CHAPTER I

PRELIMINARIES

In this chapter we shall give some notations, definitions and theorems used in this thesis. Our notations are :

$\mathbb{Z}$ is the set of all integers,

$\mathbb{Z}^+$ is the set of all positive integers,

$\mathbb{Z}_0^+ = \mathbb{Z}^+ \cup \{0\}$,

$\mathbb{Q}$ is the set of all rational numbers,

$\mathbb{Q}^+$ is the set of all positive rational numbers,

$\mathbb{Q}_0^+ = \mathbb{Q}^+ \cup \{0\}$,

$\mathbb{R}$ is the set of all real numbers,

$\mathbb{R}^+$ is the set of all positive real numbers,

$\mathbb{R}_0^+ = \mathbb{R}^+ \cup \{0\}$.

Definition 1.1. A triple $(S,+,\cdot)$ is said to be a right [left] seminear-ring if

(1) $(S,+)$ and $(S,\cdot)$ are semigroups

and (2) for all $x,y,z \in S$, $(x+y)z = xz+yz$ [$z(x+y) = zx+zy$].

The operations + and $\cdot$ are called the addition and multiplication of the right [left] seminear-ring, respectively.

If S is a right and a left seminear-ring, then we call S a semiring.

Throughout this thesis we shall only study right seminear-rings.

All definitions and theorems stated for right seminear-rings have a dual statement and proof for left seminear-rings. So from now on the word "seminear-ring" will mean a right seminear-ring.

Example 1.2.

(1) $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{Z}_o^+$, $\mathbb{Q}^+$ and $\mathbb{R}^+$ with the usual addition and multiplication are semirings and hence they are seminear-rings.

(2) Let $(S,+)$ be any semigroup. Let $M(S) = \{f : S \to S \mid f \text{ is a map}\}$. Define $+$ and $\cdot$ on $M(S)$ by $(f+g)(x) = f(x)+g(x)$ and $(f\cdot g)(x) = f(g(x))$ for all $x \in S$. Then $(M(S),+,\cdot)$ is a seminear-ring which is not in general a semiring.

Definition 1.3. An element x of a semigroup S is said to be a right [left] zero if for all $y \in S$, $yx = x$ [$xy = x$]. S is said to be a right [left] zero semigroup if for all $x \in S$, x is a right [left] zero.

Definition 1.4. Let S be a semigroup and $x \in S$. Then x is called right [left] cancellative (R.C.)[(L.C.)] if for all $y,z \in S$, $yx = zx$ [$xy = xz$] implies that $y = z$. The element x is called cancellative if x is both R.C. and L.C.. S is called R.C. [L.C.] if for all $y \in S$, y is R.C. [L.C.]. S is called cancellative if S is both R.C. and L.C..

Example 1.5. $\mathbb{Z}^+$ with the usual multiplication is a cancellative semigroup.

Proposition 1.6. Every finite cancellative semigroup is a group.

See [5], page 8.

Definition 1.7. A semigroup S is said to satisfy the right [left] Ore condition if for all $a,b \in S\setminus\{0\}$ there exist $x,y \in S\setminus\{0\}$ such that

$ax = by$ [$xa = yb$] where 0 denotes the zero of S if it exists.

<u>Remark</u>. Every commutative semigroup satisfies the right and the left Ore condition but the converse is not true.

<u>Example 1.8</u>. $S = \left\{\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \middle| x,z \in \mathbf{Z}^+ \text{ and } y \in \mathbf{Z}\right\}$ with the usual multiplication is a semigroup. Let $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$, $B = \begin{bmatrix} d & e \\ 0 & f \end{bmatrix} \in S$.

Let $X = \begin{bmatrix} ad & d+ec-bf \\ 0 & af \end{bmatrix}$ and $Y = \begin{bmatrix} a^2 & a \\ 0 & ac \end{bmatrix}$. Then $X,Y \in S$

and $AX = \begin{bmatrix} a^2d & ad+aec \\ 0 & caf \end{bmatrix} = \begin{bmatrix} da^2 & da+eac \\ 0 & fac \end{bmatrix} = BY$.

Hence $(S,\cdot)$ satisfies the right Ore condition and $(S,\cdot)$ is noncommutative.

