ผลคูณเทนเซอร์ของพีชคณิตตัวดำเนินการเหนือพีชคณิตย่อย

นางสาว สมลักษณ์ อุตุดี

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรดุษฎีบัณฑิต
สาขาวิชาคณิตศาสตร์ ภาควิชาคณิตศาสตร์
คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย
ปีการศึกษา 2547
ISBN 974-17-6945-8

TENSOR PRODUCTS OF OPERATOR ALGEBRAS OVER SUBALGEBRAS

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A Dissertation Submitted in Partial Fulfillment of the Requirements

for the Degree of Doctor of Philosophy in Mathematics

Department of Mathematics

Faculty of Science

Chulalongkorn University

Academic year 2004

ISBN 974-17-9645-8

Thesis Title Tensor products of operator algebras over subalgebras

By Miss Somlak Utudee

Filed of study Mathematics

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สมลักษณ์ อุตุดี : ผลคูณเทนเซอร์ของพีชคณิตตัวดำเนินการเหนือพีชคณิตย่อย. (TENSOR PRODUCTS OF OPERATOR ALGEBRAS OVER SUBALGEBRAS) อ. ที่ปรึกษา : ผศ.ดร. วิชาญ ลิ่วกีรติยุตกุล, อ.ที่ปรึกษาร่วม : PROF. LÁSZLÓ ZSIDÓ, Ph.D. และ DR. PAOLO BERTOZZINI, Ph.D. 79 หน้า. ISBN 974-17-6945-8.

เราสร้างความเชื่อมโยงแบบธรรมชาติระหว่างผลคูณซีสตาร์เทนเซอร์เล็กสุดของพีชคณิตซีสตาร์เหนือพีชคณิตซีสตาร์สลับที่ ซึ่งมีนิยามอยู่บนพื้นฐานของการแยกในฟีลด์ของพีชคณิตซีสตาร์ และผลคูณดับเบิลยูสตาร์เทนเซอร์เชิงปริภูมิของพีชคณิตดับเบิลยูสตาร์เหนือพีชคณิตดับเบิลยูสตาร์สลับที่ซึ่งนิยามขึ้นอยู่กับสมสัณฐานสตาร์โดยใช้ตัวแทนสตาร์ปรกติที่เหมาะสม

โดยเฉพาะอย่างยิ่ง เราได้ว่าถ้า C เป็นพีชคณิตซีสตาร์สลับที่ที่มีเอกลักษณ์ A_1, A_2 เป็นพีชคณิตซีสตาร์เหนือ C และ π_1, π_2 เป็นตัวแทนสตาร์แบบหนึ่งต่อหนึ่งและไม่ลดรูปที่เหมาะสมของ A_1 , A_2 ตามลำดับ ซึ่งมีค่าเท่ากันบน C ดังนั้นผลคูณซีสตาร์เทนเซอร์เล็กสุดของพีชคณิตซีสตาร์ของ A_1 และ A_2 เหนือ C สามารถพิสูจน์ได้ว่าเป็นสิ่งเดียวกันกับพีชคณิตซีสตาร์ที่ก่อกำเนิดโดยภาพ $\pi_1(A_1)$ และ $\pi_2(A_2)$ ในผลคูณดับเบิลยูสตาร์เทนเซอร์เชิงปริภูมิของส่วนปิดคลุมภายใต้ทอพอโลยีตัวดำเนินการแบบอ่อนของภาพทั้งสอง เทียบกับส่วนปิดคลุมภายใต้ทอพอโลยีตัวดำเนินการแบบอ่อนของภาพของ C.

ภาควิชา…………คณิตศาสตร์ …………ลายมือชื่อนิสิต…………………………………………………
สาขาวิชา…………คณิตศาสตร์…………ลายมือชื่ออาจารย์ที่ปรึกษา………………………………….
ปีการศึกษา…………2547………………ลายมือชื่ออาจารย์ที่ปรึกษาร่วม………………………………
……………………………………

4373841923 : MAJOR MATHEMATICS

KEY WORD: SPATIAL REPRESENTATION / TENSOR PRODUCT / VON NEUMANN ALGEBRA / C^*-ALGEBRA

SOMLAK UTUDEE : TENSOR PRODUCTS OF OPERATOR ALGEBRAS OVER SUBALGEBRAS. THESIS ADVISOR : ASST. PROF. WICHARN LEWKEERATIYUTKUL, Ph.D., THESIS COADVISORS : PROF. LASZLÓ ZSIDÓ, Ph.D. AND DR. PAOLO BERTOZZINI, Ph.D., 79 pp. ISBN 974-17-6945-8.

We establish natural links between minimal C^*-tensor products of C^*-algebras over abelian C^*-algebras, whose definition is based on a decomposition in fields of C^*-algebras, and spatial W^*-tensor products of W^*-algebras over abelian W^*-algebras, defined up to *-isomorphism by using appropriate normal *-representations.

In particular, we obtain that if C is a unital abelian C^*-algebra, A_1, A_2 are C^*-algebras over C and π_1 , π_2 are appropriately faithful non-degenerate *-representations of A_1 respectively A_2 , which coincide on C , then the minimal C^*-tensor product of A_1 and A_2 over C can be identified with the C^*-algebra generated by the images $\pi_1(A_1)$ and $\pi_2(A_2)$ in the spatial W^*-tensor product of their weak operator closures with respect to the weak operator closure of the image of C.

Department……..Mathematics…………….Student's signature……………………………………

Field of study……..Mathematics…………..Advisor's signature……………………………………

Academic year………..2004……………….Co-advisor's signature………………………………..

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ACKNOWLEDGEMENTS

First and foremost, I would like to express my sincere gratitude to Professor Dr. László Zsidó for a number of his encouragement and useful advice throughout the preparation and completion of this dissertaion. I also gratefully acknowledge the hospitality of Department of Mathematics, University of Rome II 'Tor Vergata', during the elaboration of this thesis.

I am greatly indebted to my thesis supervisor, Assistant Professor Dr. Wicharn Lewkeeratiyutkul, and my thesis co-advisor, Dr. Paolo Bertozzini, for their help and suggestions. I would like to express my special thanks to my thesis committee: Assistant Professor Dr. Imchit Termwuttipong (chairman), Associate Professor Dr. Kritsana Neammanee, Professor Dr. Roberto Conti, and Associate Professor Dr. Pachara Chaisuriya. Their suggestions and comments are my sincere appreciation.

My appreciation goes to the Development and Promotion of Science and Technology Talents Project (DPST) for the financial support throughout my undergraduate and graduate study and to the Thai Research Fund for financial support of this research through the Royal Golden Jubilee Ph.D. Program .

I feel very thankful to all of my teachers who have taught me my knowledge and skills. Finally, I wish to express my thankfulness to my beloved parents, my sister, my brother-in-law, and my friends for their loves and encouragement throughout my study.

CONTENTS

CHAPTER I

INTRODUCTION

For every C^*-algebra A, let $Z(A) = \{z \in A\,;\, az = za \text{ for all } a \in A\}$ be its *centre* and $M(A) = \{x \in A^{**}\,;\, A\,x \cup xA \subset A\}$ its *multiplier algebra* (see e.g. [1], 3.12 or [2], 2.2).

We recall that a $*$-representation $\pi : A \to \mathcal{B}(\mathcal{H})$ is called *non-degenerate* if for any $0 \neq \xi \in \mathcal{H}$ there is some $a \in A$ with $\pi(a)\,\xi \neq 0$, or equivalently, if the closed linear span $\mathcal{H}_e$ of $\pi(A)\mathcal{H}$ is equal to $\mathcal{H}$. To a given $*$-representation $\pi : A \to \mathcal{B}(\mathcal{H})$ we always can associate the non-degenerate $*$-representation $A \ni a \longmapsto \pi(a)\,|\,\mathcal{H}_e \in \mathcal{B}(\mathcal{H}_e)$. If A is unital and $\pi : A \to \mathcal{B}(\mathcal{H})$ is a non-degenerate $*$-representation, then π carries the unit 1_A of A to the identity map $1_{\mathcal{H}}$ on $\mathcal{H}$.

Every non-degenerate $*$-representation $\pi : A \to \mathcal{B}(\mathcal{H})$ extends to a unique unital $*$-representation $M(\pi) : M(A) \to \mathcal{B}(\mathcal{H})$, which is a $*$-isomorphism of $M(A)$ onto the C^*-subalgebra $\big\{T \in \mathcal{B}(\mathcal{H})\,;\, \pi(A)\,T \cup T\pi(A) \subset \pi(A)\big\} \subset \mathcal{B}(\mathcal{H})$ whenever π is injective (see e.g. [1], 3.12 or [2], 2.2.11, 2.2.16, 2.2.17). More precisely, $M(\pi)$ is the restriction to $M(A)$ of the normal extension $A^{**} \to \mathcal{B}(\mathcal{H})$ of π, so $\pi(A)$ and $M(\pi)\big(M(A)\big)$ generate the same von Neumann algebra.

Let now C be a unital abelian C^*-algebra and let Ω denote its Gelfand spectrum. If A is a C^*-algebra and $\iota : C \to Z\big(M(A)\big)$ is an injective unital $*$-homomorphism, then we say that $(A\,,\iota)$, or simply A if ι is clear from the context, is a *C^*-algebra over C*. In this case, for any non-degenerate $*$-representation $\pi : A \to \mathcal{B}(\mathcal{H})$, the composition $\pi \circ \iota = M(\pi) \circ \iota$ can be considered.

If $(A\,,\iota)$ is a C^*-algebra over C, then

$$I_\iota(t) = \overline{\{\,\iota(c)\,;\,c\in C\,,\,c(t)=0\,\}\,A}\,, \qquad t\in\Omega \tag{1.1}$$

are closed two-sided ideals in A. We shall call them *Glimm ideals.* Let $\pi_{\iota,t}$ denote the canonical map $A\to A/I_\iota(t)$. Then we have $\bigcap\limits_{t\in\Omega} I_\iota(t)=\{0\}$, that is $\|a\|=\sup\limits_{t\in\Omega}\|\pi_{\iota,t}(a)\|$ for all $a\in A$ (see [3], Remarks on page 232). We notice that the functions

$$\Omega\ni t\longmapsto\|\pi_{\iota,t}(a)\|\,, \qquad a\in A$$

are always upper semi-continuous (see [3], Lemma 9 or [4], Lemma 3.1 or [5], Lemma 2.3), but they are in general not continuous. If they are continuous, then $(A\,,\iota)$ will be called a *continuous* C^**-algebra over* C.

C^*-tensor products of C^*-algebras over C were already considered by G. A. Elliott [6] and G. G. Kasparov [7], 1.6, but a systematic study of such tensor products was undertaken only later by É. Blanchard [8], [9], B. Magajna [10] and T. Giordano - J. Mingo [11].

Let $(A_1\,,\iota_1)$ and $(A_2\,,\iota_2)$ be C^*-algebras over C and let us consider the $*$-homomorphisms

$$\pi_{\iota_1,t}\otimes\pi_{\iota_2,t}:A_1\otimes A_2\longrightarrow\big(A_1/I_{\iota_1}(t)\big)\otimes\big(A_2/I_{\iota_2}(t)\big)\,, \qquad t\in\Omega\,,$$

where $\otimes$ stands for the algebraic tensor product over $\mathbb{C}$. On every quotient $\big(A_1/I_{\iota_1}(t)\big)\otimes\big(A_2/I_{\iota_2}(t)\big)$ there exists the least C^*-norm $\|\cdot\|_{\min}$ (see [12] or [13], 6.4) and

$$A_1\otimes A_2\ni a\longmapsto\|(\pi_{\iota_1,t}\otimes\pi_{\iota_2,t})(a)\|_{\min}$$

is a C^*-seminorm. Following É. Blanchard, the *minimal* C^**-tensor product of* A_1 *and* A_2 *over* C is defined as the Hausdorff completion $A_1\otimes_{C,\min}A_2$ of $A_1\otimes A_2$ with respect to the C^*-seminorm

$$A_1\otimes A_2\ni a\longmapsto\|a\|_{C,\min}=\sup_{t\in\Omega}\,\|(\pi_{\iota_1,t}\otimes\pi_{\iota_2,t})(a)\|_{\min}\,, \tag{1.2}$$

that is the C^*-algebra obtained by the completion of the quotient $*$-algebra

$$(A_1 \otimes A_2)\big/\mathcal{J}_C \ \text{ with } \ \mathcal{J}_C = \{\, a \in A_1 \otimes A_2\,;\, (\pi_{\iota_1,t} \otimes \pi_{\iota_2,t})(a) = 0\,, \text{ for all } t \in \Omega\,\}$$

relative to the C^*-norm induced by $\|\cdot\|_{C,\min}$.

On the other hand, spatial tensor products of W^*-algebras over abelian W^*-algebras were considered by Ş. Strătilă and L. Zsidó. They showed in [14], Lemma 5.2 that if Z is an abelian W^*-algebra, M_1, M_2 are W^*-algebras and $\iota_1 : Z \longrightarrow Z(M_1)$, $\iota_2 : Z \longrightarrow Z(M_2)$ are injective unital normal $*$-homomorphisms, then there exist injective unital normal $*$-representations $\pi_1 : M_1 \longrightarrow \mathcal{B}(\mathcal{H})$, $\pi_2 : M_2 \longrightarrow \mathcal{B}(\mathcal{H})$ on the same Hilbert space $\mathcal{H}$, such that $\pi_1 \circ \iota_1 = \pi_2 \circ \iota_2$ and $\pi_1(M_1) \subset N$, $\pi_2(M_2) \subset N'$ for some type I von Neumann algebra $N \subset \mathcal{B}(\mathcal{H})$ with centre equal to $(\pi_j \circ \iota_j)(Z)$. On the other hand, according to [14], Lemma 5.4, if $\rho_1 : M_1 \longrightarrow \mathcal{B}(\mathcal{K})$, $\rho_2 : M_2 \longrightarrow \mathcal{B}(\mathcal{K})$ are any injective unital normal $*$-representations such that $\rho_1 \circ \iota_1 = \rho_2 \circ \iota_2$ and $\rho_1(M_1) \subset R$, $\rho_2(M_2) \subset R'$ for some type I von Neumann algebra $R \subset \mathcal{B}(\mathcal{K})$ with centre equal to $(\rho_j \circ \iota_j)(Z)$, then there is a $*$-isomorphism

$$\Theta : \pi_1(M_1) \vee \pi_2(M_2) \longrightarrow \rho_1(M_1) \vee \rho_2(M_2)$$

satisfying

$$\Theta\big(\pi_1(x_1)\pi_2(x_2)\big) = \rho_1(x_1)\rho_2(x_2) \ \text{ for all } \ x_1 \in M_1\,,\ x_2 \in M_2\,.$$

In other words, the von Neumann algebra $\pi_1(M_1) \vee \pi_2(M_2)$ is unique up to canonical $*$-isomorphism. Since in the case $Z = \mathbb{C}$ it is $*$-isomorphic to the usual spatial tensor product (over $\mathbb{C}$) $M_1 \overline{\otimes} M_2$ (see [15], Lemma 2), it is natural to call it in the general case the *spatial W^*-tensor product of M_1 and M_2 over Z*.

The goal of this thesis is to link the minimal C^*-tensor product with the spatial W^*-tensor product.

The first main result (Theorem 4.4) claims that if C is a unital abelian C^*-algebra, $(A_1\,,\,\iota_1)$ and $(A_2\,,\,\iota_2)$ are C^*-algebras over C and $\pi_j : A_j \longrightarrow \mathcal{B}(\mathcal{H})\,,\ j = 1\,,2\,,$ are non-degenerate $*$-representations such that

$$M(\pi_1)\circ\iota_1 = M(\pi_2)\circ\iota_2 \text{ and } \pi_1(A_1) \subset N\,,\ \pi_2(A_2) \subset N' \tag{1.3}$$

for some type I von Neumann algebra $N \subset \mathcal{B}(\mathcal{H})$ with centre $\big(M(\pi_j)\circ\iota_j\big)(C)''$, then there exists a $*$-representation of $A_1 \otimes_{C,\min} A_2$ on $\mathcal{H}$, which carries the canonical image $(a_1 \otimes a_2)/\mathcal{J}_C \in (A_1 \otimes A_2)/\mathcal{J}_C$ of any $a_1 \otimes a_2 \in A_1 \otimes A_2$ to $\pi_1(a_1)\,\pi_2(a_2)$. This $*$-representation is uniquely determined and we denote it by $\pi_1 \otimes_{C,\min} \pi_2$. Clearly, $\pi_1 \otimes_{C,\min} \pi_2$ maps the minimal C^*-tensor product $A_1 \otimes_{C,\min} A_2$ into the spatial W^*-tensor product $\pi_1(A_1)'' \vee \pi_2(A_2)''$ of $\pi_1(A_1)''$ and $\pi_2(A_2)''$ over $(\pi_j \circ \iota_j)(C)''$.

Chapter 5 is dedicated to tensor products of Hilbert modules occuring in the theory of spatial tensor products of W^*-algebras over abelian W^*-algebras. This chapter can be considered as belonging to the topological reduction theory of von Neumann algebras, in the spirit of [16], [17], [18], [19] and [4]. In the main result of this chapter (Theorem 5.5) we give a description of the elements in the tensor product of the considered Hilbert modules, extending a previously proved result concerning the description of the vectors in a Hilbert space tensor product (Proposition 5.1). The results of this chapter are used in Chapter 6 to reprove a result of H. Halpern about the structure of a normal conditional expectation of a type I von Neumann algebra onto its centre (Theorem 6.6).

In Chapter 7 Glimm ideals are described in terms of a faithful spatial representation. As an application, $\mathcal{J}_C$ is characterized in terms of faithful non-degenerate $*$-representations $\pi_j : A_j \to \mathcal{B}(\mathcal{H})$ satisfying (1.3) (Corollary 7.7).

Finally, in Chapter 8 we first exhibit an example of faithful π_1 and π_2 for which $\pi_1 \otimes_{C,\min} \pi_2$ is not faithful (Proposition 8.2). Subsequently we prove criteria for faithful non-degenerate $*$-representations $\pi_j : A_j \to \mathcal{B}(\mathcal{H})$ satisfying (1.3) in order that $\pi_1 \otimes_{C,\min} \pi_2$ be faithful (Theorem 8.5). It will follow that if A_1, A_2 are unital and π_1, π_2 are faithful in a stronger sense, then $\pi_1 \otimes_{C,\min} \pi_2$ will be faithful, providing thus an identification of the minimal C^*-tensor product $A_1 \otimes_{C,\min} A_2$ with the C^*-subalgebra of the spatial W^*-tensor product $\pi_1(A_1)'' \vee \pi_2(A_2)''$ generated by the images $\pi_1(A_1)$ and $\pi_2(A_2)$ (Corollary 8.7).

For the basic facts concerning C^*-algebras and von Neumann algebras we refer to the standard textbooks [20], [21], [13], [1], [22] and [23].

CHAPTER II

PRELIMINARIES RELATED WITH SPATIAL W^*-TENSOR PRODUCTS OVER ABELIAN W^*-ALGEBRAS

In [14], Lemma 2.2, the commutation theorem of M. Tomita was extended to the frame of spatial W^*-tensor products over abelian W^*-subalgebras. The proof of this general commutative theorem is based on a careful analysis of the Z_h-submodule and Z-submodule of Ne, where N is a type I W^*-algebra with centre Z and e is an abelian projection in N, performed in [14], Chapter 2. In this chapter we recall certain facts concerning such submodules, completing them when our needs require this.

We recall that if N is a type I factor and e is a minimal projection in N, then the equality

$$exe = \varphi_e(x)\, e\,, \qquad x \in N$$

defines a normal state φ_e on N, Ne becomes a Hilbert space with the inner product

$$Ne \times Ne \ni (x, y) \longmapsto \varphi_e(y^* x)$$

and, associating to every $x \in N$ the left multiplication operator

$$Ne \ni y \longmapsto L_x(y) = xy \in Ne\,,$$

we get a $*$-isomorphism $N \ni x \longmapsto L_x \in \mathcal{B}(Ne)$.

The above construction can be extended to arbitrary type I von Neumann algebras. Let N be a type I von Neumann algebra with centre Z. If e is an abelian projection in N with central support $\mathrm{z}_N(e)$, then the map

$$Z\,\mathrm{z}_N(e) \ni z\,\mathrm{z}_N(e) \longmapsto z\,\mathrm{z}_N(e)e = ze \in eNe \tag{2.1}$$

is a $*$-isomorphism. For every $x \in N$, we denote the inverse image of exe in $Z\,\mathrm{z}_N(e)$ under this isomorphism by $\Phi_e(x)$. Then $\Phi_e : N \longrightarrow Z\,\mathrm{z}_N(e)$ is a normal positive Z-module mapping with $\Phi_e(1_N) = \mathrm{z}_N(e)$, uniquely defined by the equality

$$exe = \Phi_e(x)e\,, \qquad x \in N \tag{2.2}$$

(see e.g. [16], [17]). Furthermore, since (2.1) is isometric, we have

$$\|exe\| = \|\Phi_e(x)\|\,, \qquad x \in N\,. \tag{2.3}$$

Furthermore, if $\mathrm{z}_N(e) = 1_N$, then Φ_e is a normal conditional expectation of N onto Z with support e . In this case Ne becomes a Hilbert Z-module with the Z-valued inner product

$$Ne \times Ne \ni (x, y) \longmapsto \Phi_e(y^*x)\,.$$

Let $\mathcal{B}_Z(Ne)$ denote the set of all bounded Z-module morphisms of Ne into itself and, for every $x \in N$, let us consider the left multiplication operator

$$Ne \ni y \longmapsto L_x(y) = xy \in Ne\,.$$

Then $\mathcal{B}_Z(Ne)$ becomes in a natural way a C^*-algebra and the map

$$N \ni x \longmapsto L_x \in \mathcal{B}_Z(Ne) \tag{2.4}$$

is an injective $*$-homomorphism (see [14], 1.13.(4)). Actually we have more:

Lemma 2.1. *Let N be a type I von Neumann algebra with centre Z, and $e \in N$ an abelian projection of central support 1_N . Then* (2.4) *is a $*$-isomorphism.*

Proof. By well known classical results (see e.g. [24], 7.5 – 7.6), the closed unit ball of $\mathcal{B}_Z(Ne)$ is compact with respect to the topology of the pointwise w-convergence, so the C^*-algebra $\mathcal{B}_Z(Ne)$ is the dual space of some Banach space and the corresponding weak* topology on the closed unit ball of $\mathcal{B}_Z(Ne)$ coincides with the topology of the pointwise w-convergence. Therefore $\mathcal{B}_Z(Ne)$ is a W^*-algebra and the w-topology on its closed unit ball is the topology of the pointwise w-convergence (see [22], 1.1.2, 1.13.3, 1.16.7 or [24], 8.4). In particular, since $\{L_x\,;\,x \in N\} \ni L_{1_N} = 1_{\mathcal{B}_Z(Ne)}$ is a $*$-subalgebra of $\mathcal{B}_Z(Ne)$, whose closed unit ball is closed with respect to the topology of the pointwise w-convergence, it is a W^*-subalgebra of $\mathcal{B}_Z(Ne)$.

Next we compute the relative commutant of the above W^*-subalgebra:

$$\{L_x\,;\,x \in N\}' \cap \mathcal{B}_Z(Ne) = \{L_z\,;\,z \in Z\}\,. \tag{2.5}$$

For let $T \in \{L_x\,;\,x \in N\}' \cap \mathcal{B}_Z(Ne)$ be arbitrary. Then

$$T(xy) = (T \circ L_x)(y) = (L_x \circ T)(y) = x\,T(y)\,, \qquad x \in N\,,\, y \in Ne\,. \tag{2.6}$$

In particular, $T(e) = e\,T(e) \in eNe = Ze$, hence $T(e) = z_T\,e$ for some $z_T \in Z$. Now (2.6) yields for every $x \in Ne$:

$$T(x) = T(xe) = xT(e) = xz_T\,e = z_T\,x = L_{z_T}(x)\,.$$

By (2.5), the centre of $\mathcal{B}_Z(Ne)$ is $\{L_z\,;\,z \in Z\}$. In particular, the central support of the projection L_e is $L_{1_N} = 1_{\mathcal{B}_Z(Ne)}$.

On the other hand, L_e is an abelian projection. Indeed, since $L_e(x) = ex = \Phi_e(x)e\,,\ x \in Ne$, we have for every $T \in \mathcal{B}_Z(Ne)$

$$\begin{aligned}(L_eTL_e)(x) &= (L_eT)\big(\Phi_e(x)e\big) = L_e\big(\Phi_e(x)T(e)\big) = \Phi_e(x)\,\Phi_e\big(T(e)\big)e\\ &= z_T\,\Phi_e(x)e = z_T\,L_e(x) = \big(L_{z_T}L_e\big)(x)\,,\end{aligned}$$

so $L_eTL_e = L_{z_T}L_e \in Z\big(\mathcal{B}_Z(Ne)\big)\cdot L_e$.

Consequently, $\mathcal{B}_Z(Ne)$ is a type I W^*-algebra. Since

$$Z\big(\mathcal{B}_Z(Ne)\big) = \{L_z\,;\, z \in Z\} \subset \{L_x\,;\, x \in N\} \subset \mathcal{B}_Z(Ne)\,,$$

(2.5) yields that $\{L_x\,;\, x \in N\} = \mathcal{B}_Z(Ne)$ (cf. [14], 1.7.(4)). □

The next three simple lemmas concerning abelian projections are variants of well known results. They are exposed here for further reference, for the convenience of the reader:

Lemma 2.2. *Let N be a type I von Neumann algebra. If $f\,,\, p \in N$ are projections, $f \leq p$ and f is abelian, then there exists an abelian projection $e \in N$ such that*

$$f \leq e \leq p\,, \quad \mathrm{z}_N(e) = \mathrm{z}_N(p)\,.$$

Proof. Let us first consider the case $f = 0$. Since N is of type I, so is pNp. Let e be an abelian projection in pNp with central support one, that is $\mathrm{z}_{pNp}(e) = p$. Since

$$exeye = e(pxep)(pyep) = e(pyep)(pxep) = eyexe\,, \qquad x\,,\, y \in N\,,$$

e is an abelian projection also in N. Clearly, $e \leq p$ implies $\mathrm{z}_N(e) \leq \mathrm{z}_N(p)$. On the other hand, since $e \leq p\,\mathrm{z}_N(e)p \in Z(pNp)$ and $\mathrm{z}_{pNp}(e) = p$, we have

$$p \leq p\,\mathrm{z}_N(e)p = p\,\mathrm{z}_N(e) \leq \mathrm{z}_N(e)\,.$$

Consequently also the converse inequality $\mathrm{z}_N(p) \leq \mathrm{z}_N(e)$ holds.

The case of a general f can be reduced to the above treated case. Indeed, by the above part of the proof there is an abelian projection $e_o \in N$ such that

$$e_o \leq p - p\,\mathrm{z}_N(f)\,, \quad \mathrm{z}_N(e_o) = \mathrm{z}_N\left(p - p\,\mathrm{z}_N(f)\right) = \mathrm{z}_N(p) - \mathrm{z}_N(f)$$

and then $e = f + e_o \in N$ will be an abelian projection satisfying $f \leq e \leq p$ and $\mathrm{z}_N(e) = \mathrm{z}_N(p)$. □

Lemma 2.3. *Let N be a type I von Neumann algebra. Then*

$$\|x\| = \sup\{\|xv\|\,;\, v \in N \text{ partial isometry, } v^*v \le e\}\,, \quad x \in N$$

holds for any abelian projection $e \in N$ with $\mathrm{z}_N(e) = 1_N$. On the other hand,

$$\|x\|^2 = \sup\{\|\Phi_e(x^*x)\|\,;\, e \in N \text{ abelian projection, } \mathrm{z}_N(e) = 1_N\}\,, \quad x \in N\,.$$

Proof. First we prove that

$$\|x\| = \sup\{\|xf\|\,;\, f \in N \text{ abelian projection }\}\,, \quad x \in N\,. \tag{2.7}$$

For let $x \in N$ and $\varepsilon > 0$ be arbitrary. By the spectral theorem there exists a projection $p \in N$ commuting with x^*x such that

$$x^*x\,p \ge \big(\|x^*x\| - \varepsilon\big)p\,, \tag{2.8}$$

$$x^*x\,(1_N - p) \le \big(\|x^*x\| - \varepsilon\big)(1_N - p)$$

(see e.g. [23], Corollary 2.21). Note that $p \neq 0$, because $p = 0$ would imply $x^*x \le \|x^*x\| - \varepsilon$, a contradiction. Since N is of type I, p majorizes a non-zero abelian projection $f \in N$ and (2.8) yields

$$fx^*xf = fx^*xpf \ge \big(\|x^*x\| - \varepsilon\big)f\,.$$

Consequently $\|xf\|^2 = \|fx^*xf\| \ge \big(\|x^*x\| - \varepsilon\big)\|f\| = \|x\|^2 - \varepsilon$.

