CHAPTER IV

SOLUTIONS OF CLASS 1

In this chapter we shall determine all solutions of (*) of class 1. The main result of this chapter is Theorem 4.9.

<u>Lemma 4.1.</u> Let (f,g) be any solution of (*). For each x,y in S, if $f(x) = 0 = f(y)$, then $f(xy^{-1}) = 0$.

<u>Proof.</u> Let $x,y \in S$ be such that $f(x) = 0 = f(y)$. Therefore

$$
\begin{aligned}
f(xy^{-1})^2 &= g(xy^{-1}(xy^{-1})^{-1}) - g(xy^{-1})^2 \\
&= g(xyy^{-1}x^{-1}) - g(xy^{-1})^2 \\
&= [g(xyy^{-1})g(x) + f(xyy^{-1})f(x)] - [g(x)g(y) + f(x)f(y)]^2 \\
&= g(xyy^{-1})g(x) - [g(x)g(y)]^2 \\
&= [g(xy)g(y) + f(xy)f(y)]g(x) - g(x)^2g(y)^2 \\
&= g(xy)g(y)g(x) - g(x)^2g(y)^2 \\
&= [g(x)g(y^{-1}) + f(x)f(y^{-1})]g(y)g(x) - g(x)^2g(y)^2 \\
&= g(x)g(y^{-1})g(y)g(x) - g(x)^2g(y)^2 \\
&= g(x)^2g(y)^2 - g(x)^2g(y)^2 \\
&= 0.
\end{aligned}
$$

The first equality follows from (3.3.1); the third and fifth and seventh equalities follow from (*); the fourth and sixth and eighth equalities follow from hypothesis; the ninth equality follows from (3.3.2). #

<u>Lemma 4.2.</u> Let (f,g) be any solution of (*). If e is any element in E(S) such that $g(e) = 1$, then

$$f(x) = f(xe) \quad \text{and} \quad g(x) = g(xe)$$

for all x in S.

<u>Proof</u>. Let $e \varepsilon E(S)$ be such that $g(e) = 1$. Therefore, by (3.4.2) we have that $f(e) = 0$. Thus

$$\begin{aligned} g(xe) &= g(xe^{-1}) \\ &= g(x)g(e) + f(x)f(e) \\ &= g(x) \end{aligned}$$

for all x in S.

Next we shall show that $f(x) = f(xe)$. First, consider the case $f(x) = 0$. Since $f(x) = 0$, hence, by Lemma 4.1, we have that $f(xe^{-1}) = 0$. Therefore $f(x) = f(xe^{-1}) = f(xe)$. In the case $f(x) \neq 0$, it follows from $g(x) = g(xe)$ for all x in S that

$$\begin{aligned} g(xx^{-1}) &= g(xx^{-1}e) \\ &= g(x(xe)^{-1}) \\ &= g(x)g(xe) + f(x)f(xe) \\ &= g(x)^2 + f(x)f(xe). \end{aligned}$$

Thus

$$f(x)f(xe) = g(xx^{-1}) - g(x)^2.$$

Consequenctly, using (3.3.1), we have

$$f(x)f(xe) = f(x)^2.$$

From $f(x) \neq 0$ we have that $f(xe) = f(x)$ #

Now we shall make use of the concept of the minimum group congruence mentioned in Theorem 2.2. We recall that the minimum group congruence on an inverse semigroup S is given by

$$\sigma = \{(x,y) \in S \times S \ / \ xe = ye \text{ for some } e \text{ in } E(S)\}.$$

Lemma 4.3. Let (f,g) be any solution of (*) of class 1. Then f,g are constant on each σ-class, where σ is the minimum group congruence on S.

Proof. Let x be an arbitrary element in S. Let $y \in x\sigma$. Then there exists e in E(S) such that xe = ye. Therefore

$$f(xe) = f(ye) \quad \text{and} \quad g(xe) = g(ye)$$

Thus, by Lemma 4.2, we have that

$$f(x) = f(xe) = f(ye) = f(y),$$

and

$$g(x) = g(xe) = g(ye) = g(y).$$

Hence f,g are constant on each σ-class. #

Theorem 4.4. The solutions of (*) of class 1 are those and only those (f,g) of the forms:

$$f(x) = f_0(x\sigma) \quad \text{and} \quad g(x) = g_0(x\sigma)$$

for all x in S, where σ is the minimum group congruence on S and (f_0, g_0) is a class 1 solution of (*) on S/σ.