<u>Definition 1.9</u>. A seminear-ring $(S,+,\cdot)$ is said to be <u>additively</u> [multiplicatively] <u>commutative</u> if $(S,+)$ [$(S,\cdot)$] is commutative. S is said to be <u>commutative</u> if S is both additively and multiplicatively commutative.

<u>Example 1.10</u>.

(1) Let $S = \left\{\begin{bmatrix} x & y \\ z & w \end{bmatrix} \middle| x,y,z,w \in \mathbf{Z}^+\right\}$. Then S with the usual addition and multiplication is an additively commutative semiring.

(2) Let $(S,\cdot)$ be a commutative semigroup. Define $x+y = x$ for all $x,y \in S$. Then $(S,+,\cdot)$ is a multiplicatively commutative semiring.

(3) $\mathbf{Z}^+$ with the usual addition and multiplication is a commutative semiring.

Definition 1.11. A seminear-ring $(D,+,\cdot)$ is called a ratio seminear-ring if $(D,\cdot)$ is a group.

Example 1.12.

(1) Let $D = \left\{ \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \middle/ x,z \in \mathbb{Q}^+ \text{ and } y \in \mathbb{Q} \right\}$. Then D with the usual addition and multiplication is a ratio seminear- ring.

(2) $\mathbb{Q}^+$ and $\mathbb{R}^+$ with the usual addition and multiplication are ratio semirings.

Definition 1.13. Let $(S,+,\cdot)$ be a seminear-ring and $x \in S$. Then x is called right [left] multiplicatively cancellative (R.M.C.) [(L.M.C.)] if x is R.C. [L.C.] in $(S,\cdot)$. The element x is called right [left] additively cancellative (R.A.C.) [(L.A.C.)] if x is R.C. [L.C.] in $(S,+)$. S is called R.M.C. [L.M.C.] if for all $y \in S$, y is R.M.C. [L.M.C.]. S is called R.A.C. [L.A.C.] if for all $y \in S$, y is R.A.C. [L.A.C.]. S is called M.C. if S is both R.M.C. and L.M.C.. S is called A.C. if S is both R.A.C. and L.A.C..

Example 1.14. $\mathbb{Z}^+$, $\mathbb{Q}^+$ and $\mathbb{R}^+$ with the usual addition and multiplication are A.C. and M.C..

Definition 1.15. Let $(S,+,\cdot)$ be a seminear-ring and $a \in S$. Then a is called a right [left] additive zero if a is a right [left] zero of $(S,+)$. The element a is called a right [left] multiplicative zero if a is a right [left] zero of $(S,\cdot)$. The element a is called a right [left] additive identity if for all $x \in S$, $x+a = x$ [$a+x = x$]. The element a is called a right [left] multiplicative identity if for all $x \in S$, $x \cdot a = x$ [$a \cdot x = x$]. The element a is called an additive [multiplicative] zero if it is both a right and a left additive [multiplicative] zero. The element a is called an additive [multiplicative] identity if it is both a right and a left additive [multiplicative] identity.

Let $d \in S$. Then an element $x \in S$ is called a right [left] additive identity of d if $d+x = d$ [$x+d = d$]. Let $D \subseteq S$ be nonempty. Then the set of all right [left] additive identities of d in D is denoted by $RI_D(d)$ [$LI_D(d)$].

Proposition 1.16. Let D be a ratio seminear-ring and e the multiplicative identity of D. If e is an additive zero of D, then $D = \{e\}$.

See [3], page 8.

Proposition 1.17. Let D be a ratio seminear-ring and e the multiplicative identity of D. If e is an additive identity of D, then $D = \{e\}$.

See [3], page 8.

Definition 1.18. Let $(S,+,\cdot)$ be a seminear-ring with a right [left] multiplicative zero. Then S is called a 0-right [left] multiplicatively cancellative (0-R.M.C.) [(0-L.M.C.)] if for all $x,y,z \in S$, $yx = zx$ [$xy = xz$] and x is not a right [left] multiplicative zero imply that $y = z$. S is called a 0-multiplicatively cancellative (0-M.C.) if S is both 0-R.M.C. and 0-L.M.C..

Example 1.19. Z_0^+ with the usual addition and multiplication is 0-M.C..

Definition 1.20. Let S be a semiring and $K \subseteq S$ nonempty. Then K is called a right [left] semiring-ideal of S if

(1) for all $x,y \in K$, $x+y \in K$

and (2) for all $x \in K$ and all $s \in S$, $xs \in K$ [$sx \in K$].

