor-2}$	B	$(WB)^{\frac{l-3}{2}}W$	B
R	$(WB)^{\frac{l-3}{2}}W$	B							

We see that B_1 is feasible. Since some B_2 is infeasible and $\{0, a\} \in A_0$ where 0 has color R, we can modify by changing B to R in its top row and W to R in its bottom row.

R	$(WR)^{\frac{(\lambda \mod l)-3}{2}}W$	R	W	$(BW)^{\frac{l-3}{2}}B$	W	$[W(BW)^{\frac{l-3}{2}}BW]^{\lfloor\frac{\lambda}{l}\rfloor-2}$	W	$(BW)^{\frac{l-3}{2}}B$	R
W	$(BW)^{\frac{l-3}{2}}B$	W	B	$(RB)^{\frac{(\lambda \mod l)-3}{2}}R$	B	$[B(WB)^{\frac{l-3}{2}}WB]^{\lfloor\frac{\lambda}{l}\rfloor-2}$	B	$(WB)^{\frac{l-3}{2}}W$	B
R	$(WB)^{\frac{l-3}{2}}W$	B							

Modify the remain infeasible A_0 into R.

<table>
<tr><td>R</td><td>$(WR)^{\frac{(\lambda \mod l)-3}{2}}W$</td><td>$R$</td><td>$W$</td><td colspan="3">$(BW)^{\frac{l-3}{2}}B$</td><td>$R$</td><td>$[W(BW)^{\frac{l-3}{2}}BR]^{\lfloor \frac{\lambda}{l} \rfloor -2}$</td><td>$W$</td><td>$(BW)^{\frac{l-3}{2}}B$</td><td>$R$</td></tr>
<tr><td>W</td><td colspan="3">$(BW)^{\frac{l-3}{2}}B$</td><td>W</td><td>B</td><td>$(RB)^{\frac{(\lambda \mod l)-3}{2}}R$</td><td>$B$</td><td>$[R(WB)^{\frac{l-3}{2}}WB]^{\lfloor \frac{\lambda}{l} \rfloor -2}$</td><td>$R$</td><td>$(WB)^{\frac{l-3}{2}}W$</td><td>$B$</td></tr>
<tr><td>R</td><td colspan="3">$(WB)^{\frac{l-3}{2}}W$</td><td>B</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
</table>

The coloring completes.

4.6 Assume that q is even and λ, l are odd. Then $\lambda - l$ is even.

4.6.1 Assume that $q > 2$.

Phase 1: Assign color to all entries of $S(a, \lambda a)$ according to the C^2.

<table>
<tr><td>B</td><td>$(WB)^{\frac{\lambda-l}{2}-1}$</td><td>$W$</td><td>$B$</td><td colspan="2">$(WB)^{\frac{l-3}{2}}W$</td><td>$B$</td></tr>
<tr><td>W</td><td colspan="5" rowspan="2"></td><td rowspan="2">$(WB)^{\frac{q}{2}-1}$</td></tr>
<tr><td rowspan="2">$(BW)^{\frac{q}{2}-1}$</td></tr>
<tr><td colspan="3"></td><td>B</td><td>$(WB)^{\frac{\lambda-l}{2}-1}$</td><td>$W$</td></tr>
<tr><td>B</td><td colspan="2">$(WB)^{\frac{l-3}{2}}W$</td><td>B</td><td colspan="3"></td></tr>
</table>

Phase 2: Since A_1 is infeasible, we modify 0 into R.

<table>
<tr><td>R</td><td>$(WB)^{\frac{\lambda-l}{2}-1}$</td><td>$W$</td><td>$B$</td><td colspan="2">$(WB)^{\frac{l-3}{2}}W$</td><td>$B$</td></tr>
<tr><td>W</td><td colspan="5" rowspan="2"></td><td rowspan="2">$(WB)^{\frac{q}{2}-1}$</td></tr>
<tr><td rowspan="2">$(BW)^{\frac{q}{2}-1}$</td></tr>
<tr><td colspan="3"></td><td>B</td><td>$(WB)^{\frac{\lambda-l}{2}-1}$</td><td>$W$</td></tr>
<tr><td>B</td><td colspan="2">$(WB)^{\frac{l-3}{2}}W$</td><td>B</td><td colspan="3"></td></tr>
</table>

We see that A_2 is feasible. Since B_1 is infeasible and $\{(q\lambda+l-1)a, 0\} \in A_1$ where 0 has color R, we can modify by changing W to R in its bottom row and B to R in its top row.

<table>
<tr><td>R</td><td>$(WB)^{\frac{\lambda-l}{2}-1}$</td><td>$W$</td><td>$R$</td><td colspan="2">$(WR)^{\frac{l-3}{2}}W$</td><td>$R$</td></tr>
<tr><td>W</td><td colspan="5" rowspan="2"></td><td rowspan="2">$(WB)^{\frac{q}{2}-1}$</td></tr>
<tr><td rowspan="2">$(BW)^{\frac{q}{2}-1}$</td></tr>
<tr><td colspan="3"></td><td>B</td><td>$(WB)^{\frac{\lambda-l}{2}-1}$</td><td>$W$</td></tr>
<tr><td>B</td><td colspan="2">$(RB)^{\frac{l-3}{2}}R$</td><td>B</td><td colspan="3"></td></tr>
</table>

Since some B_2 is infeasible and $\{0, a\} \in A_0$ where 0 has color R, we can modify by changing B to R in its top row and W to R in its bottom row.

<table>
<tr><td>R</td><td>$(WR)^{\frac{\lambda-l}{2}-1}$</td><td>$W$</td><td>$R$</td><td colspan="2">$(WR)^{\frac{l-3}{2}}W$</td><td>$R$</td></tr>
<tr><td>W</td><td colspan="5" rowspan="2"></td><td rowspan="2">$(WB)^{\frac{q}{2}-1}$</td></tr>
<tr><td rowspan="2">$(BW)^{\frac{q}{2}-1}$</td></tr>
<tr><td colspan="3"></td><td>B</td><td>$(RB)^{\frac{\lambda-l}{2}-1}$</td><td>$R$</td></tr>
<tr><td>B</td><td colspan="2">$(RB)^{\frac{l-3}{2}}R$</td><td>B</td><td colspan="3"></td></tr>
</table>

The coloring completes.

4.6.2 Assume that $q = 2$. Since $\lambda \neq \frac{c-1}{2}$, $2\lambda + l = q\lambda + l = c \neq 2\lambda + 1$, so $l \neq 1$. For the reason, $RW(BW)^{\frac{q}{2}-1}B = RW(BW)^0B = RWB$ and

$R(WB)^{\frac{q}{2}-1}R = R(WB)^0R = RR$ and the above complete coloring, we have

<table>
<tr><td>R</td><td>$(WR)^{\frac{\lambda-l}{2}-1}$</td><td>$W$</td><td>$R$</td><td colspan="2">$(WR)^{\frac{l-3}{2}}W$</td><td>$R$</td></tr>
<tr><td>W</td><td colspan="2">$(BW)^{\frac{l-3}{2}}B$</td><td>W</td><td>B</td><td>$(RB)^{\frac{\lambda-l}{2}-1}$</td><td>$R$</td></tr>
<tr><td>B</td><td colspan="2">$(RB)^{\frac{l-3}{2}}R$</td><td>B</td></tr>
</table>

If $l = \lambda$, then the above coloring completes, otherwise does not because $\{(\lambda-1)a, (2\lambda-1)a\} \in B_0$ and those entries have the same color R. So we cannot assign color to all entries of $S(a, \lambda a)$ according to the C^2 in Phase 1.

4.6.2.1 $l = \lambda$.

R	$(WR)^{\frac{l-3}{2}}W$	R
W	$(BW)^{\frac{l-3}{2}}B$	W
B	$(RB)^{\frac{l-3}{2}}R$	B

4.6.2.2 $2 < l < \frac{\lambda}{2}$. Since λ and l are odd, we have the following two cases.

4.6.2.2.1 If ($\lambda \mod l$) is even, then the coloring same as 4.5.2.2.1.

4.6.2.2.2 If ($\lambda \mod l$) is odd, then the coloring same as 4.5.2.2.2.

4.6.2.3 $\frac{\lambda}{2} < l < \lambda$, except $l = 3$ and $\lambda = 5$. Then $\frac{2l-\lambda-3}{2} \neq -1$.

<table>
<tr><td colspan="2">$\lambda - l$
columns</td><td colspan="2">$2l - \lambda - 1$
columns</td><td>$(l-1)a$</td><td>$\lambda - l - 1$
columns</td><td>$(\lambda - 1)a$</td></tr>
<tr><td colspan="4">$l - 1$
columns</td><td>$(\lambda + l - 1)a$</td><td colspan="2">$\lambda - l$
columns</td></tr>
<tr><td>$2l - \lambda - 1$
columns</td><td colspan="2">$(\lambda + 2l - 1)a$</td><td>$\lambda - l - 1$
columns</td><td>$(2\lambda + l - 1)a$</td><td></td><td></td></tr>
</table>

Phase 1: Assign color to all entries of $S(a, \lambda a)$ according to the SbC^2.

