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APPENDIX A

GAUGE TRANSFORMATIONS IN ELECTROMAGNETISM

Let us denote by $|\alpha\rangle$ the state ket in the presence of vector potential **A**, the state ket for the same physical situation when

$$\tilde{\mathbf{A}} = \mathbf{A} + \nabla\Lambda \tag{A.1}$$

is used in place of **A** is denoted by $|\tilde{\alpha}\rangle$. Here Λ, as well as **A**, is a function of the position operator **x**. The basis requirements are

$$\langle\alpha|\mathbf{x}|\alpha\rangle = \langle\tilde{\alpha}|\mathbf{x}|\tilde{\alpha}\rangle \tag{A.2}$$

and

$$\langle\alpha|\left(\mathbf{p} - \frac{e\mathbf{A}}{c}\right)|\alpha\rangle = \langle\tilde{\alpha}|\left(\mathbf{p} - \frac{e\tilde{\mathbf{A}}}{c}\right)|\tilde{\alpha}\rangle \tag{A.3}$$

where **p** is the canonical momentum. In addition, we require, as the usual, the norm of the state ket to be preserved

$$\langle\alpha|\alpha\rangle = \langle\tilde{\alpha}|\tilde{\alpha}\rangle . \tag{A.4}$$

We must construct an operator T that relates $|\tilde{\alpha}\rangle$ to $|\alpha\rangle$:

$$|\tilde{\alpha}\rangle = T|\alpha\rangle . \tag{A.5}$$

Invariance properties (A.2) and (A.3) are guaranteed if

$$T^{\dagger}\mathbf{x}\,T = \mathbf{x} \tag{A.6}$$

and

$$T^{\dagger}\left(\mathbf{p}-\frac{e\mathbf{A}}{c}-\frac{e\nabla\Lambda}{c}\right)T \;=\; \mathbf{p}-\frac{e\mathbf{A}}{c}\,. \qquad \text{(A.7)}$$

We assert that

$$T \;=\; \exp\left[\frac{ie\Lambda(\mathbf{x})}{\hbar c}\right] \qquad \text{(A.8)}$$

will do the job. First, T is unitary, so Eq.(A.4) is all right. Second, Eq.(A.6) is obviously satisfied because $\mathbf{x}$ commutes with any function of $\mathbf{x}$. As for Eq.(A.7), just note that

$$\begin{aligned}
\exp\left(\frac{-ie\Lambda}{\hbar c}\right)\mathbf{p}\exp\left(\frac{ie\Lambda}{\hbar c}\right) &= \exp\left(\frac{-ie\Lambda}{\hbar c}\right)\left[\mathbf{p},\exp\left(\frac{ie\Lambda}{\hbar c}\right)\right]+\mathbf{p} \\
&= -\exp\left(\frac{-ie\Lambda}{\hbar c}\right)i\hbar\nabla\left[\exp\left(\frac{ie\Lambda}{\hbar c}\right)\right]+\mathbf{p} \\
&= \mathbf{p}+\frac{e\nabla\Lambda}{c}\,. \qquad \text{(A.9)}
\end{aligned}$$

The invariance of quantum mechanics under gauge transformations can also be demonstrated by looking directly at the Schrödinger equation. Let $|\alpha, t_0 ; t\rangle$ be a solution to the Schrödinger equation in the presence of **A**.

$$\left[\frac{(\mathbf{p}-e\mathbf{A}/c)^2}{2m}+e\phi\right]|\alpha, t_0 ; t\rangle \;=\; i\hbar\frac{\partial}{\partial t}|\alpha, t_0 ; t\rangle\,. \qquad \text{(A.10)}$$

The corresponding solution in the presence of $\widetilde{\mathbf{A}}$ must satisfy

$$\left[\frac{(\mathbf{p}-e\mathbf{A}/c-e\nabla\Lambda/c)^2}{2m}+e\phi\right]|\widetilde{\alpha, t_0 ; t}\rangle \;=\; i\hbar\frac{\partial}{\partial t}|\widetilde{\alpha, t_0 ; t}\rangle\,. \qquad \text{(A.11)}$$