If K is both a right and a left semiring-ideal of S, then K is called a semiring-ideal of S.

Example 1.21. $\mathbb{Z}^+$ with the usual addition and multiplication is a semiring and for any $n \in \mathbb{Z}^+$, $n\mathbb{Z}^+$ is a semiring-ideal of $\mathbb{Z}^+$.

Definition 1.22. A seminear-ring $(K,+,\cdot)$ is called a seminear-field if there is an element $a \in K$ such that $(K\smallsetminus\{a\},\cdot)$ is a group, and such an element a is called a special element of K. If a seminear-field K is commutative, then K is called a semifield.

Example 1.23.

(1) $\mathbb{Q}_0^+$ and $\mathbb{R}_0^+$ with the usual addition and multiplication are semifields with 0 as a special element.

(2) $\left\{\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \middle/ x,z \in \mathbb{Q}^+ \text{ and } y \in \mathbb{Q}\right\} \cup \left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right\}$ with the usual addition and multiplication is a seminear-field with $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ as a special element.

Theorem 1.24. Let K be a seminear-field with a as a special element. Then exactly one of the following statements hold :

(1) $ax = xa = a$ for all $x \in K$.

(2) $ax = xa = x$ for all $x \in K$.

(3) $ax = a$ and $xa = x$ for all $x \in K$.

(4) $ax = x$ and $xa = a$ for all $x \in K$.

(5) $a^2 \neq a$ and $ae = ea = a$ where e is the identity of $(K\smallsetminus\{a\},\cdot)$.

(6) $a^2 \neq a$ and $ae = ea \neq a$ where e is the identity of $(K\smallsetminus\{a\},\cdot)$.

See [3], pages 62-63.

From Theorem 1.24 we get that there are six types of special elements a in a seminear-field K. We call a special element a satisfying (1),(2),(3),(4),(5) or (6) a <u>category I,II,III,IV,V or VI special element of K</u>, respectively. A seminear-field satisfying (1),(2),(3),(4),(5) or (6) is called a <u>seminear-field with a category I,II,III, IV,V or VI special element</u>, respectively.

<u>Theorem 1.25</u>. Let K be a seminear-field with a as a category I special element. Then K satisfies exactly one of the following properties :

(1) $a+x = x+a = x$ for all $x \in K$.

(2) $a+x = x+a = a$ for all $x \in K$.

(3) $a+x = a$ and $x+a = x$ for all $x \in K$.

(4) $a+x = x$ and $x+a = a$ for all $x \in K$.

See [3], pages 12-13.

From Theorem 1.25 we see that a category I special element a has exactly one of the following properties :

(1) $a+x = x+a = x$ for all $x \in K$. In this case we say that a is a <u>0-special element</u>.

(2) $a+x = x+a = a$ for all $x \in K$. In this case we say that a is an <u>∞-special element</u>.

(3) $a+x = a$ and $x+a = x$ for all $x \in K$. Then for any $x,y \in K$, $x+y = (x+a)+y = x+(a+y) = x+a = x$. Thus $(K,+)$ is a left zero semigroup.

(4) $a+x = x$ and $x+a = a$ for all $x \in K$. Then for any $x,y \in K$, $x+y = x+(a+y) = (x+a)+y = a+y = y$. Thus $(K,+)$ is a right zero semigroup.

If K contains a 0-special element, then K is called a <u>0-seminear-field</u>.

If K contains an ∞-special element, then K is called an <u>∞-seminear-field</u>.

If K contains a category I special element a satisfying (3), then K is called an additive left zero seminear-field with a category I special element.

If K contains a category I special element a satisfying (4), then K is called an additive right zero seminear-field with a category I special element.

Proposition 1.26. Every ratio seminear-ring can be embedded into a 0-seminear-field.

See [3], page 117.

Proposition 1.27. Every ratio seminear-ring can be embedded into an ∞-seminear-field.

See [3], page 117.

Proposition 1.28. Let $(D,+,\cdot)$ be a ratio seminear-ring such that $(D,+)$ is a left zero semigroup. Then D can be embedded into an additive left zero seminear-field with a category I special element.

See [3], pages 117-118.

Proposition 1.29. Let $(D,+,\cdot)$ be a ratio seminear-ring such that $(D,+)$ is a right zero semigroup. Then D can be embedded into an additive right zero seminear-field with a category I special element.

