CHAPTER IV

EFFECTIVE ACTION FOR ADIABATIC PROCESS

In this chapter we shall be interested in the path integral frame work in topological structure as stimulated by the recent discovery of the quantum adiabatic theorem. The problem is stated as follows. Kuratsuji and Iida find this way of formulating the Born-Oppenheimer idea much more appropriate, and easier to generalize, than the usual formulation in terms of fast and slow variables. The connection between the two is as follows. Transitions to states separated by a large energy gap require large changes in frequency, and are therefore associated with fast variables. Rapid oscillations in time accompany such transitions, and lead to cancellations in processes whose characteristic time scale is much longer than, in processes associated with motion of the slow variables. Towards the end of this chapter we shall discuss the relationship between these two approaches more precisely. It is appropriate to mention one conclusion from that discussion now, however, we shall find that quantum variables can only be slow in a very weak sense. For example, in a path integral description, the important space-time paths are not differentiable, and the typical velocity is strictly speaking infinite even for so called slow variables throughout this thesis.

Lagrangian Formulation

Often it is more convenient to work with a path integral description. Phenomenological models are typically easier to formulate in terms of a Lagrangian, where symmetries are manifest. Non equilibrium and non perturbative problems, such as calculating tunneling amplitudes may be easier to solve in the language of path integrals. In addition, as we shall see, it is much easier to incorporate corrections to the adiabatic approximation (which are higher order in time derivatives) in an effective Lagrangian.

In the derivation of effective Lagrangians, we should expect, and will find, that geometrical phases occur. This is particularly clear if we think in terms of path integrals. Then along any particular path the slow degrees of freedom can be considered as external parameters governing the state of the fast ones. Therefore, the amplitude for such a path can contain a geometrical phase factor of the classic type. Geometrical phases of this sort are connected with some of the most subtle and interesting phenomena in quantum field theory, including the occurrence of fractional quantum numbers and anomalies . Although Berry's phases have such an appealing feature, they are still concerned with the static aspect only, i.e., the time development of the external parameter space is given form the outset. Actually, the parameter space itself can be regarded as a dynamic object. For example, in the Born-Oppenheimer theory, the internuclear distance, which is frozen is regarded as a dynamical variable. Thus we are forced to inquire a dynamical meaning to Berry's and Simon's topological phases. The review of this chapter is to put forward an answer to this question. A similar dynamical argument was suggested by Mead and Truhlar before Berry and Simon. They showed that this specific phase acquires a meaning of the effective vector potential in the Schrödinger equation for the nuclear motion in molecular collisions. However, the argument based on the Schrödinger equation is of essentially local nature and does not seem to be appropriate for describing the global character of this non integrable phase. In order to push the global aspect forward, we adopt the path integral formulation for the bound state problem of two interacting systems. In this formulation, the geometrical phase naturally arises as an additional action to the conventional action function induced by adiabatic processes. Simultaneously this topological action is shown to modify the semiclassical quantization rule for the motion of the external system.

Effective action by path integral

Consider two interacting systems, which are described by variable conventionally called "internal" and "collective" coordinates; $\mathbf{r}$ and $\mathbf{R}$ respectively. We adopt a Hamiltonian

$$\hat{H} = \hat{h}(\mathbf{r}, \mathbf{R}) + \hat{H}_0(\mathbf{P}, \mathbf{R}) \tag{1}$$

where the internal Hamiltonian h is assumed to depend on $\mathbf{R}$ and not on its conjugate momentum $\mathbf{P}$. Let us consider the trace of the evolution operator

$$K(T) = \mathrm{Tr}\left(\exp\left[-\frac{i}{\hbar}\hat{H}\,T\right]\right)$$

which is written as

$$K(T) = \sum_{n} \int \left\langle n(\mathbf{R}_0), \mathbf{R}_0 \left| \exp\left[-\frac{i}{\hbar}\hat{H}\,T\right] \right| n(\mathbf{R}_0), \mathbf{R}_0 \right\rangle d\mu(\mathbf{R}_0) \tag{2}$$