Now let e be any abelian projection in N with $\mathrm{z}_N(e) = 1_N$. Let further $x \in N$ be arbitrary. Taking into account (2.7),

$$\|x\| = \sup\{\|xv\|\,;\, v \in N \text{ partial isometry, } v^*v \le e\}$$

will follow once we show that for every abelian projection $f \in N$ there exists a partial isometry $v \in N$ such that $v^*v \le e$ and $\|xf\| \le \|xv\|$.

But $\mathrm{z}_N(f) \le 1_N = \mathrm{z}_N(e)$ implies the existence of a partial isometry $v \in N$ such that $v\,v^* = f$, $v^*v \le e$ (see e.g. [23], Proposition 4.10). Then

$$\|xf\|^2 = \|xfx^*\| = \|xvv^*x^*\| = \|xv\|^2 .$$

Finally, let $x \in N$ be arbitrary. Again by (2.7),

$$\|x\|^2 = \sup\{\|\Phi_e(x^*x)\| \,;\, e \in N \text{ abelian projection, } \mathrm{z}_N(e) = 1_N\}$$

will follow once we show that for every abelian projection $f \in N$ there exists an abelian projection $e \in N$ with $\mathrm{z}_N(e) = 1_N$ such that $\|xf\|^2 \leq \|\Phi_e(x^*x)\|$.

But Lemma 2.2, applied with $p = 1_N$, implies the existence of an abelian projection $e \in N$ such that $f \leq e$ and $\mathrm{z}_N(e) = 1_N$. Then (2.2) yields

$$\|xf\|^2 \leq \|xe\|^2 = \|ex^*xe\| = \|\Phi_e(x^*x)e\| \leq \|\Phi_e(x^*x)\| .$$

□

Lemma 2.4. *Let $N \subset \mathcal{B}(\mathcal{H})$ be a type I von Neumann algebra, e an abelian projection in N , and f an abelian projection in N' . Then ef is an abelian projection in $N \vee N'$ with $\mathrm{z}_{N\vee N'}(ef) = \mathrm{z}_N(e)\,\mathrm{z}_{N'}(f)$ and*

$$\Phi_{ef}(xy) = \Phi_e(x)\Phi_f(y) , \qquad x \in N , \, y \in N' .$$

Moreover, if $\mathrm{z}_N(e) = \mathrm{z}_{N'}(f)$, then

$$\Phi_e = \Phi_{ef}|_N \text{ and } \Phi_f = \Phi_{ef}|_{N'} .$$

Proof. Let us denote for convenience $Z = Z(N) = Z(N') = Z(N \vee N')$.

Clearly, $ef = fe$ is a projection in $N \vee N'$. Since, for every $x_1 , x_2 \in N$ and $y_1 , y_2 \in N'$,

$$\begin{aligned}(efx_1y_1ef)(efx_2y_2ef) =& (ex_1ex_2e)(fy_1fy_2f)\\ =& (ex_2ex_1e)(fy_2fy_1f) = (efx_2y_2ef)(efx_1y_1ef) ,\end{aligned}$$

ef is an abelian projection in $N \vee N'$.

If $p \in Z$ is a projection such that $ef \leq p$, then it follows successively

$$ey'f\xi = y'efp\xi = py'ef\xi \in p\mathcal{H} \text{ for all } y' \in N\,,\ \xi \in \mathcal{H}\,,\ \text{i.e. } eN'f\mathcal{H} \subset p\mathcal{H}\,,$$

$$e\,\mathrm{z}_{N'}(f)\mathcal{H} \subset p\mathcal{H}\,,\ \text{i.e. } \mathrm{z}_{N'}(f)e = e\,\mathrm{z}_{N'}(f) \leq p\,,$$

$$\mathrm{z}_{N'}(f)ye\xi = ye\,\mathrm{z}_{N'}(f)\xi = ype\,\mathrm{z}_{N'}(f)\xi = py\,\mathrm{z}_{N'}(f)e\xi \in p\mathcal{H}\,, \quad y \in N, \xi \in \mathcal{H}\,, \quad \text{i.e.}$$
$$\mathrm{z}_{N'}(f)Ne\mathcal{H} \subset p\mathcal{H}\,,$$

$$\mathrm{z}_{N'}(f)\,\mathrm{z}_N(e)\mathcal{H} \subset p\mathcal{H}\,,\ \text{i.e. } \mathrm{z}_{N'}(f)\,\mathrm{z}_N(e) \leq p\,.$$

Therefore $\mathrm{z}_{N'}(f)\,\mathrm{z}_N(e) \leq \mathrm{z}_{N\vee N'}(ef)$. But the converse inequality is trivial, so we actually have

$$\mathrm{z}_{N\vee N'}(ef) = \mathrm{z}_{N'}(f)\,\mathrm{z}_N(e)\,. \tag{2.9}$$

Let $x \in N$, $y \in N'$ be arbitrary. According to (2.2), we deduce

$$efxyef = (exe)(fyf) = \Phi_e(x)e\Phi_f(y)f = \Phi_e(x)\Phi_f(y)ef\,.$$

Since, by (2.9), we have $\Phi_e(x)\Phi_f(y) \in Z\,\mathrm{z}_N(e)\,\mathrm{z}_{N'}(f) = Z\,\mathrm{z}_{N\vee N'}(ef)$, it follows that $\Phi_{ef}(xy) = \Phi_e(x)\Phi_f(y)$.

Assume now that $\mathrm{z}_N(e) = \mathrm{z}_{N'}(f) = \mathrm{z}_{N\vee N'}(ef)$. Then, for every $x \in N$, $efxef = (exe)f = \Phi_e(x)ef$ and $\Phi_e(x) \in Z\,\mathrm{z}_{N\vee N'}(ef)$ imply that $\Phi_{ef}(x) = \Phi_e(x)$. Therefore $\Phi_e = \Phi_{ef}|_N$. Similarly we deduce also $\Phi_f = \Phi_{ef}|_{N'}$. □

The following result concerning the structure of the Z-submodules of Ne, where N is a type I von Neumann algebra with centre Z and e is an abelian projection in N, will be used in the sequel:

Lemma 2.5. *Let $N \subset \mathcal{B}(\mathcal{H})$ be a type I von Neumann algebra with centre Z, and $e \in N$ an abelian projection. If $X \subset Ne$ is a Z-submodule, then there is a unique projection $p \in N$ such that*

$$\overline{X}^{s} = pNe\,, \quad \mathrm{z}_N(p) \leq \mathrm{z}_N(e)\,,$$

namely p is the orthogonal projection onto $\overline{\text{lin}}\, X\mathcal{H}$ *(the closed linear span of* $\{x\xi\,;\, x \in X,\, \xi \in \mathcal{H}\}$*). Moreover, if* $X = Me$ *, where* $Z \subset M \subset N$ *is a von Neumann subalgebra, then*

$$p \in M' \cap N\,, \quad e \leq p\,, \quad \mathrm{z}_N(e) = \mathrm{z}_N(p)\,.$$

Proof. All the above statements, except those concerning central supports, were proved in [14], 1.6 and 1.7. For $\mathrm{z}_N(e) \geq \mathrm{z}_N(p)$, let $q \in Z$ be a projection majorizing e . Then $xe = xeq = qxe$ for every $x \in M$, so $q(xe\xi) = xe\xi$ for every $\xi \in \mathcal{H}$. Since p is the projection onto $\overline{\text{lin}}\, Me\mathcal{H}$, it follows that $q \geq p$. □

We shall need also the following variant of [14], Lemma 1.2, for which we have just to reproduce the proof of [14], Lemma 1.2:

Lemma 2.6. *Let* N *be a type I von Neumann algebra with centre* Z *and* $e \in N$ *an abelian projection. For every* $*$*-subalgebra* $B \subset N$ *and* $x \in \overline{Be}^s$, $\|x\| = 1$ *, we have*

$$x \in \overline{\{y \in BeZ_1^+\,;\, \|y\| \leq 1\}}^s\,,$$

where Z_1^+ *denotes the set of all elements* $z \in Z$ *with* $0 \leq z \leq 1_N$ *.*

Proof. Let $x \in \overline{Be}^s$ be such that $\|x\| = 1$. Consider a net

$$Be \ni b_\lambda e = x_\lambda \xrightarrow{s} x\,.$$

Then

$$\Phi_e(x_\lambda^* x_\lambda)^{1/2} \xrightarrow{s} \Phi_e(x^* x)^{1/2}\,.$$

Let $f, g : [0, \infty) \to [0, 1]$ be functions such that

$$f(t) = 1 \quad \text{for } t \leq 1\,,$$

$$g(t) = 1 \quad \text{for } t \geq 1\,,$$

$$\text{and} \quad g(t) = t f(t) \quad \text{for all } t \in [0, \infty)\,.$$

Since f is operator continuous,

$$Z_h \ni f\big(\Phi_e(x_\lambda^* x_\lambda)^{1/2}\big) \stackrel{s}{\longrightarrow} f\big(\Phi_e(x^* x)^{1/2}\big) = 1_N\,,$$

$$\big\| f\big(\Phi_e(x_\lambda^* x_\lambda)^{1/2}\big)\big\| \leq 1 \quad \text{for all } \lambda\,.$$

Therefore $f\big(\Phi_e(x_\lambda^* x_\lambda)^{1/2}\big)x_\lambda \stackrel{s}{\longrightarrow} x$ with

$$\begin{aligned}
\big\| f\big(\Phi_e(x_\lambda^* x_\lambda)^{1/2}\big)x_\lambda\big\| &= \big\|\Phi_e\big(x_\lambda^* f(\Phi_e(x_\lambda^* x_\lambda)^{1/2})^2 x_\lambda\big)\big\| \\
&= \big\| f\big(\Phi_e(x_\lambda^* x_\lambda)^{1/2}\big)^2 \Phi_e(x_\lambda^* x_\lambda)\big\| \\
&= \big\| f\big(\Phi_e(x_\lambda^* x_\lambda)^{1/2}\big)\Phi_e(x_\lambda^* x_\lambda)^{1/2}\big\|^2 \\
&= \big\| g\big(\Phi_e(x_\lambda^* x_\lambda)^{1/2}\big)\big\|^2 \leq 1\,,
\end{aligned}$$

and $f\big(\Phi_e(x_\lambda^* x_\lambda)^{1/2}\big)x_\lambda \in BeZ_1^+$ because $x_\lambda = b_\lambda e\,,\ \big\| f\big(\Phi_e(x_\lambda^* x_\lambda)^{1/2}\big)\big\| \leq 1\,.$ □

CHAPTER III

PRELIMINARIES RELATED WITH MINIMAL C^*-TENSOR PRODUCTS OVER ABELIAN C^*-ALGEBRAS

Let C be a unital abelian C^*-algebra and let Ω denote its Gelfand spectrum. If $(A\,,\iota)$ is a C^*-algebra over C, then also $(M(A)\,,\iota)$ is a C^*-algebra over C. To distinguish between the ideals defined by (1.1) for $(A\,,\iota)$ and for $(M(A)\,,\iota)$, we shall keep the notation

$$I_\iota(t) = \overline{\{\,\iota(c)\,;\,c\in C\,,\,c(t)=0\,\}\,A}\,, \qquad t\in\Omega$$

for the ideals of A and shall set

$$\widetilde{I}_\iota(t) = \overline{\{\,\iota(c)\,;\,c\in C\,,\,c(t)=0\,\}\,M(A)}\,, \qquad t\in\Omega\,.$$

Similarly, we keep the notation $\pi_{\iota,t}$ for the canonical map $A\to A/I_\iota(t)$ and shall denote the canonical map $M(A)\to M(A)\big/\widetilde{I}_\iota(t)$ by $\widetilde{\pi}_{\iota,t}$.

The next proposition establishes a link between $I_\iota(t)$ and $\widetilde{I}_\iota(t)$, as well as between $\pi_{\iota,t}$ and $\widetilde{\pi}_{\iota,t}$ (cf. [4], Lemma 3.4):

Proposition 3.1. *Let C be a unital abelian C^*-algebra, Ω its Gelfand spectrum, and $(A\,,\iota)$ a C^*-algebra over C. Then*

(i) $\pi_{\iota,t}\big(\iota(c)\,a\big) = c(t)\pi_{\iota,t}(a)\,, \qquad t\in\Omega\,,\,c\in C\,,\,a\in A\,;$

(ii) $\|\pi_{\iota,t}(a)\| = \inf\limits_{\substack{c\in C\\ c(t)=1}}\|\iota(c)\,a\| = \inf\limits_{\substack{c\in C\\ 0\le c\le 1_C\\ c(t)=1}}\|\iota(c)\,a\|\,, \qquad t\in\Omega\,,\,a\in A\,;$

(iii) *for any $t \in \Omega$ we have*

$$I_\iota(t) = A \cap \widetilde{I}_\iota(t)\,, \qquad \|\pi_{\iota,t}(a)\| = \|\widetilde{\pi}_{\iota,t}(a)\|\,,\, a \in A\,.$$

Proof. (i) Since $\iota(c)a - c(t)a = \big(\iota(c) - c(t)1_{M(A)}\big)a = \iota\big(c - c(t)1_C\big)a \in I_\iota(t)$, we have $\pi_{\iota,t}\big(\iota(c)a - c(t)a\big) = 0$.

(ii) Since $\|\pi_{\iota,t}\| \leq 1$, by the above proved (i) we have

$$\|\pi_{\iota,t}(a)\| = \inf_{\substack{c\in C\\ c(t)=1}} \|c(t)\pi_{\iota,t}(a)\| = \inf_{\substack{c\in C\\ c(t)=1}} \|\pi_{\iota,t}\big(\iota(c)a\big)\| \leq \inf_{\substack{c\in C\\ c(t)=1}} \|\iota(c)a\|$$

$$\leq \inf_{\substack{c\in C\\ 0\leq c\leq 1_C\\ c(t)=1}} \|\iota(c)a\|\,.$$

For the converse inequalities, let $\varepsilon > 0$ be arbitrary. Since

$$\Big\{ \sum_{j=1}^{n} \iota(c_j)a_j\,;\, c_j \in C\,, c_j(t) = 0\,, a_j \in A\,, n \in \mathbb{N} \Big\}$$

is dense in $I_\iota(t)$ and $\|\pi_{\iota,t}(a)\| = \|a/I_\iota(t)\| = \inf\{\|a - y\|\,, y \in I_\iota(t)\}$, there exist $c_1, c_2, \ldots, c_n \in C$ and $a_1, a_2, \ldots, a_n \in A$ such that $c_j(t) = 0$ for all $j = 1, 2, \ldots, n$ and

$$\|\pi_{\iota,t}(a)\| \geq \Big\| a - \sum_{j=1}^{n} \iota(c_j)a_j \Big\| - \varepsilon$$

and then there is an open set $t \in V_o \subset \Omega$ such that

$$s \in V_0 \Longrightarrow |c_j(s)| < \frac{\varepsilon}{n\|a_j\|} \text{ for all } 1 \leq j \leq n\,.$$

By Urysohn Lemma, there is $c_o \in C$ such that $0 \leq c_o \leq 1_C$, $c_o(t) = 1$, and $c_o(s) = 0$ for every $s \in \Omega \setminus V_o$. Since $|(c_oc_j)(s)| = 0$ for $s \in \Omega \setminus V_o$ and $|(c_oc_j)(s)| \leq \frac{\varepsilon}{n\|a_j\|}$ for $s \in V_o$, we have for every $1 \leq j \leq n$:

$$\|\iota(c_oc_j)a_j\| \leq \|\iota(c_oc_j)\|\,\|a_j\| \leq \frac{\varepsilon}{n\|a_j\|}\,\|a_j\| = \frac{\varepsilon}{n}\,.$$

Therefore

$$\|\pi_{\iota,t}(a)\| + \varepsilon \geq \Big\|\, a - \sum_{j=1}^{n} \iota(c_j)a_j \,\Big\| \geq \Big\|\, \iota(c_o)a - \sum_{j=1}^{n} \iota(c_oc_j)a_j \,\Big\|$$

$$\geq \|\iota(c_o)a\| - \sum_{j=1}^{n} \|\iota(c_o c_j)a_j\| \geq \|\iota(c_o)a\| - \varepsilon\,,$$

so $\|\pi_{\iota,t}(a)\| + 2\,\varepsilon \geq \|\iota(c_o)a\| \geq \inf\limits_{\substack{c\in C\\ 0\leq c\leq 1_C\\ c(t)=1}} \|\iota(c)a\|\,.$

(iii) Let $a \in A$ be arbitrary. Applying (ii) to $\pi_{\iota,t}(a)$ and to $\widetilde{\pi}_{\iota,t}(a)$, we get

$$\|\pi_{\iota,t}(a)\| = \inf_{\substack{c\in C\\ c(t)=1}} \|\iota(c)a\| = \|\widetilde{\pi}_{\iota,t}(a)\|\,.$$

In particular, $a \in A \cap \widetilde{I}_\iota(t) \implies a \in I_\iota(t)$, hence the inclusion $A \cap \widetilde{I}_\iota(t) \subset I_\iota(t)$ holds. Since the converse inclusion is trivial, we have $I_\iota(t) = A \cap \widetilde{I}_\iota(t)$. □

Proposition 3.1.(iii) implies immediately:

Corollary 3.2. *Let C be a unital abelian C^*-algebra, Ω its Gelfand spectrum, and $(A\,,\iota)$ a C^*-algebra over C. Then, for every $t \in \Omega$, the map*

$$\rho_{\iota,t} : A/I_\iota(t) \ni \pi_{\iota,t}(a) \longmapsto \widetilde{\pi}_{\iota,t}(a) \in M(A)/\widetilde{I}_\iota(t)$$

is a well defined injective $*$-homomorphism and the diagram

$$\begin{array}{ccc} A & \xrightarrow{\text{inclusion}} & M(A) \\ {\scriptstyle \pi_{\iota,t}}\big\downarrow & & \big\downarrow{\scriptstyle \widetilde{\pi}_{\iota,t}} \\ A/I_\iota(t) & \xrightarrow{\rho_{\iota,t}} & M(A)/\widetilde{I}_\iota(t) \end{array}$$

is commutative.

Now let C be a unital abelian C^*-algebra with Gelfand spectrum Ω and let $(A_1\,,\iota_1)$, $(A_2\,,\iota_2)$ be C^*-algebras over C. For every $t \in \Omega$, Corollary 3.2 entails the existence of the injective $*$-homomorphisms $\rho_{\iota_1,t}$, $\rho_{\iota_2,t}$ and then the tensor product $*$-homomorphism

$$\rho_{\iota_1,t} \otimes_{\min} \rho_{\iota_2,t} : A_1/I_{\iota_1}(t) \otimes_{\min} A_2/I_{\iota_2}(t) \longrightarrow M(A_1)/\widetilde{I}_{\iota_1}(t) \otimes_{\min} M(A_2)/\widetilde{I}_{\iota_2}(t)$$

is injective, hence isometric, and the diagram

$$\begin{array}{ccc} A_1 \otimes A_2 & \xrightarrow{\text{inclusion}} & M(A_1) \otimes M(A_2) \\ {\scriptstyle \pi_{\iota_1,t}\otimes\pi_{\iota_2,t}}\big\downarrow & & \big\downarrow{\scriptstyle \widetilde{\pi}_{\iota_1,t}\otimes\widetilde{\pi}_{\iota_2,t}} \\ \big(A_1/I_{\iota_1}(t)\big) \otimes_{\min} \big(A_2/I_{\iota_2}(t)\big) & \xrightarrow{\rho_{\iota_1,t}\otimes_{\min}\rho_{\iota_2,t}} & \big(M(A_1)/\widetilde{I}_{\iota_1}(t)\big) \otimes_{\min} \big(M(A_2)/\widetilde{I}_{\iota_2}(t)\big) \end{array}$$

is commutative. Consequently:

Corollary 3.3. *Let C be a unital abelian C^*-algebra with Gelfand spectrum Ω and let $(A_1\,,\,\iota_1)\,,\,(A_2\,,\,\iota_2)$ be C^*-algebras over C. Then, for every $t\in\Omega$,*

$$\|(\pi_{\iota_1,t}\otimes\pi_{\iota_2,t})(a)\|_{\min}=\|(\widetilde{\pi}_{\iota_1,t}\otimes\widetilde{\pi}_{\iota_2,t})(a)\|_{\min}\,,\qquad a\in A_1\otimes A_2\,.$$

As a consequence of the above corollary, we have

$$\sup_{t\in\Omega}\|(\pi_{\iota_1,t}\otimes\pi_{\iota_2,t})(a)\|_{\min}=\sup_{t\in\Omega}\|(\widetilde{\pi}_{\iota_1,t}\otimes\widetilde{\pi}_{\iota_2,t})(a)\|_{\min}\,,\qquad a\in A_1\otimes A_2\,,$$

hence the restriction of the C^*-seminorm

$$M(A_1)\otimes M(A_2)\ni x\longmapsto\sup_{t\in\Omega}\|(\widetilde{\pi}_{\iota_1,t}\otimes\widetilde{\pi}_{\iota_2,t})(x)\|_{\min}$$

to $A_1\otimes A_2$ is equal to the C^*-seminorm

$$A_1\otimes A_2\ni a\longmapsto\sup_{t\in\Omega}\|(\pi_{\iota_1,t}\otimes\pi_{\iota_2,t})(a)\|_{\min}\,.$$

Therefore the C^*-seminorm (1.2) can be defined also by the formula

$$\|a\|_{C,\min}=\sup_{t\in\Omega}\|(\widetilde{\pi}_{\iota_1,t}\otimes\widetilde{\pi}_{\iota_2,t})(a)\|_{\min}\,,\qquad a\in A_1\otimes A_2\,.$$

Every bounded linear functional φ on a C^*-algebra A can be considered in the natural way a linear functional on A^{**}, hence also on $M(A)\subset A^{**}$: the obtained linear functional on $M(A)$, which will be still denoted by φ, is actually the strictly continuous extension of the original functional on $M(A)$ (for the strict topology see e.g. [2], 2.3).

The next result is slightly more general than [21], Proposition 4.3.14 and can be deduced from [24], Corollary 4.7:

Proposition 3.4. *Let C be a unital abelian C^*-algebra, Ω its Gelfand spectrum, $(A\,,\,\iota)$ a C^*-algebra over C, and φ a state on A. Then, for every $t\in\Omega$, the conditions*

(i) $\varphi\big(\iota(c)\,a\big) = c(t)\,\varphi(a)\,, \qquad c \in C\,,\ a \in A\,,$

(ii) $\varphi\big|_{I_\iota(t)} = 0\,,$

(iii) $\varphi\big(\iota(c)\big) = c(t)\,, \qquad c \in C$

are equivalent. Moreover, if φ is a pure state on A then the above conditions are satisfied for an appropriate $t \in \Omega$.

Proof. (i) ⇒ (ii) is obvious and (ii) ⇒ (iii) follows easily: any approximate unit $\{u_\lambda\}_\lambda$ for A is strictly convergent to $1_{M(A)}$ (see e.g. [2], Lemma 2.3.3) and the strict continuity of φ on $M(A)$ yields

$$\varphi\Big(\iota\big(c - c(t)1_C\big)u_\lambda\Big) \longrightarrow \varphi\Big(\iota\big(c - c(t)1_C\big)\Big) = \varphi\big(\iota(c)\big) - c(t)\,, \qquad c \in C\,.$$

Now let us assume that (iii) is satisfied and let $a \in A^+$, $\|a\| \le 1$, be arbitrary. For $\varphi(a) = 0$ we have by the Schwarz Inequality

$$\varphi\big(\iota(c)\,a\big) = 0 = c(t)\,\varphi(a)\,, \qquad c \in C\,,$$

while for $\varphi(1_{M(A)} - a) = 0$ we deduce, again by the Schwarz inequality,

$$\varphi\big(\iota(c)\,a\big) = \varphi\big(\iota(c)\big) - \varphi\big(\iota(c)\,(1_{M(A)} - a)\big) = c(t) = c(t)\,\varphi(a)\,, \qquad c \in C\,.$$

On the other hand, if $\varphi(a) > 0$ and $\varphi(1_{M(A)} - a) > 0$ then

$$C \ni c \overset{\psi_1}{\longmapsto} \frac{1}{\varphi(a)}\,\varphi\big(\iota(\,\cdot\,)a\big)\,, \quad C \ni c \overset{\psi_2}{\longmapsto} \frac{1}{\varphi(1_{M(A)} - a)}\,\varphi\Big(\iota(\,\cdot\,)(1_{M(A)} - a)\Big)$$

are states satisfying $\varphi \circ \iota = \varphi(a)\,\psi_1 + \varphi(1_{M(A)} - a)\,\psi_2$. Since $\varphi \circ \iota$ is by (iii) a character, hence a pure state, it follows that $\psi_1 = \psi_2 = \varphi \circ \iota$. Therefore

$$\varphi\big(\iota(c)\,a\big) = \varphi(a)\,\psi_1(c) = \varphi(a)\,\varphi\big(\iota(c)\big) = c(t)\,\varphi(a)\,, \qquad c \in C\,.$$

Finally, let us assume that φ is a pure state on A. Let $\pi_\varphi : A \to \mathcal{B}(\mathcal{H}_\varphi)$ denote the GNS representation associated to φ and let ξ_φ be its canonical cyclic vector. Then π_φ, hence also $M(\pi_\varphi)$ is irreducible and it follows that $M(\pi_\varphi)\big(\iota(C)\big) = \mathbb{C}\,1_{\mathcal{H}_\varphi}$. Therefore

$$\big(M(\pi_\varphi)\circ\iota\big)(c) = c(t)\,1_{\mathcal{H}_\varphi}\,, \qquad c\in C$$

for some $t\in\Omega$ and we obtain

$$\varphi\big(\iota(c)\big) = \Big(M(\pi_\varphi)\big(\iota(c)\big)\,\xi_\varphi\,\Big|\,\xi_\varphi\Big) = c(t)\,(\xi_\varphi\,|\,\xi_\varphi) = c(t)\,, \qquad c\in C\,.$$

□

$S(A)$ will denote the set of all states of the C^*-algebra A, while $P(A)$ will stand for the set of all pure states of A. If C and $(A\,,\iota)$ are as in Proposition 3.4, then we denote by $S_\iota(A)$ the set of all states φ of A for which $\varphi\circ\iota$ is a character on C. By Lemma 3.4, $P(A)\subset S_\iota(A)$.

As a corollary, we get the following formula for the minimal C^*-tensor product norm (see [6], Sublemma 2.1):

Corollary 3.5. *Let C be a unital abelian C^*-algebra with Gelfand spectrum Ω and let $(A_1\,,\iota_1)$, $(A_2\,,\iota_2)$ be C^*-algebras over C. Then, for any $a\in A_1\otimes A_2$,*

$$\|a\|^2_{C,\min} = \sup\left\{\frac{(\varphi_1\otimes\varphi_2)(b^*a^*ab)}{(\varphi_1\otimes\varphi_2)(b^*b)}\,;\; \begin{matrix}\varphi_j\in P(A_j)\,,\, j=1,2,\ \varphi_1\circ\iota_1=\varphi_2\circ\iota_2\\ b\in A_1\otimes A_2\,,\ (\varphi_1\otimes\varphi_2)(b^*b)>0\end{matrix}\right\}.$$

Proof. The well known formula for the spatial tensor product norm (see e.g. [24], Corollary 3/4.20 or [5], Lemma 4.7) yields that $\|(\pi_{\iota_1,t}\otimes\pi_{\iota_2,t})(a)\|^2_{\min}$ is, for every $t\in\Omega$, the supremum of

$$\frac{(\psi_1\otimes\psi_2)\big((\pi_{\iota_1,t}\otimes\pi_{\iota_2,t})(b^*a^*ab)\big)}{(\psi_1\otimes\psi_2)\big((\pi_{\iota_1,t}\otimes\pi_{\iota_2,t})(b^*b)\big)} = \frac{\big((\psi_1\circ\pi_{\iota_1,t})\otimes(\psi_2\circ\pi_{\iota_2,t})\big)(b^*a^*ab)}{\big((\psi_1\circ\pi_{\iota_1,t})\otimes(\psi_2\circ\pi_{\iota_2,t})\big)(b^*b)} \tag{3.1}$$

over all $\psi_j\in P\big(A_j/I_{\iota_j}(t)\big)$, $b\in A_1\otimes A_2$ with $(\psi_1\otimes\psi_2)\big((\pi_{\iota_1,t}\otimes\pi_{\iota_2,t})(b^*b)\big)>0$. Thus $\|a\|^2_{C,\min}$ is the supremum of (3.1) over

all $\psi_j\in P\big(A_j/I_{\iota_j}(t)\big)$, $b\in A_1\otimes A_2$ with $(\psi_1\otimes\psi_2)\big((\pi_{\iota_1,t}\otimes\pi_{\iota_2,t})(b^*b)\big)>0$

and all $t\in\Omega$.