We see that if the new ket is taken be

$$|\widetilde{\alpha, t_0 ; t}\rangle \;=\; \exp\left[\frac{ie\Lambda}{\hbar c}\right]|\alpha, t_0 ; t\rangle \qquad \text{(A.12)}$$

in accordance with (A.8), then the new Schrödinger equation (A.11) will be satisfied, all we have to note is that

$$\exp\left[\frac{-\,ie\Lambda}{\hbar c}\right]\left(\mathbf{p} - \frac{e\mathbf{A}}{c} - \frac{e\nabla\Lambda}{\hbar c}\right)^2 \exp\left[\frac{ie\Lambda}{\hbar c}\right] = \left(\mathbf{p} - \frac{e\mathbf{A}}{c}\right)^2 \tag{A.13}$$

which follows from applying (A.9) twice.

Equation (A.12) also implies that the corresponding wave equations are related via

$$\tilde{\psi}(\mathbf{x}', t) = \exp\left[\frac{ie\Lambda(\mathbf{x}')}{\hbar c}\right]\psi(\mathbf{x}', t) \tag{A.14}$$

where $\Lambda\,(\mathbf{x}')$ is now a real function of the position vector eigenvalue $\mathbf{x}'$.

APPENDIX B

MAGNETIC MONOPOLES

Suppose there is a point magnetic monopole, situated at the origin, of strength g analogous to a point electric charge. The static magnetic field is then given by

$$\mathbf{B} = g\,\frac{\hat{r}}{r^2}\,. \tag{B.1}$$

At first sight it may appear that the magnetic field Eq.(B.1) can be derived from

$$\mathbf{A} = \frac{g\left(1 - \cos\theta\right)}{r\sin\theta}\,\hat{\phi}\,. \tag{B.2}$$

Recall the expression for curl in spherical coordinates

$$\nabla \times \mathbf{A} = \hat{r}\left[\frac{1}{r\sin\theta}\,\frac{\partial}{\partial\theta}\left(A_\phi \sin\theta\right) - \frac{\partial A_\theta}{\partial\phi}\right]$$

$$+\,\hat{\theta}\,\frac{1}{r}\left[\frac{1}{\sin\theta}\,\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}\left(rA_\theta\right)\right] + \hat{\phi}\,\frac{1}{r}\left[\frac{\partial}{\partial r}\left(rA_\theta - \frac{\partial A_r}{\partial\theta}\right)\right]. \tag{B.3}$$

But vector potential Eq.(B.3) has one difficulty, it is singular on the negative z-axis ($\theta = \pi$). In fact, it turns out to be impossible to construct a singularity-free potential valid everywhere for this problem. To see this we first note "Gauss's law"

$$\oint\!\!\oint_S \mathbf{B}\cdot d\mathbf{S} = 4\pi g \tag{B.4}$$

for any surface **S** boundary enclosing the origin at which the magnetic monopole is located. On the other hand., if **A** were nonsingular, we would have

$$\nabla \cdot \left(\nabla \times \mathbf{A}\right) = 0 \qquad \text{(B.5)}$$

everywhere, hence,

$$\oiint_S \mathbf{B} \cdot d\mathbf{S} = \oiiint_V \nabla \cdot \left(\nabla \times \mathbf{A}\right) d^3x = 0 \qquad \text{(B.6)}$$

in contradiction with Eq.(B.4).

However, one might argue that because the vector potential is just a device for obtaining **B**, we need not insist on having a single expression for **A** valid every where. In order to avoid singular vector potential, Wu and Yang (1976) introduced the following construction. They covered S^2 with two patches S^+ and S^-. And construct a pair of potentials,

$$\mathbf{A}^{(I)} = \left[\frac{g\left(1 - \cos\theta\right)}{r\sin\theta}\right]\hat{\phi}, \qquad \left(\theta < \pi - \varepsilon\right) \quad \text{over } S^+ \qquad \text{(B.7)}$$

and

$$\mathbf{A}^{(II)} = -\left[\frac{g\left(1 + \cos\theta\right)}{r\sin\theta}\right]\hat{\phi}, \qquad \left(\theta > \varepsilon\right) \quad \text{over } S^- \qquad \text{(B.8)}$$

such that the potential $\mathbf{A}^{(I)}$ can be used everywhere except inside the cone defined by $\theta = \pi - \varepsilon$ around the negative z-axis, likewise, the potential $\mathbf{A}^{(II)}$ can be used everywhere except inside the cone $\theta = \varepsilon$ around the positive z-axis, see Fig. B1. Together they lead to the correct expression for **B** everywhere.(An alternative approach to this problem uses A everywhere, but taking special care of the string of singularities, known as a Dirac string, along the negative z-axis.)