In Eq.(2) one naturally picks up the transition amplitude for the quantum process starting from the initial state $|n(\mathbf{R}_0), \mathbf{R}_0\rangle$ $(= |n(\mathbf{R}_0)\rangle\!\rangle \otimes |\mathbf{R}_0\rangle)$ and returning via closed loops C to the same state. Here $|\mathbf{R}_0\rangle$ denotes the eigenstate of $\hat{H}_0(\mathbf{P}, \mathbf{R})$ and $|n(\mathbf{R}_0)\rangle\!\rangle$ is the eigenstate of $\hat{h}(\mathbf{r}, \mathbf{R})$ at $\mathbf{R} = \mathbf{R}_0$ with eigenvalue $E_n(\mathbf{R}_0)$. Then with the aid of time-discretization together with the completeness relation holding for $|\mathbf{R}\rangle$, we get

$$\left\langle n(\mathbf{R}_0), \mathbf{R}_0 \left| \exp\left[-\frac{i}{\hbar}\hat{H}\,T\right] \right| n(\mathbf{R}_0), \mathbf{R}_0 \right\rangle$$

$$= \left\langle n(\mathbf{R}_0), \mathbf{R}_0 \left| \exp\left[-\frac{i}{\hbar}\varepsilon\,\hat{H}\,N\right] \right| n(\mathbf{R}_0), \mathbf{R}_0 \right\rangle \tag{3}$$

with $\varepsilon = T/N$ and the identity

$$\exp\left[-\frac{i}{\hbar}\varepsilon\hat{H}N\right] = \lim_{N\to\infty}\left(1-\frac{i}{\hbar}\varepsilon\hat{H}\right)^N.$$

The R.H.S. of Eq.(3) is equal to

$$= \lim_{N\to\infty}\left\langle n(\mathbf{R}_0),\mathbf{R}_0\left|\left(1-\frac{i}{\hbar}\varepsilon\hat{H}\right)^N\right|n(\mathbf{R}_0),\mathbf{R}_0\right\rangle \tag{4}$$

$$= \lim_{N\to\infty}\left\langle n(\mathbf{R}_0),\mathbf{R}_0\left|\underbrace{\left(1-\frac{i}{\hbar}\varepsilon\hat{H}\right)\cdots\left(1-\frac{i}{\hbar}\varepsilon\hat{H}\right)}_{\text{N factor}}\right|n(\mathbf{R}_0),\mathbf{R}_0\right\rangle \tag{5}$$

insert complete set

$$= \lim_{N\to\infty}\int\cdots\int\left\langle n(\mathbf{R}_0),\mathbf{R}_0\left|1-\frac{i}{\hbar}\varepsilon\hat{H}\right|\mathbf{R}_{N-1}\right\rangle d\mathbf{R}_{N-1}$$

$$\left\langle\mathbf{R}_{N-1}\left|1-\frac{i}{\hbar}\varepsilon\hat{H}\right|\mathbf{R}_{N-2}\right\rangle d\mathbf{R}_{N-2}\cdots\left\langle\mathbf{R}_2\left|1-\frac{i}{\hbar}\varepsilon\hat{H}\right|\mathbf{R}_1\right\rangle d\mathbf{R}_1$$

$$\left\langle\mathbf{R}_1\left|1-\frac{i}{\hbar}\varepsilon\hat{H}\right|n(\mathbf{R}_0),\mathbf{R}_0\right\rangle$$

$$= \lim_{N\to\infty}\int\cdots\int\prod_{k=1}^{N-1}D[\mathbf{R}_k]\left\langle n(\mathbf{R}_0),\mathbf{R}_0\left|\exp\left[1-\frac{i}{\hbar}\varepsilon\hat{H}\right]\right|\mathbf{R}_{N-1}\right\rangle\cdot$$

$$\left\langle\mathbf{R}_{N-1}\left|\exp\left[1-\frac{i}{\hbar}\varepsilon\hat{H}\right]\right|\mathbf{R}_{N-2}\right\rangle\cdots\left\langle\mathbf{R}_2\left|\exp\left[1-\frac{i}{\hbar}\varepsilon\hat{H}\right]\right|\mathbf{R}_1\right\rangle$$

$$\left\langle\mathbf{R}_1\left|\exp\left[1-\frac{i}{\hbar}\varepsilon\hat{H}\right]\right|n(\mathbf{R}_0),\mathbf{R}_0\right\rangle \tag{6}$$