Proof. Assume that (f,g) is a class 1 solution of (*). Define $f_0, g_0 : S/\sigma \to F$ by

$$f_0(x\sigma) = f(x) \quad \text{and} \quad g_0(x\sigma) = g(x)$$

for all x in S. By Lemma 4.3, f_0, g_0 are well-defined. Since (f,g) satisfies (*) on S, so (f_0, g_0) satisfies (*) on S/σ. Since E(S) is contained in the σ-class respersenting the identity of the group S/σ and (f,g) is of class 1 on S, then (f_0, g_0) is of class 1 on S/σ.

Conversely, assume (f_o, g_o) is a class 1 solution of (*) on S/σ. Define $f,g: S \to F$ by

$$f(x) = f_o(x\sigma) \qquad \text{and} \qquad g(x) = g_o(x\sigma)$$

Since (f_o, g_o) satisfies (*) on S/σ, (f,g) satisfies (*) on S. Because for $e \in E(S)$, $e\sigma$ is the identity of S/σ, we have $g(e) = g_o(e\sigma) = 1$ for all $e \in E(S)$. #

Remark 4.5. From Theorem 4.4, we see that if (f,g) is a class 1 negative-type solution, then (f_o, g_o) is also a class 1 negative-type solution; and if (f,g) is a class 1 positive-type solution, then (f_o, g_o) is also a class 1 positive-type solution. Hence to determine all solutions (f,g) of (*) of class 1, we need to determine all solutions (f_o, g_o) of (*) on the abelian group S/σ such that (f_o, g_o) is of class 1. This problem is solved in [1] (see Theorem 3.20 and Theorem 3.29). We state these results of [1] in our terminologies in the following theorems:

Theorem 4.6. ([1]) (f,g) is a class 1 negative-type solution of (*) on an abelian group G if and only if f,g are of the forms:

$$f(x) = \frac{h(x) - h(x^{-1})}{2i}, \qquad g(x) = \frac{h(x) + h(x^{-1})}{2}$$

where h is a homomorphism from G into M(F).

Theorem 4.7. ([1]) (f,g) is a class 1 positive-type solution of (*) on an abelian group G if and only if f,g are of the forms:

$$(4.7.1) \qquad f(x) = \begin{cases} 0 & , x \in H \\ d & , x \notin H \end{cases} \quad , \quad g(x) = \begin{cases} 1 & , x \in H \\ c & , x \notin H \end{cases}$$

where H is a subgroup of index 1 or 2 in G and $c,d \in F$ are such that $c \neq \pm 1$, $c^2 + d^2 = 1$; or

$$(4.7.2) \qquad f(x) = \begin{cases} 0 & , x \in H \text{ or } x_1H \\ d & , x \in x_2H \\ -d & , x \in x_3H \end{cases} \quad , \quad g(x) = \begin{cases} 1, & x \in H \\ -1, & x \in x_1H \\ c, & x \in x_2H \\ -c, & x \in x_3H \end{cases}$$

where H is a subgroup of index 4 in G such that $G/H = \{H, x_1H, x_2H, x_3H\}$ is the Klein four group and $c,d \in F$ are such that $c \neq \pm 1$, $c^2 + d^2 = 1$.

Theorem 4.8. (f,g) is a class 1 positive-type solution of (*) on an abelian group G if and only if f,g are of the forms:

$$(4.84.1) \qquad f(x) = \begin{cases} 0 & , x \in H \\ dh(x), & x \notin H \end{cases} \quad , \quad g(x) = \begin{cases} h(x) & , x \in H \\ ch(x), & x \notin H \end{cases}$$

where H is a subgroup of index 1 or 2 in G and h is a homomorphism from G into {1,-1} and $c,d \in F$ are such that $c \neq \pm 1$, $c^2 + d^2 = 1$.

Proof. By straight forward verification it can be shown that if $f,g: G \to F$ are of the form (4.8.1) , then (f,g) is a class 1 positive-type solution of (*) on G.