<table>
<tr><td>B</td><td colspan="2">$(WB)^{\frac{\lambda-l}{2}-1}$</td><td>$W$</td><td>$B$</td><td>$(WB)^{\frac{2l-\lambda-3}{2}}$</td><td>$W$</td><td>$W$</td><td>$B$</td><td>$(WB)^{\frac{\lambda-l}{2}-2}W$</td><td>$B$</td><td>$W$</td></tr>
<tr><td>W</td><td colspan="5">$(BW)^{\frac{l-3}{2}}$</td><td>B</td><td>B</td><td>W</td><td colspan="2">$(BW)^{\frac{\lambda-l}{2}-1}$</td><td>$B$</td></tr>
<tr><td>B</td><td>$(WB)^{\frac{2l-\lambda-3}{2}}$</td><td>$W$</td><td>$B$</td><td>$W$</td><td>$(BW)^{\frac{\lambda-l}{2}-2}B$</td><td>$W$</td><td>$W$</td><td></td><td></td><td></td><td></td></tr>
</table>

We see that A_1 is feasible.

Phase 2: Since A_2 is infeasible, we can modify by changing B to R in its left column and W to R in its right column.

<table>
<tr><td>B</td><td colspan="2">$(WB)^{\frac{\lambda-l}{2}-1}$</td><td>$W$</td><td>$B$</td><td>$(WB)^{\frac{2l-\lambda-3}{2}}$</td><td>$W$</td><td>$W$</td><td>$B$</td><td>$(WB)^{\frac{\lambda-l}{2}-2}W$</td><td>$B$</td><td>$R$</td></tr>
<tr><td>W</td><td colspan="5">$(BW)^{\frac{l-3}{2}}$</td><td>B</td><td>B</td><td>W</td><td colspan="2">$(BW)^{\frac{\lambda-l}{2}-1}$</td><td>$B$</td></tr>
<tr><td>R</td><td>$(WB)^{\frac{2l-\lambda-3}{2}}$</td><td>$W$</td><td>$B$</td><td>$W$</td><td>$(BW)^{\frac{\lambda-l}{2}-2}B$</td><td>$W$</td><td>$W$</td><td></td><td></td><td></td><td></td></tr>
</table>

Since some B_1 is infeasible in the first $2l - \lambda - 1$ columns, we can suitably modify by changing B to R in its bottom row and W to R in its top row.

<table>
<tr><td>B</td><td colspan="2">$(WB)^{\frac{\lambda-l}{2}-1}$</td><td>$W$</td><td>$B$</td><td>$(RB)^{\frac{2l-\lambda-3}{2}}$</td><td>$R$</td><td>$W$</td><td>$B$</td><td>$(WB)^{\frac{\lambda-l}{2}-2}W$</td><td>$B$</td><td>$R$</td></tr>
<tr><td>W</td><td colspan="5">$(BW)^{\frac{l-3}{2}}$</td><td>B</td><td>B</td><td>W</td><td colspan="2">$(BW)^{\frac{\lambda-l}{2}-1}$</td><td>$B$</td></tr>
<tr><td>R</td><td>$(WR)^{\frac{2l-\lambda-3}{2}}$</td><td>$W$</td><td>$B$</td><td>$W$</td><td>$(BW)^{\frac{\lambda-l}{2}-2}B$</td><td>$W$</td><td>$W$</td><td></td><td></td><td></td><td></td></tr>
</table>

We see that B_2 is feasible. Modify the remain infeasible A_0 into R.

<table>
<tr><td>B</td><td colspan="2">$(WB)^{\frac{\lambda-l}{2}-1}$</td><td>$W$</td><td>$B$</td><td>$(RB)^{\frac{2l-\lambda-3}{2}}$</td><td>$R$</td><td>$W$</td><td>$B$</td><td>$(WB)^{\frac{\lambda-l}{2}-2}W$</td><td>$B$</td><td>$R$</td></tr>
<tr><td>W</td><td colspan="5">$(BW)^{\frac{l-3}{2}}$</td><td>B</td><td>R</td><td>W</td><td colspan="2">$(BW)^{\frac{\lambda-l}{2}-1}$</td><td>$B$</td></tr>
<tr><td>R</td><td>$(WR)^{\frac{2l-\lambda-3}{2}}$</td><td>$W$</td><td>$B$</td><td>$W$</td><td>$(BW)^{\frac{\lambda-l}{2}-2}B$</td><td>$R$</td><td>$W$</td><td></td><td></td><td></td><td></td></tr>
</table>

The coloring completes.

We illustrate Theorem 3.2.1 by the following examples.

Example 3.2.2. The graph $C_{\mathbb{Z}_5\times\mathbb{Z}_5}((0,1),(0,2))$ has $\frac{|\mathbb{Z}_5\times\mathbb{Z}_5|}{|\langle(0,1),(0,2)\rangle|} = \frac{25}{5} = 5$ connected components, each of which isomorphic to $C_H((0,1),(0,2))$ where $H = \langle(0,1),(0,2)\rangle$. Since $(0,2) = 2(0,1)$, $\lambda = 2$. Since $c = o((0,1)) = 5 = 2\lambda+1$, $S(a,2a)$ has 3 rows and 2 columns. The last row contains 1 entry.

$(0,0)$	$(0,1)$
$(0,2)$	$(0,3)$
$(0,4)$	

$(1,0)$	$(1,1)$
$(1,2)$	$(1,3)$
$(1,4)$	

$(2,0)$	$(2,1)$
$(2,2)$	$(2,3)$
$(2,4)$	

$(3,0)$	$(3,1)$
$(3,2)$	$(3,3)$
$(3,4)$	

$(4,0)$	$(4,1)$
$(4,2)$	$(4,3)$
$(4,4)$	

By Case 1, we have the vertex coloring for $C_H((0,1),(0,2))$ and the other component in the following table.

B	W
R	G
Y	

Example 3.2.3. The graph $C_{\mathbb{Z}_{26}}(2,10)$ has $\frac{|\mathbb{Z}_{26}|}{|\langle 2,10\rangle|} = \frac{26}{13} = 2$ connected components, each of which isomorphic to $C_H(2,10)$ where $H = \langle 2,10\rangle$. Since $10 = 5(2)$, $\lambda = 5$. Since $c = o(2) = 13 = 2\lambda+3$, $S(a,5a)$ has 3 rows and 5 columns. The last row contains 3 entries.

0	2	4	6	8
10	12	14	16	18
20	22	24		

1	3	5	7	9
11	13	15	17	19
21	23	25		

By Case 2, we have the vertex coloring for $C_H(2,10)$ and the another component in the following table.

B	W	R	B	W
R	B	W	R	B
W	R	G		

Example 3.2.4. The graph $C_{\mathbb{Z}_2\times\mathbb{Z}_8}((1,1),(0,2))$ has $\frac{|\mathbb{Z}_2\times\mathbb{Z}_8|}{|\langle(1,1),(0,2)\rangle|} = \frac{16}{8} = 2$ connected components, each of which isomorphic to $C_H((1,1),(0,2))$ where $H = \langle(1,1),(0,2)\rangle$. Since $(0,2) = 2(1,1)$, $\lambda = 2$. Since $c = o((1,1)) = 8 = 3\lambda+2$, $S(a,2a)$ has 4 rows and 2 columns. Since $l = \lambda = 2$, the last row of $S(a,2a)$ is full.

(0,0)	(1,1)
(0,2)	(1,3)
(0,4)	(1,5)
(0,6)	(1,7)

(1,0)	(0,1)
(1,2)	(0,3)
(1,4)	(0,5)
(1,6)	(0,7)

Since $3 \nmid c$ and $c \equiv 0 \mod 4$, by Subcase 3.3.2, we have the vertex coloring for $C_H((1,1),(0,2))$ and the another component in the following table.

B	W
R	G
B	W
R	G

Example 3.2.5. Since $\mathbb{Z}_{13} = \langle 1, 2\rangle$, the graph $C_{\mathbb{Z}_{13}}(1,2)$ is connected. Since $2 = 2(1)$, $\lambda = 2$. Since $c = o(1) = 13 = 6\lambda + 1$, $S(a, 2a)$ has 7 rows and 2 columns. The last row contains 1 entry.

0	1
2	3
4	5
6	7
8	9
10	11
12	

Since $3 \nmid c$ and $c \equiv 1 \mod 4$, by Subcase 3.3, we have the vertex coloring for $C_{\mathbb{Z}_{13}}(1,2)$ in the following table.

B	W
R	G
B	W
R	G
W	R
G	W
R	

Example 3.2.6. Since $\mathbb{Z}_{25} = \langle 1, 11\rangle$, the graph $C_{\mathbb{Z}_{25}}(1,11)$ is connected. Since $11 = 11(1)$, $\lambda = 11$. Since $c = o(1) = 25 = 2\lambda + 3$, $q = 2$ and $l = 3$, so ($\lambda \mod l$) $= 2$ and $\lfloor \frac{\lambda}{l} \rfloor = 3$. We obtain $S(a, 11a)$ has $q + 1 = 3$ rows and $\lambda = 11$ columns. The last row contains $l = 3$ entries.

0	1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20	21
22	23	24								

Since $2 < \lambda < \frac{c}{2}$, $S(a, 11a)$ is in Case 4. Since $q = 2$ and λ, l are odd where $2 < l < \frac{\lambda}{2}$ and $(\lambda \mod l)$ is even, by Subcase 4.6.2.2.1, we have the vertex coloring for $C_{\mathbb{Z}_{25}}(1, 11)$ in the following table.

