CHAPTER V

MAGNETIC MONOPOLES

Spins in Magnetic Fields

In this chapter we will show the relationship between the geometrical phase and the magnetic monopole (Berry, 1984; Holstein, 1989). The geometrical phase provides an elegant explanation of quantum mechanical phenomena in systems whose environment undergoes a cyclic change. Now, consider spin 1/2 charged particles circling an isolated magnetic field **R**(t) [Note we have written the external magnetic field as **R**(t) so as not to be confused with the curl of the Berry potential which we have denoted by the symbol **B**(t).], and pick **R** along the z axis being the case in which the degeneracy is a spin-type degeneracy. The fast parameter is the spin state of the system, and the slow ones are the polar angles that describe the orientation of the magnetic field. The relevant Hamiltonian is (Stone, 1986)

$$\hat{h}(\mathbf{R}(t)) = -\frac{\mu}{2}\,\sigma\cdot\mathbf{R}(t) \tag{1}$$

$$= -\frac{\mu}{2}\begin{pmatrix} z(t) & x(t) - iy(t) \\ x(t) + iy(t) & -z(t) \end{pmatrix} \tag{2}$$

where μ is the magnetic moment and $\boldsymbol{\sigma}$ are the Pauli spin matrices. The eigenvalues are given, of course, by

$$E_{+}(\mathbf{R}) = -E_{-}(\mathbf{R}) \tag{3}$$

$$= -\frac{\mu}{2}\left[x^2(t) + y^2(t) + z^2(t)\right]^{1/2} \tag{4}$$

so that there exists a degeneracy when **R** = 0. The state function for the fast parameter satisfies

$$\mu\, \sigma \cdot \mathbf{R}\,(t)|\psi(t)\rangle = i\frac{d}{dt}\,|\psi(t)\rangle \tag{5}$$

where the magnitude B of the magnetic field has been absorbed into μ . We consider the large μ limit , so that the slow changes in **R** do not cause transitions between the two spin eigenstates $|\uparrow ; \mathbf{R}\,(t)\rangle$ (spin up eigenstate) and $|\downarrow ; \mathbf{R}\,(t)\rangle$ (spin down eigenstate), in the adiabatic approximation. Suppose at t = 0 we start with $|\uparrow ; \mathbf{R}\,(0)\rangle$, where

$$\mathbf{R}(0) = (\sin\theta_0 \cos\phi_0, \sin\theta_0 \sin\phi_0, \cos\theta_0) \tag{6}$$

At a general time t,

$$\mathbf{R}(t) = (\sin\theta\ \cos\phi\ , \sin\theta\ \sin\phi\ , \cos\theta\ ) \tag{7}$$

where the slow parameters θ and ϕ (polar angle) will vary on the surface of a sphere (S^2). The Berry phase $\gamma_\uparrow$ (t) from Eq.(20) in chapter IV for the spin up ($\uparrow$) eigenstate is

$$\gamma_\uparrow(t) = i\oint_C \langle\uparrow ; \mathbf{R}\,(t)|\nabla_\mathbf{R}|\uparrow ; \mathbf{R}\,(t)\rangle \cdot d\mathbf{R} \tag{8}$$

$$= \oint_C \mathbf{A}\,(\mathbf{R})\cdot d\mathbf{R} \tag{9}$$

We shall calculate this directly using an explicit wave function for $|\uparrow ; \mathbf{R}(t)\rangle$, already there are crucial features that emerge.

The spin up wave function is

$$|\uparrow ; \mathbf{R}\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\, e^{i\phi} \end{pmatrix} \tag{10}$$

for a spinor along the direction **R** specified by spherical coordinate angles θ and ϕ. Then from Eq.(8) the associated **A(R)** for the spin up state is given by

$$\mathbf{A}_{\uparrow}(\mathbf{R}) = i\langle\uparrow ; \mathbf{R}|\nabla_{R}|\uparrow ; \mathbf{R}\rangle \tag{11}$$

Substituting the state function Eq.(10) into Eq.(11) we have that

$$\mathbf{A}_{\uparrow}(\mathbf{R}) = i\begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2}\, e^{-i\phi} \end{pmatrix} \nabla_{\mathbf{R}} \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\, e^{i\phi} \end{pmatrix} \tag{12}$$

where $\nabla_{\mathbf{R}}$ in spherical form is

$$\nabla_{\mathbf{R}} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} \tag{13}$$

We get the vector potential

$$\mathbf{A}_{\uparrow}(\mathbf{R}) = -\hat{\phi}\frac{1}{r\sin\theta}\sin^{2}\frac{\theta}{2} \tag{14}$$