See [3], page 118.

Proposition 1.30. Every ratio seminear-ring can be embedded into a seminear-field with a category II special element.

See [3], page 118.

Proposition 1.31. Every ratio seminear-ring can be embedded into a seminear-field with a category VI special element.

See [3], page 118.

Theorem 1.32. Let K be a seminear-field with a as a special element. Then a is a category VI special element of K if and only if there is a unique $d \in K \setminus \{a\}$ such that $ax = dx$ and $xa = xd$ for all $x \in K$.

See [3], pages 66-67.

Proposition 1.33. Let K be a seminear-field with a as a category VI special element. Then $xy \neq a$ for all $x,y \in K$.

See [3], page 67.

Proposition 1.34. If K is a seminear-field with a category III or IV special element, then $|K| = 2$.

See [5], page 67.

Proposition 1.35. If K is a seminear-field with a category V special element, then $|K| = 2$.

See [3], pages 63-64.

Definition 1.36. Let S be a seminear-ring with a multiplicative zero 0 such that $|S| > 1$. Then a seminear-field K is called a seminear-field of right [left] quotients of S if there exists a monomorphism $i : S \to K$ such that for all $x \in K$ there exist $a \in S$, $b \in S \setminus \{0\}$ such that $x = i(a)(i(b)^{-1})[i(b)^{-1}i(a)]$. A monomorphism i satisfying the above property is called a right [left] quotient embedding of S into K.

Example 1.37. Let $S = \left\{ \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \middle| \, x,z \in \mathbb{Z}^+ \text{ and } y \in \mathbb{Z} \right\} \cup \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$

and $K = \left\{ \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \middle| \, x,z \in \mathbb{Q}^+ \text{ and } y \in \mathbb{Q} \right\} \cup \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$.

Then S and K with the usual addition and multiplication are a seminear-ring with a multiplicative zero and a seminear-field, respectively.

To show that K is a seminear-field of right quotients of S. Let $X \in K$. If $X = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, then let $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $B \in S \setminus \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$. So $X = AB^{-1}$. Suppose that $X = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Then $x = \frac{p}{q}$, $y = \frac{m}{n}$ and $z = \frac{u}{v}$ where $p,q,n,u,v \in \mathbb{Z}^+$ and $m \in \mathbb{Z}$.

Let $A = \begin{bmatrix} p & mv+p \\ 0 & un \end{bmatrix}$ and $B = \begin{bmatrix} q & q \\ 0 & vn \end{bmatrix}$. Then $A, B \in S \setminus \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$ and $AB^{-1} = \begin{bmatrix} p & mv+p \\ 0 & un \end{bmatrix} \begin{bmatrix} \frac{1}{q} & \frac{1}{vn} \\ 0 & \frac{1}{vn} \end{bmatrix} = X$. Hence K is a seminear-field of right quotients of S.

<u>Theorem 1.38</u>. Let S be a seminear-ring with a multiplicative zero such that $|S| > 1$. Then a seminear-field of right [left] quotients of S exists if and only if S is 0-M.C. and $(S,\cdot)$ satisfies the right [left] Ore condition.

<u>Proof</u>. The proof of this theorem is given in [4], pages 18-24. The left distributive law in S was only used to show that the left distributive law in $\frac{S\times(S\setminus\{0\})}{\sim}$ holds. #

We shall now review the construction used in the proof of Theorem 1.38.

Assume that S is 0-M.C. and $(S,\cdot)$ satisfies the right Ore condition. Define a binary relation $\sim$ on $S\times(S\setminus\{0\})$ by $(a,b) \sim (c,d)$ if and only if there exist $x,y \in S\setminus\{0\}$ such that $ax = cy$ and $bx = dy$

for all $a,c \in S$ and all $b,d \in S\setminus\{0\}$. In [4] it was shown that $\sim$ is an equivalence relation.

Let $\alpha,\beta \in K = \frac{S\times(S\setminus\{0\})}{\sim}$. Define + and $\cdot$ on $S\times(S\setminus\{0\})$ in the following way : Choose $(a,b) \in \alpha$ and $(c,d) \in \beta$. Then there exist $x \in S$ and $y,u,v \in S\setminus\{0\}$ such that $bx = cy$ and $bu = dv$. Define $\alpha\cdot\beta = [(ax,dy)]$ and $\alpha+\beta = [(au+cv,bu)]$. In [4] it was shown that $(K,+,\cdot)$ is a seminear-field of right quotients of S.