To show the converse, assume that (f,g) is a class 1 positive-type solution of (*) on G. Therefore, by Theorem 4.7 we have that f,g are of the forms:

$$(4.8.2)\qquad f(x) = \begin{cases} 0, & x \in H \\ d, & x \notin H \end{cases}, \qquad g(x) = \begin{cases} 1, & x \in H \\ c, & x \notin H \end{cases}$$

where H is a subgroup of index 1 or 2 in G and $c, d \in F$ are such that $c \neq \pm 1$, $c^2 + d^2 = 1$; or

$$(4.8.3)\qquad f(x) = \begin{cases} 0, & x \in H \text{ or } x_1H \\ d, & x \in x_2H \\ -d, & x \in x_3H \end{cases}, \quad g(x) = \begin{cases} 1, & x \in H \\ -1, & x \in x_1H \\ c, & x \in x_2H \\ -c, & x \in x_3H \end{cases}$$

where H is a subgroup of index 4 in G such that $G/H = \{H, x_1H, x_2H\ x_3H\}$ is the Klein four group and $c, d \in F$ are such that $c \neq \pm 1$, $c^2+d^2 = 1$. Observe that f,g in (4.8.2) can be written as

$$f(x) \quad \begin{cases} 0, & x \in H \\ dh(x), & x \notin H \end{cases}, \quad g(x) = \begin{cases} h(x), & x \in H \\ ch(x), & x \notin H \end{cases}$$

where h is given by $h(x) = 1$ for all x in G.

To see that f,g of the form (4.8.3) can be written in the form (4.8.1), let $K = H \cup x_1H$ and define $h: G \to \{1, -1\}$ by

$$h(x) = \begin{cases} 1, & x \in H \cup x_2H \\ -1, & x \in x_1H \cup x_3H \end{cases}$$

Since G/H is the Klein four group, it follows that K is a subgroup of index 2 in G and h is a homomorphism. Observe that f,g can be written in terms of K and h as follows:

$$f(x) = \begin{cases} 0, & x \in K \\ dh(x), & x \notin K \end{cases}, \quad g(x) = \begin{cases} h(x), & x \in K \\ ch(x), & x \notin K \end{cases}$$

where $c, d \in F$ are such that $c \neq \pm 1$, $c^2 + d^2 = 1$. Thus f, g are of the form (4.8.1). #

Theorem 4.9. The solutions of (*) on S of class 1 are those and only those (f,g) of the forms:

$$(4.9.1) \qquad f(x) = \begin{cases} 0 & , x \in \bar{1} \\ dh(x), & x \in x_1\eta \end{cases} , \quad g(x) = \begin{cases} h(x) & , x \in \bar{1} \\ ch(x) & , x \in x_1\eta \end{cases}$$

where η is a $\mathcal{G}_{1,2}$- congruence on S such that $S/\eta = \{\bar{1}\}$ or $\{\bar{1}, x_1\eta\}$, $\bar{1} \neq x_1\eta$ and h is a homomorphism from S into $\{1, -1\}$ and $c, d \in F$ are such that $c \neq \pm 1$, $c^2 + d^2 = 1$; or

$$(4.9.2) \qquad f(x) = \frac{h(x) - h(x^{-1})}{2i} , \quad g(x) = \frac{h(x) + h(x^{-1})}{2}$$

where h is a homomorphism from S into $M(F)$.

Proof. By straight forward verification it can be shown that if $f, g: S \to F$ are of the forms (4.9.1) or (4.9.2) then (f,g) is a class 1 solution of (*).

To shown the converse, we assume that (f,g) is a class 1 solution of (*). Then, by Theorem 4.4 we have that

$$(4.9.3) \qquad f(x) = f_o(x\sigma) \qquad \text{and} \qquad g(x) = g_o(x\sigma)$$

where σ is the minimum group congruence on S and (f_o, g_o) is a class 1 solution of (*) on S/σ. By Theorem 3.9 and hypothesis, we have that (f,g) must be a class 1 negative-type solution or a class 1 positive-type solution.