R	W	B	W	B	R	W	B	R	W	R
W	B	W	B	R	W	B	R	W	B	W
B	R	B								

Example 3.2.7. The graph $C_{\mathbb{Z}_5 \times \mathbb{Z}_{25}}((3, 1), (3, 6))$ has $\frac{|\mathbb{Z}_5 \times \mathbb{Z}_{25}|}{|\langle (3,1),(3,6) \rangle|} = \frac{125}{25} = 5$ connected components, each of which isomorphic to $C_H((3, 1), (3, 6))$ where $H = \langle (3, 1), (3, 6) \rangle$. Since $(3, 6) = 6(3, 1)$, $\lambda = 6$. Since $c = o((3, 1)) = 25 = 4\lambda + 1$, $q = 4$ and $l = 1$, that is, $S(a, 6a)$ has $q + 1 = 5$ rows and $\lambda = 6$ columns. The last row contains $l = 1$ entry.

$(0, 0)$	$(3, 1)$	$(1, 2)$	$(4, 3)$	$(2, 4)$	$(0, 5)$
$(3, 6)$	$(1, 7)$	$(4, 8)$	$(2, 9)$	$(0, 10)$	$(3, 11)$
$(1, 12)$	$(4, 13)$	$(2, 14)$	$(0, 15)$	$(3, 16)$	$(1, 17)$
$(4, 18)$	$(2, 19)$	$(0, 20)$	$(3, 21)$	$(1, 22)$	$(4, 23)$
$(2, 24)$					

$(1, 0)$	$(4, 1)$	$(2, 2)$	$(0, 3)$	$(3, 4)$	$(1, 5)$
$(4, 6)$	$(2, 7)$	$(0, 8)$	$(3, 9)$	$(1, 10)$	$(4, 11)$
$(2, 12)$	$(0, 13)$	$(3, 14)$	$(1, 15)$	$(4, 16)$	$(2, 17)$
$(0, 18)$	$(3, 19)$	$(1, 20)$	$(4, 21)$	$(2, 22)$	$(0, 23)$
$(3, 24)$					

$(2, 0)$	$(0, 1)$	$(3, 2)$	$(1, 3)$	$(4, 4)$	$(2, 5)$
$(0, 6)$	$(3, 7)$	$(1, 8)$	$(4, 9)$	$(2, 10)$	$(0, 11)$
$(3, 12)$	$(1, 13)$	$(4, 14)$	$(2, 15)$	$(0, 16)$	$(3, 17)$
$(1, 18)$	$(4, 19)$	$(2, 20)$	$(0, 21)$	$(3, 22)$	$(1, 23)$
$(4, 24)$					

$(3,0)$	$(1,1)$	$(4,2)$	$(2,3)$	$(0,4)$	$(3,5)$
$(1,6)$	$(4,7)$	$(2,8)$	$(0,9)$	$(3,10)$	$(1,11)$
$(4,12)$	$(2,13)$	$(0,14)$	$(3,15)$	$(1,16)$	$(4,17)$
$(2,18)$	$(0,19)$	$(3,20)$	$(1,21)$	$(4,22)$	$(2,23)$
$(0,24)$					

$(4,0)$	$(2,1)$	$(0,2)$	$(3,3)$	$(1,4)$	$(4,5)$
$(2,6)$	$(0,7)$	$(3,8)$	$(1,9)$	$(4,10)$	$(2,11)$
$(0,12)$	$(3,13)$	$(1,14)$	$(4,15)$	$(2,16)$	$(0,17)$
$(3,18)$	$(1,19)$	$(4,20)$	$(2,21)$	$(0,22)$	$(3,23)$
$(1,24)$					

Since $2 < \lambda < \frac{c}{2}$, $S(a, 11a)$ is in Case 4. Since q, λ are even with $q > 2$ and $l = 1$, by Subcase 4.5.1.2, we have the vertex coloring for $C_{\mathbb{Z}_5 \times \mathbb{Z}_{25}}((3,1),(3,6))$ and the other component in the following table.

W	B	R	W	R	W
B	R	W	B	W	B
R	W	B	W	B	R
W	B	W	B	R	B
R					

3.3 The case $b \notin \langle a \rangle$

When $b \notin \langle a \rangle$, we have $\chi(C_H(a,b)) = 2$ (Theorem 3.1.1) and $\chi(C_H(a,b)) = 3$ with vertex coloring algorithm in the next theorem.

Theorem 3.3.1. *Let $a, b \in G \setminus \{0\}$ be such that $a \neq \pm b, o(b) \leq o(a)$ and $b \notin \langle a \rangle$. If $r + \lambda$ is odd or c is odd, then $\chi(C_H(a,b)) = 3$.*

Proof. Assume that $r + \lambda$ is odd or c is odd. Then $\chi(C_H(a,b)) > 2$ by Theorem 3.1.1. We can prove this theorem by to finding an assignment of three colors to the vertices of graph $C_H(a,b)$.

Next we shall describe the vertex coloring algorithm for the graph $C_H(a,b)$ with $b \notin \langle a \rangle$, we use 2-phases method. In Phase 1, we start by suitably coloring $M_G(a,b)$ with C^2 and BC^2. After that we suitably modify into R the color of one entry of each infeasible edge, which we shall consider from boundary edges A_1, B_1 and B_2 of $M_G(a,b)$ respectively, which these coloring do not make the other a and b-edges become infeasible. Thus, we have an approach for $C_H(a,b)$ as follows.

0	$\dots$	$(\lambda-1)a$	λa	$\dots$	$(c-1)a$
$\vdots$					$\vdots$
$(r-1)b$	$\dots$	$(r-1)b$ $+(c-\lambda-1)a$	$(r-1)b$ $+(c-\lambda)a$	$\dots$	$(r-1)b$ $+(c-1)a$

Case 1. *r is odd and c, λ are even.* Then $c-\lambda$ is even.

Phase 1: Assign color to all entries of $M_G(a,b)$ according to the C^2.

B	$(WB)^{\frac{\lambda}{2}-1}$	W	B	$(WB)^{\frac{c-\lambda}{2}-1}$	W
$(WB)^{\frac{r-3}{2}}W$					$(BW)^{\frac{r-3}{2}}B$
B	$(WB)^{\frac{c-\lambda}{2}-1}$	W	B	$(WB)^{\frac{\lambda}{2}-1}$	W

We see that A_1 is feasible.

Phase 2: Since B_1 is infeasible, we can modify by changing B to R in its bottom row and W to R in its top row.

B	$(WB)^{\frac{\lambda}{2}-1}$	W	B	$(RB)^{\frac{c-\lambda}{2}-1}$	R
$(WB)^{\frac{r-3}{2}}W$					$(BW)^{\frac{r-3}{2}}B$
R	$(WR)^{\frac{c-\lambda}{2}-1}$	W	B	$(WB)^{\frac{\lambda}{2}-1}$	W

Since B_2 is infeasible and $\{(r-1)b+(c-1)a, (r-1)b\} \in A_1$ where $(r-1)b$ has color R, we can modify by changing B to R in its bottom row and W to R in its top row.

B	$(RB)^{\frac{\lambda}{2}-1}$	R	B	$(RB)^{\frac{c-\lambda}{2}-1}$	R
$(WB)^{\frac{r-3}{2}}W$					$(BW)^{\frac{r-3}{2}}B$
R	$(WR)^{\frac{c-\lambda}{2}-1}$	W	R	$(WR)^{\frac{\lambda}{2}-1}$	W

The coloring completes.

Case 2. *r, c are odd and λ is even.* Then $c-\lambda$ is odd.

Phase 1: Assign color to all entries of $M_G(a,b)$ according to the C^2.

B	$(WB)^{\frac{\lambda}{2}-1}$	W	B	$(WB)^{\frac{c-\lambda-3}{2}}W$	B
$(WB)^{\frac{r-3}{2}}W$					$(WB)^{\frac{r-3}{2}}W$
B	$(WB)^{\frac{c-\lambda-3}{2}}W$	B	W	$(BW)^{\frac{\lambda}{2}-1}$	B

Phase 2: Since A_1 is infeasible, we can modify by changing B to R in its left column and W to R in its right column.

<table>
<tr><td>R</td><td>$(WB)^{\frac{\lambda}{2}-1}$</td><td>$W$</td><td>$B$</td><td colspan="4">$(WB)^{\frac{c-\lambda-3}{2}}W$</td><td>$B$</td></tr>
<tr><td>$(WR)^{\frac{r-3}{2}}W$</td><td colspan="7"></td><td>$(RB)^{\frac{r-3}{2}}R$</td></tr>
<tr><td>R</td><td colspan="4">$(WB)^{\frac{c-\lambda-3}{2}}W$</td><td>$B$</td><td>$W$</td><td>$(BW)^{\frac{\lambda}{2}-1}$</td><td>$B$</td></tr>
</table>

Since some B_1 is infeasible and $\{(c-1)a, 0\} \in A_1$ where 0 has color R, we can modify by changing W to R in its top row and B to R in its bottom row.