In [4] it was shown that $[(0,c)] = [(0,d)]$ for all $c,d \in S\setminus\{0\}$. We denote $[(0,c)]$ by 0 where $c \in S\setminus\{0\}$. In [4] it was shown that 0 is a multiplicative zero of K, $[(c,c)]$ is the identity of $(K\setminus\{0\},\cdot)$ where $c \in S\setminus\{0\}$ and $[(a,b)]^{-1} = [(b,a)]$ where $[(a,b)] \in K\setminus\{0\}$.

In the proof of Theorem 1.38, P. Satravaha defined $\theta : S \to K$ by $\theta(x) = [(xc,c)]$ for fixed $c \in S\setminus\{0\}$ and for all $x \in S$ and he showed that θ is a monomorphism. This is not necessary as we shall show now.

Define $i : S \to K$ by $i(x) = \begin{cases} 0 & \text{if } x = 0 \\ [(x^2,x)] & \text{if } x \neq 0. \end{cases}$

We shall show that i is a right quotient embedding of S into K.

(1) We must show that i is a homomorphism. Let $c,d \in S$. If $c = 0$ or $d = 0$, then $i(cd) = 0 = i(c)i(d)$. Assume that $c \neq 0$ and $d \neq 0$. There exist $x,y \in S\setminus\{0\}$ such that $cx = d^2y$. Then $i(c)i(d) = [(c^2,c)][(d^2,d)] = [(c^2x,dy)] = [(cd^2y,dy)]$. Since $cd,dy \in S\setminus\{0\}$, there exist $z,w \in S\setminus\{0\}$ such that $cdz = dyw$. Therefore $cdcdz = cd^2yw$. Hence $i(cd) = [(cdcd,cd)] = [(cd^2y,dy)] = i(c)i(d)$.

We shall show that $i(c+d) = i(c)+i(d)$. There exist $x,y \in S\setminus\{0\}$ such that $cx = dy$. Then $i(c)+i(d) = [(c^2,c)] + [(d^2,d)] = [(c^2x+d^2y,cx)] =$

$[(c^2x+dcx,cx)] = [((c+d)cx,cx)]$.

Case 1. $c+d = 0$. Then $i(c)+i(d) = [(0(cx),cx)] = [(0,cx)] = 0 = i(0) = i(c+d)$.

Case 2. $c+d \neq 0$. If $c = 0$, then $i(c+d) = i(d) = i(0)+i(d) = i(c)+i(d)$. Assume that $c \neq 0$. Since $c+d$, $cx \in S\setminus\{0\}$, there exist $z,w \in S\setminus\{0\}$ such that $(c+d)z = cxw$. Therefore $(c+d)(c+d)z = (c+d)cxw$. Hence $i(c)+i(d) = [((c+d)cx,cx)] = [((c+d)(c+d),c+d)] = i(c+d)$.

(2) We must show that i is an injection. Let $c,d \in S$ be such that $i(c) = i(d)$. If $c = 0$, then $d = 0$. Hence $c = d$, so done. Suppose that $c \neq 0$. Then $d \neq 0$ and $[(c^2,c)] = [(d^2,d)]$. There exist $x,y \in S\setminus\{0\}$ such that $c^2x = d^2y$ and $cx = dy$. Since $cx \neq 0$ and S is 0-M.C., $c = d$.

(3) We must show that for all $\alpha \in K$ there exist $c,d \in S\setminus\{0\}$ such that $\alpha = i(c)i(d)^{-1}$. Let $\alpha \in K$. Choose $(c,d) \in \alpha$. There exist $x \in S$ and $y \in S\setminus\{0\}$ such that $cx = dy$. Then $\alpha = [(c,d)] = [(c^2x,d^2y)] = [(c^2,c)][(d,d^2)] = i(c)i(d)^{-1}$.

Hence i is a right quotient embedding of S into K.

Remark. Let S be a seminear-ring having K as its seminear-field of right or left quotients. Then the following statements hold:

(1) If S is additively commutative, then K is additively commutative.

(2) If S is multiplicatively commutative, then K is multiplicatively commutative.

(3) If S is commutative, then K is a semifield and we shall call it the semifield of quotients of S.

Remark.