But, taking into account Proposition 3.4, it is easy to see that this supremum is equal to that one in the statement. □

We can consider on the quotients $\big(A_1/I_{\iota_1}(t)\big)\otimes\big(A_2/I_{\iota_2}(t)\big)$ also the greatest C^*-norm $\|\cdot\|_{\max}$ (see e.g. [13], 6.3) and define the C^*-seminorm

$$A_1\otimes A_2\ni a\longmapsto \|a\|_{C,\max}=\sup_{t\in\Omega}\|(\pi_{\iota_1,t}\otimes\pi_{\iota_2,t})(a)\|_{\max}\,.$$

Following É. Blanchard, the *maximal C^*-tensor product of A_1 and A_2 over C* is defined as the Hausdorff completion $A_1\otimes_{C,\max}A_2$ of $A_1\otimes A_2$ with respect to the above C^*-seminorm, that is the C^*-algebra obtained by the completion of the quotient $*$-algebra $(A_1\otimes A_2)/\mathcal{J}_C$ relative to the C^*-norm induced by $\|\cdot\|_{C,\max}$.

The subscripts max and min for the seminorms $\|\cdot\|_{C,\max}$ and $\|\cdot\|_{C,\min}$ are explained by the following extremality properties proved by G. A. Elliott (see [6], Sublemma 2.1) and É. Blanchard (see [8], Propositions 2.4 and 2.8):

Proposition 3.6. *Let C be a unital abelian C^*-algebra and let $(A_1\,,\,\iota_1)$, $(A_2\,,\,\iota_2)$ be C^*-algebras over C. If $p(\cdot)$ is a C^*-seminorm on $A_1\otimes A_2$, then*

$$\mathcal{J}_C\subset\{a\in A_1\otimes A_2\,;\,p(a)=0\}\implies p(a)\le\|a\|_{C,\max}\,,\ a\in A_1\otimes A_2\,,$$

$$\mathcal{J}_C=\{a\in A_1\otimes A_2\,;\,p(a)=0\}\implies p(a)\ge\|a\|_{C,\min}\,,\ a\in A_1\otimes A_2\,.$$

We recall that the algebraic tensor product $A_1\otimes_C A_2$ is the quotient $*$-algebra $(A_1\otimes A_2)/\mathcal{I}_C$, where $\mathcal{I}_C$ is the self-adjoint two-sided ideal of $A_1\otimes A_2$ equal to the linear span

$$\mathrm{lin}\Big(\big\{\,\big(\iota_1(c)\,a_1\big)\otimes a_2-a_1\otimes\big(\iota_2(c)\,a_2\big)\,;\,a_1\in A_1\,,\,a_2\in A_2\,,\,c\in C\,\big\}\Big)\,.$$

Since $\mathcal{I}_C$ is clearly contained in

$$\mathcal{J}_C=\{\,a\in A_1\otimes A_2\,;\,\|a\|_{C,\min}=0\,\}=\{\,a\in A_1\otimes A_2\,;\,\|a\|_{C,\max}=0\,\}\,,$$

the seminorms $\|\cdot\|_{C,\min}$ and $\|\cdot\|_{C,\max}$ factorize to C^*-seminorms on $A_1 \otimes_C A_2$, still denoted by $\|\cdot\|_{C,\min}$ and $\|\cdot\|_{C,\max}$. These C^*-seminorms are not always C^*-norms, because in general $\mathcal{I}_C \neq \mathcal{J}_C$ (see [8], Section 3).

Nevertheless, according to [8], Propositions 2.2 and 3.1, we have:

Proposition 3.7. *Let C be a unital abelian C^*-algebra and let (A_1, ι_1), (A_2, ι_2) be C^*-algebras over C. Then any C^*-seminorm on $A_1 \otimes A_2$, which vanishes on $\mathcal{I}_C$, will vanish on whole $\mathcal{J}_C$. Moreover, if (A_1, ι_1) or (A_2, ι_2) is continuous, then even $\mathcal{I}_C = \mathcal{J}_C$ holds.*

We remark that T. Giordano and J. A. Mingo studied the case when A_1, A_2 and C are von Neumann algebras and the mappings $c \mapsto \iota_1(c)$ and $c \mapsto \iota_2(c)$ are normal (see [11], Section 3). They showed that in this case, for given spatial representations $A_1 \subset \mathcal{B}(\mathcal{H})$ and $A_2 \subset \mathcal{B}(\mathcal{K})$, one gets a faithful representation of $A_1 \otimes_C A_2$ on the Hilbert space $\mathcal{H} \otimes_C \mathcal{K}$ constructed by J.-L. Sauvageot [25], such that $\|x\|_{C,\min}$ is the operator norm on $\mathcal{H} \otimes_C \mathcal{K}$ for all $x \in A_1 \otimes_C A_2$. In particular, $\|\cdot\|_{C,\min}$ is a norm on $A_1 \otimes_C A_2$, that is $\mathcal{I}_C = \mathcal{J}_C$. None the less, since in this case (A_1, ι_1) and (A_2, ι_2) are continuous (see [3], Lemma 10), the above equality follows also from Proposition 3.7.

A *proper C^*-algebra over C* is a C^*-algebra (A, ι) over C such that, for some faithful unital $*$-representation $\pi : M(A) \longrightarrow \mathcal{B}(\mathcal{H})$, $(\pi \circ \iota)(C)$ is weak operator closed, i.e. $(\pi \circ \iota)(C) \subset \mathcal{B}(\mathcal{H})$ is a von Neumann algebra. B. Magajna extended the above quoted result of Giordano and Mingo to the case when (A_1, ι_1) and (A_2, ι_2) are proper C^*-algebras over C (see [10], Section 3). We notice that proper C^*-algebras over C are still continuous.

CHAPTER IV

TENSOR PRODUCTS OF $*$-REPRESENTATIONS OVER ABELIAN C^*-ALGEBRAS

In this chapter we prove that if C is a unital abelian C^*-algebra, $(A_1\,,\,\iota_1)$ and $(A_2\,,\,\iota_2)$ are C^*-algebras over C and $\pi_j : A_j \to \mathcal{B}(\mathcal{H})\,,\ j = 1\,,\,2\,,$ are non-degenerate $*$-representations such that

$$M(\pi_1) \circ \iota_1 = M(\pi_2) \circ \iota_2 \ \text{ and } \ \pi_1(A_1) \subset N\,,\, \pi_2(A_2) \subset N'$$

for some type I von Neumann algebra $N \subset \mathcal{B}(\mathcal{H})$ with centre $\big(M(\pi_j) \circ \iota_j\big)(C)''$, then the $*$-homomorphism $\pi : A_1 \otimes A_2 \to \mathcal{B}(\mathcal{H})$ defined by

$$\pi(a_1 \otimes a_2) = \pi_1(a_1)\pi_2(a_2)\,, \qquad a_1 \in A_1\,,\, a_2 \in A_2\,.$$

can be factored through $A_1 \otimes_{C,\min} A_2$ and so gives rise to a $*$-representation $A_1 \otimes_{C,\min} A_2 \to \mathcal{B}(\mathcal{H})$, the C^*-tensor product over C of π_1 and π_2.

Lemma 4.1. *Let $N \subset \mathcal{B}(\mathcal{H})$ be a type I von Neumann algebra of centre Z, $Z \subset M_1 \subset N\,,\ Z \subset M_2 \subset N'$ von Neumann subalgebras, $B_1 \subset M_1\,,\ B_2 \subset M_2$ s-dense $*$-subalgebras, and $e\,,\,f$ abelian projections in $N\,,\,N'$, respectively. Let further $p \in M_1' \cap N$ and $q \in M_2' \cap N'$ be the projections such that*

$$\overline{M_1 e}^{\,s} = pNe\,, \quad e \le p\,, \quad \mathrm{z}_N(e) = \mathrm{z}_N(p)\,,$$

$$\overline{M_2 f}^{\,s} = qN'f\,, \quad f \le q\,, \quad \mathrm{z}_{N'}(f) = \mathrm{z}_{N'}(q)$$

(*such $p\,,\,q$ exist and are unique by* Lemma 2.5). *Then*

(i) *ef is an abelian projection of central support pq in $pq(N \vee N')pq$;*

(ii) $\overline{(M_1 \vee M_2)ef}^{\,s} = pq(N \vee N')ef$;

(iii) *for every $x \in N \vee N'$, we have*

$$\|xpq\| = \sup\, \big\{ \|xy\| \;;\; y \in \operatorname{lin}(B_1B_2)\, ef Z_1^+ \,,\, \|y\| \le 1 \big\} \,.$$

Proof. (i) By Lemma 2.4, ef is an abelian projection in $N \vee N'$. Since $ef \le pq$, it is an abelian projection also in $pq(N \vee N')pq$.

On the other hand, since the centre of the reduced algebra $pq(N \vee N')pq$ is equal to $pqZ(N \vee N') = pqZ$, the central support $\mathrm{z}_{pq(N\vee N')pq}(ef)$ is of the form pqz_o for some projection $z_o \in Z$. Now, taking into account Lemma 2.4, we deduce successively

$$ef \le \mathrm{z}_{pq(N\vee N')pq}(ef) = pqz_o \le z_o \,,$$

$$pq \le \mathrm{z}_N(p)\, \mathrm{z}_{N'}(q) = \mathrm{z}_N(e)\, \mathrm{z}_{N'}(f) = \mathrm{z}_{N\vee N'}(ef) \le z_o \,,$$

$$pq = pqz_o = \mathrm{z}_{pq(N\vee N')pq}(ef) \,.$$

(ii) Since

$$x_1x_2ef = x_1ex_2f = px_1eqx_2f = pqx_1x_2ef \,, \qquad x_1 \in M_1 \,,\, x_2 \in M_2 \,,$$

we have $\overline{(M_1 \vee M_2)ef}^{\,s} \subset pq(N \vee N')ef$.

To prove the reverse inclusion, let $y \in N$, $y' \in N'$ be arbitrary. Then $pye \in \overline{M_1e}^{\,s}$ and $qy'f \in \overline{M_2f}^{\,s}$, so by Lemma 2.6 there exist nets $\{a_\lambda e\}_\lambda \subset M_1e$ and $\{b_\mu f\}_\mu \subset M_2f$ such that

$$a_\lambda e \xrightarrow{\ s\ } pye \ \text{ and } \ \|a_\lambda e\| \le \|pye\| \quad \text{for every } \ \lambda \,,$$

$$b_\mu f \xrightarrow{\ s\ } qy'f \ \text{ and } \ \|b_\mu f\| \le \|qy'f\| \ \text{ for every } \ \mu \,.$$

It follows that $a_\lambda b_\mu ef \xrightarrow[\lambda,\,\mu]{s} pqyy'ef$, hence $pqyy'ef \in \overline{(M_1 \vee M_2)ef}^{\,s}$.

(iii) Let $x \in N \vee N'$ be arbitrary.

According to (i), ef is an abelian projection of central support pq in the type I von Neumann algebra $pq(N \vee N')pq$. Thus Lemma 2.3 and (ii) yield

$$\begin{aligned}\|xpq\|^2 &= \|pqx^*xpq\| \\ &= \sup\left\{\|pqx^*xv\| \,;\, v \in pq(N \vee N')pq \text{ partial isometry}\,,\, v^*v \le ef\right\} \\ &\le \|xpq\| \sup\left\{\|xv\| \,;\, v \in pq(N \vee N')pq \text{ partial isometry}\,,\, v^*v \le ef\right\},\end{aligned}$$

so

$$\begin{aligned}\|xpq\| &= \sup\left\{\|xv\| \,;\, v \in pq(N \vee N')pq \text{ partial isometry}\,,\, v^*v \le ef\right\} \\ &= \sup\left\{\|xv\| \,;\, v \in pq(N \vee N')pq \text{ partial isometry}\right\} \\ &= \sup\left\{\|xy\| \,;\, y \in pq(N \vee N')ef\,,\, \|y\| \le 1\right\} \\ &= \sup\left\{\|xy\| \,;\, y \in \overline{(M_1 \vee M_2)ef}^{\,s}\,,\, \|y\| \le 1\right\}.\end{aligned}$$

Since $\operatorname{lin}(B_1B_2)$ is a $*$-subalgebra of $N \vee N'$ and

$$\overline{\operatorname{lin}(B_1B_2)ef}^{\,s} = \overline{\operatorname{lin}(M_1M_2)ef}^{\,s} = \overline{(M_1 \vee M_2)ef}^{\,s},$$

Lemma 2.6 entails that

$$\{y \in \overline{(M_1 \vee M_2)ef}^{\,s}\,,\, \|y\| \le 1\} = \overline{\{y \in \operatorname{lin}(B_1B_2)\, efZ_1^+\,,\, \|y\| \le 1\}}^{\,s}.$$

Consequently

$$\begin{aligned}\|xpq\| &= \sup\left\{\|xy\| \,;\, y \in \overline{(M_1 \vee M_2)ef}^{\,s}\,,\, \|y\| \le 1\right\} \\ &= \sup\left\{\|xy\| \,;\, y \in \operatorname{lin}(B_1B_2)\, efZ_1^+\,,\, \|y\| \le 1\right\}. \qquad \square\end{aligned}$$

Lemma 4.2. *Let C be a unital abelian C^*-algebra with Gelfand spectrum Ω and let $(A_1\,,\,\iota_1)\,,\,(A_2\,,\,\iota_2)$ be C^*-algebras over C. Let further $\pi_j : A_j \to \mathcal{B}(\mathcal{H})\,,\ j = 1, 2,$ be non-degenerate $*$-representations, such that*

$$M(\pi_1) \circ \iota_1 = M(\pi_2) \circ \iota_2 \ \text{ and } \ \pi_1(A_1) \subset N\,,\, \pi_2(A_2) \subset N'$$

for a type I von Neumann algebra $N \subset \mathcal{B}(\mathcal{H})$ *with centre* $Z = \big(M(\pi_j) \circ \iota_j\big)(C)''$, $\widetilde{\Omega}$ *the Gelfand spectrum of* Z , *and* $\pi : A_1 \otimes A_2 \to \mathcal{B}(\mathcal{H})$ *the* $*$*-homomorphism defined by*

$$\pi(a_1 \otimes a_2) = \pi_1(a_1)\pi_2(a_2)\,, \qquad a_1 \in A_1\,,\, a_2 \in A_2\,.$$

If $p \in \pi_1(A_1)' \cap N$, $q \in \pi_2(A_2)' \cap N'$ *are projections such that*

$$pNe = \overline{\pi_1(A_1)e}^{\,s}\,, \qquad qN'f = \overline{\pi_2(A_2)f}^{\,s}$$

for some abelian projections $e \in N$ *and* $f \in N'$ *satisfying*

$$e \le p\,,\ \mathrm{z}_N(e) = \mathrm{z}_N(p)\,, \qquad f \le q\,,\ \mathrm{z}_{N'}(f) = \mathrm{z}_{N'}(q)\,,$$

then, denoting $z_o = \mathrm{z}_{N \vee N'}(ef) = \mathrm{z}_N(e)\,\mathrm{z}_{N'}(f)$, *we have for all* $a \in A_1 \otimes A_2$:

$$\begin{aligned} &\|\pi(a)pq\| = \\ &= \sup\left\{ \chi(z)(\chi \circ \Phi_{ef} \circ \pi)(b^*a^*ab)^{1/2}\,;\ \begin{array}{l} b \in A_1 \otimes A_2\,,\ z \in Z_1^+\,,\ \chi \in \widetilde{\Omega} \\ \|\pi(b)efz\| \le 1 \end{array} \right\} &(4.1)\\ &= \sup\Big\{ \chi(z)\Big((\chi \circ \Phi_{ez_o} \circ \pi_1) \otimes (\chi \circ \Phi_{fz_o} \circ \pi_2)\Big)(b^*a^*ab)^{1/2}\,; \\ &\qquad\qquad \begin{array}{l} b \in A_1 \otimes A_2\,,\ z \in Z_1^+\,,\ \chi \in \widetilde{\Omega} \\ \|\pi(b)efz\| \le 1 \end{array} \Big\} &(4.2)\\ &\le \sup_{t \in \Omega} \|(\pi_{\iota_1,t} \otimes \pi_{\iota_2,t})(a)\|_{\min}\,. &(4.3) \end{aligned}$$

Proof. We notice that the equality $\mathrm{z}_{N \vee N'}(ef) = \mathrm{z}_N(e)\,\mathrm{z}_{N'}(f)$ in the definition of z_o holds by Lemma 2.4.

Set

$$M_j = \pi_j(A_j)'' = \overline{\pi_j(A_j)}\,s\,, \qquad j = 1\,,\,2\,.$$

Applying Lemma 4.1(iii) with $B_j = \pi_j(A_j)$, $j = 1, 2$, we obtain for every $x \in N \vee N'$:

$$\begin{aligned}\|xpq\| &= \sup\ \left\{\|xy\|\ ;\ y\in \operatorname{lin}\big(\pi_1(A_1)\pi_2(A_2)\big)\, efZ_1^+\,,\ \|y\|\le 1\right\}\\ &= \sup\ \left\{\|xy\|\ ;\ y\in \pi(A_1\otimes A_2)\, efZ_1^+\,,\ \|y\|\le 1\right\}\\ &= \sup\ \left\{\|x\pi(b)efz\|\ ;\ b\in A_1\otimes A_2\,,\ z\in Z_1^+\,,\ \|\pi(b)efz\|\le 1\right\}.\end{aligned}$$

Let $a\in A_1\otimes A_2$ be arbitrary. Using the above equality with $x=\pi(a)$, as well as (2.3), we deduce (4.1) :

$$\begin{aligned}&\|\pi(a)pq\|^2 =\\ &= \sup\ \left\{\|\pi(ab)efz\|^2\ ;\ b\in A_1\otimes A_2\,,\ z\in Z_1^+\,,\ \|\pi(b)efz\|\le 1\right\}\\ &= \sup\ \left\{\|efz^2\pi(b^*a^*ab)ef\|\ ;\ b\in A_1\otimes A_2\,,\ z\in Z_1^+\,,\ \|\pi(b)efz\|\le 1\right\}\\ &= \sup\ \left\{\|\Phi_{ef}\big(z^2\pi(b^*a^*ab)\big)\|\ ;\ b\in A_1\otimes A_2\,,\ z\in Z_1^+\,,\ \|\pi(b)efz\|\le 1\right\}\\ &= \sup\ \left\{\|z^2(\Phi_{ef}\circ\pi)(b^*a^*ab)\|\ ;\ b\in A_1\otimes A_2\,,\ z\in Z_1^+\,,\ \|\pi(b)efz\|\le 1\right\}\\ &= \sup\ \left\{\chi(z)^2(\chi\circ\Phi_{ef}\circ\pi)(b^*a^*ab)\ ;\ \begin{matrix} b\in A_1\otimes A_2\,,\ z\in Z_1^+\,,\ \chi\in\widetilde{\Omega}\\ \|\pi(b)efz\|\le 1\end{matrix}\right\}.\end{aligned}$$

By Lemma 2.4, we have for every $\chi\in\widetilde{\Omega}$ and $a_1\in A_1\,,\ a_2\in A_2$:

$$\begin{aligned}(\chi\circ\Phi_{ef}\circ\pi)(a_1\otimes a_2) &= \chi\Big(\Phi_{efz_o}\big(\pi_1(a_1)\pi_2(a_2)\big)\Big)\\ &= \chi\Big(\Phi_{ez_o}\big(\pi_1(a_1)\big)\Phi_{fz_o}\big(\pi_2(a_2)\big)\Big)\\ &= \big(\chi\circ\Phi_{ez_o}\circ\pi_1\big)(a_1)\big(\chi\circ\Phi_{fz_o}\circ\pi_2\big)(a_2)\\ &= \Big((\chi\circ\Phi_{ez_o}\circ\pi_1)\otimes(\chi\circ\Phi_{fz_o}\circ\pi_2)\Big)(a_1\otimes a_2)\,.\end{aligned}$$

Therefore

$$\chi\circ\Phi_{ef}\circ\pi = (\chi\circ\Phi_{ez_o}\circ\pi_1)\otimes(\chi\circ\Phi_{fz_o}\circ\pi_2)\,,\qquad \chi\in\widetilde{\Omega} \tag{4.4}$$

and (4.2) follows.

According to Corollary 3.3, for the proof of (4.3) we can assume without loss of generality that both A_1 and A_2 are unital. (4.3) will follow once we show that, for every $b \in A_1 \otimes A_2$, $z \in Z_1^+$ and $\chi \in \widetilde{\Omega}$ with $\|\pi(b)efz\| \leq 1$,

$$\chi(z)^2\Big((\chi \circ \Phi_{ez_o} \circ \pi_1) \otimes (\chi \circ \Phi_{fz_o} \circ \pi_2)\Big)(b^* a^* ab) \leq \sup_{t\in\Omega} \|(\pi_{\iota_1,t} \otimes \pi_{\iota_2,t})(a)\|^2_{\min} . \quad (4.5)$$

If $\chi(z_o) = 0$, then $\chi \circ \Phi_{ez_o} \circ \pi_1 = \chi \circ \Phi_{fz_o} \circ \pi_2 = 0$ and (4.5) holds trivially. Therefore we shall assume in the sequel that $\chi(z_o) \neq 0$. Since $\chi(z_o)\chi(z_o) = \chi(z_o^2) = \chi(z_o)$, then $\chi(z_o) = 1$.

Let us denote, for convenience,

$$\varphi_1 = \chi \circ \Phi_{ez_o} \circ \pi_1 , \quad \varphi_2 = \chi \circ \Phi_{fz_o} \circ \pi_2 .$$

φ_1 and φ_2 are positive linear functionals and $\|\varphi_j\| = \varphi_j(1_{A_j}) = \chi(z_o) = 1$, so they are states. Furthermore, since

$$(\varphi_j \circ \iota_j)(c) = \chi\big(z_o(\pi_j \circ \iota_j)(c)\big) = \chi(z_o)\chi\big((\pi_j \circ \iota_j)(c)\big) = (\chi \circ \pi_j \circ \iota_j)(c) , \quad c \in C ,$$

$\varphi_1 \circ \iota_1 = \chi \circ \pi_j \circ \iota_j = \varphi_2 \circ \iota_2$ is a multiplicative state on C, that is a character $t_\chi \in \Omega$.

We claim that φ_1 vanishes on $I_{\iota_1}(t_\chi)$. Indeed, for every $c \in C$, $c(t_\chi) = 0$, and $a_1 \in A_1$,

$$\varphi_1\big(\iota_1(c)\, a_1\big) = \chi\Big((\pi_1 \circ \iota_1)(c)\, \Phi_{ez_o}\big(\pi_1(a_1)\big)\Big) = c(t_\chi)\, \varphi_1(a_1) = 0 .$$

Consequently there exists a state ψ_1 on $A_1/I_{\iota_1}(t_\chi)$ such that $\varphi_1 = \psi_1 \circ \pi_{\iota_1,t_\chi}$. Similarly, φ_2 vanishes on $I_{\iota_2}(t_\chi)$ and so $\varphi_2 = \psi_2 \circ \pi_{\iota_2,t_\chi}$ for some state ψ_2 on $A_2/I_{\iota_2}(t_\chi)$. Then $\varphi_1 \otimes \varphi_2$ factors by the tensor product state $\psi_1 \otimes_{\min} \psi_2$ on $\big(A_1/I_{\iota_1}(t_\chi)\big) \otimes_{\min} \big(A_2/I_{\iota_2}(t_\chi)\big)$:

$$\varphi_1 \otimes \varphi_2 = (\psi_1 \otimes_{\min} \psi_2) \circ (\pi_{\iota_1,t_\chi} \otimes \pi_{\iota_2,t_\chi}) . \quad (4.6)$$

Now, the norm of the positive linear functional

$$\theta = \chi(z)^2\, (\psi_1 \otimes_{\min} \psi_2)\Big((\pi_{\iota_1,t_\chi} \otimes \pi_{\iota_2,t_\chi})(b)^* \cdot (\pi_{\iota_1,t_\chi} \otimes \pi_{\iota_2,t_\chi})(b)\Big)$$

on $\big(A_1/I_{\iota_1}(t_\chi)\big) \otimes_{\min} \big(A_2/I_{\iota_2}(t_\chi)\big)$ is ≤ 1. Indeed, since $\|\theta\|$ is equal to the value of θ in the unit of $\big(A_1/I_{\iota_1}(t_\chi)\big) \otimes_{\min} \big(A_2/I_{\iota_2}(t_\chi)\big)$, by (4.6) and (4.4) we obtain

$$\begin{aligned}
\|\theta\| &= \chi(z)^2\,(\psi_1 \otimes_{\min} \psi_2)\Big((\pi_{\iota_1,t_\chi} \otimes \pi_{\iota_2,t_\chi})(b^*b)\Big)\\
&= \chi(z)^2\,(\varphi_1 \otimes \varphi_2)(b^*b) = \chi(z)^2\,(\chi \circ \Phi_{ef} \circ \pi)(b^*b)\\
&= \chi\Big(\Phi_{ef}\big(z^2\pi(b^*b)\big)\Big) = \chi\Big(\Phi_{ef}\big(zef\pi(b)^*\pi(b)efz\big)\Big)\\
&\le \|\pi(b)efz\|^2 \le 1\,.
\end{aligned}$$

Thus, by (4.6),

$$\begin{aligned}
&\chi(z)^2\Big((\chi \circ \Phi_{ez_o} \circ \pi_1) \otimes (\chi \circ \Phi_{fz_o} \circ \pi_2)\Big)(b^*a^*ab) \;=\\
&= \chi(z)^2(\varphi_1 \otimes \varphi_2)(b^*a^*ab)\\
&= \chi(z)^2\Big((\psi_1 \otimes_{\min} \psi_2) \circ (\pi_{\iota_1,t_\chi} \otimes \pi_{\iota_2,t_\chi})\Big)(b^*a^*ab)\\
&= \theta\big((\pi_{\iota_1,t_\chi} \otimes \pi_{\iota_2,t_\chi})(a^*a)\big)\\
&\le \|(\pi_{\iota_1,t_\chi} \otimes \pi_{\iota_2,t_\chi})(a^*a)\|_{\min} = \|(\pi_{\iota_1,t_\chi} \otimes \pi_{\iota_2,t_\chi})(a)\|^2_{\min}
\end{aligned}$$

and (4.5) follows. □

Lemma 4.3. *Let $N \neq \{0\}$ be a type I von Neumann algebra with centre Z, and $Z \subset M \subset N$ a von Neumann subalgebra. Then there exists a set $\mathcal{P}$ of mutually orthogonal non-zero projections in $M' \cap N$ such that $\sum_{p \in \mathcal{P}} p = 1_N$ and, for every $p \in \mathcal{P}$,*

$$pNe = \overline{Me}^{\,s}$$

for some abelian projection $e \in N$ satisfying $e \le p$, $\mathrm{z}_N(e) = \mathrm{z}_N(p)$.

Proof. Let $\mathcal{P}$ be a maximal set of mutually orthogonal non-zero projections in $M' \cap N$ such that, for every $p \in \mathcal{P}$,

$$pNe_p = \overline{Me_p}^{\,s}$$

for some abelian projection $e_p \in N$ satisfying $e_p \le p$, $\mathrm{z}_N(e_p) = \mathrm{z}_N(p)$. Such family $\mathcal{P}$ exists by Lemma 2.5 and by Zorn Lemma. We will show that $\sum_{p \in \mathcal{P}} p = 1_N$.