<table>
<tr><td>R</td><td>$(WB)^{\frac{\lambda}{2}-1}$</td><td>$W$</td><td>$B$</td><td colspan="4">$(RB)^{\frac{c-\lambda-3}{2}}R$</td><td>$B$</td></tr>
<tr><td>$(WR)^{\frac{r-3}{2}}W$</td><td colspan="7"></td><td>$(RB)^{\frac{r-3}{2}}R$</td></tr>
<tr><td>R</td><td colspan="4">$(WR)^{\frac{c-\lambda-3}{2}}W$</td><td>$R$</td><td>$W$</td><td>$(BW)^{\frac{\lambda}{2}-1}$</td><td>$B$</td></tr>
</table>

We see that B_2 is feasible. The coloring completes.

Case 3. *r, c and λ are odd.* Then $c-\lambda$ is even.

Phase 1: Assign color to all entries of $M_G(a,b)$ according to the C^2.

<table>
<tr><td>B</td><td>$(WB)^{\frac{\lambda-3}{2}}W$</td><td>$B$</td><td>$W$</td><td colspan="4">$(BW)^{\frac{c-\lambda}{2}-1}$</td><td>$B$</td></tr>
<tr><td>$(WB)^{\frac{r-3}{2}}W$</td><td colspan="7"></td><td>$(WB)^{\frac{r-3}{2}}W$</td></tr>
<tr><td>B</td><td colspan="4">$(WB)^{\frac{c-\lambda}{2}-1}$</td><td>$W$</td><td>$B$</td><td>$(WB)^{\frac{\lambda-3}{2}}W$</td><td>$B$</td></tr>
</table>

Phase 2: Since A_1 is infeasible, we can modify by changing B to R in its left column and W to R in its right column.

<table>
<tr><td>R</td><td>$(WB)^{\frac{\lambda-3}{2}}W$</td><td>$B$</td><td>$W$</td><td colspan="4">$(BW)^{\frac{c-\lambda}{2}-1}$</td><td>$B$</td></tr>
<tr><td>$(WR)^{\frac{r-3}{2}}W$</td><td colspan="7"></td><td>$(RB)^{\frac{r-3}{2}}R$</td></tr>
<tr><td>R</td><td colspan="4">$(WB)^{\frac{c-\lambda}{2}-1}$</td><td>$W$</td><td>$B$</td><td>$(WB)^{\frac{\lambda-3}{2}}W$</td><td>$B$</td></tr>
</table>

We see that B_1 is feasible. Since some B_2 is infeasible and $\{0, a\} \in A_0$ where 0 has color R, we can modify by changing B to R in its top row and W to R in its bottom row.

<table>
<tr><td>R</td><td>$(WR)^{\frac{\lambda-3}{2}}W$</td><td>$R$</td><td>$W$</td><td colspan="4">$(BW)^{\frac{c-\lambda}{2}-1}$</td><td>$B$</td></tr>
<tr><td>$(WR)^{\frac{r-3}{2}}W$</td><td colspan="7"></td><td>$(RB)^{\frac{r-3}{2}}R$</td></tr>
<tr><td>R</td><td colspan="4">$(WB)^{\frac{c-\lambda}{2}-1}$</td><td>$W$</td><td>$B$</td><td>$(RB)^{\frac{\lambda-3}{2}}R$</td><td>$B$</td></tr>
</table>

The coloring completes.

Case 4. *r is even, c is odd and λ is even.* Then $c-\lambda$ is odd.

4.1 Assume that $r > 2$.

Phase 1: Assign color to all entries of $M_G(a,b)$ according to the C^2.

B	$(WB)^{\frac{\lambda}{2}-1}$	W	B	$(WB)^{\frac{c-\lambda-3}{2}}W$	B
$(WB)^{\frac{r}{2}-1}$					$(WB)^{\frac{r}{2}-1}$
W	$(BW)^{\frac{c-\lambda-3}{2}}B$	W	B	$(WB)^{\frac{\lambda}{2}-1}$	W

Phase 2: Since A_1 is infeasible, we can modify by changing B to R in its left column and W to R in its right column.

R	$(WB)^{\frac{\lambda}{2}-1}$	W	B	$(WB)^{\frac{c-\lambda-3}{2}}W$	B
$(WR)^{\frac{r}{2}-1}$					$(RB)^{\frac{r}{2}-1}$
W	$(BW)^{\frac{c-\lambda-3}{2}}B$	W	B	$(WB)^{\frac{\lambda}{2}-1}$	R

We see that B_1 is feasible. Since some B_2 is infeasible and $\{0,a\} \in A_0$ where 0 has color R, we can modify by changing B to R in its top row and W to R in its bottom row.

R	$(WR)^{\frac{\lambda}{2}-1}$	W	B	$(WB)^{\frac{c-\lambda-3}{2}}W$	B
$(WR)^{\frac{r}{2}-1}$					$(RB)^{\frac{r}{2}-1}$
W	$(BW)^{\frac{c-\lambda-3}{2}}B$	W	B	$(RB)^{\frac{\lambda}{2}-1}$	R

The coloring completes.

4.2 Assume that $r = 2$.

4.2.1 $\lambda < \frac{c}{2}$. Since $R(WR)^{\frac{r}{2}-1}W = R(WR)^0W = RW$ and $B(RB)^{\frac{r}{2}-1}R = B(RB)^0R = BR$ and the above complete coloring, we have

R	$(WR)^{\frac{\lambda}{2}-1}$	W	B	$(WB)^{\frac{c-\lambda-3}{2}}W$	B
W	$(BW)^{\frac{c-\lambda-3}{2}}B$	W	B	$(RB)^{\frac{\lambda}{2}-1}$	R

These coloring is feasible for this case.

4.2.2 $\lambda > \frac{c}{2}$.

R	$(WR)^{\frac{\lambda}{2}-1}$	W	B	$(WB)^{\frac{c-\lambda-3}{2}}W$	B
W	$(BW)^{\frac{c-\lambda-3}{2}}B$	W	B	$(RB)^{\frac{\lambda}{2}-1}$	R

Since there exists at least one consecutive entries in the same column have color R, the above coloring is infeasible. So we adjust these assigning color in Phase 1 according to the BC^2. Since c is odd and $o(a + \langle b \rangle) \mid c$, $o(a + \langle b \rangle)$ is odd.

Phase 1: Assign color to all entries of $M_G(a, b)$ according to the BC^2.

B	$(WB)^{\frac{o(a+\langle b \rangle)-3}{2}}W$	B	$[B(WB)^{\frac{o(a+\langle b \rangle)-3}{2}}WB]^{\frac{c}{o(a+\langle b \rangle)}-2}$	B	$(WB)^{\frac{o(a+\langle b \rangle)-3}{2}}W$	B
W	$(BW)^{\frac{o(a+\langle b \rangle)-3}{2}}B$	W	$[W(BW)^{\frac{o(a+\langle b \rangle)-3}{2}}BW]^{\frac{c}{o(a+\langle b \rangle)}-2}$	W	$(BW)^{\frac{o(a+\langle b \rangle)-3}{2}}B$	W

Phase 2: Since A_1 is infeasible, we can modify by changing B to R in its left column and W to R in its right column.

R	$(WB)^{\frac{o(a+\langle b \rangle)-3}{2}}W$	B	$[B(WB)^{\frac{o(a+\langle b \rangle)-3}{2}}WB]^{\frac{c}{o(a+\langle b \rangle)}-2}$	B	$(WB)^{\frac{o(a+\langle b \rangle)-3}{2}}W$	B
W	$(BW)^{\frac{o(a+\langle b \rangle)-3}{2}}B$	W	$[W(BW)^{\frac{o(a+\langle b \rangle)-3}{2}}BW]^{\frac{c}{o(a+\langle b \rangle)}-2}$	W	$(BW)^{\frac{o(a+\langle b \rangle)-3}{2}}B$	R

We see that B_1 and B_2 are feasible. Since A_0 is infeasible and $\{(r-1)b + (c-1)a, (\lambda - 1)a\} \in B_2$ and $(r-1)b + (c-1)a$ has color R, we can suitably modify by changing W to R in the last column of a block and B to R in the first column of consecutive block.

R	$(WB)^{\frac{o(a+\langle b \rangle)-3}{2}}W$	B	$[R(WB)^{\frac{o(a+\langle b \rangle)-3}{2}}WB]^{\frac{c}{o(a+\langle b \rangle)}-2}$	R	$(WB)^{\frac{o(a+\langle b \rangle)-3}{2}}W$	B
W	$(BW)^{\frac{o(a+\langle b \rangle)-3}{2}}B$	R	$[W(BW)^{\frac{o(a+\langle b \rangle)-3}{2}}BR]^{\frac{c}{o(a+\langle b \rangle)}-2}$	W	$(BW)^{\frac{o(a+\langle b \rangle)-3}{2}}B$	R

The coloring completes.

Case 5. *r is even, c and λ are odd.* Then $c-\lambda$ is even.

5.1 Assume that $r>2$.

Phase 1: Assign color to all entries of $M_G(a,b)$ according to the C^2.