(1) If S is a seminear-ring having K as its seminear-field of right or left quotients, then K is a seminear-field with a category I special element.

(2) If S is a commutative semiring, then the construction given in the above theorem is the same as the following construction :

Define a binary relation $\sim$ on $S\times(S\setminus\{0\})$ by $(a,b) \sim (c,d)$ if and only if $ad = bc$ for all $(a,b), (c,d) \in S\times(S\setminus\{0\})$. It is easily shown that $\sim$ is an equivalence relation.

Let $\alpha,\beta \in K' = \frac{S\times(S\setminus\{0\})}{\sim}$. Define + and $\cdot$ on K' in the following way : Choose $(a,b) \in \alpha$ and $(c,d) \in \beta$. Define $\alpha+\beta = [(ad+bc,bd)]$ and $\alpha\cdot\beta = [(ac,bd)]$. Then $(K',+,\cdot)$ is a semifield of quotients of S.

(3) Let S be a commutative semiring sucht that $|S| > 1$. Then a semifield of quotients of S exists if and only if S is 0-M.C..

Corollary 1.39. Let S be a seminear-ring with a multiplicative zero sucht that $|S| > 1$ and $(S,\cdot)$ satisfies the right [left] Ore condition. If S is 0-M.C., then S can be embedded into a seminear-field with a category I special element.

Proposition 1.40. Let S be a seminear-ring with a multiplicative zero a which is 0-M.C. and $|S| > 1$. If $(S,\cdot)$ satisfies the right [left] Ore condition, then either a is an additive identity or a is an additive zero or $(S,+)$ is a right zero semigroup or $(S,+)$ is a left zero semigroup.

Proof. This proposition follows from Corollary 1.39 and Theorem 1.25.

#

<u>Proposition 1.41</u>. Let S be a seminear-ring having K as its seminear-field of right [left] quotients, $i : S \to K$ the right [left] quotient embedding, L a seminear-field with 0 as a category I special element and $f : S \to L$ a homomorphism such that $f(x) = 0$ if and only if $x = 0$. Then there exists a unique homomorphism $g : K \to L$ such that $g \circ i = f$. Furthermore, if f is a monomorphism, then g is a monomorphism.

<u>Proof</u>. The proof of this proposition was given in [4], pages 25-26. The left distributive law is not used in the proof.

#

<u>Corollary 1.42</u>. If S is a seminear-ring having K and K' as seminear-fields of right or left quotients, then $K \cong K'$.

<u>Proof</u>. The proof of this proposition is similar to the proof in [4], page 26.

#

<u>Definition 1.43</u>. Let S be a seminear-ring such that $|S| > 1$. Then a ratio seminear-ring D is said to be a <u>ratio seminear-ring of right</u> [left] <u>quotients of S</u> if there exists a monomorphism $i : S \to D$ such that for all $x \in D$ there exist $a, b \in S$ such that $x = i(a)(i(b)^{-1})$ $[i(b)^{-1}i(a)]$. A monomorphism i satisfying the above property is said to be a <u>right</u> [left] <u>quotient embedding</u> of S into D.

<u>Example 1.44</u>. Let $S = \left\{ \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \middle| \; x, z \in \mathbb{Z}^+ \text{ and } y \in \mathbb{Z} \right\}$ and

$D = \left\{ \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \middle| \; x, z \in \mathbb{Q}^+ \text{ and } y \in \mathbb{Q} \right\}$. Then S and D with the usual addition and multiplication are seminear-rings and D is a ratio seminear-ring of right quotients of S with the inclusion map $i : S \to D$

as the right quotient embedding.

The construction of a ratio seminear-ring of right [left] quotients of a seminear-ring S is the same as the construction of a seminear-field of right [left] quotients of S, so we have the following theorem :

Theorem 1.45. Let S be a seminear-ring such that $|S| > 1$. Then a ratio seminear-ring of right [left] quotients of S exists if and only if S is M.C. and $(S,\cdot)$ satisfies the right [left] Ore condition.

Remark. Let S be a seminear-ring having D as its ratio seminear-ring of right or left quotients. Then the following statements hold:

(1) If S is additively commutative, then D is additively commutative.

(2) If S is multiplicatively commutative, then D is multiplicatively commutative.

(3) If S is commutative, then D is a commutative ratio semiring and we shall call it the commutative ratio semiring of quotients of S.

Remark.