Suppose the contrary, that is $1_N - \sum_{p\in\mathcal{P}} p \neq 0$. By Lemma 2.2 there exists an abelian projection $e \in N$ such that

$$e \leq 1_N - \sum_{p\in\mathcal{P}} p\,, \quad \mathrm{z}_N(e) = \mathrm{z}_N\Big(1_N - \sum_{p\in\mathcal{P}} p\Big)\,.$$

In particular, $e \neq 0$. Further, by Lemma 2.5

$$\overline{Me}^{\,s} = p_o Ne \ \text{ for some projection } \ p_o \in M' \cap N \ \text{ with } \ e \leq p_o\,.$$

Let $y \in N$ be arbitrary. Since $p_o ye \in p_o Ne = \overline{Me}^{\,s}$, there is a net $\{x_\lambda\}_\lambda$ in M such that $x_\lambda e \xrightarrow{\ s\ } p_o ye$. Since $\mathcal{P} \subset M' \cap N$, it follows that

$$x_\lambda e = x_\lambda\Big(1_N - \sum_{p\in\mathcal{P}} p\Big)e = \Big(1_N - \sum_{p\in\mathcal{P}} p\Big)x_\lambda e \xrightarrow{\ s\ } \Big(1_N - \sum_{p\in\mathcal{P}} p\Big)p_o ye\,.$$

Consequently $p_o ye = \Big(1_N - \sum_{p\in\mathcal{P}} p\Big)p_o ye$, i.e. $\sum_{p\in\mathcal{P}} p\, p_o ye = 0$.

We conclude that $\sum_{p\in\mathcal{P}} p\, p_o Ne = \{0\}$ and so, since $\mathrm{z}_N(e)$ is the orthogonal projection onto the closed linear span of $Ne\mathcal{H}$, $\sum_{p\in\mathcal{P}} p\, p_o\, \mathrm{z}_N(e) = 0$. Thus

$$M' \cap N \ni p_o\, \mathrm{z}_N(e) = \Big(1_N - \sum_{p\in\mathcal{P}} p\Big)p_o\, \mathrm{z}_N(e) \leq 1_N - \sum_{p\in\mathcal{P}} p\,.$$

Furthermore, $\mathrm{z}_N(e) \geq p_o\, \mathrm{z}_N(e) p_o \geq p_o e p_o = e \neq 0$ implies that $p_o\, \mathrm{z}_N(e) \neq 0$ and $\mathrm{z}_N\big(p_o\, \mathrm{z}_N(e)\big) = \mathrm{z}_N(e)$.

Thus $p_o\, \mathrm{z}_N(e)$ is a non-zero projection in $M' \cap N$ such that $p_o\, \mathrm{z}_N(e)Ne = p_o Ne = \overline{Me}^{\,s}$ with e an abelian projection in N satisfying $e \leq p_o\, \mathrm{z}_N(e)$ and $\mathrm{z}_N(e) = \mathrm{z}_N\big(p_o\, \mathrm{z}_N(e)\big)$. But, since $p_o\, \mathrm{z}_N(e) \leq 1_N - \sum_{p\in\mathcal{P}} p$, this contradicts the maximality of $\mathcal{P}$. □

Theorem 4.4. *Let C be a unital abelian C^*-algebra with Gelfand spectrum Ω and let $(A_1\,,\,\iota_1)\,,\,(A_2\,,\,\iota_2)$ be C^*-algebras over C. Let further $\pi_j : A_j \to \mathcal{B}(\mathcal{H})$, $j = 1, 2$, be non-degenerate $*$-representations, such that*

$$M(\pi_1) \circ \iota_1 = M(\pi_2) \circ \iota_2 \ \text{ and } \ \pi_1(A_1) \subset N\,,\ \pi_2(A_2) \subset N'$$

for a type I von Neumann algebra $N \subset \mathcal{B}(\mathcal{H})$ with centre $Z = \big(M(\pi_j) \circ \iota_j\big)(C)''$, and $\pi : A_1 \otimes A_2 \to \mathcal{B}(\mathcal{H})$ the $*$-homomorphism defined by

$$\pi(a_1 \otimes a_2) = \pi_1(a_1)\pi_2(a_2)\,, \qquad a_1 \in A_1\,,\, a_2 \in A_2\,.$$

Then

$$\|\pi(a)\| \leq \sup_{t\in\Omega} \|(\pi_{\iota_1,t} \otimes \pi_{\iota_2,t})(a)\|_{\min} = \|a\|_{C,\min}\,, \qquad a \in A_1 \otimes A_2 \tag{4.7}$$

and thus there is a unique $*$*-representation* $\widetilde{\pi} : A_1 \otimes_{C,\min} A_2 \to \mathcal{B}(\mathcal{H})$ *such that*

$$\pi(a) = \widetilde{\pi}\big(a/\mathcal{J}_C\big)\,, \qquad a \in A_1 \otimes A_2\,,$$

where $a/\mathcal{J}_C$ *denotes the natural image of* $a \in A_1 \otimes A_2$ *in the quotient* $*$*-algebra* $(A_1 \otimes A_2)/\mathcal{J}_C \subset A_1 \otimes_{C,\min} A_2$.

Proof. If $\mathcal{H} = \{0\}$, then (4.7) holds trivially. It remains to prove it in the case $\mathcal{H} \neq \{0\}$.

By Lemma 4.3 there exists a set $\mathcal{P} \subset \pi_1(A_1)' \cap N$ of mutually orthogonal non-zero projections such that $\sum\limits_{p\in\mathcal{P}} p = 1_{\mathcal{H}}$ and, for every $p \in \mathcal{P}$,

$$pNe_p = \overline{\pi_1(A_1)'' e_p}^{\,s}$$

for some abelian projection $e_p \in N$ satisfying $e_p \leq p$, $\mathrm{z}_N(e_p) = \mathrm{z}_N(p)$.

Similarly, there exists a set $\mathcal{Q} \subset \pi_2(A_2)' \cap N'$ of mutually orthogonal non-zero projections such that $\sum\limits_{q\in\mathcal{Q}} q = 1_{\mathcal{H}}$ and, for every $q \in \mathcal{Q}$,

$$qN'f_q = \overline{\pi_2(A_2)'' f_q}^{\,s}$$

for some abelian projection $f_q \in N'$ satisfying $f_q \leq q$, $\mathrm{z}_{N'}(f_q) = \mathrm{z}_{N'}(q)$.

Let $a \in A_1 \otimes A_2$ be arbitrary. By Lemma 4.2 we have

$$\|\pi(a)pq\| \leq \|a\|_{C,\min} \text{ for every } p \in \mathcal{P}\,,\, q \in \mathcal{Q}\,.$$

Since $\sum\limits_{p\in\mathcal{P}} p = \sum\limits_{q\in\mathcal{Q}} q = 1_{\mathcal{H}}$ and $\mathcal{P} \cup \mathcal{Q} \subset \pi_1(A_1)' \cap \pi_2(A_2)' \subset \pi(A_1 \otimes A_2)'$, we have $\pi(a^*a) = \sum\limits_{p,q} \pi(a^*a)pq$, where the operators $\pi(a^*a)pq$ are positive and mutually orthogonal. Consequently:

$$\|\pi(a)\|^2 = \|\pi(a^*a)\| = \sup_{p,q} \|\pi(a^*a)pq\| = \sup_{p,q} \|\pi(a)pq\|^2 \le \|a\|^2_{C,\min} .$$

□

We will denote $\widetilde{\pi}$ in Theorem 4.4 by $\pi_1 \otimes_{C,\min} \pi_2$ and call it the *tensor product of* π_1 *and* π_2 *over* C. We notice that the $*$-representation $\pi_1 \otimes_{C,\min} \pi_2$ maps $A_1 \otimes_{C,\min} A_2$ onto the C^*-subalgebra $\overline{\mathrm{lin}\big(\pi_1(A_1)\pi_2(A_2)\big)} \subset \mathcal{B}(\mathcal{H})$ and it is non-degenerate. Indeed, if $\{u_\lambda\}_\lambda$ is an increasing approximate unit for A_1 and $\{v_\mu\}_\mu$ is an increasing approximate unit for A_2, then we have

$$\pi_1(u_\lambda) \xrightarrow{so} 1_{\mathcal{H}} \text{ and } \pi_2(v_\mu) \xrightarrow{so} 1_{\mathcal{H}}$$

(see e.g. [24], Lemma 3/4.1), so

$$(\pi_1 \otimes_{C,\min} \pi_2)\big((u_\lambda \otimes v_\mu)/\mathcal{J}_C\big) = \pi_1(u_\lambda)\pi_2(v_\mu) \xrightarrow{so} 1_{\mathcal{H}} .$$

Therefore $M\Big(\overline{\mathrm{lin}\big(\pi_1(A_1)\pi_2(A_2)\big)}\Big)$ can be identified with

$$\Big\{T \in \mathcal{B}(\mathcal{H})\,;\, \pi_1(A_1)\pi_2(A_2)\,T \cup T\pi_1(A_1)\pi_2(A_2) \subset \overline{\mathrm{lin}\big(\pi_1(A_1)\pi_2(A_2)\big)}\Big\}.$$

With this identification,

$$\begin{aligned} &\pi_1(A_1) \cup \pi_2(A_2) \subset M\Big(\overline{\mathrm{lin}\big(\pi_1(A_1)\pi_2(A_2)\big)}\Big) \text{ and} \\ &\pi_1(a_1)\pi_2(v_\mu) \xrightarrow{\text{strictly}} \pi_1(a_1)\,, \qquad a_1 \in A_1\,, \\ &\pi_1(u_\lambda)\pi_2(a_2) \xrightarrow{\text{strictly}} \pi_2(a_2)\,, \qquad a_2 \in A_2\,. \end{aligned} \tag{4.8}$$

Indeed, we have for every $b_1 \in A_1$, $b_2 \in A_2$:

$$\begin{aligned} &\|\big(\pi_1(a_1) - \pi_1(a_1)\pi_2(v_\mu)\big)\,\pi_1(b_1)\pi_2(b_2)\| = \|\pi_1(a_1b_1)\,\pi_2(b_2 - v_\mu b_2)\| \longrightarrow 0\,, \\ &\|\pi_1(b_1)\pi_2(b_2)\,\big(\pi_1(a_1) - \pi_1(a_1)\pi_2(v_\mu)\big)\| = \|\pi_1(b_1a_1)\,\pi_2(b_2 - b_2v_\mu)\| \longrightarrow 0\,, \\ &\|\big(\pi_2(a_2) - \pi_1(u_\lambda)\pi_2(a_2)\big)\,\pi_1(b_1)\pi_2(b_2)\| = \|\pi_1(b_1 - u_\lambda b_1)\,\pi_2(a_2b_2)\| \longrightarrow 0\,, \\ &\|\pi_1(b_1)\pi_2(b_2)\,\big(\pi_2(a_2) - \pi_1(u_\lambda)\pi_2(a_2)\big)\| = \|\pi_2(b_2a_2)\,\pi_1(b_1 - b_1u_\lambda)\| \longrightarrow 0\,. \end{aligned}$$

We notice that it can happen that, for given non-zero C^*-algebras $(A_1\,,\,\iota_1)$, $(A_2\,,\,\iota_2)$ over C, only the $*$-representations $\pi_1 : A_1 \to \{0\}$ and $\pi_2 : A_2 \to \{0\}$ satisfy the assumptions in Theorem 4.4. Let, for example, $(A_1\,,\,\iota_2)$, $(A_2\,,\,\iota_2)$ be the

C^*-algebras over $C([0,1])$ defined in [8] before Proposition 3.3, for which $A_1 \otimes_{C([0,1]),\min} A_2 = \{0\}$. Then, if $\pi_j : A_j \to \mathcal{B}(\mathcal{H})$, $j = 1, 2$, are any non-degenerate $*$- representations satisfying the conditions in Theorem 4.4, then the $*$-representation $\pi_1 \otimes_{C,\min} \pi_2$ can be non-degenerate only if $\mathcal{H} = \{0\}$. Nevertheless, this pathology is possible only in the case of non-unital A_1 and A_2 (cf. Corollary 8.8).

Criteria for the faithfulness of $\pi_1 \otimes_{C,\min} \pi_2$ will be proved in Chapter 8.

CHAPTER V

ON TENSOR PRODUCTS OF HILBERT MODULES

We shall denote the support of a self-adjoint element a of a von Neumann algebra M by $\mathrm{s}_M(a)$.

Let $N \subset \mathcal{B}(\mathcal{H})$ be a type I von Neumann algebra with center Z and $e \in N$, $f \in N'$ be abelian projections of central support $1_{\mathcal{H}}$. Then, by Lemma 2.4, the Z-valued inner product

$$(N \vee N')ef \times (N \vee N')ef \ni (w_1, w_2) \longmapsto \Phi_{ef}(w_2^* w_1)$$

defines a Hilbert Z-module structure on $(N \vee N')ef = Z'ef$, the mappings

$$Ne \ni x \longmapsto xf \in (N \vee N')ef\,, \quad N'f \ni y \longmapsto ye \in (N \vee N')ef$$

are isometric Hilbert module imbeddings of

$$Ne \text{ endowed with } Ne \times Ne \ni (x_1, x_2) \mapsto \Phi_e(x_2^* x_1) \text{ and}$$
$$N'f \text{ endowed with } N'f \times N'f \ni (y_1, y_2) \mapsto \Phi_f(y_2^* y_1)$$

in the above Hilbert module, which can be considered, up to the above imbeddings, the W^*-tensor product over Z of the Hilbert modules Ne and $N'f$. The aim of this chapter is to develop the elements of $(N \vee N')ef$ in series of products of elements of Ne and $N'f$ (elementary tensors).

Let us first consider the case when N is a factor, that is $\mathcal{H} = \mathcal{H}_1 \overline{\otimes} \mathcal{H}_2$, $N = \mathcal{B}(\mathcal{H}_1) \overline{\otimes} 1_{\mathcal{H}_2}$, $N' = 1_{\mathcal{H}_1} \overline{\otimes} \mathcal{B}(\mathcal{H}_2)$ and $N \vee N' = \mathcal{B}(\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2)$:

Proposition 5.1. *For $\mathcal{H}_1, \mathcal{H}_2$ Hilbert spaces and $0 \neq \zeta \in \mathcal{H}_1 \overline{\otimes} \mathcal{H}_2$, there are*

- $\nu \in \{2, 3, \dots\} \cup \{\infty\}$,

- *orthonormal sequences* $\{\xi_k\}_{1\le k<\nu} \subset \mathcal{H}_1$ *and* $\{\eta_k\}_{1\le k<\nu} \subset \mathcal{H}_2$,

- *real numbers* $\{\lambda_k\}_{1\le k<\nu}$ *with* $\lambda_1 \ge \lambda_2 \ge \dots > 0$ *and* $\sum\limits_{1\le k<\nu} \lambda_k = \|\zeta\|^2$

such that

$$\zeta = \sum_{1\le k<\nu} \sqrt{\lambda_k}\, \xi_k \otimes \eta_k \, .$$

Consequently,

$$\big((x \otimes 1_{\mathcal{H}_2})\zeta|\zeta\big) = \sum_{1\le k<\nu} \lambda_k (x\xi_k|\xi_k)\, , \qquad x \in \mathcal{B}(\mathcal{H}_1)\, .$$

Proof. For every $\theta \in \mathcal{H}_1 \overline{\otimes} \mathcal{H}_2$, let $T_\theta : \mathcal{H}_1 \to \mathcal{H}_2$ be the anti-linear map defined by

$$\big(T_\theta(\xi) \,\big|\, \eta\big) = \big(\theta \,\big|\, \xi \otimes \eta\big)\, , \quad \xi \in \mathcal{H}_1\, , \, \eta \in \mathcal{H}_2\, .$$

Then T_θ is compact.

Indeed, if $\xi \in \mathcal{H}_1$ and $\eta \in \mathcal{H}_2$, then $T_{\xi\otimes\eta} = (\xi|\,\cdot\,)\eta$, hence $T_{\xi\otimes\eta}$ has rank ≤ 1 . Consequently, T_θ is of finite-dimensional rank for any θ in the algebraic tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$. Now, any $\theta \in \mathcal{H}_1 \overline{\otimes} \mathcal{H}_2$ is the limit of a sequence $\{\theta_n\}_{n\ge 1} \subset \mathcal{H}_1 \otimes \mathcal{H}_2$ and, since $\|T_\theta - T_{\theta_n}\| = \|T_{\theta-\theta_n}\| \le \|\theta - \theta_n\| \longrightarrow 0$, T_θ is the operator norm limit of anti-linear operators of finite-dimensional rank.

It follows that $T_\zeta^* T_\zeta : \mathcal{H}_1 \to \mathcal{H}_1$ is a non-zero positive compact linear operator. So there exist

- $\nu \in \{2, 3, \dots\} \cup \{\infty\}$,

- orthonormal sequences $\{\xi_k\}_{1\le k<\nu} \subset \mathcal{H}_1$

- real numbers $\{\lambda_k\}_{1\le k<\nu}$ with $\lambda_1 \ge \lambda_2 \ge \dots > 0$, $\lambda_k \longrightarrow 0$ if $\nu = \infty$,

such that

$$T_\zeta^* T_\zeta = \sum_{1\le k<\nu} \lambda_k p_{\xi_k}\,,$$

where p_{ξ_k} is the orthogonal projection from $\mathcal{H}_1$ onto the subspace $\mathbb{C}\,\xi_k$ and the sum is convergent in the operator norm (see e.g. [21], 2.8.26, 2.8.29).

For any $1 \le k, k' < \nu$, we have

$$\big(T_\zeta(\xi_k)|T_\zeta(\xi_{k'})\big) = \big(\xi_{k'}|T_\zeta^* T_\zeta(\xi_k)\big) = \lambda_k(\xi_{k'}|\xi_k)\,.$$

In other words, the vectors $T_\zeta(\xi_k)$, $1 \le k < \nu$, are mutually orthogonal and $\|T_\zeta(\xi_k)\|^2 = \lambda_k > 0$. Setting $\eta_k = \dfrac{1}{\|T_\zeta(\xi_k)\|}\,T_\zeta(\xi_k) = \dfrac{1}{\sqrt{\lambda_k}}\,T_\zeta(\xi_k)$, $\{\eta_k\}_{1\le k<\nu}$ is an orthonormal sequence in $\mathcal{H}_2$.

For every $1 \le k < \nu$, $V_{\xi_k} : \mathcal{H}_2 \ni \eta \longmapsto \xi_k \otimes \eta \in \mathcal{H}_1 \overline{\otimes} \mathcal{H}_2$ is a linear isometry such that $V_{\xi_k} V_{\xi_k}^*$ is the orthogonal projection P_{ξ_k} from $\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2$ onto $(\mathbb{C}\,\xi_k) \overline{\otimes} \mathcal{H}_2$. Since

$$\big(V_{\xi_k}^*(\zeta)\big|\eta\big) = (\zeta|\xi_k \otimes \eta) = \big(T_\zeta(\xi_k)\big|\eta\big)\,, \quad \eta \in \mathcal{H}_2\,,$$

we have $V_{\xi_k}^*(\zeta) = T_\zeta(\xi_k)$. In particular,

$$\lambda_k = \|T_\zeta(\xi_k)\|^2 = \|V_{\xi_k}^*(\zeta)\|^2 = \|V_{\xi_k} V_{\xi_k}^*(\zeta)\|^2 = \|P_{\xi_k}(\zeta)\|^2\,.$$

Now, by the Bessel Inequality, we get $\sum\limits_{1\le k<\nu} \lambda_k = \sum\limits_{1\le k<\nu} \|P_{\xi_k}(\zeta)\|^2 \le \|\zeta\|^2$, so we can consider the vector $\zeta_o = \sum\limits_{1\le k<\nu} \sqrt{\lambda_k}\,\xi_k \otimes \eta_k \in \mathcal{H}_1 \overline{\otimes} \mathcal{H}_2$. Then

$$T_{\zeta_o}(\xi_k) = \sum_{1\le k'<\nu} \sqrt{\lambda_{k'}}\,T_{\xi_{k'}\otimes\eta_{k'}}(\xi_k) = \sqrt{\lambda_k}\,\eta_k = T_\zeta(\xi_k)\,, \qquad 1 \le k < \nu$$

and

$$\|T_\zeta(\xi)\|^2 = \big(\xi\big|T_\zeta^* T_\zeta \xi\big) = \Big(\xi\Big|\sum_{1\le k<\nu} \lambda_k(\xi|\xi_k)\xi_k\Big) = 0\,,$$
$$T_{\zeta_o}(\xi) = \sum_{1\le k<\nu} \sqrt{\lambda_k}\,T_{\xi_k\otimes\eta_k}(\xi) = \sum_{1\le k<\nu} \sqrt{\lambda_k}\,(\xi_k|\xi)\eta_k = 0$$

for all $\xi \in \{\xi_k\,;\,1 \le k < \nu\}^\perp$, so $T_\zeta = T_{\zeta_o}$. Consequently,

$$(\zeta|\xi \otimes \eta) = \big(T_\zeta(\xi)\big|\eta\big) = \big(T_{\zeta_o}(\xi)\big|\eta\big) = (\zeta_o|\xi \otimes \eta)\,, \qquad \xi \in \mathcal{H}_1\,,\ \eta \in \mathcal{H}_2$$

and we conclude that $\zeta = \zeta_o$, in particular

$$\|\zeta\|^2 = \sum_{1\le k<\nu} \big\|\sqrt{\lambda_k}\,\xi_k \otimes \eta_k\big\|^2 = \sum_{1\le k<\nu} \lambda_k\,.$$

Finally, for any $x \in \mathcal{B}(\mathcal{H}_1)$,

$$\big((x \otimes 1_{\mathcal{H}_2})\zeta\big|\zeta\big) = \sum_{1\le k<\nu} \sqrt{\lambda_k}\,(x\xi_k \otimes \eta_k|\zeta) = \sum_{1\le k<\nu} \lambda_k(x\xi_k|\xi_k)\,. \qquad \square$$

For the treatment of the general case we follow the same idea as in the proof of the above proposition, but replace the used Hilbert space theory with the Hilbert module methods developed in [26], [27], [14]. In the next 3 lemmas $N \subset \mathcal{B}(\mathcal{H})$ will be a type I von Neumann algebra with center Z, while $e \in N$, $f \in N'$ will stand for fixed abelian projections of central support $1_{\mathcal{H}}$.

Lemma 5.2. *For any $w \in Z'ef$, there exist*

(i) *a unique Z-module antimorphism $T_w : Ne \longrightarrow N'f$ such that*

$$\Phi_{ef}(x^*y^*w) = \Phi_f\big(y^*T_w(x)\big)\,, \qquad x \in Ne\,,\ y \in N'f\,, \tag{5.1}$$

(ii) *a unique Z-module antimorphism $T_w^* : N'f \longrightarrow Ne$ such that*

$$\Phi_f\big(y^*T_w(x)\big) = \Phi_e\big(x^*T_w^*(y)\big)\,, \qquad x \in Ne\,,\ y \in N'f\,, \tag{5.2}$$

(iii) *a unique $a(w) \in N$ such that*

$$\big(T_w^*T_w\big)(x) = a(w)\,x\,, \qquad x \in Ne\,. \tag{5.3}$$

Moreover, $a(w) \ge 0$ and $\|a(w)\| = \|T_w^*T_w\| = \|T_w\|^2 = \|T_w^*\|^2 \le \|w\|^2$.

Proof. For every $x \in Ne$, $N'f \ni y \longmapsto \Phi_{ef}(x^*y^*w) \in Z$ is a Z-module antimorphism bounded by $\|w\|\,\|x\|$ and thus, by [26], Theorem 5 or by [14], Lemma 1.11.(b), there exists a unique $T_w(x) \in N'f$ such that the equality in (5.1) holds for all $y \in N'f$. Moreover, $\|T_w(x)\|$ is equal to the norm of the above Z-module antimorphism, hence $\|T_w(x)\| \le \|w\|\,\|x\|$.

Therefore, $T_w : Ne \ni x \longmapsto T_w(x) \in N'f$ is the mapping required in (i) and $\|T_w\| \le \|w\|$. Now, by [26], Theorem 6 or by [14], Lemma 1.12, 1.13.(3), there exists a unique Z-module antimorphism $T_w^* : N'f \to Ne$ satisfying (5.2) (the Hilbert module adjoint of T_w) and it has the same norm as T_w.

Finally, $T_w^* T_w : Ne \longrightarrow Ne$ is a Z-module morphism of norm $\|T_w\|^2$, so by Lemma 2.1 there exists a unique $a(w) \in N$ such that (5.3) holds and it clearly satisfies $a(w) \geq 0$, $\|a(w)\| = \|T_w^* T_w\| = \|T_w\|^2$. □

Lemma 5.3. *For any* $w \in Z'ef$ *and mutually orthogonal abelian projections* $e_1, \ldots, e_n \in N$, *if* $a(w)$ *is the element of* N *defined in* Lemma 5.2, *then*

$$\sum_{1 \leq k \leq n} \Phi_{e_k}\big(a(w)\big) \leq \Phi_{ef}(w^* w). \tag{5.4}$$

Proof. Let $1 \leq k \leq n$ be arbitrary.

Since $\mathrm{z}_N(e) = 1_{\mathcal{H}} \geq \mathrm{z}_N(e_k)$, we have $e_k \prec e$, that is $u_k u_k^* = e_k$, $u_k^* u_k \leq e$ for some partial isometry $u_k \in N$. Then $u_k \in Ne$ so $u_k N'f$ is a Z-submodule of $Z'ef$. We prove that this submodule is s-closed.

Indeed, if $z' \in Z'ef$ belongs to the s-closure of $u_k N'f$, then by [14], Lemma 1.2 there is a net $\{y_\lambda\}_\lambda \subset N'f$ with $u_k y_\lambda \overset{s}{\longrightarrow} z'$ and $\|u_k y_\lambda\| \leq \|z'\|$ for all λ. Since $\|u_k y_\lambda\| = \|u_k u_k^* u_k y_\lambda\| = \|u_k u_k^* y_\lambda u_k\| \leq \|e_k y_\lambda\| = \|u_k y_\lambda u_k^*\| \leq \|u_k y_\lambda\|$, we have $\|e_k y_\lambda\| = \|u_k y_\lambda\| \leq \|z'\|$ for all λ. Using the fact that the induction map $N' \mathrm{z}_N(e_k) \longrightarrow \big(N' \mathrm{z}_N(e_k)\big) e_k = N' e_k$ is a $*$-isomorphism, it follows that $\|\mathrm{z}_N(e_k) y_\lambda\| = \|e_k y_\lambda\| \leq \|z'\|$ for all λ and therefore a subnet of $\{\mathrm{z}_N(e_k) y_\lambda\}_\lambda$ is w-convergent to some $y \in N'f$. But then a subnet of $\{u_k \mathrm{z}_N(e_k) y_\lambda\}_\lambda$ is w-convergent to $u_k y$ and, since $u_k \mathrm{z}_N(e_k) y_\lambda = \mathrm{z}_N(e_k) u_k u_k^* u_k y_\lambda = e_k u_k y_\lambda = u_k y_\lambda$, it follows that $z' = u_k y \in u_k N'f$.

Applying now [26], Theorem 3 or [14], 1.5, 1.6, there is a unique Z-module morphism $P_{u_k N'f} : Z'ef \longrightarrow u_k N'f \subset Z'ef$ acting identically on $u_k N'f$ and vanishing in any $z' \in Z'ef$ with $(z')^* u_k N'f = \{0\}$.

Let us consider the bounded Z-module map $U_k : N'f \ni y \longmapsto u_k y \in Z'ef$. Using

Lemma 2.4 we deduce for every $y_1, y_2 \in N'f$:

$$\begin{aligned}\Phi_{ef}\big(U_k(y_1)^* U_k(y_2)\big) &= \Phi_{ef}\big(u_k^* u_k y_1^* y_2\big) = \Phi_e\big(\,\mathrm{z}_N(u_k^* u_k)e\big)\,\Phi_f(y_1^* y_2) \\ &= \mathrm{z}_N(e_k)\,\Phi_f(y_1^* y_2)\,.\end{aligned} \tag{5.5}$$

Let $U_k^* : Z'ef \longrightarrow N'f$ be the Z-module map defined by

$$\Phi_{ef}\big((z')^* U_k(y)\big) = \Phi_f\big(U_k^*(z')^* y\big)\,, \qquad z' \in Z'ef\,,\, y \in N'f\,, \tag{5.6}$$

whose existence is guaranted by [26], Theorem 6 or by [14], Lemma 1.12, 1.13.(2). Then $U_k\, U_k^* = P_{u_k N'f}$. Indeed, since

$$\begin{aligned}\Phi_f\big(U_k^*(u_k y_1)^* y_2\big) &\overset{(5.6)}{=} \Phi_{ef}\big((u_k y_1)^* U_k(y_2)\big) \\ &\overset{(5.5)}{=} \mathrm{z}_N(e_k)\,\Phi_f(y_1^* y_2) = \Phi_f\big((\mathrm{z}_N(e_k)\,y_1)^* y_2\big)\,, \quad y_1, y_2 \in N'f\,,\end{aligned}$$

we get for every $y \in N'f$ first $U_k^*(u_k y) = \mathrm{z}_N(e_k)\,y$, and then

$$(U_k\,U_k^*)(u_k y) = u_k\,\mathrm{z}_N(e_k)\,y = \mathrm{z}_N(e_k)\,u_k u_k^* u_k\,y = u_k\,y\,.$$

On the other hand, if $z' \in Z'ef$ satisfies $(z')^* u_k N'f = \{0\}$, then (5.6) implies that $\Phi_f\big(U_k^*(z')^* y\big) = 0$ for all $y \in N'f$ and so $U_k^*(z') = 0$, $(U_k\,U_k^*)(z') = 0$.