<table>
<tr><td>B</td><td colspan="2">$(WB)^{\frac{\lambda-3}{2}}W$</td><td>$B$</td><td>$W$</td><td>$(BW)^{\frac{c-\lambda}{2}-1}$</td><td>$B$</td></tr>
<tr><td>$(WB)^{\frac{r}{2}-1}$</td><td colspan="5"></td><td>$(WB)^{\frac{r}{2}-1}$</td></tr>
<tr><td>W</td><td>$(BW)^{\frac{c-\lambda}{2}-1}$</td><td>$B$</td><td>$W$</td><td colspan="2">$(BW)^{\frac{\lambda-3}{2}}B$</td><td>$W$</td></tr>
</table>

Phase 2: Since A_1 is infeasible, we can modify by changing B to R in its left column and W to R in its right column.

<table>
<tr><td>R</td><td colspan="2">$(WB)^{\frac{\lambda-3}{2}}W$</td><td>$B$</td><td>$W$</td><td>$(BW)^{\frac{c-\lambda}{2}-1}$</td><td>$B$</td></tr>
<tr><td>$(WR)^{\frac{r}{2}-1}$</td><td colspan="5"></td><td>$(RB)^{\frac{r}{2}-1}$</td></tr>
<tr><td>W</td><td>$(BW)^{\frac{c-\lambda}{2}-1}$</td><td>$B$</td><td>$W$</td><td colspan="2">$(BW)^{\frac{\lambda-3}{2}}B$</td><td>$R$</td></tr>
</table>

Since B_1 is infeasible, we can modify by changing B to R in its bottom row and W to R in its top row.

<table>
<tr><td>R</td><td colspan="2">$(WB)^{\frac{\lambda-3}{2}}W$</td><td>$B$</td><td>$R$</td><td>$(BR)^{\frac{c-\lambda}{2}-1}$</td><td>$B$</td></tr>
<tr><td>$(WR)^{\frac{r}{2}-1}$</td><td colspan="5"></td><td>$(RB)^{\frac{r}{2}-1}$</td></tr>
<tr><td>W</td><td>$(RW)^{\frac{c-\lambda}{2}-1}$</td><td>$R$</td><td>$W$</td><td colspan="2">$(BW)^{\frac{\lambda-3}{2}}B$</td><td>$R$</td></tr>
</table>

We see that B_2 is feasible. The coloring completes.

5.2 Assume that $r=2$.

5.2.1 $\lambda>\frac{c}{2}$. Since $R(WR)^{\frac{r}{2}-1}W=R(WR)^0W=RW$ and $B(RB)^{\frac{r}{2}-1}R=B(RB)^0R=BR$ and the above complete coloring, we have

<table>
<tr><td>R</td><td colspan="2">$(WB)^{\frac{\lambda-3}{2}}W$</td><td>$B$</td><td>$R$</td><td>$(BR)^{\frac{c-\lambda}{2}-1}$</td><td>$B$</td></tr>
<tr><td>W</td><td>$(RW)^{\frac{c-\lambda}{2}-1}$</td><td>$R$</td><td>$W$</td><td colspan="2">$(BW)^{\frac{\lambda-3}{2}}B$</td><td>$R$</td></tr>
</table>

These coloring is feasible for this case.

5.2.2 $\lambda<\frac{c}{2}$.

<table>
<tr><td>R</td><td>$(WB)^{\frac{\lambda-3}{2}}W$</td><td>$B$</td><td>$R$</td><td colspan="4">$(BR)^{\frac{c-\lambda}{2}-1}$</td><td>$B$</td></tr>
<tr><td>W</td><td colspan="4">$(RW)^{\frac{c-\lambda}{2}-1}$</td><td>$R$</td><td>$W$</td><td>$(BW)^{\frac{\lambda-3}{2}}B$</td><td>$R$</td></tr>
</table>

Since there exists at least one consecutive entries in the same column have color R, the above coloring is infeasible. So we adjust these assigning color in Phase 1 according to the BC^2. Since c is odd and $o(a+\langle b\rangle)\mid c$, $o(a+\langle b\rangle)$ is odd. The feasible coloring for this case is same as 4.2.2.

Case 6. *r,c are even and λ is odd.* Then $c-\lambda$ is odd.

6.1 Assume that $r > 2$.

Phase 1: Assign color to all entries of $M_G(a, b)$ according to the C^2.

B	$(WB)^{\frac{\lambda-3}{2}}W$	B	W	$(BW)^{\frac{c-\lambda-3}{2}}B$	W
$(WB)^{\frac{r}{2}-1}$					$(BW)^{\frac{r}{2}-1}$
W	$(BW)^{\frac{c-\lambda-3}{2}}B$	W	B	$(WB)^{\frac{\lambda-3}{2}}W$	B

We see that A_1 is feasible.

Phase 2: Since B_1 is infeasible, we can modify by changing W to R in its bottom row and B to R in its top row.

B	$(WB)^{\frac{\lambda-3}{2}}W$	B	W	$(RW)^{\frac{c-\lambda-3}{2}}R$	W
$(WB)^{\frac{r}{2}-1}$					$(BW)^{\frac{r}{2}-1}$
R	$(BR)^{\frac{c-\lambda-3}{2}}B$	R	B	$(WB)^{\frac{\lambda-3}{2}}W$	B

Since B_2 is infeasible and $\{(r-1)b+(c-\lambda-1)a, (r-1)b+(c-\lambda)a\} \in A_0$ and $(r-1)b + (c-\lambda-1)a$ has color R, we can modify by changing W to R in its bottom row and B to R in its top row.

R	$(WR)^{\frac{\lambda-3}{2}}W$	R	W	$(RW)^{\frac{c-\lambda-3}{2}}R$	W
$(WB)^{\frac{r}{2}-1}$					$(BW)^{\frac{r}{2}-1}$
R	$(BR)^{\frac{c-\lambda-3}{2}}B$	R	B	$(RB)^{\frac{\lambda-3}{2}}R$	B

The coloring completes.

6.2 Assume that $r = 2$. Since $R(WB)^{\frac{r}{2}-1}R = R(WB)^0R = RR$ and $W(BW)^{\frac{r}{2}-1}B = W(BW)^0B = WB$ and the above complete coloring, we have

R	$(WR)^{\frac{\lambda-3}{2}}W$	R	W	$(RW)^{\frac{c-\lambda-3}{2}}R$	W
R	$(BR)^{\frac{c-\lambda-3}{2}}B$	R	B	$(RB)^{\frac{\lambda-3}{2}}R$	B

Since $\{0, b\} \in B_0$ and those entries have the same color R, the above coloring is infeasible. So we adjust these assigning color in Phase 1 according to the BC^2. Since λ is odd and $o(a + \langle b\rangle) \mid \lambda$, $o(a + \langle b\rangle)$ is odd. The feasible coloring for this case is same as 4.2.2.

This completes Theorem 3.3.1 and the coloring algorithm. □

We give three examples to demonstrate Theorem 3.3.1.

Example 3.3.2. The graph $C_{\mathbb{Z}_6\times\mathbb{Z}_{18}}((0,2),(3,3))$ has $\frac{|\mathbb{Z}_6\times\mathbb{Z}_{18}|}{|\langle(0,2),(3,3)\rangle|} = \frac{108}{18} = 6$ connected components, each of which isomorphic to $C_H((0,2),(3,3))$ where $H =$

$\langle(0,2),(3,3)\rangle$. Since $r = o((3,3) + \langle(0,2)\rangle) = 2$ and $(0,6) = r(3,3) = \lambda_G(0,2)$, we have $\lambda = 3$. Since $c = o((0,2)) = 9$, $M_G(a,b)$ has $r = 2$ rows and $c = 9$ columns.

$(0,0)$	$(0,2)$	$(0,4)$	$(0,6)$	$(0,8)$	$(0,10)$	$(0,12)$	$(0,14)$	$(0,16)$
$(3,3)$	$(3,5)$	$(3,7)$	$(3,9)$	$(3,11)$	$(3,13)$	$(3,15)$	$(3,17)$	$(3,1)$

$(1,0)$	$(1,2)$	$(1,4)$	$(1,6)$	$(1,8)$	$(1,10)$	$(1,12)$	$(1,14)$	$(1,16)$
$(4,3)$	$(4,5)$	$(4,7)$	$(4,9)$	$(4,11)$	$(4,13)$	$(4,15)$	$(4,17)$	$(4,1)$

$(2,0)$	$(2,2)$	$(2,4)$	$(2,6)$	$(2,8)$	$(2,10)$	$(2,12)$	$(2,14)$	$(2,16)$
$(5,3)$	$(5,5)$	$(5,7)$	$(5,9)$	$(5,11)$	$(5,13)$	$(5,15)$	$(5,17)$	$(5,1)$

$(3,0)$	$(3,2)$	$(3,4)$	$(3,6)$	$(3,8)$	$(3,10)$	$(3,12)$	$(3,14)$	$(3,16)$
$(0,3)$	$(0,5)$	$(0,7)$	$(0,9)$	$(0,11)$	$(0,13)$	$(0,15)$	$(0,17)$	$(0,1)$

$(4,0)$	$(4,2)$	$(4,4)$	$(4,6)$	$(4,8)$	$(4,10)$	$(4,12)$	$(4,14)$	$(4,16)$
$(1,3)$	$(1,5)$	$(1,7)$	$(1,9)$	$(1,11)$	$(1,13)$	$(1,15)$	$(1,17)$	$(1,1)$

$(5,0)$	$(5,2)$	$(5,4)$	$(5,6)$	$(5,8)$	$(5,10)$	$(5,12)$	$(5,14)$	$(5,16)$
$(2,3)$	$(2,5)$	$(2,7)$	$(2,9)$	$(2,11)$	$(2,13)$	$(2,15)$	$(2,17)$	$(2,1)$

Since r is even and c, λ are odd, $M_G(a,b)$ is in Case 5. Since $r = 2, \lambda < \frac{c}{2}$ and $o(a + \langle b\rangle) = o((0,2) + \langle(3,3)\rangle) = 3$, by Subcase 5.2.2, we have the vertex coloring for $C_{\mathbb{Z}_6\times\mathbb{Z}_{18}}((0,2),(3,3))$ and the other component in the following table.