(1) If S is a commutative semiring, then the construction given in the above theorem is the same as the following construction :

Define a binary relation $\sim$ on $S\times S$ by $(a,b) \sim (c,d)$ if and only if $ad = bc$ for all $(a,b),(c,d) \in S\times S$. It is easily shown that $\sim$ is an equivalence relation.

Let $\alpha,\beta \in \frac{S\times S}{\sim}$. Define + and $\cdot$ on $\frac{S\times S}{\sim}$ in the following way : Choose $(a,b) \in \alpha$ and $(c,d) \in \beta$. Define $\alpha+\beta = [(ad+bc,bd)]$ and $\alpha\cdot\beta = [(ac,bd)]$. Then $(\frac{S\times S}{\sim},+,\cdot)$ is a commutative ratio semiring of quotients of S.

(2) Let S be a commutative semiring such that $|S| > 1$. Then a commutative ratio semiring of quotients of S exists if and only if S is M.C.

(3) If a seminear-ring S has a ratio seminear-ring of right or left quotients, then S cannot contain a multiplicative zero.

Proposition 1.46. Let S be a seminear-ring having D as its ratio seminear-ring of right [left] quotients, $i : S \to D$ the right [left] quotient embedding, E a ratio seminear-ring and $f : S \to E$ a homomorphism. Then there exists a unique homomorphism $g : D \to E$ such that $g \circ i = f$. Furthermore, if f is a monomorphism, then g is a monomorphism.

Proof. The proof of this proposition is similar to the proof of Proposition 1.41.

#

Corollary 1.47. If S is a seminear-ring having K and K' as ratio seminear-rings of right or left quotients, then $K \cong K'$.

Definition 1.48. Let $\mathcal{S}$ be a category whose objects are seminear-rings and whose morphisms are seminear-ring homomorphisms. Let $\mathcal{K}$ be a category whose objects are seminear-fields and whose morphisms are seminear-field homomorphisms. Let S be an object in $\mathcal{S}$. Then a quotient seminear-field of S w.r.t. the category $\mathcal{K}$ is a triple (S,f,K) where K is a seminear-field in $\mathcal{K}$ and $f : S \to K$ a seminear-ring monomorphism such that for each seminear-field $K' \in \mathcal{K}$ and for each seminear-ring homomorphism $i : S \to K'$ there exists a unique $g \in Mor_{\mathcal{K}}(K,K')$ such that $g \circ f = i$.

Remark. It is clear that if a seminear-ring S has a quotient seminear-field w.r.t. the category $\mathcal{K}$, then S can be embedded into a seminear-field in $\mathcal{K}$.

We shall study the problem of the existence of a quotient seminear-field with respect to a given category of seminear-fields in Chapter III.

Definition 1.49. A semiring $(R,+,\cdot)$ is said to be a skew ring if $(R,+)$ is a group.

We shall always denote the identity element of $(R,+)$ by 0. Note that 0 is a multiplicative zero.

Example 1.50. Let $(R,+)$ be an arbitrary group. Define $\cdot$ on R by $x\cdot y = 0$ for all $x,y \in R$. Then $(R,+,\cdot)$ is a skew ring.

In [4] and [6] the definition of an ideal in a skew ring was given. We shall now generalize this definition.

Definition 1.51. Let R be a skew ring and $J \subseteq R$ nonempty. Then J is called a right [left] weak ideal of R if

(1) for all $x,y \in J$, $x-y \in J$

and (2) for all $x \in J$ and all $r \in R$, $xr \in J$ [$rx \in J$].

If J is both a right and a left weak ideal of R, then J is called a weak ideal of R. A weak ideal J of R is called an ideal of R if J is an additive normal subgroup of R.

Example 1.52.

(1) Let R be a skew ring. Then R and $\{0\}$ are ideals of R.

(2) Let $(R,+,\cdot)$ and $(T,+,\cdot)$ be skew rings. Define $(x,y)\oplus(z,w) = (x+z,y+w)$ and $(x,y)\odot(z,w) = (x\cdot z,y\cdot w)$ for all $(x,y),(z,w) \in R\times T$. Then $(R\times T,\oplus,\odot)$ is a skew ring. Let $I = R\times\{0\}$ and $J = \{0\}\times T$. Then I and J are ideals of $R\times T$.