Further noting the relation for $\varepsilon \approx 0$ and that ε and $|\mathbf{R}_k - \mathbf{R}_{k-1}|$ are small

$$\left\langle \mathbf{R}_k \middle| \exp\left[1 - \frac{i}{\hbar}\hat{H}\,\varepsilon\right] \middle| \mathbf{R}_{k-1} \right\rangle$$

$$= \left\langle \mathbf{R}_k \middle| \exp\left[1 - \frac{i}{\hbar}\hat{H}_0\,\varepsilon\right] \middle| \mathbf{R}_{k-1} \right\rangle e^{-\frac{i}{\hbar}\hat{h}(\mathbf{R}_k)\varepsilon}$$

$$= \int d\mathbf{P}_k \langle \mathbf{R}_k | \mathbf{P}_k \rangle \left\langle \mathbf{P}_k \middle| \exp\left[1 - \frac{i}{\hbar}\hat{H}_0\,\varepsilon\right] \middle| \mathbf{R}_{k-1} \right\rangle e^{-\frac{i}{\hbar}\hat{h}(\mathbf{R}_k)\varepsilon}$$

$$= \int d\mathbf{P}_k \langle \mathbf{R}_k | \mathbf{P}_k \rangle \langle \mathbf{P}_k | \mathbf{R}_{k-1} \rangle\, e^{-\frac{i}{\hbar}\hat{H}_0(\mathbf{R}_k)\varepsilon}\, e^{-\frac{i}{\hbar}\hat{h}(\mathbf{R}_k)\varepsilon} \qquad (7)$$

From quantum mechanics, the momentum eigenfunctions $\langle \mathbf{r} | \mathbf{p} \rangle$ are

$$\langle \mathbf{r} | \mathbf{p} \rangle = \frac{1}{(2\pi\hbar)^{3/2}} \exp\left[\frac{i}{\hbar}\ \mathbf{p}\cdot\mathbf{r}\right]$$

Therefore

$$\left\langle \mathbf{R}_k \middle| \exp\left[-\frac{i}{\hbar}\hat{H}\,\varepsilon\right] \middle| \mathbf{R}_{k-1} \right\rangle$$

$$= \frac{1}{(2\pi\hbar)^3} \int \exp\left[\frac{i}{\hbar}\mathbf{P}_k\cdot(\mathbf{R}_k - \mathbf{R}_{k-1}) - \frac{i}{\hbar}\hat{H}_0(\mathbf{R}_k, \mathbf{P}_k)\,\varepsilon\right] e^{-\frac{i}{\hbar}\hat{h}(\mathbf{R}_k)\varepsilon}\, d\mathbf{P}_k \qquad (8)$$

Eq.(2) can be expressed as

$$K(T) = \sum_n \int T_{nn}(C) \exp\left[\frac{i}{\hbar} S_0(C)\right] \prod_t d\mu(\mathbf{R}_t, \mathbf{P}_t) \qquad (9)$$

where

$$S_0(C) = \int \left(\mathbf{P} \cdot \dot{\mathbf{R}} - \hat{H}_0\right) dt$$

is the action for the collective motion along closed loops C. T_{nn} (C) is just the internal transition amplitude and given by

$$T_{nn}(C) = \left\langle n(\mathbf{R}_0) \middle| \exp\left[-\frac{i}{\hbar}\hat{h}(N)\varepsilon\right] \cdots \exp\left[-\frac{i}{\hbar}\hat{h}(1)\varepsilon\right] \middle| n(\mathbf{R}_0) \right\rangle \qquad (10)$$

i.e., the time ordered product, where $\hat{h}(k)$ denotes the internal Hamiltonian at the point $\mathbf{R} = \mathbf{R}_k$ on the loop C (Pechukas, 1969). Namely, if we denote $|\varnothing_n(T)\rangle$ as a solution of the time-dependent Schrödinger equation

$$\left(i\hbar\frac{\partial}{\partial t} - \hat{h}(\mathbf{r}, \mathbf{R})\right)|\varnothing_n(t)\rangle = 0$$

with the boundary condition $|\varnothing_n(0)\rangle = |n(\mathbf{R}_0)\rangle$, T_{nn} (C) is written as

$$T_{nn}(C) = \langle n(\mathbf{R}_0)|\varnothing_n(t)\rangle$$

Under the above prescription we turn to the case of the adiabatic motion where the period T is large. By inserting the completeness relation holding for the internal state on each point of external variables $\mathbf{R}_k$;

$$\sum_{m_k} |m_k\rangle\langle m_k| = 1$$

Eq. (10) is written as

$$T_{nn}(C) = \left\langle n(\mathbf{R}_0) \middle| e^{-\frac{i}{\hbar}\hat{h}(N)\varepsilon} e^{-\frac{i}{\hbar}\hat{h}(N-1)\varepsilon} \cdots e^{-\frac{i}{\hbar}\hat{h}(2)\varepsilon} e^{-\frac{i}{\hbar}\hat{h}(1)\varepsilon} \middle| n(\mathbf{R}_0) \right\rangle \qquad (11)$$