R	W	B	R	W	B	R	W	B
W	B	R	W	B	R	W	B	R

Example 3.3.3. The graph $C_{\mathbb{Z}_6\times\mathbb{Z}_{12}}((0,2),(3,9))$ has $\frac{|\mathbb{Z}_6\times\mathbb{Z}_{12}|}{|\langle(0,2),(3,9)\rangle|} = \frac{72}{12} = 6$ connected components, each of which isomorphic to $C_H((0,2),(3,9))$ where $H = \langle(0,2),(3,9)\rangle$. Since $r = o((3,9) + \langle(0,2)\rangle) = 2$ and $(0,6) = r(3,9) = \lambda_G(0,2)$, we have $\lambda = 3$. Since $c = o((0,2)) = 6$, $M_G(a,b)$ has $r = 2$ rows and $c = 6$ columns.

$(0,0)$	$(0,2)$	$(0,4)$	$(0,6)$	$(0,8)$	$(0,10)$
$(3,9)$	$(3,11)$	$(3,1)$	$(3,3)$	$(3,5)$	$(3,7)$

$(1,0)$	$(1,2)$	$(1,4)$	$(1,6)$	$(1,8)$	$(1,10)$
$(4,9)$	$(4,11)$	$(4,1)$	$(4,3)$	$(4,5)$	$(4,7)$

$(2,0)$	$(2,2)$	$(2,4)$	$(2,6)$	$(2,8)$	$(2,10)$
$(5,9)$	$(5,11)$	$(5,1)$	$(5,3)$	$(5,5)$	$(5,7)$

$(3,0)$	$(3,2)$	$(3,4)$	$(3,6)$	$(3,8)$	$(3,10)$
$(0,9)$	$(0,11)$	$(0,1)$	$(0,3)$	$(0,5)$	$(0,7)$

$(4,0)$	$(4,2)$	$(4,4)$	$(4,6)$	$(4,8)$	$(4,10)$
$(1,9)$	$(1,11)$	$(1,1)$	$(1,3)$	$(1,5)$	$(1,7)$

$(5,0)$	$(5,2)$	$(5,4)$	$(5,6)$	$(5,8)$	$(5,10)$
$(2,9)$	$(2,11)$	$(2,1)$	$(2,3)$	$(2,5)$	$(2,7)$

Since r, c are even and λ is odd, $M_G(a,b)$ is in Case 6. Since $r = 2$ and $o(a + \langle b\rangle) = o((0,2) + \langle(3,9)\rangle) = 3$, by Subcase 6.2, we have the vertex coloring for $C_{\mathbb{Z}_6\times\mathbb{Z}_{12}}((0,2),(3,9))$ and the other component in the following table.

R	W	B	R	W	B
W	B	R	W	B	R

Example 3.3.4. The graph $C_{\mathbb{Z}_6\times\mathbb{Z}_{18}}((0,2),(2,4))$ has $\frac{|\mathbb{Z}_6\times\mathbb{Z}_{18}|}{|\langle(0,2),(2,4)\rangle|} = \frac{108}{27} = 4$ connected components, each of which isomorphic to $C_H((0,2),(2,4))$ where $H = \langle(0,2),(2,4)\rangle$. Since $r = o((2,4)+\langle(0,2)\rangle) = 3$ and $(0,12) = r(2,4) = \lambda_G(0,2)$, we have $\lambda = 6$. Since $c = o((0,2)) = 9$, $M_G(a,b)$ has $r = 3$ rows and $c = 9$ columns.

$(0,0)$	$(0,2)$	$(0,4)$	$(0,6)$	$(0,8)$	$(0,10)$	$(0,12)$	$(0,14)$	$(0,16)$
$(2,4)$	$(2,6)$	$(2,8)$	$(2,10)$	$(2,12)$	$(2,14)$	$(2,16)$	$(2,0)$	$(2,2)$
$(4,8)$	$(4,10)$	$(4,12)$	$(4,14)$	$(4,16)$	$(4,0)$	$(4,2)$	$(4,4)$	$(4,6)$

$(0,1)$	$(0,3)$	$(0,5)$	$(0,7)$	$(0,9)$	$(0,11)$	$(0,13)$	$(0,15)$	$(0,17)$
$(2,5)$	$(2,7)$	$(2,9)$	$(2,11)$	$(2,13)$	$(2,15)$	$(2,17)$	$(2,1)$	$(2,3)$
$(4,9)$	$(4,11)$	$(4,13)$	$(4,15)$	$(4,17)$	$(4,1)$	$(4,3)$	$(4,5)$	$(4,7)$

$(1,0)$	$(1,2)$	$(1,4)$	$(1,6)$	$(1,8)$	$(1,10)$	$(1,12)$	$(1,14)$	$(1,16)$
$(3,4)$	$(3,6)$	$(3,8)$	$(3,10)$	$(3,12)$	$(3,14)$	$(3,16)$	$(3,0)$	$(3,2)$
$(5,8)$	$(5,10)$	$(5,12)$	$(5,14)$	$(5,16)$	$(5,0)$	$(5,2)$	$(5,4)$	$(5,6)$

$(1,1)$	$(1,3)$	$(1,5)$	$(1,7)$	$(1,9)$	$(1,11)$	$(1,13)$	$(1,15)$	$(1,17)$
$(3,5)$	$(3,7)$	$(3,9)$	$(3,11)$	$(3,13)$	$(3,15)$	$(3,17)$	$(3,1)$	$(3,3)$
$(5,9)$	$(5,11)$	$(5,13)$	$(5,15)$	$(5,17)$	$(5,1)$	$(5,3)$	$(5,5)$	$(5,7)$

Since r, c are odd and λ is even, by Case 2, we have the vertex coloring for $C_{\mathbb{Z}_6\times\mathbb{Z}_{18}}((0,2),(2,4))$ and the other component in the following table.

R	W	B	W	B	W	B	R	B
W	B	W	B	W	B	W	B	R
R	W	R	W	B	W	B	W	B

3.4 Conclusions

Let $r = o(b + \langle a \rangle), c = o(a)$ and $\lambda \in \{0, 1, \ldots, c-1\}$ be such that $rb = \lambda a$. We shall conclude the vertex coloring algorithm for $C_H(a, b)$ developing in the previous sections in the following table.

Theorem 3.1.1 : $r + \lambda$ and c are even.

<table>
<tr><th>Conditions</th><th>Assignments</th></tr>
<tr><td>Case 1. r and λ are even</td><td>
<table>
<tr><td>B</td><td>$(WB)^{\frac{\lambda}{2}-1}$</td><td>$W$</td><td>$B$</td><td>$(WB)^{\frac{c-\lambda}{2}-1}$</td><td>$W$</td></tr>
<tr><td>$(WB)^{\frac{r}{2}-1}$</td><td colspan="4">BW-schema</td><td>$(BW)^{\frac{r}{2}-1}$</td></tr>
<tr><td>W</td><td>$(BW)^{\frac{c-\lambda}{2}-1}$</td><td>$B$</td><td>$W$</td><td>$(BW)^{\frac{\lambda}{2}-1}$</td><td>$B$</td></tr>
</table>
</td></tr>
<tr><td>Case 2. r and λ are odd</td><td>
<table>
<tr><td>B</td><td>$(WB)^{\frac{\lambda-3}{2}}W$</td><td>$B$</td><td>$W$</td><td>$(BW)^{\frac{c-\lambda-3}{2}}B$</td><td>$W$</td></tr>
<tr><td>$(WB)^{\frac{r-3}{2}-1}W$</td><td colspan="4">BW-schema</td><td>$(BW)^{\frac{r-3}{2}}B$</td></tr>
<tr><td>B</td><td>$(WB)^{\frac{c-\lambda-3}{2}}W$</td><td>$B$</td><td>$W$</td><td>$(BW)^{\frac{\lambda-3}{2}}B$</td><td>$W$</td></tr>
</table>
</td></tr>
</table>