We notice also that, since

$$\Phi_f\big(y^* U_k^*(w)\big) \overset{(5.6)}{=} \Phi_{ef}\big(U_k(y)^* w\big) = \Phi_{ef}(u_k^* y^* w) \overset{(5.1)}{=} \Phi_f\big(y^* T_w(u_k)\big)\,, \;\; y \in N'f\,,$$

we have

$$T_w(u_k) = U_k^*(w)\,. \tag{5.7}$$

Now we recall that $\Phi_{e_k}\big(a(w)\big) \in Z\,\mathrm{z}_N(e_k)$ and $\Phi_{e_k}\big(a(w)\big)\,e_k = e_k a(w) e_k$, so we have

$$\Phi_{e_k}\big(a(w)\big)\,u_k^* u_k = u_k^* \Phi_{e_k}\big(a(w)\big) e_k u_k = u_k^* e_k a(w) e_k u_k = u_k^* a(w) u_k\,,$$

$$\Phi_{e_k}\big(a(w)\big) u_k^* u_k = \Phi_{e_k}\big(a(w)\big)\,\mathrm{z}_N(u_k^* u_k) e = \Phi_{e_k}\big(a(w)\big)\,\mathrm{z}_N(e_k) e = \Phi_{e_k}\big(a(w)\big) e\,.$$

Consequently, $\Phi_{e_k}\big(a(w)\big)e = u_k^* a(w) u_k$ and we obtain

$$\begin{aligned}\Phi_{e_k}\big(a(w)\big) &= \Phi_e\Big(\Phi_{e_k}\big(a(w)\big)e\Big) = \Phi_e\big(u_k^* a(w) u_k\big)\\ &\overset{(5.3)}{=} \Phi_e\big(u_k^*\,(T_w^* T_w)(u_k)\big) \overset{(5.2)}{=} \Phi_f\big(T_w(u_k)^* T_w(u_k)\big)\\ &\overset{(5.7)}{=} \Phi_f\big(U_k^*(w)^* U_k^*(w)\big) \overset{(5.6)}{=} \Phi_{ef}\big(w^*\,(U_k U_k^*)(w)\big)\\ &= \Phi_{ef}\big(w^* P_{u_k N'f}(w)\big)\end{aligned}$$

Now it is easy to see that $\mathcal{X} = \sum\limits_{1\le k\le n} u_k N'f$ is an s-closed submodule of $Z'ef$ and the Z-orthogonal projection $P_{\mathcal{X}} : Z'ef \longrightarrow \mathcal{X} \subset Z'ef$ (see [26], Theorem 3 or [14], 1.5, 1.6) is equal to $\sum\limits_{1\le k\le n} P_{u_k N'f}$. Therefore

$$\begin{aligned}\sum_{1\le k\le n} \Phi_{e_k}\big(a(w)\big) &= \sum_{1\le k\le n} \Phi_{ef}\big(w^* P_{u_k N'f}(w)\big) = \Phi_{ef}\big(w^* P_{\mathcal{X}}(w)\big) = \Phi_{ef}\big(|P_{\mathcal{X}}(w)|^2\big)\\ &\le \Phi_{ef}\big(|P_{\mathcal{X}}(w)|^2 + |w - P_{\mathcal{X}}(w)|^2\big) = \Phi_{ef}(w^* w)\,. \qquad \square\end{aligned}$$

The next lemma is an immediate consequence of Lemma 5.3 and [27]:

Lemma 5.4. *For any $0 \ne w \in Z'ef$, if $a(w)$ is defined as in* Lemma 5.2*, then there exist*

- $\nu \in \{2, 3, \ldots\} \cup \{\infty\}$,
- *a sequence $\{e_k\}_{1\le k<\nu}$ of mutually orthogonal abelian projections in N,*
- $\{z_k\}_{1\le k<\nu} \subset Z^+$, $z_1 \ge z_2 \ge \ldots$, $\|z_k\| \longrightarrow 0$ *if* $\nu = \infty$, $s\text{-}\sum\limits_{1\le k<\nu} z_k \le \Phi_{ef}(w^* w)$

such that

$$\mathrm{z}_N(e_k) = \mathrm{s}_Z(z_k) \ne 0 \text{ for all } 1 \le k < \nu\,,$$

$$a(w) = \sum_{1\le k<\nu} z_k\, e_k\,, \tag{5.8}$$

where the series converges in the operator norm.

Proof. By Lemma 5.3 and by the second half of the proof of [27], Proposition 4.2, $a(w)$ belongs to the norm-closed two-sided ideal of N, generated by the abelian projections. Therefore the spectral theorem [27], Theorems 2.2 and 2.3 (cf. [19], Theorem 6.14) can be applied to $a(w)$ and it follows the existence of ν, $\{e_k\}_{1\leq k<\nu}$ and $\{z_k\}_{1\leq k<\nu}$ satisfying all the required conditions except $s-\sum\limits_{1\leq k<\nu} z_k \leq \Phi_{ef}(w^*w)$. But, since $\Phi_{e_k}\big(a(w)\big) = z_k$, Lemma 5.3 yields that

$$\sum_{1\leq k\leq n} z_k = \sum_{1\leq k\leq n} \Phi_{e_k}\big(a(w)\big) \leq \Phi_{ef}(w^*w) \text{ for all } 1\leq n<\nu\,. \qquad \square$$

We notice that, according to [27], Theorem 2.3, ν and $\{z_k\}_{1\leq k<\nu}$ are uniquely determined by w. Also, (5.8) implies that

$$\mathrm{s}_N\left(a(w)\right) = s-\sum_{1\leq k<\nu} e_k\,. \tag{5.9}$$

Now we prove the module version of Proposition 5.1:

Theorem 5.5. *Let $N\subset\mathcal{B}(\mathcal{H})$ be a type I von Neumann algebra with centre Z, $e\in N$, $f\in N'$ abelian projections of central support $1_{\mathcal{H}}$, and $0\neq w\in Z'ef$. Then there are*

- $\nu\in\{2,3,\ldots\}\cup\{\infty\}$,
- *partial isometries* $\{u_k\}_{1\leq k<\nu}\subset N$ *with* $u_k^*u_k\leq e$, $1\leq k<\nu$, *and mutually orthogonal* $u_ku_k^*$, $1\leq k<\nu$,
- *partial isometries* $\{v_k\}_{1\leq k<\nu}\subset N'$ *with* $v_k^*v_k\leq f$, $1\leq k<\nu$, *and mutually orthogonal* $v_kv_k^*$, $1\leq k<\nu$,
- $\{z_k\}_{1\leq k<\nu}\subset Z^+$, $z_1\geq z_2\geq\ldots$, $\|z_k\|\longrightarrow 0$ *if* $\nu=\infty$ *and* $s-\sum\limits_{1\leq k<\nu} z_k = \Phi_{ef}(w^*w)$

such that

$$\mathrm{z}_N(u_k u_k^*) = \mathrm{z}_{N'}(v_k v_k^*) = \mathrm{s}_Z(z_k) \neq 0 \text{ \textit{for all} } 1 \leq k < \nu\,,$$

$$w = s\text{-}\sum_{1\leq k<\nu} z_k^{1/2}\, u_k v_k\,, \tag{5.10}$$

Consequently,

$$w^* x\, w = s\text{-}\sum_{1\leq k<\nu} z_k\, u_k^* x\, u_k\, f\,, \qquad x \in N\,. \tag{5.11}$$

Proof. Let T_w and $a(w)$ be as defined in Lemma 5.2, and ν, $\{e_k\}_{1\leq k<\nu}$ and $\{z_k\}_{1\leq k<\nu}$ as in Lemma 5.4. Choose for every $1 \leq k < \nu$ a partial isometry $u_k \in N$ such that $u_k^* u_k \leq e$ and $u_k u_k^* = e_k$. We notice that

$$u_k^* u_k = \mathrm{z}_N(u_k^* u_k)\, e = \mathrm{z}_N(e_k)\, e\,, \text{ so } \Phi_e(u_k^* u_k) = \mathrm{z}_N(e_k) = \mathrm{s}_Z(z_k)\,. \tag{5.12}$$

Clearly, $a(w)^{1/2} Ne \subset Ne$ is a Z-submodule with s-closure

$$\mathrm{s}_N\left(a(w)\right) Ne \overset{(5.9)}{=} \big(s\text{-}\sum_{1\leq k<\nu} e_k\big) Ne\,.$$

For every x_1, $x_2 \in Ne$ we have

$$\Phi_f\big(T_w(x_1)^* T_w(x_2)\big) \overset{(5.2)}{=} \Phi_e\Big(x_2^*\, T_w^*\big(T_w(x_1)\big)\Big) \overset{(5.3)}{=} \Phi_e\Big(\big(a(w)^{1/2} x_2\big)^* a(w)^{1/2} x_1\Big)\,,$$

so $a(w)^{1/2} Ne \ni a(w)^{1/2} x \longmapsto T_w(x) \in N'f$ is a well defined Z-isometric Z-module antimorphism. Furthermore, by [14], Lemma 1.3 it can be extended to an s-continuous Z-module antimorphism $V_w : \big(s\text{-}\sum_{1\leq k<\nu} e_k\big) Ne \longrightarrow N'f$, which is still Z-isometric. Since $u_k \in e_k Ne$, we can consider the elements

$$v_k = V_w(u_k) \in N'f\,, \qquad 1 \leq k < \nu$$

and we have

$$\Phi_f(v_{k_1}^* v_{k_2}) = \Phi_e(u_{k_2}^* u_{k_1}) = \begin{cases} \Phi_e(u_{k_1}^* u_{k_1}) & \text{for } k_1 = k_2 \\ 0 & \text{for } k_1 \neq k_2 \end{cases}\,. \tag{5.13}$$

In particular,

$$\Phi_f(v_k^* v_k) = \Phi_e(u_k^* u_k) \overset{(5.12)}{=} \mathrm{s}_Z(z_k)\,. \tag{5.14}$$

Since $v_k^*v_k \in fN'f$, we have $\|v_k^*v_k\| \overset{(2.3)}{=} \|\Phi_f(v_k^*v_k)\| \overset{(5.14)}{=} \|\mathrm{s}_Z(z_k)\| = 1$ and so $v_k^*v_k \le f$. On the other hand, $\mathrm{z}_N(u_k^*u_k) = \mathrm{z}_N(e_k) = \mathrm{s}_Z(z_k)$ yields

$$v_k\,\mathrm{s}_Z(z_k) = V_w\big(u_k\,\mathrm{s}_Z(z_k)\big) = V_w(u_k) = v_k\,,$$
$$\text{so } v_k^*v_k = v_k^*v_k\,\mathrm{s}_Z(z_k) \le \|v_k^*v_k\|\,\mathrm{s}_Z(z_k) = \mathrm{s}_Z(z_k)\,.$$

Consequently $v_k^*v_k \le \mathrm{s}_Z(z_k)f$, that is $\mathrm{s}_Z(z_k)f - v_k^*v_k \ge 0$. But

$$\Phi_f\big(\mathrm{s}_Z(z_k)f - v_k^*v_k\big) = \mathrm{s}_Z(z_k) - \Phi_f(v_k^*v_k) \overset{(5.14)}{=} 0\,,$$

so $v_k^*v_k = \mathrm{s}_Z(z_k)f$. In particular, v_k is a partial isometry with $v_k^*v_k \le f$ and $\mathrm{z}_{N'}(v_kv_k^*) = \mathrm{z}_{N'}(v_k^*v_k) = \mathrm{s}_Z(z_k)$.

The projections $v_k^*v_k$, $1 \le k < \nu$, are mutually orthogonal: if $k_1 \ne k_2$ then (5.13) implies $\Phi_f(v_{k_1}^*v_{k_2}) = 0$, so $v_{k_1}^*v_{k_2} = fv_{k_1}^*v_{k_2}f \overset{(2.2)}{=} \Phi_f(v_{k_1}^*v_{k_2})f = 0$.

Since the series $\sum\limits_{1\le k<\nu} z_k$ is w-convergent and, for any $1 \le n < m < \nu$,

$$\begin{aligned}\Big|\sum_{1\le k\le m} z_k^{1/2}\,u_kv_k - \sum_{1\le k\le n} z_k^{1/2}\,u_kv_k\Big|^2 &= \sum_{n<k_1,\,k_2\le m} z_{k_1}^{1/2}z_{k_2}^{1/2}\,u_{k_1}^*u_{k_2}\,v_{k_1}^*v_{k_2}\\ &= \sum_{n<k\le m} z_k\,u_k^*u_kv_k^*v_k\\ &\le \sum_{1\le k\le m} z_k - \sum_{1\le k\le n} z_k\,,\end{aligned}$$

$\Big\{\sum\limits_{1\le k\le n} z_k^{1/2}\,u_kv_k\Big\}_{1\le n<\nu}$ is a Cauchy sequence with respect to the s-topology. Taking into account that $\Big|\sum\limits_{1\le k\le n} z_k^{1/2}\,u_kv_k\Big|^2 \le \sum\limits_{1\le k\le n} z_k \le \Phi_{ef}(w^*w)$ and the closed balls of Z' are s-complete, it follows that $w_o = s\text{-}\sum\limits_{1\le k<\nu} z_k^{1/2}\,u_kv_k \in Z'ef$ exists.

Next we show that $T_{w_o} = T_w$. Since T_{w_o} and T_w are by (5.1) s-continuous, according to [26], Theorem 3 or [14], 1.5, 1.6 it is enough to prove that

$$T_{w_o}(u_k) = T_w(u_k) \text{ for all } 1 \le k < \nu$$
$$\text{and } T_{w_o}(x) = 0 = T_w(x) \text{ for all } x \in Ne \text{ with } x^*u_k = 0\,,\ 1 \le k < \nu\,.$$

Let $1 \le k < \nu$ be arbitrary. For every $y \in N'f$ we have

$$\Phi_f\big(y^*T_{w_o}(u_k)\big) \overset{(5.1)}{=} \Phi_{ef}(u_k^*y^*w_o) = z_k^{1/2}\,\Phi_{ef}(u_k^*u_ky^*v_k)$$

$$\overset{\text{Lemma 2.4}}{=} z_k^{1/2}\Phi_e(u_k^*u_k)\Phi_f(y^*v_k) \overset{(5.12)}{=} \Phi_f(y^* z_k^{1/2} v_k)\,.$$

Thus $T_{w_o}(u_k) = z_k^{1/2} v_k = V_w(z_k^{1/2} u_k) \overset{(5.8)}{=} V_w\big(a(w)^{1/2} u_k\big)$ and the definition of V_w yields $T_{w_o}(u_k) = T_w(u_k)$.

On the other hand, let $x \in Ne$ be such that $x^*u_k = 0\,,\ 1 \le k < \nu$. Since

$$\Phi_f\big(y^*T_{w_o}(x)\big) \overset{(5.1)}{=} \Phi_{ef}(x^*y^*w_o) = s\text{-}\sum_{1\le k<\nu} z_k^{1/2}\,\Phi_{ef}(x^*u_k\,y^*v_k) = 0\,, \quad y \in N'f,$$

we have $T_{w_o}(x) = 0$. But $e_kx = u_ku_k^*x = 0\,,\ 1 \le k < \nu$, implies $a(w)^{1/2}\,x = 0$ and by the definition of V_w we get also $T_w(x) = V_w\big(a(w)^{1/2}\,x\big) = 0$.

Using $T_{w_o} = T_w$, we obtain for all $x \in Ne\,,\ y \in N'f$:

$$\Phi_{ef}(x^*y^*w_o) \overset{(5.1)}{=} \Phi_f\big(y^*T_{w_o}(x)\big) = \Phi_f\big(y^*T_w(x)\big) \overset{(5.1)}{=} \Phi_{ef}(x^*y^*w)\,.$$

Therefore $w_o = w$, so that (5.10) holds.

Since $u_k^*u_k = \mathrm{s}_Z(z_k)e$ and $v_k^*v_k = \mathrm{s}_Z(z_k)f$, by (5.10) we get successively

$$\sum_{1\le k\le n} z_kef = \sum_{1\le k\le n} z_ku_k^*u_kv_k^*v_k = \Big|\sum_{1\le k\le n} z_k^{1/2}\,u_kv_k\Big|^2 \xrightarrow{\ w\ } w^*w\,,$$
$$\sum_{1\le k\le n} z_k = \Phi_{ef}\Big(\sum_{1\le k\le n} z_kef\Big) \xrightarrow{\ w\ } \Phi_{ef}(w^*w)\,,$$
$$s\text{-}\sum_{1\le k<\nu} z_k = \Phi_{ef}(w^*w)\,.$$

Finally, let $x \in N$ be arbitrary. Since the s-topology and the s^*-topology coincide on $Z'ef$ (see [14], 1.1), the bounded sequence $\Big\{\sum_{1\le k\le n} z_k^{1/2}\,u_kv_k\Big\}_{1\le n<\nu}$ is s^*-convergent to w. Consequently

$$\Big(\sum_{1\le k_1\le n} z_{k_1}^{1/2}\,u_{k_1}^*v_{k_1}^*\Big)x\Big(\sum_{1\le k_2\le n} z_{k_2}^{1/2}\,u_{k_2}v_{k_2}\Big) \xrightarrow{\ s\ } w^*xw\,.$$

But

$$\begin{aligned}\Big(\sum_{1\le k_1\le n} z_{k_1}^{1/2}\,u_{k_1}^*v_{k_1}^*\Big)x\Big(\sum_{1\le k_2\le n} z_{k_2}^{1/2}\,u_{k_2}v_{k_2}\Big) &= \sum_{1\le k_1,k_2\le n} z_{k_1}^{1/2}\,z_{k_2}^{1/2}\,u_{k_1}^*xu_{k_2}v_{k_1}^*v_{k_2}\\ &= \sum_{1\le k\le n} z_ku_k^*xu_k\,\mathrm{s}_Z(z_k)f\\ &= \sum_{1\le k\le n} z_ku_k^*xu_kf\,,\end{aligned}$$

so $w^*x\,w = s\text{-}\sum_{1\le k<\nu} z_k\,u_k^*x\,u_k\,f\,.$ □

Corollary 5.6. *Let $\{0\} \neq N \subset \mathcal{B}(\mathcal{H})$ be a type I von Neumann algebra with centre Z, and $p \in Z'$ an abelian projection of central support $1_{\mathcal{H}}$. Then there exist*

- $\nu \in \{2, 3, \dots\} \cup \{\infty\}$,
- *mutually orthogonal abelian projections $\{e_k\}_{1 \leq k < \nu}$ in N,*
- $\{z_k\}_{1 \leq k < \nu} \subset Z^+,\ z_1 \geq z_2 \geq \dots,\ \|z_k\| \longrightarrow 0$ *if* $\nu = \infty$, $s\text{-}\sum\limits_{1 \leq k < \nu} z_k = 1_{\mathcal{H}}$

such that

$$\mathrm{z}_N(e_k) = \mathrm{s}_Z(z_k) \neq 0 \text{ for all } 1 \leq k < \nu\,,$$

$$\Phi_p(x) = s\text{-}\sum_{1 \leq k < \nu} z_k\, \Phi_{e_k}(x)\,, \qquad x \in N\,. \tag{5.15}$$

Proof. Let $e \in N$, $f \in N'$ be abelian projections of central support $1_{\mathcal{H}}$. By Lemma 2.4 ef is an abelian projection of central support $1_{\mathcal{H}}$ in $N \vee N' = Z'$, so there exists a partial isometry $w \in Z'$ such that $w^*w = ef$, $ww^* = p$. Let ν, $\{u_k\}_{1 \leq k < \nu}$, $\{v_k\}_{1 \leq k < \nu}$, $\{z_k\}_{1 \leq k < \nu}$ be as in Theorem 5.5.

Now let $x \in N$ be arbitrary. By (2.2) we have $\Phi_p(x)ww^* = ww^*xww^*$, so, using (5.11) and Lemma 2.4, we deduce successively

$$\Phi_p(x)\, ef = w^*\big(\Phi_p(x)ww^*\big)w = w^*w = s\text{-}\sum_{1 \leq k < \nu} z_k\, u_k^* x\, u_k\, f\,,$$

$$\Phi_p(x) = \Phi_{ef}\big(\Phi_p(x)\, ef\big) = s\text{-}\sum_{1 \leq k < \nu} z_k\, \Phi_{ef}(u_k^* x\, u_k\, f) = s\text{-}\sum_{1 \leq k < \nu} z_k\, \Phi_e(u_k^* x u_k)\,.$$

But, using (2.2), it is easily seen that $\Phi_e(u_k^* x u_k) = \Phi_{e_k}(x)$ and (5.15) follows. □

CHAPTER VI

CONDITIONAL EXPECTATIONS ONTO W^*-SUBALGEBRAS OF THE CENTRE

We shall denote the support of a normal linear functional φ on a von Neumann algebra M by $\mathrm{s}_M(\varphi)$.

The proof of [17], Theorem 3.1 works to prove the following theorem (cf. [16], Theorem 1 and [18], Proposition 1.4):

Theorem 6.1. *Let M be a von Neumann algebra, $Z \subset Z(M)$ a von Neumann subalgebra, and φ a positive linear functional on M such that $\varphi|_Z$ is normal. Then there exists a unique positive Z-module mapping $E : M \to Z$ such that*

$$\varphi = \varphi \circ E \ \ \textit{and} \ \ \mathrm{s}_M\big(E(1_M)\big) \le \mathrm{s}_Z\big(\varphi|_Z\big)\,.$$

Moreover, then $E(1_M) = \mathrm{s}_Z\big(\varphi|_Z\big)$ and E is normal whenever φ is normal.

If A is a C^*-algebra, $C \subset \mathcal{B}(\mathcal{K})$ is an abelian C^*-algebra and $\Phi : A \to C$ is a positive linear mapping, then

$$\Phi(yx)^*\Phi(yx) \le \Phi(y\,y^*)\,\Phi(x^*x)\,, \qquad x\,,\, y \in A$$

(see e.g. [18], Proposition 1.1). In particular, if A is unital then

$$\Phi(x)^*\Phi(x) \le \Phi(1_A)\,\Phi(x^*x) \le \|x\|^2\,\Phi(1_M)^2\,, \qquad x \in A \tag{6.1}$$

and so $\|\Phi\| = \|\Phi(1_A)\|$. Moreover, Φ is necessarily completely positive (see e.g. [24], Proposition 5.5). Therefore, by the Stinespring Theorem (see e.g. [24], Theorem 5.3), there exist a $*$-representation $\pi : A \to \mathcal{B}(\mathcal{H})$ and a bounded linear map $V : \mathcal{K} \to \mathcal{H}$ such that

$$\Phi(x) = V^*\pi(x)\,V\,, \qquad x \in A\,,$$

$$\mathcal{H} \text{ is the closed linear span of } \pi(A)V\mathcal{K} \text{ (hence } \pi \text{ is non-degenerate)},$$

$$\|V\| = \|\Phi\|^{1/2}\,.$$

The pair (π, V) is uniquely determined up to natural equivalence and is called the *Stinespring dilation* of Φ. We notice that if A is a von Neumann algebra, C is a von Neumann subalgebra of $\mathcal{B}(\mathcal{K})$ and Φ is normal, then also π is normal (see e.g. [24], Corollary 4/8.4).

Corollary 6.2. *Let $M \neq \{0\}$ be a von Neumann algebra, and $Z \subset Z(M)$ a von Neumann subalgebra. Then there exists a normal conditional expectation $E : M \to Z$.*

Proof. Let $\mathcal{P}$ be a maximal set of mutually orthogonal non-zero countably decomposable projections in Z. Then $\sum_{p\in\mathcal{P}} p = 1_M$.

For every $p \in \mathcal{P}$ there exists a normal state ψ_p on Z such that $\mathrm{s}_Z(\psi_p) = p$. Let φ_p be a normal state on M which extends ψ_p. By Theorem 6.1 there exists a normal positive Z-module mapping $E_p : M \to Z$ such that

$$\varphi_p = \psi_p \circ E_p \ \text{ and } \ E_p(1_M) = \mathrm{s}_Z(\psi_p) = p\,.$$

By (6.1) we have, $\|E_p\| = 1$ and $\mathrm{s}_Z\big(E_p(x)\big) \leq p$ for all $x \in M$.

Now it is easy to verify that $E : M \ni x \longmapsto \sum_{p\in\mathcal{P}} E_p(x) \in Z$ is a normal positive Z-module mapping with $E(1_M) = 1_M$, that is a normal conditional expectation. $\square$

Theorem 6.3. *Let M be a von Neumann algebra, $Z \subset Z(M)$ a von Neumann subalgebra, and $\Phi : M \to Z$ a positive Z-module mapping. Then there exists a unique positive Z-module mapping $E : M \to Z$ such that*

$$\mathrm{s}_Z\big(E(1_M)\big) \le \mathrm{s}_Z\big(\Phi(1_M)\big) \ \ and \ \ \Phi(x) = E(x)\,\Phi(1_M)\,,\ x \in M\,.$$

Moreover, then $E(1_M) = \mathrm{s}_Z\big(\Phi(1_M)\big)$ *and* E *is normal whenever* Φ *is normal.*

Proof. Let Z be imbedded in $\mathcal{B}(\mathcal{K})$ as a von Neumann subalgebra and let (π, V) be the Stinespring dilation of Φ. Since $V^*V = V^*\pi(1_M)\,V = \Phi(1_M)$, by the polar decomposition of V we have $V = U\,\Phi(1_M)^{1/2}$, where $U : \mathcal{K} \to \mathcal{H}$ is a partial isometry with $U^*U = \mathrm{s}_Z\big(\Phi(1_M)\big)$. Define $E : M \to \mathcal{B}(\mathcal{K})$ by

$$E(x) = U^*\pi(x)\,U\,, \qquad x \in M\,.$$

Then E is a positive linear mapping with $E(1_M) = U^*U = \mathrm{s}_Z\big(\Phi(1_M)\big)$.

Let $x \in M$ and $T \in Z'$ be arbitrary. Since

$$\begin{aligned}\Phi(1_M)^{1/2}E(x)\,\Phi(1_M)^{1/2} &= \Phi(1_M)^{1/2}U^*\pi(x)\,U\,\Phi(1_M)^{1/2} = V^*\pi(x)\,V \\ &= \Phi(x)\,,\end{aligned} \tag{6.2}$$

we obtain successively

$$\begin{gathered}\Phi(1_M)^{1/2}T\,E(x)\,\Phi(1_M)^{1/2} = T\,\Phi(x) = \Phi(x)\,T = \Phi(1_M)^{1/2}E(x)\,T\,\Phi(1_M)^{1/2}\,, \\ \Phi(1_M)^{1/2}\big(T\,E(x) - E(x)\,T\big)\Phi(1_M)^{1/2} = 0\,, \\ \mathrm{s}_Z\big(\Phi(1_M)\big)\,\big(T\,E(x) - E(x)\,T\big)\,\mathrm{s}_Z\big(\Phi(1_M)\big) = 0\,, \\ T\,\mathrm{s}_Z\big(\Phi(1_M)\big)\,E(x) = E(x)\,\mathrm{s}_Z\big(\Phi(1_M)\big)\,T\,.\end{gathered}$$

But $\mathrm{s}_Z\big(\Phi(1_M)\big) = U^*U$ yields

$$\mathrm{s}_Z\big(\Phi(1_M)\big)\,E(x) = U^*U\,U^*\pi(x)\,U = U^*\pi(x)\,U = E(x) \tag{6.3}$$

and, similarly, $E(x)\,\mathrm{s}_Z\big(\Phi(1_M)\big) = E(x)$. Consequently, $T\,E(x) = E(x)\,T$.