Definition 1.53. Let S be a semiring such that $|S| > 1$. A skew ring R is said to be a skew ring of right [left] differences of S if there exists a monomorphism $i : S \to R$ such that for all $x \in R$ there exist $a,b \in S$ such that $x = i(a)-i(b)$ [$-i(b)+i(a)$]. A monomorphism i satisfying the above property is said to be a right [left] difference embedding of S into R.

Example 1.54. Let $S = \left\{\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \middle| \, x,z \in \mathbb{Z}^+ \text{ and } y \in \mathbb{Z}\right\}$ and $A,B \in S$. Define $A \oplus B = AB$ and $A \odot B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then $(S,\oplus,\odot)$ is a semiring. Let $R = \left\{\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \middle| \, x,z \in \mathbb{Q}^+ \text{ and } y \in \mathbb{Q}\right\}$ and $X,Y \in R$. Define $X \oplus Y = XY$ and $X \odot Y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then $(R,\oplus,\odot)$ is a skew ring of right differences of S.

Theorem 1.55. Let S be a semiring such that $|S| > 1$. Then a skew ring of right [left] differences of S exists if and only if S is A.C. and $(S,+)$ satisfies the right [left] Ore condition.

See [4], pages 52-55.

We shall now review the construction used in the proof of Theorem 1.55. Assume that S is A.C. and $(S,+)$ satisfies the right Ore condition. Define a binary relation $\sim$ on $S \times S$ by $(a,b) \sim (c,d)$ if and only if there exist $x,y \in S$ such that $a+x = c+y$ and $b+x = d+y$ for all $a,b,c,d \in S \times S$. In Theorem 1.55 it was shown that $\sim$ is an equivalence relation.

Let $\alpha,\beta \in K = \frac{S \times S}{\sim}$. Define $+$ and $\cdot$ on K in the following ways : Choose $(a,b) \in \alpha$ and $(c,d) \in \beta$. There exist $x,y \in S$ such that $b+x = c+y$. Define $\alpha+\beta = [(a+x,d+y)]$ and $\alpha\cdot\beta = [(ac+bd,ad+bc)]$. Theorem 1.55 has shown that $(K,+,\cdot)$ is a skew ring having $[(z,z)]$, where $z \in S$, as an additive identity which we denote by 0. We see that $[(a,b)] = -[(b,a)]$ and $[(a,b)] = 0$ if and only if $a = b$ where $a,b \in S$.

Define $i : S \to K$ by $i(x) = [(x+x,x)]$ for all $x \in S$. Theorem 1.55 has shown that i is a right difference embedding of S into K.

Remark. Let S be a semiring having R as its skew ring of right or left differences. In the proof of Theorem 1.55 we get that:

(1) If S is multiplicatively commutative, then R is multiplicatively commutative.

(2) If S is additively commutative, then R is a ring and we shall call it the ring of differences of S.

Remark.

(1) If a semiring S has a skew ring of right or left differences, then S cannot contain an additive zero and $ab+cd = cd+ab$ for all $a,b,c,d \in S$.

(2) Let S be a commutative semiring such that $|S| > 1$. Then a ring of differences of S exists if and only if S is A.C..

(3) If S is a commutative semiring, then the construction given in the above theorem is the same as the following construction :

Define a binary relation $\sim$ on $S\times S$ by $(a,b) \sim (c,d)$ if and only if $ad = bc$ for all $(a,b),(c,d) \in S\times S$. It is easily shown that $\sim$ is an equivalence relation.

Let $\alpha,\beta \in \frac{S\times S}{\sim}$. Define + and $\cdot$ on $\frac{S\times S}{\sim}$ in the following way : Choose $(a,b) \in \alpha$ and $(c,d) \in \beta$. Define $\alpha+\beta = [(a+c,b+d)]$ and

$\alpha \cdot \beta = [(ac+bd, ad+bc)]$. Then $(\frac{S \times S}{\sim}, +, \cdot)$ is a ring of differences of S.

Proposition 1.56. Let S be a semiring having R as its skew ring of right [left] differences, i the right [left] difference embedding of S into R, T a skew ring and $f : S \to T$ a homomorphism. Then there exists a unique homomorphism $g : R \to T$ such that $g \circ i = f$. Furthermore, if f is a monomorphism, then g is a monomorphism.

See [4], pages 55-56.

Corollary 1.57. If S is a semiring having K and K' as skew rings of right or left differences, then $K \cong K'$.