Theorem 3.2.1 : λ is even or c is odd with $2 \leq \lambda \leq \frac{c}{2}$

<table>
<tr><th>Conditions</th><th>Assignments</th></tr>
<tr><td>Case 1. $c = 5$</td><td><table><tr><td>B</td><td>W</td></tr><tr><td>R</td><td>G</td></tr><tr><td>Y</td><td></td></tr></table></td></tr>
<tr><td>Case 2. $c = 13$ and $\lambda = 5$</td><td><table><tr><td>B</td><td>W</td><td>R</td><td>B</td><td>W</td></tr><tr><td>R</td><td>B</td><td>W</td><td>R</td><td>B</td></tr><tr><td>W</td><td>R</td><td>G</td><td></td><td></td></tr></table></td></tr>
<tr><td>Case 3. $c \neq 5$ and $\lambda \in \{2, \frac{c-1}{2}\}$
3.1 $3 \mid c$</td><td></td></tr>
<tr><td>3.1.1 $c = 6k - 3$ where $k \in \mathbb{N}$</td><td><table><tr><td>B</td><td>W</td></tr><tr><td>R</td><td>B</td></tr><tr><td>W</td><td>R</td></tr><tr><td>$(BRW)^{k-2}$</td><td>$(WBR)^{k-2}$</td></tr><tr><td>B</td><td>W</td></tr><tr><td>R</td><td></td></tr></table></td></tr>
<tr><td>3.1.2 $c = 6k$ where $k \in \mathbb{N}$</td><td><table><tr><td>B</td><td>W</td></tr><tr><td>R</td><td>B</td></tr><tr><td>W</td><td>R</td></tr><tr><td>$(BRW)^{k-2}B$</td><td>$(WBR)^{k-2}W$</td></tr><tr><td>R</td><td>B</td></tr><tr><td>W</td><td>R</td></tr></table></td></tr>
</table>

Conditions	Assignments

3.2 $3 \nmid c$ and $c \equiv 0$ or $3 \mod 4$

3.2.1 $c = 6k - 1$ or $6k - 5$ where k is even

B	W
R	G
B	W
R	G
$(BR)^{\frac{3k}{2}-3}$	$(WG)^{\frac{3k}{2}-3}$
B	W
R	

3.2.2 $c = \begin{cases} 6k-2 & \text{if } k \text{ is odd;} \\ 6k-4 & \text{if } k \text{ is even.} \end{cases}$

B	W
R	G
B	W
R	G
$(BR)^{\frac{3k-7}{2}}$	$(WG)^{\frac{3k-7}{2}}$
B	W
R	G

3.3 $c = 6k - 5$ or $6k - 1$ where k is odd

B	W
R	G
B	W
R	G
$(BR)^{\frac{3k-9}{2}}B$	$(WG)^{\frac{3k-9}{2}}W$
R	G
W	R
G	W
R	

3.4 $c = \begin{cases} 6k-4, & \text{if } k \text{ is odd;} \\ 6k-2, & \text{if } k \text{ is even.} \end{cases}$

B	W
R	G
B	W
R	G
$(BR)^{\frac{3k-7}{2}}B$	$(WG)^{\frac{3k-7}{2}}W$
R	G
W	R

Conditions	Assignments
Case 4. $2 < \lambda \leq \frac{c}{2}$ and $\lambda \neq \frac{c-1}{2}$, except $c = 13$ and $\lambda = 5$	
4.1 q, λ and l are even	<table><tr><td>B</td><td>$(WB)^{\frac{\lambda-l}{2}-1}$</td><td>$W$</td><td>$B$</td><td>$(RB)^{\frac{l}{2}-1}$</td><td>$R$</td></tr><tr><td>$W$</td><td colspan="4" rowspan="2">BW-schema</td><td rowspan="2">$(BR)^{\frac{q}{2}-1}$</td></tr><tr><td rowspan="2">$(RW)^{\frac{q}{2}-1}$</td></tr><tr><td colspan="2"></td><td>W</td><td>$(BW)^{\frac{\lambda-l}{2}-1}$</td><td>$B$</td></tr><tr><td>$R$</td><td>$(WR)^{\frac{l}{2}-1}$</td><td>$W$</td></tr></table>
4.2 q is odd and λ, l are even	<table><tr><td>R</td><td>$(WR)^{\frac{\lambda-l}{2}-1}$</td><td>$W$</td><td>$B$</td><td>$(WB)^{\frac{l}{2}-1}$</td><td>$R$</td></tr><tr><td>$W$</td><td colspan="4" rowspan="2">BW-schema</td><td rowspan="2">$(BR)^{\frac{q-3}{2}}B$</td></tr><tr><td rowspan="2">$(RW)^{\frac{q-3}{2}}R$</td></tr><tr><td colspan="2"></td><td>B</td><td>$(RB)^{\frac{\lambda-l}{2}-1}$</td><td>$R$</td></tr><tr><td>$W$</td><td>$(BW)^{\frac{l}{2}-1}$</td><td>$B$</td></tr></table>
4.3 q, l are odd and λ is even	<table><tr><td>B</td><td>$(WB)^{\frac{\lambda-l-3}{2}}W$</td><td>$B$</td><td>$W$</td><td>$(RW)^{\frac{l-3}{2}}R$</td><td>$W$</td></tr><tr><td>$R$</td><td colspan="4" rowspan="2">BW-schema</td><td rowspan="2">$(RW)^{\frac{q-3}{2}}R$</td></tr><tr><td rowspan="2">$(BR)^{\frac{q-3}{2}}B$</td></tr><tr><td colspan="2"></td><td>W</td><td>$(BW)^{\frac{\lambda-l-3}{2}}B$</td><td>$W$</td></tr><tr><td>$R$</td><td>$(BR)^{\frac{l-3}{2}}B$</td><td>$R$</td></tr></table>

<table>
<tr><th>Conditions</th><th>Assignments</th></tr>
<tr><td>4.4 q, λ are odd and l is even</td><td>
<table>
<tr><td>R</td><td>$(WR)^{\frac{\lambda-l-3}{2}}W$</td><td>$R$</td><td>$W$</td><td colspan="2">$(RW)^{\frac{l}{2}-1}$</td><td>$R$</td></tr>
<tr><td>W</td><td colspan="5" rowspan="2">BW-schema</td><td rowspan="2">$(WB)^{\frac{q-3}{2}}W$</td></tr>
<tr><td rowspan="2">$(BW)^{\frac{q-3}{2}}B$</td></tr>
<tr><td colspan="3"></td><td>B</td><td>$(RB)^{\frac{\lambda-l-3}{2}}R$</td><td>$B$</td></tr>
<tr><td>R</td><td colspan="2">$(BR)^{\frac{l}{2}-1}$</td><td>B</td></tr>
</table>
</td></tr>
<tr><td>4.5 q, λ are even and l is odd
4.5.1 $q > 2$
4.5.1.1 $2 < l < \lambda$</td><td>
<table>
<tr><td>R</td><td>$(WR)^{\frac{\lambda-l-3}{2}}W$</td><td>$R$</td><td>$W$</td><td colspan="3">$(BW)^{\frac{l-3}{2}}B$</td><td>$R$</td></tr>
<tr><td>W</td><td colspan="6" rowspan="2">BW-schema</td><td rowspan="2">$(BR)^{\frac{q}{2}-1}$</td></tr>
<tr><td rowspan="2">$(RW)^{\frac{q}{2}-1}$</td></tr>
<tr><td colspan="4"></td><td>B</td><td>$(RB)^{\frac{\lambda-l-3}{2}}R$</td><td>$B$</td></tr>
<tr><td>R</td><td colspan="3">$(WB)^{\frac{l-3}{2}}W$</td><td>B</td></tr>
</table>
</td></tr>
<tr><td>4.5.1.2 $l = 1$</td><td>
<table>
<tr><td>W</td><td>B</td><td>R</td><td>$(WR)^{\frac{\lambda}{2}-2}$</td><td>$W$</td></tr>
<tr><td>B</td><td>R</td><td colspan="2"></td><td>B</td></tr>
<tr><td>R</td><td colspan="3" rowspan="2">BW-schema</td><td rowspan="2">$(RB)^{\frac{q}{2}-2}R$</td></tr>
<tr><td rowspan="2">$(WR)^{\frac{q}{2}-2}W$</td></tr>
<tr><td>B</td><td>W</td><td>$(BR)^{\frac{\lambda}{2}-2}$</td><td>$B$</td></tr>
<tr><td>R</td></tr>
</table>
</td></tr>
</table>

Conditions	Assignments

4.5.2 $q = 2$

4.5.2.1 $\frac{\lambda}{2} < l < \lambda$

R	$(WR)^{\frac{\lambda-l-3}{2}}W$	R	W	$(BW)^{\frac{l-3}{2}}B$	R
W	$(BW)^{\frac{l-3}{2}}B$	W	B	$(RB)^{\frac{\lambda-l-3}{2}}R$	B
R	$(WB)^{\frac{l-3}{2}}W$	B			

4.5.2.2 $2 < l < \frac{\lambda}{2}$

4.5.2.2.1 $(\lambda \mod l)$ is even

R	$(WR)^{\frac{\lambda \mod l}{2}-1}$	W	B	$(WB)^{\frac{l-3}{2}}W$	B	$[R(WB)^{\frac{l-3}{2}}WB]^{\lfloor\frac{\lambda}{l}\rfloor-2}$	R	$(WR)^{\frac{l-3}{2}}W$	R
W	$(BW)^{\frac{l-3}{2}}B$	W	B	$(RB)^{\frac{\lambda \mod l}{2}-1}$	R	$[W(BW)^{\frac{l-3}{2}}BR]^{\lfloor\frac{\lambda}{l}\rfloor-2}$	W	$(BW)^{\frac{l-3}{2}}B$	W
B	$(RB)^{\frac{l-3}{2}}R$	B							