We conclude that $E(x) \in (Z')' = Z$ for all $x \in M$, hence E maps M into Z. Thus (6.2) entails

$$\Phi(x) = \Phi(1_M)^{1/2}E(x)\,\Phi(1_M)^{1/2} = E(x)\,\Phi(1_M)\,, \qquad x \in M\,.$$

It follows also the Z-linearity of E . For let $x \in M$ and $z \in Z$ be arbitrary. Using the above formula and (6.3), we deduce succesively

$$\big(E(zx) - z\,E(x)\big)\Phi(1_M) = \Phi(zx) - z\,\Phi(x) = 0\,,$$
$$E(zx) - z\,E(x) = \big(E(zx) - z\,E(x)\big)\,\mathrm{s}_Z\big(\Phi(1_M)\big) = 0\,.$$

For the uniqueness, let $F : M \to Z$ be any Z-module mapping such that

$$\mathrm{s}_Z\big(F(1_M)\big) \leq \mathrm{s}_Z\big(\Phi(1_M)\big) \text{ and } \Phi(x) = F(x)\,\Phi(1_M)\,,\ x \in M$$

and let $x \in M$ be arbitrary. Then $\big(F(x) - E(x)\big)\,\Phi(1_M) = \Phi(x) - \Phi(x) = 0$, so $\big(F(x) - E(x)\big)\,\mathrm{s}_Z\big(\Phi(1_M)\big) = 0$. But, by $\mathrm{s}_Z\big(F(1_M)\big) \leq \mathrm{s}_Z\big(\Phi(1_M)\big)$ and by (6.1) we have $F(x)\,\mathrm{s}_Z\big(\Phi(1_M)\big) = F(x)$, hence, taking into account (6.3), we conclude that $F(x) - E(x) = \big(F(x) - E(x)\big)\,\mathrm{s}_Z\big(\Phi(1_M)\big) = 0$.

Finally, if Φ is normal, then also π is normal and the normality of E follows from its definition. □

Corollary 6.4. *Let M be a von Neumann algebra, $Z \subset Z(M)$ a von Neumann subalgebra, and $\Phi : M \to Z$ a positive Z-module mapping. Then there exists a conditional expectation $E : M \to Z$ such that*

$$\Phi(x) = E(x)\,\Phi(1_M)\,, \qquad x \in M\,.$$

Moreover, if Φ is normal then E can be chosen normal.

Proof. By Theorem 6.3 there exists a positive Z-module mapping $E_1 : M \to Z$ such that $E_1(1_M) = \mathrm{s}_Z\big(\Phi(1_M)\big)$ and $\Phi(x) = E_1(x)\,\Phi(1_M)$, $x \in M$. Further, by Corollary 6.2, there exists a normal conditional expectation

$$E_2 : M\left(1_M - \mathrm{s}_Z\big(\Phi(1_M)\big)\right) \longrightarrow Z\left(1_M - \mathrm{s}_Z\big(\Phi(1_M)\big)\right).$$

Now $E : M \ni x \longmapsto E_1(x) + E_2\Big(x\big(1_M - \mathrm{s}_Z\big(\Phi(1_M)\big)\big)\Big) \in Z$ is a conditional expectation such that $E(x)\,\Phi(1_M) = E_1(x)\,\Phi(1_M) = \Phi(x)$ for all $x \in M$.

If Φ is normal, then E_1 is normal by Theorem 6.3 and the normality of E follows from its definition. □

The next result about the 'GNS-representation' associated to a conditional expectation onto a von Neumann subalgebra of the centre is a variant of [17], Proposition 4.2:

Lemma 6.5. *Let M be a von Neumann algebra, $Z \subset Z(M)$ a von Neumann subalgebra, $E : M \to Z$ a conditional expectation, and $\pi_o : Z \to \mathcal{B}(\mathcal{K})$ an injective normal unital $*$-representation such that $\pi_o(Z)$ is a maximal abelian von Neumann subalgebra of $\mathcal{B}(\mathcal{K})$. If $\big(\pi : M \to \mathcal{B}(\mathcal{H}), V\big)$ is the Stinespring dilation of $\pi_o \circ E$, then*

$$\pi|_Z \text{ is normal and injective,}$$

$$VV^* \text{ is an abelian projection of central support } 1_{\mathcal{H}} \text{ in } \pi(Z)',$$

$$\pi\big(E(x)\big) = \Phi_{VV^*}\big(\pi(x)\big), \qquad x \in M.$$

Moreover, if E is normal and $E|_{Z(M)}$ is faithful, then π is normal and injective.

Proof. Since E acts identically on Z, we have

$$\pi_o(z) = (\pi_o \circ E)(z) = V^*\pi(z)V, \qquad z \in Z. \tag{6.4}$$

Therefore $1_{\mathcal{K}} = \pi_o(1_M) = V^*\pi(1_M)V = V^*V$, so $V : \mathcal{K} \to \mathcal{H}$ is an isometry. In particular, $VV^* \in \mathcal{B}(\mathcal{H})$ is a projection. We notice also that

$$V^*\pi(z) = \pi_o(z)\, V^*, \qquad z \in Z. \tag{6.5}$$

Indeed, if $z \in Z$, then we have for every $x \in M$ and $\eta \in \mathcal{K}$:

$$V^*\pi(z)\pi(x)V\eta = (\pi_o \circ E)(zx)\,\eta = \pi_o(z)\,(\pi_o \circ E)(x)\,\eta = \pi_o(z)\,V^*\pi(x)V\eta.$$

Since $\mathcal{H}$ is the closed linear span of $\pi(M)V\mathcal{K}$, (6.5) follows.

The projection VV^* belongs to the commutant $\pi(Z)'$. Indeed, using (6.5) and (6.4), we get successively for every $z \in Z$:

$$VV^*\pi(z) = V\pi_o(z)V^* = VV^*\pi(z)VV^*\,,$$

$$\pi(z)VV^* = \big(VV^*\pi(z^*)\big)^* = \big(VV^*\pi(z^*)VV^*\big)^* = VV^*\pi(z)VV^*\,,$$

$$VV^*\pi(z) = VV^*\pi(z)VV^* = \pi(z)VV^*\,.$$

Taking now into account that $VV^* \in \pi(Z)'$, (6.4) yields

$$V\pi_o(z)V^* = VV^*\pi(z)VV^* = \pi(z)VV^*\,, \qquad z \in Z\,. \tag{6.6}$$

Next we show that $VV^*\pi(Z)'VV^* \subset \pi(Z)VV^*$ and so VV^* is an abelian projection in $\pi(Z)'$. Indeed, if $T \in \pi(Z)'$ then we have for every $z \in Z$:

$$\begin{aligned} V^*TV\pi_o(z) &= V^*TV\pi_o(z)V^*V \overset{(6.6)}{=} V^*T\pi(z)VV^*V = V^*T\pi(z)V \\ &= V^*\pi(z)TV = V^*VV^*\pi(z)TV \\ &= V^*\pi(z)VV^*TV \overset{(6.4)}{=} \pi_o(z)V^*TV\,. \end{aligned}$$

Consequently, $V^*TV \in \pi_o(Z)' = \pi_o(Z)$ and, taking into account (6.6), we conclude that $VV^*TVV^* \in V\pi_o(Z)V^* \subset \pi(Z)VV^*$.

The injectivity of $\pi|_Z$ is easy to see: if z is in the kernel of $\pi|_Z$, then (6.5) implies $\pi_o(z) = \pi_o(z)V^*V = V^*\pi(z)V = 0$ and the injectivity of π_o entails that $z = 0$.

For the normality of $\pi|_Z$, let us consider a net $z_\lambda \nearrow z$ in Z^+. Then we have, for every $x_1, x_2 \in M$ and $\eta_1, \eta_2 \in \mathcal{K}$,

$$\begin{aligned} \Big(\pi(z - z_\lambda)\pi(x_1)V\eta_1 \,\Big|\, \pi(x_2)V\eta_2\Big) &= \Big(V^*\pi\big(x_2^*(z - z_\lambda)x_1\big)V\eta_1 \,\Big|\, \eta_2\Big) \\ &= \Big((\pi_o \circ E)\big(x_2^*(z - z_\lambda)x_1\big)\eta_1 \,\Big|\, \eta_2\Big) \\ &= \Big((\pi_o \circ E)\big(x_2^*x_1\big)\pi_o(z - z_\lambda)\eta_1 \,\Big|\, \eta_2\Big) \\ &= \Big(\pi_o(z - z_\lambda)\eta_1 \,\Big|\, (\pi_o \circ E)\big(x_1^*x_2\big)\eta_2\Big) \longrightarrow 0\,. \end{aligned}$$

Since $\mathcal{H}$ is the closed linear span of $\pi(M)V\mathcal{K}$, it follows that $\pi(z_\lambda) \nearrow \pi(z)$.

The normality of $\pi|_Z$ implies, in particular, that $\pi(Z) \subset \mathcal{B}(\mathcal{H})$ is a von Neumann algebra. Since $\pi(Z) \subset \pi(Z)'$, we actually have

$$Z\big(\pi(Z)'\big) = \pi(Z)' \cap \pi(Z)'' = \pi(Z)' \cap \pi(Z) = \pi(Z)\,.$$

Therefore the central support of $VV^* \in \pi(Z)'$ is of the form $\pi(p)$ with $p \in Z$ a projection. Using (6.4) and $VV^* \leq \pi(p)$, we obtain

$$\pi_o(p) = V^*\pi(p)V = V^*\pi(p)VV^*V = V^*VV^*V = 1_{\mathcal{K}}\,.$$

Now the faithfulness of π_o yields $p = 1_M$ and so $\mathrm{z}_{\pi(Z)'}(VV^*) = \pi(1_M) = 1_{\mathcal{H}}$.

For every $x \in M$, using (6.6) we deduce

$$VV^*\pi(x)VV^* = V(\pi_o \circ E)(x)V^* = \pi\big(E(x)\big)VV^*$$

and by (2.2) it follows that $\Phi_{VV^*}\big(\pi(x)\big) = \pi\big(E(x)\big)$.

Finally, let us assume that E is normal. Since Stinespring dilation preserves normality, then also π is normal. In particular, the kernel of π is of the form Mp_π for some projection $p_\pi \in Z(M)$. If we assume also that $E|_{Z(M)}$ is faithful, then $\big(\pi_o \circ E\big)(p_\pi) = V^*\pi(p_\pi)V = 0$ implies that $p_\pi = 0$, i.e. π is injective. $\square$

Lemma 6.5 and Corollary 5.6 imply [16], Corollary of Theorem 2:

Theorem 6.6. *Let $N \neq \{0\}$ be a type I von Neumann algebra, and $E : N \to Z(N)$ a normal conditional expectation. Then there exist*

- $\nu \in \{2, 3, \ldots\} \cup \{\infty\}$,
- *mutually orthogonal abelian projections* $\{e_k\}_{1 \leq k < \nu} \subset N$,
- $\{z_k\}_{1 \leq k < \nu} \subset Z(N)^+$, $z_1 \geq z_2 \geq \ldots$, $\|z_k\| \longrightarrow 0$ *if* $\nu = \infty$, $s\text{-}\sum\limits_{1 \leq k < \nu} z_k = 1_N$

such that

$$\mathrm{z}_N(e_k) = \mathrm{s}_{Z(N)}(z_k) \neq 0 \textit{ for all } 1 \leq k < \nu\,,$$

$$E(x) = s\text{-}\sum_{1 \leq k < \nu} z_k\, \Phi_{e_k}(x)\,, \qquad x \in N\,.$$

Proof. Let $\pi_o : Z(N) \to \mathcal{B}(\mathcal{K})$ be an injective normal unital $*$-representation such that $\pi_o\big(Z(N)\big)$ is a maximal abelian von Neumann subalgebra of $\mathcal{B}(\mathcal{K})$ and let $\big(\pi : N \to \mathcal{B}(\mathcal{H}), V\big)$ denote the Stinespring dilation of $\pi_o \circ E$. By Lemma 6.5, VV^* is an abelian projection of central support $1_{\mathcal{H}}$ in $\pi\big(Z(N)\big)'$, $\pi\big(E(x)\big) = \Phi_{VV^*}\big(\pi(x)\big)$, $x \in N$, and π is normal and injective.

By Corollary 5.6 there exist

- $\nu \in \{2, 3, \ldots\} \cup \{\infty\}$,
- mutually orthogonal abelian projections $\{e_k\}_{1 \le k < \nu}$ in N,
- $\{z_k\}_{1 \le k < \nu} \subset Z(N)^+$, $z_1 \ge z_2 \ge \ldots$, $\|z_k\| \longrightarrow 0$ if $\nu = \infty$, $s\text{-}\sum_{1 \le k < \nu} z_k = 1_N$

such that

$$\mathrm{z}_N(e_k) = \mathrm{s}_{Z(N)}(z_k) \ne 0 \text{ for all } 1 \le k < \nu,$$

$$\pi\big(E(x)\big) = \Phi_{VV^*}\big(\pi(x)\big) = s\text{-}\sum_{1 \le k < \nu} \pi(z_k)\, \Phi_{\pi(e_k)}\big(\pi(x)\big), \qquad x \in N.$$

But $\Phi_{\pi(e_k)}\big(\pi(x)\big) = \pi\big(\Phi_{e_k}(x)\big)$ and so the injectivity of π yields

$$E(x) = s\text{-}\sum_{1 \le k < \nu} z_k\, \Phi_{e_k}(x), \qquad x \in N. \qquad \square$$

CHAPTER VII

DESCRIPTION OF THE GLIMM IDEALS IN SPATIALLY REPRESENTED C^*-ALGEBRAS

If A is a unital C^*-algebra and $1_A \in C \subset Z(A)$ is a C^*-subalgebra with Gelfand spectrum Ω, then we shall denote by $I_{C\subset A}(t)$ the ideal $I_\iota(t)$, where ι is the inclusion map of C in $Z(A)$. In other words,

$$I_{C\subset A}(t) = \overline{\{\, c \in C\,;\, c(t) = 0\,\}\, A}\,, \qquad t \in \Omega\,. \tag{7.1}$$

Proposition 3.1.(ii) implies the following dependence of $I_{C\subset A}(t)$ on A: If M is a unital C^*-algebra and $1_M \in C \subset A \subset M$ are C^*-subalgebras such that $C \subset Z(M)$, then

$$I_{C\subset A}(t) = A \cap I_{C\subset M}(t)\,, \qquad t \in \Omega\,. \tag{7.2}$$

The dependence of $I_{C\subset A}(t)$ on C is described in the following lemma:

Lemma 7.1. *Let M be a unital C^*-algebra, $1_M \in Z \subset Z(M)$ a C^*-subalgebra with Gelfand spectrum $\widetilde{\Omega}$, and $1_M \in C \subset Z$ a C^*-subalgebra with Gelfand spectrum Ω. Then*

$$I_{C\subset M}(t) = \bigcap \left\{ I_{Z\subset M}(\chi)\,;\, \chi \in \widetilde{\Omega}\,,\, \chi(c) = c(t) \text{ for all } c \in C \right\}, \qquad t \in \Omega\,.$$

Proof. Let $t \in \Omega$ be arbitrary and let us denote

$$\widetilde{\Omega}_t = \left\{ \chi \in \widetilde{\Omega}\,;\, \chi(c) = c(t) \text{ for all } c \in C \right\} = \left\{ \chi \in \widetilde{\Omega}\,;\, \chi|_{I_{C\subset Z}(t)} = 0 \right\}.$$

The inclusion $I_{C\subset M}(t) \subset \bigcap\limits_{\chi \in \widetilde{\Omega}_t} I_{Z\subset M}(\chi)$ follows at once from definition (7.1): if $c \in C$, $c(t) = 0$ and $\chi \in \widetilde{\Omega}_t$, then $\chi(c) = c(t) = 0$, so $cM \subset I_{Z\subset M}(\chi)$. Thus it remains to show the converse inclusion.

According to (7.2) $I_{C\subset Z}(t) = Z \cap I_{C\subset M}(t)$, so

$$Z_t = Z/I_{C\subset Z}(t) \ni z/I_{C\subset Z}(t) \longmapsto z/I_{C\subset M}(t) \in M/I_{C\subset M}(t) = M_t$$

is an injective $*$-homomorphism, through which we can identify Z_t with a C^*- subalgebra of M_t . On the other hand, the map which associates to $\chi \in \widetilde{\Omega}_t$ the character $\chi_t : Z_t \ni z/I_{C\subset Z}(t) \mapsto \chi(z)$, is a homeomorphism of $\widetilde{\Omega}_t$ onto the Gelfand spectrum of Z_t . Thus

$$\bigcap_{\chi\in\widetilde{\Omega}_t} I_{Z_t\subset M_t}(\chi_t) = \{0\} .$$

Now let $x \in \bigcap\limits_{\chi\in\widetilde{\Omega}_t} I_{Z\subset M}(\chi)$ be arbitrary. For every $\chi \in \widetilde{\Omega}_t$, the quotient map $M \to M_t$ brings $I_{Z\subset M}(\chi)$ into $I_{Z_t\subset M_t}(\chi_t)$: if $z \in Z$, $\chi(z) = 0$ and $y \in M$, then we have

$$\begin{gathered}(zy)/I_{C\subset M}(t) = \big(z/I_{C\subset Z}(t)\big)\big(y/I_{C\subset M}(t)\big) \\ \text{with } \chi_t\big(z/I_{C\subset Z}(t)\big) = \chi(z) = 0 ,\end{gathered}$$

hence $(zy)/I_{C\subset M}(t) \in I_{Z_t\subset M_t}(\chi_t)$. Consequently,

$$x/I_{C\subset M}(t) \in \bigcap_{\chi\in\widetilde{\Omega}_t} I_{Z_t\subset M_t}(\chi_t) = \{0\} ,$$

that is $x \in I_{C\subset M}(t)$. □

Lemma 7.1 enables us to prove the following extension of [19], Theorem 4.2 (see also [4], Theorem 4.17):

Theorem 7.2. *Let $M \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, $Z \subset Z(M)$ a von Neumann subalgebra with Gelfand spectrum $\widetilde{\Omega}$, $1_M \in C \subset Z$ a C^*-subalgebra with Gelfand spectrum Ω , and $C \subset A \subset M$ an intermediate C^*-algebra. Then*

$$I_{C\subset A}(t) = \left\{ a \in A\, ; \begin{array}{c} \chi\big(E(a)\big) = 0 \ \text{ for every} \\ \text{normal conditional expectation } E : M \to Z \\ \text{and } \chi \in \widetilde{\Omega} \text{ with } \chi(c) = c(t) = 0 ,\, c \in C \end{array} \right\}, \quad t \in \Omega .$$

Proof. Let $t \in \Omega$ be arbitrary.

By [19], Theorem 4.2 and by Corollary 6.4 we have for every $\chi \in \widetilde{\Omega}$:

$$I_{Z\subset Z'}(\chi) = \left\{ x \in Z'\,;\ \begin{matrix} \chi\big(\Phi(x)\big) = 0 \ \text{ for every} \\ \text{normal positive } Z\text{-module mapping } \Phi : Z' \to Z \end{matrix} \right\}$$

$$= \left\{ x \in Z'\,;\ \begin{matrix} \chi\big(E(x)\big) = 0 \ \text{ for every} \\ \text{normal conditional expectation } E : Z' \to Z \end{matrix} \right\}.$$

Using Lemma 7.1, it follows

$$I_{C\subset Z'}(t) = \bigcap \big\{ I_{Z\subset Z'}(\chi)\,;\ \chi \in \widetilde{\Omega}\,,\ \chi(c) = c(t) \text{ for all } c \in C \big\}$$

$$= \left\{ x \in Z'\,;\ \begin{matrix} \chi\big(E(x)\big) = 0 \ \text{ for every} \\ \text{normal conditional expectation } E : Z' \to Z \\ \text{and } \chi \in \widetilde{\Omega} \text{ with } \chi(c) = c(t) = 0\,,\ c \in C \end{matrix} \right\}$$

and by (7.2) we conclude that

$$I_{C\subset A}(t) = A \cap I_{C\subset Z'}(t)$$

$$= \left\{ a \in A\,;\ \begin{matrix} \chi\big(E(a)\big) = 0 \ \text{ for every} \\ \text{normal conditional expectation } E : Z' \to Z \\ \text{and } \chi \in \widetilde{\Omega} \text{ with } \chi(c) = c(t) = 0\,,\ c \in C \end{matrix} \right\}$$

$$\supset \left\{ a \in A\,;\ \begin{matrix} \chi\big(E(a)\big) = 0 \ \text{ for every} \\ \text{normal conditional expectation } E : M \to Z \\ \text{and } \chi \in \widetilde{\Omega} \text{ with } \chi(c) = c(t) = 0\,,\ c \in C \end{matrix} \right\} \supset I_{C\subset A}(t)\,.$$

□

The next simple result should be known, but we have no reference for it:

Lemma 7.3. *Let N be a type I von Neumann algebra with centre Z, $e_o \in N$ an abelian projection of central support 1_N, and $b \in N$. Then there exists an abelian* *projection $e \in N$ of central support 1_N such that*

$$\Phi_{e_o}(b^* x\, b) = \Phi_{e_o}(b^* b)\, \Phi_e(x)\,, \qquad x \in N\,. \tag{7.3}$$

Proof. Let $b\,e_o = w\,|\,b\,e_o|$ be the polar decomposition of be_o and let p denote the central support of b^*b. Then $|\,b\,e_o| = (e_o b^* b\, e_o)^{1/2} = z\,e_o$ with $0 \le z \in Zp$ and $w^*w = \mathrm{s}_N(e_o b^* b\, e_o) \le e_o$, so that $w^*w = \mathrm{z}_N(w^*w)e_o = p\,e_o$.

Since $p\,e_o$ is an abelian, hence finite projection in N, there is a unitary $\widetilde{w} \in N$ such that $w = \widetilde{w}pe_o$ (see e.g. [23], E.4.9 or [21], 6.9.7). Then $e = \widetilde{w}e_o\widetilde{w}^*$ is an abelian projection of central support 1_N in N. For every $x \in N$, since

$$exe = \widetilde{w}(e_o\widetilde{w}^* x\, \widetilde{w}e_o)\widetilde{w}^* = \Phi_{e_o}(\widetilde{w}^* x\, \widetilde{w})\widetilde{w}e_o\widetilde{w}^* = \Phi_{e_o}(\widetilde{w}^* x\, \widetilde{w})e\,,$$

we have

$$\Phi_{e_o}(\widetilde{w}^* x\, \widetilde{w}) = \Phi_e(x)\,, \tag{7.4}$$

hence

$$\begin{aligned}\Phi_{e_o}(b^* x\, b) &= \Phi_{e_o}\big((be_o)^* x\, be_o\big) = \Phi_{e_o}(e_o z w^* x\, w z e_o) = z^2\, \Phi_{e_o}(w^* x\, w)\\ &= z^2\, \Phi_{e_o}(e_o p\widetilde{w}^* x \widetilde{w} p e_o) \overset{(7.4)}{=} z^2\, p\, \Phi_e(x) = z^2\, \Phi_e(x)\,.\end{aligned}$$

In particular, for $x = 1_N$, $\Phi_{e_o}(b^*b) = z^2\, \Phi_e(1_N) = z^2$ and so (7.3) holds. $\square$

To prove a variant of Theorem 7.2 for type I von Neumann algebras, in which only normal conditional expectations of form Φ_e, e abelian projection, occur, we need the following result, which is essentially [4], Lemma 5.13:

Lemma 7.4. *Let N be a type I von Neumann algebra with centre Z, $\widetilde{\Omega}$ the Gelfand spectrum of Z, e_o an abelian projection of central support 1_N in N, and $\chi \in \widetilde{\Omega}$. Then*

$$\begin{aligned} I_{Z\subset N}(\chi) &= \big\{x \in N\,;\, \chi\big(\Phi_{e_o}(b^* x\, b)\big) = 0 \ \text{ for every } \ b \in N\big\}\\ &= \left\{ x \in N\,;\ \begin{array}{l} \chi\big(\Phi_e(x)\big) = 0 \ \text{ for every} \\ \text{abelian projection } e \in N \text{ with } \ \mathrm{z}_N(e) = 1_N \end{array} \right\}. \end{aligned}$$

Proof. Clearly, $\{x \in N\,;\ \chi\big(\Phi_{e_o}(b^*xb)\big) = 0$ for every $b \in N\}$ is a norm-closed two-sided ideal $\mathcal{J}$ of N, which contains $I_{Z\subset N}(\chi)$. Let us assume that this inclusion is strict. Then there exists a positive element in $\mathcal{J} \setminus I_{Z\subset N}(\chi)$ and an appropriate spectral projection f of it will still belong to $\mathcal{J} \setminus I_{Z\subset N}(\chi)$. Since $\mathrm{z}_N(f)e_o \prec f$, there exists $u \in N$ such that $u^*u = \mathrm{z}_N(f)e_o$ and $uu^* \leq f$. Thus $\mathrm{z}_N(f)e_o = u^*fu \in \mathcal{J}$ and it follows that $\chi\big(\mathrm{z}_N(f)\big) = \Phi_{e_o}\big(\mathrm{z}_N(f)e_o\big) = 0$. But then, by definition (7.1), $f = \mathrm{z}_N(f)f \in I_{Z\subset N}(\chi)$, in contradiction with the assumption $f \in \mathcal{J} \setminus I_{Z\subset N}(\chi)$.

To complete the proof, we have to prove that

$$\mathcal{J} = \left\{ x \in N\,; \begin{array}{l} \chi\big(\Phi_e(x)\big) = 0 \text{ for every} \\ \text{abelian projection } e \in N \text{ with } \mathrm{z}_N(e) = 1_N \end{array} \right\}.$$

If $x \in \mathcal{J}$ and $e \in N$ is an abelian projection, then there exists $v \in N$ with $v^*v \leq e_o$, $vv^* = e$ and, taking into account that $v^*v = \mathrm{z}_N(v^*v)e_o$ and $\Phi_e(x) \in Z\,\mathrm{z}_N(e) = Z\,\mathrm{z}_N(v^*v)$, we obtain successively

$$v^*xv = v^*(exe)v \overset{(2.2)}{=} v^*\big(\Phi_e(x)e\big)v = \Phi_e(x)v^*v = \Phi_e(x)\,\mathrm{z}_N(v^*v)e_o = \Phi_e(x)e_o\,,$$

$$\chi\big(\Phi_e(x)\big) = \chi\big(\Phi_{e_o}(v^*xv)\big) = 0\,.$$

This proves the inclusion $\subset$.

For the converse inclusion, let $x \in N$ be such that $\chi\big(\Phi_e(x)\big) = 0$ for every abelian projection $e \in N$ of central support 1_N. For every $b \in N$, according to Lemma 7.3, there exists an abelian projection $e \in N$ with central support 1_N such that $\Phi_{e_o}(b^*x\,b) = \Phi_{e_o}(b^*b)\,\Phi_e(x)$. Then

$$\chi\big(\Phi_{e_o}(b^*x\,b)\big) = \chi\big(\Phi_{e_o}(b^*b)\big)\chi\big(\Phi_e(x)\big) = 0\,. \qquad \square$$

Now we improve Theorem 7.2 in the case of type I von Neumann algebras:

Theorem 7.5. *Let N be a type I von Neumann algebra with centre Z, $\widetilde{\Omega}$ the Gelfand spectrum of Z, $1_N \in C \subset Z$ a C^*-subalgebra with Gelfand spectrum Ω, and $C \subset$*

$A \subset N$ an intermediate C^*-algebra. Then

$$I_{C\subset A}(t) = \left\{ a \in A\,;\ \begin{array}{c} \chi\big(\Phi_e(a)\big) = 0 \ \text{ for every} \\ \text{abelian projection } e \in N \text{ with } \ \mathrm{z}_N(e) = 1_N \\ \text{and } \chi \in \widetilde{\Omega} \text{ with } \chi(c) = c(t) = 0\,,\ c \in C \end{array} \right\}, \quad t \in \Omega\,.$$

Proof. Let $t \in \Omega$ be arbitrary.