From the trigonometric formula

$$\sin^2\frac{\theta}{2} = \frac{1}{2}\left(1 - \cos\theta\right)$$

this vector potential can be rewritten as

$$\mathbf{A}_{\uparrow}(\mathbf{R}) = -\hat{\phi}\,\frac{\left(1 - \cos\theta\right)}{2r\sin\theta} \tag{15}$$

which is the "vector potential" of a magnetic monopole of strength 1/2 located at $\mathbf{R} = 0$ (the center of the sphere). Hence indeed, we obtain the Berry phase

$$\gamma_{\uparrow}(\mathbf{R}) = -\oint \hat{\phi}\,\frac{\left(1 - \cos\theta\right)}{2r\sin\theta}\cdot d\mathbf{R} \tag{16}$$

$$= 2\pi\left(1 - \cos\theta\right) \tag{17}$$

However, the wave function in Eq.(10) is ill-defined at $\theta = \pi$. An alternative choice of spin up ($\uparrow$) wave function which is well defined at $\theta = \pi$ is

$$\left|\uparrow;\mathbf{R}\right\rangle = \begin{pmatrix} \cos\frac{\theta}{2}\,e^{-i\phi} \\ \sin\frac{\theta}{2} \end{pmatrix} \tag{18}$$

Repeating the above calculations we find the vector potential

$$\mathbf{A}_{\uparrow}(\mathbf{R}) = \hat{\phi}\,\frac{\left(1 + \cos\theta\right)}{2r\sin\theta} \tag{19}$$

which is equivalent to the Berry Phase

$$\gamma_{\uparrow}(\mathbf{R}) = \oint \hat{\phi}\,\frac{(1+\cos\theta)}{2r\sin\theta}\cdot d\mathbf{R} \tag{20}$$

so that

$$\gamma_{\uparrow}(t) = -2\pi(1+\cos\theta)\,. \tag{21}$$

Though good at $\theta = \pi$, Eq(18) is ill-defined at $\theta = 0$ and in fact we are hitting here the famous problem that, for a monopole field, no single vector potential exists which is singularity-free over the entire manifold S^2 (Wu and Yang, 1976). The $\mathbf{A}_+(\mathbf{R})$ which followed from the choice in Eq. (10) has a singularity along z = -r, i.e. the negative z-axis, or $\theta = \pi$. This line of singularities is called a Dirac string [Appendix B]. Likewise, the $\mathbf{A}_-(\mathbf{R})$choice has a string along $\theta = 0$. But, comparing Eq.(15) and Eq.(19) we see that $\mathbf{A}_+$ and $\mathbf{A}_-$ differ by a gradient

$$\mathbf{A}_+(\mathbf{R}) - \mathbf{A}_-(\mathbf{R}) = \nabla\Lambda \tag{22}$$

that is, by a gauge transformation [Appendix A]. Correspondingly,

$$\gamma_{\uparrow}^{+} - \gamma_{\uparrow}^{-} = -[\Lambda(t) - \Lambda(0)] \tag{23}$$

so that for a closed path on S^2, $\gamma_{\uparrow}$ is unique

Magnetic Monopole Fields

In this section we will show how to obtain a magnetic field due to the monopole charge at center of a sphere. We consider the vector potential shown in Eq.(15) and Eq.(21). But these must yield the same curvature. Hence

$$\mathbf{B}(\mathbf{R}) = \nabla_{\mathbf{R}} \times \mathbf{A}(\mathbf{R}) \tag{24}$$

$$= -\hat{r}\,\frac{1}{2r^2} \tag{25}$$

which corresponds to a magnetic monopole of strength 1/2 located at the origin (i.e., at the place of the degeneracy). According to Eq.(9), Berry's phase is simply the line integral of the vector potential which, by Stokes' theorem can be rewritten in terms of the surface integral of the magnetic fields

$$\gamma_{\uparrow}(t) = \oiint_S \mathbf{B}(\mathbf{R}) \cdot d\mathbf{S} \tag{26}$$

$$= \pm\,\pi\left(1 - \cos\theta\right) \tag{27}$$

The solid angle swept out by this trajectory is

$$\Omega(C) = \int_0^{\theta} \sin\theta\, d\theta \int_0^{2\pi} d\phi = 2\pi\left(1 - \cos\theta\right) \tag{28}$$

so that the Berry phase is the flux associated with this monopole through the surface which is circumnavigated in parameter space, and is given by

$$\gamma_{\uparrow} = \pm\frac{\Omega(C)}{2} \tag{29}$$

where $\Omega(C)$ is the solid angle subtended by the closed path as seen from the origin of parameter space and the sign refers to the direction in which the line integral is traversed.