$$T_{nn}(C) = \sum_{m_1} \left\langle n(\mathbf{R}_0) \left| e^{-\frac{i}{\hbar}\hat{h}(N)\varepsilon} e^{-\frac{i}{\hbar}\hat{h}(N-1)\varepsilon} \cdots e^{-\frac{i}{\hbar}\hat{h}(3)\varepsilon} e^{-\frac{i}{\hbar}\hat{h}(2)\varepsilon} \right| m_1 \right\rangle$$

$$\left\langle m_1 \left| e^{-\frac{i}{\hbar}\hat{h}(1)\varepsilon} \right| n(\mathbf{R}_0) \right\rangle$$

$$T_{nn}(C) = \sum_{m_1} \sum_{m_2} \left\langle n(\mathbf{R}_0) \left| e^{-\frac{i}{\hbar}\hat{h}(N)\varepsilon} e^{-\frac{i}{\hbar}\hat{h}(N-1)\varepsilon} \cdots e^{-\frac{i}{\hbar}\hat{h}(3)\varepsilon} \right| m_2 \right\rangle$$

$$\left\langle m_2 \left| e^{-\frac{i}{\hbar}\hat{h}(2)\varepsilon} \right| m_1 \right\rangle \left\langle m_1 \left| e^{-\frac{i}{\hbar}\hat{h}(1)\varepsilon} \right| n(\mathbf{R}_0) \right\rangle$$

$$T_{nn}(C) = \sum_{m_1} \sum_{m_2} \cdots \sum_{m_{N-3}} \left\langle n(\mathbf{R}_0) \left| e^{-\frac{i}{\hbar}\hat{h}(N)\varepsilon} e^{-\frac{i}{\hbar}\hat{h}(N-1)\varepsilon} e^{-\frac{i}{\hbar}\hat{h}(N-2)\varepsilon} \right| m_{N-3} \right\rangle$$

$$\left\langle m_{N-3} \left| e^{-\frac{i}{\hbar}\hat{h}(N-3)\varepsilon} \right| m_{N-4} \right\rangle \cdots \left\langle m_1 \left| e^{-\frac{i}{\hbar}\hat{h}(1)\varepsilon} \right| n(\mathbf{R}_0) \right\rangle$$

$$T_{nn}(C) = \sum_{m_1} \sum_{m_2} \cdots \sum_{m_{N-3}} \sum_{m_{N-2}} \left\langle n(\mathbf{R}_0) \left| e^{-\frac{i}{\hbar}\hat{h}(N)\varepsilon} e^{-\frac{i}{\hbar}\hat{h}(N-1)\varepsilon} \right| m_{N-2} \right\rangle$$

$$\left\langle m_{N-2} \left| e^{-\frac{i}{\hbar}\hat{h}(N-2)\varepsilon} \right| m_{N-3} \right\rangle \left\langle m_{N-3} \left| e^{-\frac{i}{\hbar}\hat{h}(N-3)\varepsilon} \right| m_{N-4} \right\rangle$$

$$\cdots \left\langle m_1 \left| e^{-\frac{i}{\hbar}\hat{h}(1)\varepsilon} \right| n(\mathbf{R}_0) \right\rangle$$

$$T_{nn}(C) = \sum_{m_1} \cdots \sum_{m_{N-1}} \left\langle n(\mathbf{R}_0) \right| e^{-\frac{i}{\hbar}\hat{h}(N)\varepsilon} \left| m_{N-1} \right\rangle \left\langle m_{N-1} \right| e^{-\frac{i}{\hbar}\hat{h}(N-1)\varepsilon} \left| m_{N-2} \right\rangle$$

$$\cdots \left\langle m_1 \right| e^{-\frac{i}{\hbar}\hat{h}(1)\varepsilon} \left| n(\mathbf{R}_0) \right\rangle \qquad (12)$$

Now consider the term $\left\langle m_1 \middle| 1 - \frac{i}{\hbar}\hat{h}(1)\varepsilon \middle| n(\mathbf{R}_0) \right\rangle$. In the adiabatic approximation, we pick up the quantum transition only between the states with the same quantum number n ; $\left\langle n_k \middle| \exp\left[-\frac{i}{\hbar}\hat{h}(1)\varepsilon\right] \middle| n_{k-1} \right\rangle$. Then using the relation

$$\hat{h}(k)| n_k\rangle = E_n(k)| n_k\rangle$$

where $E_n(k)$ is an energy of an adiabatic level n at $\mathbf{R} = \mathbf{R}_k$,