By Lemmas 7.1 and 7.4 we have

$$\begin{aligned} I_{C\subset N}(t) &= \bigcap \big\{ I_{Z\subset N}(\chi)\,;\ \chi \in \widetilde{\Omega}\,,\ \chi(c) = c(t) \text{ for all } c \in C \big\} \\ &= \left\{ x \in N\,;\ \begin{array}{c} \chi\big(\Phi_e(x)\big) = 0 \ \text{ for every} \\ \text{abelian projection } e \in N \text{ with } \ \mathrm{z}_N(e) = 1_N \\ \text{and } \chi \in \widetilde{\Omega} \text{ with } \chi(c) = c(t) = 0\,,\ c \in C \end{array} \right\} \end{aligned}$$

and, using (7.2), we conclude that

$$\begin{aligned} I_{C\subset A}(t) &= A \cap I_{C\subset N}(t) \\ &= \left\{ a \in A\,;\ \begin{array}{c} \chi\big(\Phi_e(a)\big) = 0 \ \text{ for every} \\ \text{abelian projection } e \in N \text{ with } \ \mathrm{z}_N(e) = 1_N \\ \text{and } \chi \in \widetilde{\Omega} \text{ with } \chi(c) = c(t) = 0\,,\ c \in C \end{array} \right\}. \end{aligned}$$

□

Corollary 7.6. *Let N be a type I von Neumann algebra with centre Z, $\widetilde{\Omega}$ the Gelfand spectrum of Z, $1_N \in C \subset Z$ a C^*-subalgebra with Gelfand spectrum Ω, $C \subset A \subset N$ an intermediate C^*-algebra and $t \in \Omega$. Then every pure state φ on A with $\varphi(c) = c(t)$, $c \in C$, belongs to the weak* closure of*

$$\left\{ \chi \circ \Phi_e\,;\ \begin{array}{c} e \in N \text{ abelian projection with } \ \mathrm{z}_N(e) = 1_N \\ \chi \in \widetilde{\Omega} \text{ with } \chi(c) = c(t) = 0 \text{ for all } c \in C \end{array} \right\}.$$

Proof. For every abelian projection $e \in N$ with $\mathrm{z}_N(e) = 1_N$ and every $\chi \in \widetilde{\Omega}$ with $\chi(c) = c(t) = 0$, $c \in C$, let $\pi_{e,\chi} : A \to \mathcal{B}(\mathcal{H}_{e,\chi})$ be the GNS representation associated to the restriction of $\chi \circ \Phi_e$ to A and let $\xi_{e,\chi}$ denote its canonical cyclic vector. By Theorem 7.5 and Proposition 3.4 we have

$$\bigcap_{e,\chi} \ker(\pi_{e,\chi}) = I_{C\subset A}(t) \subset \ker(\varphi)\,,$$

so we can apply [20], Proposition 3.4.2 or [13], Theorem 5.1.15, deducing that φ belongs to the weak* closure of the states

$$\bigcup_{e,\chi} \Big\{ A \ni a \longmapsto \big(\pi_{e,\chi}(a)\xi \,\big|\, \xi\big)\ ;\ \xi \in \mathcal{H}_{e,\chi}\,,\ \|\xi\| = 1 \Big\}\,.$$

Since every $\xi \in \mathcal{H}_{e,\chi}$ with $\|\xi\| = 1$ is norm-limit in $\mathcal{H}_{e,\chi}$ of unit vectors of the form $\pi_{e,\chi}(b)\xi_{e,\chi}$ and then $\chi\big(\Phi_e(b^*b)\big) = \big(\pi_{e,\chi}(b^*b)\xi_{e,\chi}|\xi_{e,\chi}\big) = 1$, it follows that φ is in the weak* closure of the linear functionals

$$A \ni a \longmapsto \big(\pi_{e,\chi}(a)\pi_{e,\chi}(b)\xi_{e,\chi} \,\big|\, \pi_{e,\chi}(b)\xi_{e,\chi}\big) = \chi\big(\Phi_e(b^*a\,b)\big)$$

with $\chi\big(\Phi_e(b^*b)\big) = 1$.

But, according to Lemma 7.3, for every abelian projection $e \in N$ of central support 1_N and every $b \in N$, there exists an abelian projection $e(b) \in N$ of central support 1_N such that $\Phi_e(b^*x\,b) = \Phi_e(b^*b)\,\Phi_{e(b)}(x)$, $x \in N$. Therefore every linear functional $A \ni a \longmapsto \chi\big(\Phi_e(b^*a\,b)\big)$ with $\chi\big(\Phi_e(b^*b)\big) = 1$ is of the form $A \ni a \longmapsto \chi\big(\Phi_{e(b)}(a)\big) = (\chi \circ \Phi_{e(b)})(a)$. □

Corollary 7.6 implies the following description of $\mathcal{J}_C$ in terms of appropriate spatial representation:

Corollary 7.7. *Let $(A_1\,,\,\iota_1)$, $(A_2\,,\,\iota_2)$ be C^*-algebras over a unital abelian C^*-algebra C, and $\pi_j : A_j \longrightarrow \mathcal{B}(\mathcal{H})$, $j = 1, 2$, two faithful non-degenerate $*$-representations such that*

$$M(\pi_1) \circ \iota_1 = M(\pi_2) \circ \iota_2 \ \text{ and } \ \pi_1(A_1) \subset N\,,\ \pi_2(A_2) \subset N'$$

for a type I von Neumann algebra $N \subset \mathcal{B}(\mathcal{H})$ with centre $Z = \big(M(\pi_j) \circ \iota_j\big)(C)''$. Let $\widetilde{\Omega}$ denote the Gelfand spectrum of Z. Then $a \in A_1 \otimes A_2$ belongs to $\mathcal{J}_C$ if and only if

$$\Big((\chi_1\circ \Phi_e \circ \pi_1) \otimes (\chi_2\circ \Phi_f \circ \pi_2)\Big)(a) = 0$$

for all

$$\text{abelian projections } e \in N\,,\ f \in N' \text{ with } \mathrm{z}_N(e) = \mathrm{z}_{N'}(f) = 1_{\mathcal{H}}\,,$$
$$\chi_1\,,\ \chi_2 \in \widetilde{\Omega} \text{ with } \chi_1\circ M(\pi_1)\circ \iota_1 = \chi_2\circ M(\pi_2)\circ \iota_2\,.$$

Proof. Let Ω denote the Gelfand spectrum.

Assume first that $a \in \mathcal{J}_C$ and let $e \in N$, $f \in N'$ be abelian projections with $\mathrm{z}_N(e) = \mathrm{z}_{N'}(f) = 1_{\mathcal{H}}$, while $\chi_1\,,\ \chi_2 \in \widetilde{\Omega}$ with $\chi_1\circ M(\pi_1)\circ \iota_1 = \chi_2\circ M(\pi_2)\circ \iota_2$. Then $\chi_j \circ M(\pi_j)\circ \iota_j$ is $C \ni c \longmapsto c(t)$ for some $t \in \Omega$. Since

$$(\chi_1\circ \Phi_e \circ \pi_1)\big(\iota_1(c)\, a\big) = \chi_1\Big(\big(M(\pi_1)\circ \iota_1\big)(c)\, \Phi_e\big(\pi_1(a)\big)\Big) = c(t)\, (\chi_1\circ \Phi_e \circ \pi_1)(a)$$

for all $a \in A_1$ and $c \in C$, Proposition 3.4 yields $\chi_1\circ \Phi_e \circ \pi_1\big|_{I_{\iota_1}(t)} = 0$. Similarly, $\chi_2\circ \Phi_f \circ \pi_2\big|_{I_{\iota_2}(t)} = 0$. Thus $\chi_1\circ \Phi_e \circ \pi_1 = \theta_1 \circ \pi_{\iota_1,t}$ for some state θ_1 on $A_1/I_{\iota_1}(t)$ and $\chi_2\circ \Phi_f \circ \pi_2 = \theta_2 \circ \pi_{\iota_2,t}$ for some state θ_2 on $A_2/I_{\iota_2}(t)$. Consequently

$$\Big|\Big((\chi_1\circ \Phi_e \circ \pi_1) \otimes (\chi_2\circ \Phi_f \circ \pi_2)\Big)(a)\Big| \leq \|(\pi_{\iota_1,t} \otimes \pi_{\iota_2,t})(a)\|_{\min} \leq \|a\|_{C,\min} = 0\,.$$

Now let us assume that $a \in A_1 \otimes A_2$ is such that

$$\Big((\chi_1\circ \Phi_e \circ \pi_1) \otimes (\chi_2\circ \Phi_f \circ \pi_2)\Big)(a) = 0$$

for all abelian projections $e \in N$, $f \in N'$ with $\mathrm{z}_N(e) = \mathrm{z}_{N'}(f) = 1_{\mathcal{H}}$ and all χ_1, $\chi_2 \in \widetilde{\Omega}$ with $\chi_1\circ M(\pi_1)\circ \iota_1 = \chi_2\circ M(\pi_2)\circ \iota_2$. Taking into account that π_1, π_2 are injective and using Corollary 7.6, we obtain that $(\varphi_1 \otimes \varphi_2)(a) = 0$ for all $\varphi_1 \in P(A_1)$, $\varphi_2 \in P(A_2)$ with $\varphi_1 \circ \iota_1 = \varphi_2 \circ \iota_2$. In other words,

$$(\psi_1 \otimes \psi_2)\big((\pi_{\iota_1,t} \otimes \pi_{\iota_2,t})(a)\big) = 0\,, \quad \psi_j \in P\big(A_j/I_{\iota_j}(t)\big)\,,\ j = 1, 2\,, \quad t \in \Omega\,.$$

It follows that $(\pi_{\iota_1,t} \otimes \pi_{\iota_2,t})(a) = 0$ for every $t \in \Omega$, that is $a \in \mathcal{J}_C$. $\square$

CHAPTER VIII

FAITHFUL TENSOR PRODUCTS OF

$*$-REPRESENTATIONS OVER ABELIAN C^*-ALGEBRAS

Let C be a unital abelian C^*-algebra, $(A_1\,,\,\iota_1)$ and $(A_2\,,\,\iota_2)$ C^*-algebras over C, and $\pi_j : A_j \to \mathcal{B}(\mathcal{H})\,,\ j = 1\,,2$ non-degenerate $*$-representations such that

$$M(\pi_1) \circ \iota_1 = M(\pi_2) \circ \iota_2 \ \text{ and } \ \pi_1(A_1) \subset N\,,\ \pi_2(A_2) \subset N'$$

for some type I von Neumann algebra $N \subset \mathcal{B}(\mathcal{H})$ with centre $\big(M(\pi_j) \circ \iota_j\big)(C)''$. In this chapter we prove criteria for the faithfulness of $\pi_1 \otimes_{C,\min} \pi_2$.

We notice that $\pi_1 \otimes_{C,\min} \pi_2$ can be faithful without that $\pi_1\,,\ \pi_2$ be faithful. Indeed, in [8], before Proposition 3.3, an example of non-zero $A_1\,,\ A_2$ is given such that $\mathcal{J}_C = A_1 \otimes A_2$, that is $A_1 \otimes_{C,\min} A_2 = \{0\}$. Then, choosing for π_1 and π_2 the zero $*$-representation, $\pi_1 \otimes_{C,\min} \pi_2$ is faithful, while π_1 and π_2 are not. Nevertheless:

Proposition 8.1. *Let C be a unital abelian C^*-algebra with Gelfand spectrum Ω, $(A_1\,,\,\iota_1)\,,\,(A_2\,,\,\iota_2)$ C^*-algebras over C, and $\pi_j : A_j \longrightarrow \mathcal{B}(\mathcal{H})$, $j = 1, 2$, non-degenerate $*$-representations such that*

$$M(\pi_1) \circ \iota_1 = M(\pi_2) \circ \iota_2 \ \textit{ and } \ \pi_1(A_1) \subset N\,,\ \pi_2(A_2) \subset N'$$

for a type I von Neumann algebra $N \subset \mathcal{B}(\mathcal{H})$ with centre $Z = \big(M(\pi_j) \circ \iota_j\big)(C)''$. If $\pi_1 \otimes_{C,\min} \pi_2$ is faithful and $I_{\iota_2}(t) \neq A_2$ for all $t \in \Omega$, then π_1 is faithful. In particular, if $M(\pi_1) \otimes_{C,\min} M(\pi_2)$ is faithful and $A_2 \neq \{0\}$, then π_1 is faithful.

Proof. Let us assume that $\pi_1 \otimes_{C,\min} \pi_2$ is faithful, $I_{\iota_2}(t) \neq A_2$ for every $t \in \Omega$, and $a_1 \in A_1\,,\ \pi_1(a_1) = 0$.

Let $a_2 \in A_2$ be arbitrary. The injectivity of $\pi_1 \otimes_{C,\min} \pi_2$ and

$$(\pi_1 \otimes_{C,\min} \pi_2)\big((a_1 \otimes a_2)/\mathcal{J}_C\big) = \pi_1(a_1)\,\pi_2(a_2) = 0$$

imply that $a_1 \otimes a_2 \in \mathcal{J}_C$, that is $\pi_{\iota_1,t}(a_1) \otimes \pi_{\iota_2,t}(a_2) = 0$ for all $t \in \Omega$. Since, for any $t \in \Omega$, $\pi_{\iota_2,t}(a_2) \neq 0$ for some $a_2 \in A_2$, it follows that $\pi_{\iota_1,t}(a_1) = 0$, $t \in \Omega$. Consequently, $\|a_1\| = \sup\limits_{t\in\Omega} \|\pi_{\iota_1,t}(a_1)\| = 0$, that is $a_1 = 0$.

Now, if $A_2 \neq \{0\}$, then $1_{M(A_2)} \notin \widetilde{I}_{\iota_2}(t)$, so $\widetilde{I}_{\iota_2}(t) \neq M(A_2)$ for all $t \in \Omega$. Therefore, by the above part of the proof,

$$M(\pi_1) \otimes_{C,\min} M(\pi_2) \text{ faithful} \implies M(\pi_1) \text{ faithful.} \qquad \square$$

According to Proposition 8.1, by looking for the faithfulness of $\pi_1 \otimes_{C,\min} \pi_2$ it is natural to assume the faithfulness of π_1 and π_2. However, the faithfulness of π_1 and π_2 alone does not imply the faithfulness of $\pi_1 \otimes_{C,\min} \pi_2$, as the next proposition will show.

We shall denote by $l^\infty(\mathbb{N})$ the C^*-algebra of all bounded complex sequences, by $c(\mathbb{N})$ the C^*-subalgebra of $l^\infty(\mathbb{N})$ consisting of all convergent sequences, and by $l^2(\mathbb{N})$ the Hilbert space of all square-summable complex sequences.

Proposition 8.2. *Let us consider the unital abelian C^*-algebras $C = c(\mathbb{N})$, $A_1 = A_2 = l^\infty(\mathbb{N})$ and the inclusion maps $\iota_j : C \to A_j$, $j = 1, 2$. Let further π_j denote the faithful unital $*$-homomorphisms $A_j \to \mathcal{B}\big(l^2(\mathbb{N})\big)$ which associates to every $a \in l^\infty(\mathbb{N})$ the multiplication operator with a on $l^2(\mathbb{N})$. Then $\pi_1 \otimes_{C,\min} \pi_2$ is not faithful.*

Proof. We notice that the Gelfand spectrum of $c(\mathbb{N})$ can be identified with the one-point compactification $\widehat{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ of $\mathbb{N}$.

Let $\chi_{\text{odds}} \in l^\infty(\mathbb{N})$ denote the characteristic function of all odd natural numbers, and χ_{evens} the characteristic function of all even natural numbers. Then

$$(\pi_1 \otimes_{C,\min} \pi_2)\big((\chi_{\text{odds}} \otimes \chi_{\text{evens}})/\mathcal{J}_C\big) = \pi_1(\chi_{\text{odds}})\,\pi_2(\chi_{\text{evens}}) = 0\,.$$

We shall show that $\|\chi_{\text{odds}} \otimes \chi_{\text{evens}}\|_{C,\min} = 1$, hence $(\chi_{\text{odds}} \otimes \chi_{\text{evens}})/\mathcal{J}_C \neq 0$, which completes the proof of the non-injectivity of $\pi_1 \otimes_{C,\min} \pi_2$.

Let ev_n denote the evaluation map $l^\infty(\mathbb{N}) \ni a \longmapsto a(n)$. Then every ev_n is a state on $l^\infty(\mathbb{N})$. Let φ_1 be a weak*limit point of $\{\text{ev}_n\}_{n \text{ odd}}$, and φ_2 a weak*limit point of $\{\text{ev}_n\}_{n \text{ even}}$. Clearly, $\varphi_1(\chi_{\text{odds}}) = 1$ and φ_1 carries $c \in C$ to $c(\infty)$, so by Proposition 3.4 we have $\varphi_1|_{I_{\iota_1}(\infty)} = 0$. Therefore $\varphi_1 = \psi_1 \circ \pi_{\iota_1,\infty}$ for some state ψ_1 on $A_1/I_{\iota_1}(\infty)$. Similarly, $\varphi_2(\chi_{\text{evens}}) = 1$ and $\varphi_2 = \psi_2 \circ \pi_{\iota_2,\infty}$ for some state ψ_2 on $A_2/I_{\iota_2}(\infty)$. Since

$$\begin{aligned} 1 = (\varphi_1 \otimes \varphi_2)(\chi_{\text{odds}} \otimes \chi_{\text{evens}}) &= (\psi_1 \otimes \psi_2)\Big((\pi_{\iota_1,\infty} \otimes \pi_{\iota_2,\infty})(\chi_{\text{odds}} \otimes \chi_{\text{evens}})\Big) \\ &\leq \big\|(\pi_{\iota_1,\infty} \otimes \pi_{\iota_2,\infty})(\chi_{\text{odds}} \otimes \chi_{\text{evens}})\big\|_{\min} \\ &\leq \|\chi_{\text{odds}} \otimes \chi_{\text{evens}}\|_{C,\min} \leq 1, \end{aligned}$$

we conclude that $\|\chi_{\text{odds}} \otimes \chi_{\text{evens}}\|_{C,\min} = 1$. $\square$

In the sequel we shall prove criteria in order that the tensor product of two faithful $*$-representations over a unital abelian C^*-algebra be still faithful.

Let $\mathcal{H}$ be a Hilbert space, $A, B \subset \mathcal{B}(\mathcal{H})$ C^*-subalgebras with B containing $1_{\mathcal{H}}$, and $\varphi \in S(A)$. If $C^*(A \cup B)$ denotes the C^*-algebra generated by $A \cup B$, then

$$\big\{\theta \in S\big(C^*(A \cup B)\big)\,;\, \theta|_A = \varphi\big\}$$

is a weak*closed convex subset of $S\big(C^*(A \cup B)\big)$, so the subset

$$K(A, B; \varphi) = \big\{\theta|_B\,;\, \theta \in S\big(C^*(A \cup B)\big),\, \theta|_A = \varphi\big\} \subset S(B)$$

is convex and weak*closed.

Let X be a non-empty convex set in some vector space. We recall that $x \in X$ is an extreme point of X if and only if $x = \frac{1}{2}(x_1 + x_2)$, $x_1, x_2 \in X$, is possible only for $x_1 = x_2$ (cf. [4], Theorem 5.2). We denote the set of all extreme points of X (the extreme boundary of X) by $\partial_e X$.

Lemma 8.3. *Let $\mathcal{H}$ be a Hilbert space, $A\,, B \subset \mathcal{B}(\mathcal{H})$ C^*-subalgebras with B containing $1_{\mathcal{H}}$, and $\varphi \in P(A)$. Then*

$$\partial_e K(A\,, B\,; \varphi) \subset \big\{\theta|_B\,; \theta \in P\big(C^*(A \cup B)\big)\,, \theta|_A = \varphi\big\}\,.$$

If additionally $B \subset A'$, then

$$\big\{\theta|_B\,; \theta \in P\big(C^*(A \cup B)\big)\,, \theta|_A = \varphi\big\} \subset P(B)\,,$$

hence also the converse inclusion holds.

Proof. Let $\psi \in \partial_e K(A\,, B\,; \varphi)$ be arbitrary. Then

$$K_\psi = \big\{\theta \in S\big(C^*(A \cup B)\big)\,; \theta|_A = \varphi\,, \theta|_B = \psi\big\}$$

is a non-empty weak*compact convex set, so by the Krein-Milman Theorem it has an extreme point θ_o. We claim that $\theta_o \in P\big(C^*(A \cup B)\big)$.

For let us assume that $\theta_o = \frac{1}{2}(\theta_1 + \theta_2)$ with $\theta_1, \theta_2 \in S\big(C^*(A \cup B)\big)$. Since $\varphi \in P(A) = \partial_e S(A)$ and $\varphi = \theta_o|_A = \frac{1}{2}(\theta_1|_A + \theta_2|_A)$, we have $\theta_1|_A = \theta_2|_A = \varphi$. Therefore $\theta_1|_B$ and $\theta_2|_B$ belong to $K(A\,, B\,; \varphi)$. But $\psi = \theta_o|_B = \frac{1}{2}(\theta_1|_B + \theta_2|_B)$, so, using that $\psi \in \partial_e K(A\,, B\,; \varphi)$, we obtain $\theta_1|_B = \theta_2|_B = \psi$. Consequently $\theta_1, \theta_2 \in K_\psi$ and the extremality of θ_o in K_ψ yields $\theta_1 = \theta_2 = \theta_o$.

Now let us assume that $B \subset A'$ and $\psi = \theta|_B$ for some $\theta \in P\big(C^*(A \cup B)\big)$ with $\theta|_A = \varphi$. Let $\pi_\theta : C^*(A \cup B) \longrightarrow \mathcal{B}(\mathcal{H}_\theta)$ be the GNS representation associated to θ, and ξ_θ its canonical cyclic vector. Since θ is a pure state, π_θ is irreducible.

Let P_o denote the unit of the weak operator closed $*$-subalgebra $\overline{\pi_\theta(A)}^{wo}$ of $\mathcal{B}(\mathcal{H}_\theta)$. Then $P_o \in \pi_\theta(A)' \cap \pi_\theta(B)' = \pi_\theta\big(C^*(A \cup B)\big)' = \mathbb{C}\,1_{\mathcal{H}_\theta}$. Moreover, since $\theta|_A = \varphi \neq 0$, P_o is non-zero. Consequently $P_o = 1_{\mathcal{H}_\theta}$, and so $\overline{\pi_\theta(A)}^{wo}$ is a von Neumann algebra. In particular, ξ_θ belongs to $\mathcal{H}_{\theta,\varphi} = \overline{\pi_\theta(A)\xi_\theta} \subset \mathcal{H}_\theta$.

The orthogonal projection P' onto $\mathcal{H}_{\theta,\varphi}$ clearly belongs to the commutant $\pi_\theta(A)'$ of $\overline{\pi_\theta(A)}^{wo}$. The central support of P' is the orthogonal projection on

$$\overline{\text{lin}\,\big(\pi_\theta(A)'P'\mathcal{H}_\theta\big)} \supset \overline{\text{lin}\,\big(\pi_\theta(B)\pi_\theta(A)\xi_\theta\big)} = \overline{\text{lin}\,\big(\pi_\theta\big(C^*(A\cup B)\big)\xi_\theta\big)} = \mathcal{H}_\theta\,,$$

so $\mathrm{z}_{\pi_\theta(A)'}(P') = 1_{\mathcal{H}_\theta}$. Therefore the induction $*$-homomorphism

$$\rho_{\theta,\varphi} : \overline{\pi_\theta(A)}^{wo} \ni T \longmapsto T|_{\mathcal{H}_{\theta,\varphi}} \in \mathcal{B}(\mathcal{H}_{\theta,\varphi})$$

is injective. But the $*$-representation $\pi_{\theta,\varphi} : A \ni a \longmapsto \pi_\theta(a)|_{\mathcal{H}_{\theta,\varphi}} \in \mathcal{B}(\mathcal{H}_{\theta,\varphi})$ is unitarily equivalent to the GNS representation $\pi_\varphi : A \longrightarrow \mathcal{B}(\mathcal{H}_\varphi)$ of φ and $\varphi \in P(A)$, so $\pi_{\theta,\varphi}$ is irreducible and consequently the range of $\rho_{\theta,\varphi}$ is equal to $\overline{\pi_{\theta,\varphi}(A)}^{wo} = \mathcal{B}(\mathcal{H}_{\theta,\varphi})$. Therefore $N = \overline{\pi_\theta(A)}^{wo} = \rho_{\theta,\varphi}^{-1}\big(\mathcal{B}(\mathcal{H}_{\theta,\varphi})\big)$ is a type I factor.

Now, $\pi_\theta(B) \subset N'$ and the relative commutant of $\pi_\theta(B)$ in N' is

$$\pi_\theta(B)' \cap N' = \pi_\theta(B)' \cap \pi_\theta(A)' = \pi_\theta\big(C^*(A\cup B)\big)' = \mathbb{C}\,1_{\mathcal{H}_\theta}\,.$$

Since the bicommutant theorem holds in type I factors, we get $\overline{\pi_\theta(B)}^{wo} = N'$. We claim that P' is a minimal projection of N'.

For let $T' \in N'$, $0 \le T' \le 1_{\mathcal{H}_\theta}$, be arbitrary. Since

$$\big(\pi_\theta(a)T'\xi_\theta\big|\xi_\theta\big) \le \big(\pi_\theta(a)\xi_\theta\big|\xi_\theta\big) = \varphi(a)\,, \qquad a \in A^+$$

and $\varphi \in P(A)$, there exists $0 \le \lambda \le 1$ such that $\big(\pi_\theta(a)T'\xi_\theta\big|\xi_\theta\big) = \lambda\,\varphi(a)$ for all $a \in A$ (see e.g. [24], 4.7). Consequently

$$\big((T' - \lambda\,1_{\mathcal{H}_\theta})\,\pi_\theta(a_1)\xi_\theta\big|\pi_\theta(a_2)\xi_\theta\big) = \big(\pi_\theta(a_2^*a_1)T'\xi_\theta\big|\xi_\theta\big) - \lambda\,\varphi(a_2^*a_1) = 0$$

for all $a_1, a_2 \in A$ and it follows that $P'(T' - \lambda\,1_{\mathcal{H}_\theta})P' = 0$, i.e. $P'T'P' = \lambda\,P'$.

By the minimality of P' in N', for every $b \in B$ there exists $\lambda_b \in \mathbb{C}$ such that $P'\pi_\theta(b)P' = \lambda_b\,P'$. Since $\lambda_b = (\lambda_b\,P'\xi_\theta|\xi_\theta) = \big(P'\pi_\theta(b)P'\xi_\theta|\xi_\theta\big) = \theta(b) = \psi(b)$, we have $P'\pi_\theta(b)P' = \psi(b)\,P'$.

Let π be a $*$-isomorphism of the type I factor N' onto some $\mathcal{B}(\mathcal{K})$. Then $\pi(P')$ is an one-dimensional projection and, choosing a vector $\eta \in \pi(P')\mathcal{K}$, $\|\eta\| = 1$, we have $\psi(b) = \big((\pi\circ\pi_\theta)(b)\eta\big|\eta\big)$, $b \in B$. Since $(\pi\circ\pi_\theta)(B)$ is weak operator dense in $\mathcal{B}(\mathcal{K})$, we conclude that ψ is a pure state. □

Now we study the extreme points of the intersection of $K(A_1\,,B\,;\varphi_1)$ and $K(A_2\,,B\,;\varphi_2)$:

Lemma 8.4. *Let $\mathcal{H}$ be a Hilbert space, $A_1\,,\,A_2\,,B\subset\mathcal{B}(\mathcal{H})$ C^*-subalgebras with B abelian and $1_{\mathcal{H}}\in B\subset {A_1}'\cap {A_2}'$, and $\varphi_1\in P(A_1)\,,\,\varphi_2\in P(A_2)$. If*

$$\psi\in\partial_e\Big(K(A_1\,,B\,;\varphi_1)\cap K(A_2\,,B\,;\varphi_2)\Big)$$

then, for $j=1,2$, there exists $\tau_j\in P\big(C^*(A_j\cup B)\big)$ such that

$$\tau_j|_{A_j}=\varphi_j\,,\ \tau_j|_B=\psi\ \text{ and }\ \tau_j(ab)=\tau_j(a)\,\tau_j(b)\,,\ a\in C^*(A_j\cup B)\,,\ b\in B\,.$$

In particular,

$$\partial_e\Big(K(A_1\,,B\,;\varphi_1)\cap K(A_2\,,B\,;\varphi_2)\Big)=\partial_eK(A_1\,,B\,;\varphi_1)\cap\partial_eK(A_2\,,B\,;\varphi_2)\,.$$

Proof. Let us denote, for convenience,

$$K_1=K(A_1\,,B\,;\varphi_1)\,,\ K_2=K(A_2\,,B\,;\varphi_2)$$

and set

$$K_\psi=\left\{(\theta_1,\theta_2)\in S\big(C^*(A_1\cup B)\big)\times S\big(C^*(A_2\cup B)\big)\,;\ \begin{array}{l}\theta_j|_{A_j}=\varphi_j\,,\ \theta_j|_B=\psi\\ \text{for }\ j=1,2\end{array}\right\},$$

$$K=\Big\{(\theta_1,\theta_2)\in S\big(C^*(A_1\cup B)\big)\times S\big(C^*(A_2\cup B)\big)\,;\ \theta_1|_B=\theta_2|_B\Big\}\,.$$

Since $K_\psi\neq\varnothing$ is convex and compact with respect to the product of the weak* topologies, by the Krein-Milman Theorem it has an extreme point (τ_1,τ_2).