Case 1 (f,g) is a class 1 negative-type solution. Therefore, by (4.9.3), we have that $(f_\circ, g_\circ)$ is also a class 1 negative-type solution. Thus, by Theorem 4.6, we have that

$$f_0(x\sigma) = \frac{h_\circ(x\sigma) - h_\circ(x^{-1}\sigma)}{2i} \quad , \quad g_0(x\sigma) = \frac{h_\circ(x\sigma) + h_\circ(x^{-1}\sigma)}{2}$$

where $h_\circ$ is a homomorphism from S/σ into M(F).

Define $h: S \to M(F)$ by

$$h(x) = (h_0 \circ \sigma^{\#})(x)$$

where $\sigma^{\#}$ is the natural homomorphism from S onto S/σ Therefore h is a homomorphism and $h(x) = h_\circ(x\sigma)$ for all $x \in S$. Thus

$$f(x) = f_\circ(x\sigma) = \frac{h_\circ(x\sigma) - h_\circ(x^{-1}\sigma)}{2i} = \frac{h(x) - h(x^{-1})}{2i}$$

and

$$g(x) = g_\circ(x\sigma) = \frac{h_\circ(x\sigma) + h_\circ(x^{-1}\sigma)}{2} = \frac{h(x) + h(x^{-1})}{2}$$

for all x in S. Therefore f,g are of the forms (4.9.2).

Case 2 (f,g) is a class 1 positive-type solution. Therefore, by (4.9.3) we have that $(f_\circ, g_\circ)$ is a class 1 positive-type solution of (*) on S/σ. Thus, by Theorem 4.8 we have that

$$f_0(x\sigma) = \begin{cases} 0 & , x \in H \\ dh_\circ(x\sigma) & , x \notin H \end{cases} \quad , \quad g_0(x\sigma) = \begin{cases} h_\circ(x\sigma) & , x \in H \\ ch_\circ(x\sigma) & , x \notin H \end{cases}$$

where $H_\circ$ is a subgroup of index 1 or 2 in S/σ and $h_\circ$ is a homomorphism from S/σ into {1, -1} and $c,d \in F$ are such that $c \neq \pm 1$, $c^2 + d^2 = 1$.

Let

$$\eta = \{(x,y) \in S \times S \; / \; xy^{-1}\sigma \in H_c\},$$

and let h: $S \to \{1,-1\}$ be defined by

$$h(x) = (h_0 \circ \sigma^{\#})(x)$$

for all x in S, where $\sigma^{\#}$ is the natural homomorphism from S onto S/σ. Then, it is clear that h is a homomorphism and η is a congruence on S such that $S/\eta \cong (S/\sigma)/H_c$. Hence η is a $\mathcal{G}_{1,2}$-congruence and

$$f(x) = \begin{cases} 0 & , x \in \bar{1} \\ dh(x) & , x \in x_1\eta \end{cases} , \quad g(x) = \begin{cases} h(x) & , x \in \bar{1} \\ ch(x) & , x \in x_1\eta \end{cases}$$

where $\{\bar{1}, x_1\eta\} \cong \{H_c, (x_1\sigma)H_c\}$. Hence f,g are of the form (4.9.1). #

Remark 4.10. In the above proof of Theorem 4.9, we see that (f,g) is a class 1 positive-type solution if and only if f,g are of the forms

$$f(x) = \begin{cases} 0 & , x \in \bar{1} \\ dh(x), & x \in x_1\eta \end{cases} , \quad g(x) = \begin{cases} h(x) & , x \in \bar{1} \\ ch(x) & , x \in x_1\eta \end{cases}$$

where η is a $\mathcal{G}_{1,2}$-congruence on S such that $S/\eta = \{\bar{1}\}$ or $\{\bar{1}, x_1\eta\}$, $\bar{1} \neq x_1\eta$ and h is a homomorphism from S into $\{1, -1\}$ and $c,d \in F$ are such that $c \neq \pm 1$, $c^2 + d^2 = 1$.

In the case (f,g) is a class 1 negative-type solution, f and g are of the forms

$$f(x) = \frac{h(x) - h(x^{-1})}{2i} \quad , \quad g(x) = \frac{h(x) + h(x^{-1})}{2}$$

where h is a homomorphism from S into M(F).