4.5.2.2.2 $(\lambda \mod l)$ is odd

R	$(WR)^{\frac{(\lambda \mod l)-3}{2}}W$	R	W	$(BW)^{\frac{l-3}{2}}B$	R	$[W(BW)^{\frac{l-3}{2}}BR]^{\lfloor\frac{\lambda}{l}\rfloor-1}$
W	$(BW)^{\frac{l-3}{2}}B$	W	B	$(RB)^{\frac{(\lambda \mod l)-3}{2}}R$	B	$[R(WB)^{\frac{l-3}{2}}WB]^{\lfloor\frac{\lambda}{l}\rfloor-1}$
R	$(WB)^{\frac{l-3}{2}}W$	B				

Conditions	Assignments
4.6 q is even and λ, l are odd	
4.6.1 $q > 2$	see table 4.6.1 below
4.6.2 $q = 2$	
4.6.2.1 $l = \lambda$	see table 4.6.2.1 below
4.6.2.2 $2 < l < \frac{\lambda}{2}$	
4.6.2.2.1 ($\lambda \mod l$) is even	same as 4.5.2.2.1
4.6.2.2.2 ($\lambda \mod l$) is odd	same as 4.5.2.2.2
4.6.2.3 $\frac{\lambda}{2} < l < \lambda$	see table 4.6.2.3 below

4.6.1 $q > 2$

R	$(WR)^{\frac{\lambda-l}{2}-1}$	W	R	$(WR)^{\frac{l-3}{2}}W$		R
W	BW-schema					$(WB)^{\frac{q}{2}-1}$
$(BW)^{\frac{q}{2}-1}$				B	$(RB)^{\frac{\lambda-l}{2}-1}$	R
B	$(RB)^{\frac{l-3}{2}}R$		B			

4.6.2.1 $l = \lambda$

R	$(WR)^{\frac{l-3}{2}}W$	R
W	$(BW)^{\frac{l-3}{2}}B$	W
B	$(RB)^{\frac{l-3}{2}}R$	B

4.6.2.3 $\frac{\lambda}{2} < l < \lambda$

B	$(WB)^{\frac{\lambda-l}{2}-1}$		W	B	$(RB)^{\frac{2l-\lambda-3}{2}}$	R	W	B	$(WB)^{\frac{\lambda-l}{2}-2}W$	B	R
W	$(BW)^{\frac{l-3}{2}}$					B	R	W	$(BW)^{\frac{\lambda-l}{2}-1}$		B
R	$(WR)^{\frac{2l-\lambda-3}{2}}$	W	B	W	$(BW)^{\frac{\lambda-l}{2}-2}B$	R	W				

Theorem 3.3.1 : $r + \lambda$ is odd or c is odd with $b \notin \langle a \rangle$

<table>
<tr><th>Conditions</th><th>Assignments</th></tr>
<tr><td>Case 1. r is odd and c, λ are even</td><td>
<table>
<tr><td>B</td><td>$(RB)^{\frac{\lambda}{2}-1}$</td><td>$R$</td><td>$B$</td><td colspan="3">$(RB)^{\frac{c-\lambda}{2}-1}$</td><td>$R$</td></tr>
<tr><td>$(WB)^{\frac{r-3}{2}}W$</td><td colspan="6">BW-schema</td><td>$(BW)^{\frac{r-3}{2}}B$</td></tr>
<tr><td>R</td><td colspan="3">$(WR)^{\frac{c-\lambda}{2}-1}$</td><td>$W$</td><td>$R$</td><td>$(WR)^{\frac{\lambda}{2}-1}$</td><td>$W$</td></tr>
</table>
</td></tr>
<tr><td>Case 2. r, c are odd and λ is even</td><td>
<table>
<tr><td>R</td><td>$(WB)^{\frac{\lambda}{2}-1}$</td><td>$W$</td><td>$B$</td><td colspan="4">$(RB)^{\frac{c-\lambda-3}{2}}R$</td><td>$B$</td></tr>
<tr><td>$(WR)^{\frac{r-3}{2}}W$</td><td colspan="7">BW-schema</td><td>$(RB)^{\frac{r-3}{2}}R$</td></tr>
<tr><td>R</td><td colspan="4">$(WR)^{\frac{c-\lambda-3}{2}}W$</td><td>$R$</td><td>$W$</td><td>$(BW)^{\frac{\lambda}{2}-1}$</td><td>$B$</td></tr>
</table>
</td></tr>
<tr><td>Case 3 r, c and λ are odd</td><td>
<table>
<tr><td>R</td><td>$(WR)^{\frac{\lambda-3}{2}}W$</td><td>$R$</td><td>$W$</td><td colspan="4">$(BW)^{\frac{c-\lambda}{2}-1}$</td><td>$B$</td></tr>
<tr><td>$(WR)^{\frac{r-3}{2}}W$</td><td colspan="7">BW-schema</td><td>$(RB)^{\frac{r-3}{2}}R$</td></tr>
<tr><td>R</td><td colspan="4">$(WB)^{\frac{c-\lambda}{2}-1}$</td><td>$W$</td><td>$B$</td><td>$(RB)^{\frac{\lambda-3}{2}}R$</td><td>$B$</td></tr>
</table>
</td></tr>
</table>

Conditions	Assignments
Case 4. r, λ are even and c is odd	
4.1 $r > 2$	see table 4.1 below
4.2 $r = 2$	
4.2.1 $\lambda < \frac{c}{2}$	see table 4.2.1 below
4.2.2 $\lambda > \frac{c}{2}$	see table 4.2.2 below

4.1 $r > 2$

R	$(WR)^{\frac{\lambda}{2}-1}$	W	B	$(WB)^{\frac{c-\lambda-3}{2}}W$		B
$(WR)^{\frac{r}{2}-1}$	BW-schema					$(RB)^{\frac{r}{2}-1}$
W	$(BW)^{\frac{c-\lambda-3}{2}}B$		W	B	$(RB)^{\frac{\lambda}{2}-1}$	R

4.2.1 $\lambda < \frac{c}{2}$

R	$(WR)^{\frac{\lambda}{2}-1}$	W	B	$(WB)^{\frac{c-\lambda-3}{2}}W$	B
W	$(BW)^{\frac{c-\lambda-3}{2}}B$	W	B	$(RB)^{\frac{\lambda}{2}-1}$	R

4.2.2 $\lambda > \frac{c}{2}$

R	$(WB)^{\frac{o(a+\langle b\rangle)-3}{2}}W$	B	$[R(WB)^{\frac{o(a+\langle b\rangle)-3}{2}}WB]^{\frac{c}{o(a+\langle b\rangle)}-2}$	R	$(WB)^{\frac{o(a+\langle b\rangle)-3}{2}}W$	B
W	$(BW)^{\frac{o(a+\langle b\rangle)-3}{2}}B$	R	$[W(BW)^{\frac{o(a+\langle b\rangle)-3}{2}}BR]^{\frac{c}{o(a+\langle b\rangle)}-2}$	W	$(BW)^{\frac{o(a+\langle b\rangle)-3}{2}}B$	R

Conditions	Assignments
Case 5. r is even and c, λ are odd	
5.1 $r > 2$	see table 5.1 below
5.2 $r = 2$	
5.2.1 $\lambda > \frac{c}{2}$	see table 5.2.1 below
5.2.2 $\lambda < \frac{c}{2}$	same as 4.2.2
Case 6. r, c are even and λ is odd	
6.1 $r > 2$	see table 6.1 below
6.2 $r = 2$	same as 4.2.2

5.1 $r > 2$

R	$(WB)^{\frac{\lambda-3}{2}}W$	B	R	$(BR)^{\frac{c-\lambda}{2}-1}$	B
$(WR)^{\frac{r}{2}-1}$	BW-schema				$(RB)^{\frac{r}{2}-1}$
W	$(RW)^{\frac{c-\lambda}{2}-1}$	R	W	$(BW)^{\frac{\lambda-3}{2}}B$	R

5.2.1 $\lambda > \frac{c}{2}$

R	$(WB)^{\frac{\lambda-3}{2}}W$	B	R	$(BR)^{\frac{c-\lambda}{2}-1}$	B
W	$(RW)^{\frac{c-\lambda}{2}-1}$	R	W	$(BW)^{\frac{\lambda-3}{2}}B$	R

6.1 $r > 2$

R	$(WR)^{\frac{\lambda-3}{2}}W$	R	W	$(RW)^{\frac{c-\lambda-3}{2}}R$	W
$(WB)^{\frac{r}{2}-1}$	BW-schema				$(BW)^{\frac{r}{2}-1}$
R	$(BR)^{\frac{c-\lambda-3}{2}}B$	R	B	$(RB)^{\frac{\lambda-3}{2}}R$	B

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[1] J. B. Fraleigh, A First Course in Abstract Algebra, 7th Ed. Kingston: *Pearson Education*, 2003.

[2] S. Nicoloso and U. Pietropaoli, Isomorphism testing for circulant graphs, *Tech. Rep*, **664**/2007, IASI-CNR, Rome, Italy, 2007.

[3] S. Nicoloso and U. Pietropaoli, Vertex coloring of circulant graphs, *Tech. Rep*, **669**/2007, IASI-CNR, Rome, Italy, 2007.

VITA

Name	Mr. Worrawate Leela-apiradee
Date of Birth	2 August 1987
Place of Birth	Surin, Thailand
Education	B.Sc. (Mathematics) (First Class Honours), Chulalongkorn University, 2009
Scholarship	Science Achievement Scholarship of Thailand