First we show that $(\tau_1,\tau_2)\in\partial_eK$. For let $({\theta_1}',{\theta_2}')\,,\,({\theta_1}'',{\theta_2}'')\in K$ be such that

$$(\tau_1,\tau_2)=\frac{1}{2}\Big(({\theta_1}',{\theta_2}')+({\theta_1}'',{\theta_2}'')\Big)\,. \tag{8.1}$$

Then, for $j=1,2$, we have

$$\varphi_j=\tau_j|_{A_j}=\frac{1}{2}\Big({\theta_j}'|_{A_j}+{\theta_j}''|_{A_j}\Big)$$

and, since $\varphi_j\in P(A_j)$, it follows that

$${\theta_j}'|_{A_j}={\theta_j}''|_{A_j}=\varphi_j\,,\ \text{ hence }\ {\theta_j}'|_B\,,\,{\theta_j}''|_B\in K_j\,.$$

But $\theta_1{}'|_B = \theta_2{}'|_B$ and $\theta_1{}''|_B = \theta_2{}''|_B$, so actually $\theta_1{}'|_B = \theta_2{}'|_B \in K_1 \cap K_2$ and $\theta_1{}''|_B = \theta_2{}''|_B \in K_1 \cap K_2$. Now

$$\psi = \tau_1|_B \overset{(8.1)}{=} \frac{1}{2}\Big(\theta_1{}'|_B + \theta_1{}''|_B\Big) \text{ and } \psi \in \partial_e(K_1 \cap K_2)\,,$$

yields $\theta_j{}'|_B = \theta_j{}''|_B = \psi$, $j = 1, 2$, and therefore $(\theta_1{}', \theta_2{}')$, $(\theta_1{}'', \theta_2{}'') \in K_\psi$. So, by the extremality of (τ_1, τ_2) in K_ψ, we conclude that

$$(\theta_1{}', \theta_2{}') = (\theta_1{}'', \theta_2{}'') = (\tau_1, \tau_2)\,.$$

Next we prove

$$\tau_j(ab) = \tau_j(a)\,\tau_j(b) = \varphi_j(a)\,\psi(b)\,,\ a \in C^*(A_j \cup B)\,,\, b \in B\,, \quad j = 1, 2\,. \tag{8.2}$$

Clearly, it is enough to prove (8.2) in the case that $\varepsilon\, 1_{\mathcal{H}} \le b \le (1-\varepsilon)\, 1_{\mathcal{H}}$ for some $\varepsilon > 0$. Set for $j = 1, 2$:

$$\theta_j{}' = \frac{1}{\psi(b)}\,\tau_j(\,\cdot\, b)\,,\ \theta_j{}'' = \frac{1}{\psi(1_{\mathcal{H}} - b)}\,\tau_j(\,\cdot\,(1_{\mathcal{H}} - b)) \in S\big(C^*(A_j \cup B)\big)\,.$$

Since $\tau_1|_B = \psi = \tau_2|_B$, both pairs $(\theta_1{}', \theta_2{}')$ and $(\theta_1{}'', \theta_2{}'')$ belong to K. Thus

$$(\tau_1, \tau_2) = \psi(b)\,(\theta_1{}', \theta_2{}') + \psi(1_{\mathcal{H}} - b)\,(\theta_1{}'', \theta_2{}'') \text{ and } (\tau_1, \tau_2) \in \partial_e K$$

imply that $(\theta_1{}', \theta_2{}') = (\tau_1, \tau_2)$, i.e. (8.2).

Finally we prove that $\tau_j \in P\big(C^*(A_j \cup B)\big)$, $j = 1, 2$. Then, by Lemma 8.3, we have also $\psi \in \partial_e K(A_1\,, B\,; \varphi_1) \cap \partial_e K(A_2\,, B\,; \varphi_2)$.

For $\tau_1 \in P\big(C^*(A_1 \cup B)\big)$, let us assume that

$$\tau_1 = \frac{1}{2}\,(\theta' + \theta'') \text{ for some } \theta', \theta'' \in S\big(C^*(A_1 \cup B)\big)\,.$$

By (8.2) τ_1 is multiplicative on B, so $\tau_1|_B$ is a pure state on B. Therefore the above relation implies $\theta'|_B = \theta''|_B = \tau_1|_B = \psi = \tau_2|_B$ and it follows that

$$(\tau_1, \tau_2) = \frac{1}{2}\Big((\theta', \tau_2) + (\theta'', \tau_2)\Big)\,, \text{ where } (\theta', \tau_2), (\theta'', \tau_2) \in K\,.$$

Using $(\tau_1, \tau_2) \in \partial_e K$, we get $(\theta', \tau_2) = (\theta'', \tau_2) = (\tau_1, \tau_2)$, hence $\theta' = \theta'' = \tau_1$.

The proof of $\tau_2 \in P\big(C^*(A_2 \cup B)\big)$ is completely similar. □

The main result of this chapter is the next theorem, which yields faithfulness criteria for $\pi_1 \otimes_{C,\min} \pi_2$:

Theorem 8.5. *Let C be a unital abelian C^*-algebra with Gelfand spectrum Ω and let $(A_1\,,\,\iota_1)\,,\,(A_2\,,\,\iota_2)$ be C^*-algebras over C. Let further $\pi_j : A_j \to \mathcal{B}(\mathcal{H})\,,\ j = 1, 2,$ be faithful non-degenerate $*$-representations, such that*

$$M(\pi_1) \circ \iota_1 = M(\pi_2) \circ \iota_2 \ \text{ and } \ \pi_1(A_1) \subset N\,,\ \pi_2(A_2) \subset N'$$

for a type I von Neumann algebra $N \subset \mathcal{B}(\mathcal{H})$ with centre $Z = \big(M(\pi_j) \circ \iota_j\big)(C)''$, $\widetilde{\Omega}$ the Gelfand spectrum of Z, and $\pi : A_1 \otimes A_2 \to \mathcal{B}(\mathcal{H})$ the $*$-homomorphism defined by

$$\pi(a_1 \otimes a_2) = \pi_1(a_1)\pi_2(a_2)\,, \qquad a_1 \in A_1\,,\ a_2 \in A_2\,.$$

Then the following statements are equivalent:

(i) *$\pi_1 \otimes_{C,\min} \pi_2$ is faithful ;*

(ii) *the kernel of π is equal to $\mathcal{J}_C$;*

(iii) *if $T_{j,k} \in \pi_j(A_j)\,,\ j = 1, 2\,,\ 1 \le k \le n\,,$ and $\sum\limits_{1\le k\le n} T_{1,k}\, T_{2,k} = 0\,,$ then*

$$\sum_{1\le k\le n} (\chi_1 \circ \Phi_e)(T_{1,k})\,(\chi_2 \circ \Phi_f)(T_{2,k}) = 0$$

for all abelian projections $e \in N\,,\ f \in N'$ with $\mathrm{z}_N(e) = \mathrm{z}_{N'}(f) = 1_{\mathcal{H}}$ and all $\chi_1\,,\ \chi_2 \in \widetilde{\Omega}$ with $\chi_1 \circ M(\pi_1) \circ \iota_1 = \chi_2 \circ M(\pi_2) \circ \iota_2$;

(iv) *for any $\varphi_1 \in P(A_1)$ and $\varphi_2 \in P(A_2)$ with $\varphi_1 \circ \iota_1 = \varphi_2 \circ \iota_2$ we have*

$$K\big(\pi_1(A_1)\,, Z\,; \varphi_1 \circ \pi_1^{-1}\big) \cap K\big(\pi_2(A_2)\,, Z\,; \varphi_2 \circ \pi_2^{-1}\big) \neq \varnothing\,.$$

Proof. By the definition of $\pi_1 \otimes_{C,\min} \pi_2$, (ii) is equivalent to the injectivity of the restriction of $\pi_1 \otimes_{C,\min} \pi_2$ to $(A_1 \otimes A_2)/\mathcal{J}_C$, so (i) implies (ii). Conversely, if (ii) is satisfied, then the C^*-seminorm $A_1 \otimes A_2 \ni a \longmapsto \|\pi(a)\|$ vanishes exactly on $\mathcal{J}_C$, so Proposition 3.6 entails that $\|\pi(a)\| \ge \|a\|_{C,\min}$ for all $a \in A_1 \otimes A_2$. Taking into account (4.7), it follows that $\pi_1 \otimes_{C,\min} \pi_2$ is isometric on $(A_1 \otimes A_2)/\mathcal{J}_C$, hence on the whole $A_1 \otimes_{C,\min} A_2$.

By the aboves we have (i) $\Leftrightarrow$ (ii). Next we prove that (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii).

Let us assume that (i) is satisfied and $T_{j,k} \in \pi_j(A_j)$, $j = 1, 2$, $1 \leq k \leq n$ are such that $\sum\limits_{1\leq k\leq n} T_{1,k}\, T_{2,k} = 0$. Then $T_{j,k} = \pi_j(a_{j,k})$ for some $a_{j,k} \in A_j$ and, setting $a = \sum\limits_{1\leq k\leq n} a_{1,k} \otimes a_{2,k} \in A_1 \otimes A_2$, we have

$$(\pi_1 \otimes_{C,\min} \pi_2)\big(a/\mathcal{J}_C\big) = \pi(a) = \sum_{1\leq k\leq n} T_{1,k}\, T_{2,k} = 0\,,$$

and by (i) it follows that $a \in \mathcal{J}_C$. Using Corollary 7.7, we conclude that, for any abelian projections $e \in N$, $f \in N'$ with $\mathrm{z}_N(e) = \mathrm{z}_{N'}(f) = 1_{\mathcal{H}}$, and any χ_1 , $\chi_2 \in \widetilde{\Omega}$ satisfying $\chi_1 \circ M(\pi_1) \circ \iota_1 = \chi_2 \circ M(\pi_2) \circ \iota_2$,

$$\begin{aligned}\sum_{1\leq k\leq n} (\chi_1 \circ \Phi_e)(T_{1,k})\, (\chi_2 \circ \Phi_f)(T_{2,k}) &= \sum_{1\leq k\leq n} (\chi_1 \circ \Phi_e \circ \pi_1)(a_{1,k})\, (\chi_2 \circ \Phi_f \circ \pi_2)(a_{2,k}) \\ &= \Big((\chi_1 \circ \Phi_e \circ \pi_1) \otimes (\chi_2 \circ \Phi_f \circ \pi_2)\Big)(a) = 0\,.\end{aligned}$$

Now we assume that (iii) is satisfied and $a \in A_1 \otimes A_2$ is such that $\pi(a) = 0$. Then $a = \sum\limits_{1\leq k\leq n} a_{1,k} \otimes a_{2,k}$ with $a_{j,k} \in A_j$, so $\sum\limits_{1\leq k\leq n} \pi_1(a_{1,k})\, \pi_2(a_{2,k}) = \pi(a) = 0$. By (iii) it follows that

$$\begin{aligned}\Big((\chi_1 \circ \Phi_e \circ \pi_1) \otimes (\chi_2 \circ \Phi_f \circ \pi_2)\Big)(a) &= \sum_{1\leq k\leq n} (\chi_1 \circ \Phi_e)\big(\pi_1(a_{1,k})\big)\, (\chi_2 \circ \Phi_f)\big(\pi_2(a_{2,k})\big) \\ &= 0\end{aligned}$$

for all abelian projections $e \in N$, $f \in N'$ with $\mathrm{z}_N(e) = \mathrm{z}_{N'}(f) = 1_{\mathcal{H}}$ and all χ_1 , $\chi_2 \in \widetilde{\Omega}$ satisfying $\chi_1 \circ M(\pi_1) \circ \iota_1 = \chi_2 \circ M(\pi_2) \circ \iota_2$. By Corollary 7.7 it follows that $a \in \mathcal{J}_C$.

Finally we prove that (i) $\Rightarrow$ (iv) $\Rightarrow$ (ii).

Let us assume that (i) holds and let $\varphi_1 \in P(A_1)$ and $\varphi_2 \in P(A_2)$ be such that $\varphi_1 \circ \iota_1 = \varphi_2 \circ \iota_2$. Then there is $t \in \Omega$ such that $\varphi_1\big(\iota_1(c)\big) = \varphi_2\big(\iota_2(c)\big) = c(t)$ for all $c \in C$ and by Proposition 3.4 it follows that $\varphi_1|_{I_{\iota_1}(t)} = 0$, $\varphi_2|_{I_{\iota_2}(t)} = 0$. Therefore

$$|(\varphi_1 \otimes \varphi_2)(a)| \leq \|(\pi_{\iota_1,t} \otimes \pi_{\iota_2,t})(a)\|_{\min} \leq \|a\|_{C,\min}\,, \qquad a \in A_1 \otimes A_2$$

and so there exists a state $\widetilde{\varphi}$ on $A_1 \otimes_{C,\min} A_2$ such that

$$(\varphi_1 \otimes \varphi_2)(a) = \widetilde{\varphi}(a/\mathcal{J}_C)\,, \qquad a \in A_1 \otimes A_2\,.$$

Then $\tau = \widetilde{\varphi}\circ(\pi_1 \otimes_{C,\min} \pi_2)^{-1}$ is a state on $\overline{\operatorname{lin}\pi_1(A_1)\pi_2(A_2)}$, which can be extended by strict continuity to a state on $M\Big(\overline{\operatorname{lin}\pi_1(A_1)\pi_2(A_2)}\Big)$, still denoted by τ. We notice that, by (4.8), $C^*\big(\pi_1(A_1) \cup \pi_2(A_2)\big) \subset M\Big(\overline{\operatorname{lin}\pi_1(A_1)\pi_2(A_2)}\Big)$. Since

$$\tau\big(\pi(a)\big) = \tau\big((\pi_1 \otimes_{C,\min} \pi_2)(a/\mathcal{J}_C)\big) = \widetilde{\varphi}(a/\mathcal{J}_C) = (\varphi_1 \otimes \varphi_2)(a)$$

for all $a \in A_1 \otimes A_2$, choosing some increasing approximate units $\{u_\lambda\}_\lambda, \{v_\mu\}_\mu$ for A_1 respectively A_2 and using (4.8), we obtain

$$\begin{aligned}
\tau\big(\pi_1(a_1)\big) &= \lim_\mu \tau\big(\pi_1(a_1)\pi_2(v_\mu)\big) = \lim_\mu \varphi_1(a_1)\,\varphi_2(v_\mu) = \varphi_1(a_1)\,, \quad a_1 \in A_1\,,\\
\tau\big(\pi_2(a_2)\big) &= \lim_\mu \tau\big(\pi_1(u_\lambda)\pi_2(a_2)\big) = \lim_\mu \varphi_1(u_\lambda)\,\varphi_2(a_2) = \varphi_2(a_2)\,, \quad a_2 \in A_2
\end{aligned}$$

$\big($for $\varphi_2(v_\mu) \longrightarrow \|\varphi_2\| = 1$ and $\varphi_1(u_\lambda) \longrightarrow \|\varphi_1\| = 1$ see, for example [24], Theorem 4.5.(i)$\big)$. Consequently, if θ is an extension of $\tau|_{C^*\left(\pi_1(A_1)\cup\pi_2(A_2)\right)}$ to a state on $C^*\big(\pi_1(A_1) \cup Z \cup \pi_2(A_2)\big)$, then $\theta|_{\pi_j(A_j)} = \varphi_j \circ \pi_j^{-1}$, $j = 1, 2$, and so

$$\theta|_Z \in K\big(\pi_1(A_1)\,, Z\,; \varphi_1 \circ \pi_1^{-1}\big) \cap K\big(\pi_2(A_2)\,, Z\,; \varphi_2 \circ \pi_2^{-1}\big)\,.$$

Now let us assume that (iv) holds and let $a \in A_1 \otimes A_2$ with $\pi(a) = 0$ and $\varphi_1 \in P(A_1)$, $\varphi_2 \in P(A_2)$ with $\varphi_1 \circ \iota_1 = \varphi_2 \circ \iota_2$ be arbitary.

By (iv) the weak*compact convex set

$$K\big(\pi_1(A_1)\,, Z\,; \varphi_1 \circ \pi_1^{-1}\big) \cap K\big(\pi_2(A_2)\,, Z\,; \varphi_2 \circ \pi_2^{-1}\big)$$

is not empty, so by the Krein-Milman Theorem it has some extreme point ψ. Now, by Lemma 8.4, there exist $\theta_j \in P\Big(C^*\big(\pi_j(A_j) \cup Z\big)\Big)$, $j = 1, 2$, such that

$$\begin{aligned}
&\theta_j|_{\pi_j(A_j)} = \varphi_j \circ \pi_j^{-1}\,, \qquad \theta_j|_Z = \psi\,,\\
&\theta_j(T\,z) = \theta_j(T)\,\theta_j(z)\,, \quad T \in C^*\big(\pi_j(A_j) \cup Z\big)\,, z \in Z\,.
\end{aligned} \tag{8.3}$$

On the other hand, if $a = \sum_{1\le k\le n} a_{1,k}\otimes a_{2,k}$ with $a_{1,k}\in A_1$, $a_{2,k}\in A_2$, then

$$\sum_{1\le k\le n} \pi_1(a_{1,k})\,\pi_2(a_{2,k}) = \pi(a) = 0 \text{ and } \pi_1(a_{1,k})\in N\,,\ \pi_2(a_{2,k})\in N'\,.$$

By a classical result of Murray, von Neumann and Kadison (see e.g. [22], Theorem 1.20.5, or [21], Theorem 5.5.4, or [24], Proposition 7.20) it follows that there are $z_{j,k}\in Z$, $1\le j,k\le n$, such that

$$\sum_{1\le j\le n} \pi_1(a_{1,j})\, z_{jk} = 0 \text{ for every } 1\le k\le n\,,$$
$$\sum_{1\le k\le n} z_{j,k}\,\pi_2(a_{2,k}) = \pi_2(a_{2,j}) \text{ for every } 1\le j\le n\,.$$

Using (8.3) and the above equalities, we deduce that

$$\begin{aligned}\sum_{1\le j\le n} \varphi_1(a_{1,j})\,\psi(z_{j,k}) &= \sum_{1\le j\le n} \theta_1\big(\pi_1(a_{1,j})\big)\,\theta_1(z_{j,k}) = \theta_1\Big(\sum_{1\le j\le n} \pi_1(a_{1,j})\, z_{j,k}\Big)\\ &= 0 \text{ for every } 1\le k\le n\,,\end{aligned}$$

$$\begin{aligned}\sum_{1\le k\le n} \psi(z_{j,k})\,\varphi_2(a_{2,k}) &= \sum_{1\le k\le n} \theta_2(z_{j,k})\,\theta_2\big(\pi_2(a_{2,k})\big) = \theta_2\Big(\sum_{1\le k\le n} z_{j,k}\,\pi_2(a_{2,k})\Big)\\ &= \theta_2\big(\pi_2(a_{2,j})\big) = \varphi_2(a_{2,j}) \text{ for every } 1\le j\le n\,.\end{aligned}$$

Consequently

$$\begin{aligned}(\varphi_1\otimes\varphi_2)(a) &= \sum_{1\le j\le n} \varphi_1(a_{1,j})\,\varphi_2(a_{2,j})\\ &= \sum_{1\le j\le n} \varphi_1(a_{1,j})\Big(\sum_{1\le k\le n} \psi(z_{j,k})\,\varphi_2(a_{2,k})\Big)\\ &= \sum_{1\le k\le n}\Big(\sum_{1\le j\le n} \varphi_1(a_{1,j})\,\psi(z_{j,k})\Big)\varphi_2(a_{2,k}) = 0\,.\end{aligned}$$

But if a belongs to the kernel of π, then all $b^*a\,b$, $b\in A_1\otimes A_2$, belong to the kernel of π, so by the aboves we have

$$(\varphi_1\otimes\varphi_2)(b^*a\,b) = 0$$

for all $\varphi_1\in P(A_1)$, $\varphi_2\in P(A_2)$ with $\varphi_1\circ\iota_1 = \varphi_2\circ\iota_2$ and all $b\in A_1\otimes A_2$. By Corollary 3.5 it follows that $a/\mathcal{J}_C = 0$, that is $a\in\mathcal{J}_C$. □

A first application concerns the proper C^*-algebras over C:

Corollary 8.6. *Let C be a unital abelian C^*-algebra and let $(A_1\,,\,\iota_1)\,,\,(A_2\,,\,\iota_2)$ be C^*-algebras over C. If $\pi_1 : A_1 \longrightarrow \mathcal{B}(\mathcal{H})$ and $\pi_2 : A_2 \longrightarrow \mathcal{B}(\mathcal{H})$ are faithful non-degenerate $*$-representations and*

$$M(\pi_1)\circ\iota_1 = M(\pi_2)\circ\iota_2 \ \text{ and } \ \pi_1(A_1)\subset N\,,\ \pi_2(A_2)\subset N'$$

for a type I von Neumann algebra $N \subset \mathcal{B}(\mathcal{H})$ with centre $\big(M(\pi_j)\circ\iota_j\big)(C)$, then $\pi_1 \otimes_{C,\min} \pi_2$ is faithful.

Proof. Since $M(\pi_j)\circ\iota_j$ is injective and $\big(M(\pi_j)\circ\iota_j\big)(C) = \big(M(\pi_j)\circ\iota_j\big)(C)''$, any characters $\chi_1\,,\,\chi_2$ on $\big(M(\pi_j)\circ\iota_j\big)(C)''$ with $\chi_1\circ M(\pi_1)\circ\iota_1 = \chi_2\circ M(\pi_2)\circ\iota_2$ are equal. Thus condition (iii) in Theorem 8.5 is trivially satisfied, by Lemma 2.4. □

The next application of Theorem 8.5 concerns unital $*$-representations, whose normal extension on a substantial part of the second dual is faithful:

Corollary 8.7. *Let C be a unital abelian C^*-algebra and let $(A_1\,,\,\iota_1)\,,\,(A_2\,,\,\iota_2)$ be unital C^*-algebras over C. If $\pi_j : A_j \longrightarrow \mathcal{B}(\mathcal{H})$, $j = 1,2$, are unital $*$-representations, such that the normal extension $\widetilde{\pi}_j : A_j^{**} \longrightarrow \mathcal{B}(\mathcal{H})$ of π_j is faithful on $C^*\big(A_j \cup \iota_j(C)^{**}\big)$, and*

$$\pi_1\circ\iota_1 = \pi_2\circ\iota_2 \ \text{ and } \ \pi_1(A_1)\subset N\,,\ \pi_2(A_2)\subset N'$$

for a type I von Neumann algebra $N \subset \mathcal{B}(\mathcal{H})$ with centre $(\pi_j\circ\iota_j)(C)''$, then $\pi_1\otimes_{C,\min}\pi_2$ is faithful.

Proof. Let Ω denote the Gelfand spectrum of C and set $Z = (\pi_j\circ\iota_j)(C)''$. We shall verify that condition (iv) in Theorem 8.5 is satisfied.

For let $\varphi_1 \in P(A_1)$ and $\varphi_2 \in P(A_2)$ be such that $\varphi_1\circ\iota_1 = \varphi_2\circ\iota_2$. Then $C \ni c \mapsto (\varphi_j\circ\iota_j)(c)$ is a character of C, whose normal extension to C^{**} is equal to the composition $\varphi_j\circ\iota_j^{**}$ of the normal state φ_j on A_j^{**} with the second transposed map ι_j^{**}. Since $\widetilde{\pi}_j\circ\iota_j^{**} : C^{**} \to \mathcal{B}(\mathcal{H})$ is a faithful normal $*$-representation with range Z, which does not depend on $j=1,2$, we can consider the character $\chi=(\varphi_j\circ\iota_j^{**})\circ(\widetilde{\pi}_j\circ\iota_j^{**})^{-1}$ of Z.

Now let $j=1,2$ be arbitrary. Let θ_j denote the composition of the normal state φ_j of A_j^{**} with $\big(\widetilde{\pi}_j|_{C^*(A_j\cup\iota_j(C)^{**})}\big)^{-1}$. Then θ_j is a state on

$$\widetilde{\pi}_j\Big(C^*\big(A_j\cup\iota_j(C)^{**}\big)\Big)=C^*\Big(\pi_j(A_j)\cup(\widetilde{\pi}_j\circ\iota_j^{**})(C^{**})\Big),$$

whose restrictions to $\pi_j(A_j)$ and to $Z=(\widetilde{\pi}_j\circ\iota_j^{**})(C^{**})$ are $\varphi_j\circ\pi_j^{-1}$ and χ, respectively.

Consequently $K\big(\pi_1(A_1)\,,Z\,;\varphi_1\circ\pi_1^{-1}\big)\cap K\big(\pi_2(A_2)\,,Z\,;\varphi_2\circ\pi_2^{-1}\big)\ni\chi$. □

The situation in Corollary 8.7 can occur for any pair of unital C^*-algebras $(A_1\,,\iota_1)$, $(A_2\,,\iota_2)$ over C. Indeed, then $\iota_j^{**}:C^{**}\longrightarrow Z(A_j^{**})$, $j=1,2$, are injective unital normal $*$-homomorphisms, so by [14], Lemma 5.2 there exist injective unital normal $*$-representations $\widetilde{\pi}_j:A_j^{**}\longrightarrow\mathcal{B}(\mathcal{H})$, $j=1,2$, such that $\widetilde{\pi}_1\circ\iota_1^{**}=\widetilde{\pi}_2\circ\iota_2^{**}$ and $\widetilde{\pi}_1(A_1^{**})\subset N$, $\widetilde{\pi}_2(A_2^{**})\subset N'$ for some type I von Neumann algebra $N\subset\mathcal{B}(\mathcal{H})$ with centre equal to $(\widetilde{\pi}_j\circ\iota_j^{**})(C^{**})$ and, denoting $\pi_j=\widetilde{\pi}_j|_{A_j}$, $j=1,2$, the normal extension $\widetilde{\pi}_j$ of π_j to A_j^{**} is faithful and

$$\pi_1\circ\iota_1=\pi_2\circ\iota_2\,,\quad\pi_1(A_1)\subset N\,,\,\pi_2(A_2)\subset N'\,,\quad Z(N)=(\pi_j\circ\iota_j)(C)''.$$

The above remarks and Corollary 8.7 imply immediately:

Corollary 8.8. *Let C be a unital abelian C^*-algebra and let $(A_1\,,\iota_1)$, $(A_2\,,\iota_2)$ be C^*-algebras over C. Then there exist faithful unital $*$-representations $\rho_j:M(A_j)\longrightarrow\mathcal{B}(\mathcal{H})$, $j=1,2$, such that*

$$\rho_1\circ\iota_1=\rho_2\circ\iota_2\ \textit{ and }\ \rho_1\big(M(A_1)\big)\subset N\,,\,\rho_2\big(M(A_2)\big)\subset N'$$

for some type I von Neumann algebra $N\subset\mathcal{B}(\mathcal{H})$ with centre $(\rho_j\circ\iota_j)(C)''$ and $\rho_1\otimes_{C,\min}\rho_2$ is faithful.

According to Corollary 3.3, if ρ_1, ρ_2 are as in Corollary 8.8, then $\rho_1\otimes_{C,\min}\rho_2$ is faithful on $A_1\otimes_{C,\min}A_2\subset M(A_1)\otimes_{C,\min}M(A_2)$. However, in general we don't have

$\rho_j = M(\pi_j)$, and so $(\rho_1 \otimes_{C,\min} \rho_2)|_{A_1 \otimes_{C,\min} A_2} = \pi_1 \otimes_{C,\min} \pi_2$, for appropriate non-degenerate $*$-representations $\pi_j : A_j \longrightarrow \mathcal{B}(\mathcal{H})$, because $(\rho_1 \otimes_{C,\min} \rho_2)|_{A_1 \otimes_{C,\min} A_2}$ is not always non-degenerate. Taking, for example, for A_1, A_2 the non-zero C^*-algebras over $C([0,1])$ with $A_1 \otimes_{C([0,1]),\min} A_2 = \{0\}$, given in [8] before Proposition 3.3, we will have $\rho_1 \neq 0$ and $\rho_2 \neq 0$, hence $(\rho_1 \otimes_{C([0,1]),\min} \rho_2) \neq 0$, while $(\rho_1 \otimes_{C([0,1]),\min} \rho_2)|_{A_1 \otimes_{C([0,1]),\min} A_2} = 0$.

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