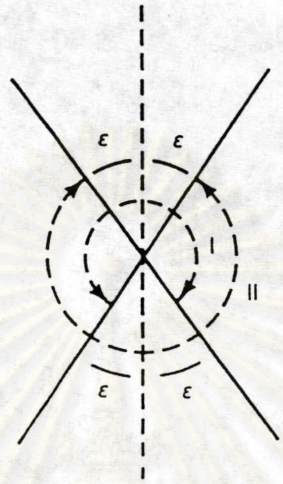

Fig. B1 Regions of validity for the potentials $\mathbf{A}^{(I)}$ and $\mathbf{A}^{(II)}$.

Consider now what happens in the overlab region $\varepsilon < \theta < \pi - \varepsilon$, where we may use either $\mathbf{A}^{(I)}$ or $\mathbf{A}^{(II)}$, Because the two potentials lead to the same magnetic field, they must be related to each other by a gauge transformation. To find Λ appropriate for this problem we first note that

$$\mathbf{A}^{(II)} - \mathbf{A}^{(I)} = -\left(\frac{2g}{r \sin\theta}\right)\hat{\phi} \ . \tag{B.9}$$

Recalling the expression for gradient in spherical coordinates,

$$\nabla\Lambda = \hat{r}\,\frac{\partial\Lambda}{\partial r} + \hat{\theta}\,\frac{1}{r}\,\frac{\partial\Lambda}{\partial\theta} + \hat{\phi}\,\frac{1}{r\sin\theta}\,\frac{\partial\Lambda}{\partial\phi} \tag{B.10}$$

we deduce that

$$\Lambda = -2g\phi \ . \tag{B.11}$$

Now, we get the relation

$$\mathbf{A}^{(II)} - \mathbf{A}^{(I)} = \nabla\Lambda \ . \qquad \text{(B.12)}$$

Next, we consider the wave function of an electrically charged particle of charge e subjected to magnetic field Eq.(B.1) . As we emphasized earlier, the particular form of the wave function depends on the particular gauge used. In the overlap region where we may use either $\mathbf{A}^{(I)}$ or $\mathbf{A}^{(II)}$, the corresponding wave function are

$$\psi^{(II)} = \exp\left[\frac{-2ieg\phi}{\hbar c}\right]\psi^{(I)} \ . \qquad \text{(B.13)}$$

Wave function $\psi^{(I)}$ and $\psi^{(II)}$ must be each be single-valued because once we choose particular gauge, the expansion of the state ket in terms of the position eigenkets must be unique. After all, as we have repeatedly emphasized, the wave function is simply an expansion coefficient for the state ket in term of the position eigenkets.

Let us now examine the behavior of wave function $\psi^{(II)}$ on the equator $\theta = \pi/2$ with some definite radius r, which is a constant. When we increase the azimuthal angle along the equator and go around once, say from $\phi = 0$ to $\phi = 2\pi$, $\psi^{(II)}$, as well as $\psi^{(I)}$, must return to its original value because each is single-valued. According to Eq.(B.13), this is possible only if

$$\frac{2eg}{\hbar c} = \pm p \qquad , p = 0, 1, 2, \cdots \ . \qquad \text{(B.14)}$$

So we reach a very far-reaching conclusion. The magnetic charge must be quantized in units of

$$\frac{\hbar c}{2|e|} \approx \frac{137}{2}|e| \ . \qquad \text{(B.15)}$$

The smallest magnetic charge possible is $\hbar c/2|e|$, where e is the electronic charge.

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