$$\left\langle m_1 \middle| 1 - \frac{i}{\hbar}\hat{h}(1)\varepsilon \middle| n(\mathbf{R}_0) \right\rangle = \left\langle m_1 \middle| \left(1 - \frac{i}{\hbar}\hat{h}(1)\varepsilon\right) \middle| n(\mathbf{R}_0) \right\rangle$$

$$= \langle m_1 | n(\mathbf{R}_0)\rangle - \frac{i}{\hbar} \langle m_1 | \hat{h}(1) | n(\mathbf{R}_0)\rangle \, \varepsilon$$

$$= \langle m_1 | n(\mathbf{R}_0)\rangle - \frac{i}{\hbar} E_m(1) \langle m_1 | n(\mathbf{R}_0)\rangle \, \varepsilon$$

$$= \left(1 - \frac{i}{\hbar} E_m(1)\right) \langle m_1 | n(\mathbf{R}_0)\rangle$$

$$= \exp\left[-\frac{i}{\hbar} E_m(1)\varepsilon\right] \langle m_1 | n(\mathbf{R}_0)\rangle \qquad (13)$$

consider Eq.(12), we used Eq.(13) so that , Eq.(12) can be rewritten as

$$T_{nn}(C) = \sum_{m_1}\cdots\sum_{m_{N-1}} e^{-\frac{i}{\hbar}E_m(N)\varepsilon}\langle n(\mathbf{R}_0)|m_{N-1}\rangle e^{-\frac{i}{\hbar}E_m(N-1)\varepsilon}\langle m_{N-1}|m_{N-2}\rangle$$

$$\cdots e^{-\frac{i}{\hbar}E_m(3)\varepsilon}\langle m_3|m_2\rangle e^{-\frac{i}{\hbar}E_m(2)\varepsilon}\langle m_2|m_1\rangle e^{-\frac{i}{\hbar}E_m(1)\varepsilon}\langle m_1|n(\mathbf{R}_0)\rangle \qquad (14)$$

we now obtain

$$T_{nn}(C) = \exp\left[-\frac{i}{\hbar}\int_0^T E_n(\mathbf{R}_t)\,dt\right]\langle n(\mathbf{R}_0)|\,n(\mathbf{R}_T)\rangle_C \qquad (15)$$

Here the overlap function $\langle n(\mathbf{R}_0)|\,n(\mathbf{R}_T)\rangle_C$ is given as an infinite product

$$\langle n(\mathbf{R}_0)|\,n(\mathbf{R}_T)\rangle_C = \lim_{N\to\infty}\prod_{k=1}^{N}\langle n(\mathbf{R}_k)|\,n(\mathbf{R}_{k-1})\rangle \qquad (16)$$

where we adopt a phase convention $|n(\mathbf{R}_0)\rangle = |N(\mathbf{R}_T)\rangle_C$. This overlap function naturally involves the history of excursion in the **R**-space which is indicated by the suffix C. Each factor $\langle n(\mathbf{R}_k)|\,n(\mathbf{R}_{k-1})\rangle$ in Eq.(16) defines a connection between two infinitesimally separated points $\mathbf{R}_k$ and $\mathbf{R}_{k-1}$, hence Eq.(16) gives a finite connection along circuit C given by a set of division points $\{\mathbf{R}_k\}$. Thus, by using the approximate relation

$$\langle n(\mathbf{R}_k)|\,n(\mathbf{R}_{k-1})\rangle \approx 1 - \left\langle n\left|\frac{\partial}{\partial \mathbf{R}_i}\right|n\right\rangle \Delta\mathbf{R}_i$$

$$\approx \exp\left[-\left\langle n\left|\frac{\partial}{\partial \mathbf{R}_i}\right|n\right\rangle \Delta\mathbf{R}_i\right]$$

$$\approx \exp\left[i \left\langle n \middle| i \frac{\partial}{\partial R_i} \middle| n \right\rangle \Delta R_i \right] \tag{17}$$

Eq(16). is written as

$$\langle n(R_0) | n(R_T) \rangle_C = \exp\left[i \oint i \left\langle n \middle| \frac{\partial}{\partial R_k} \middle| n \right\rangle \cdot dR_k \right]$$

$$= \exp[i\gamma_n(C)] \tag{18}$$

with

$$\gamma_n(C) = i \oint_C \left\langle n \middle| \frac{\partial}{\partial R_k} \middle| n \right\rangle \cdot dR_k \tag{19}$$

or

$$\gamma_n(C) = i \oint_C \langle n(R) | \nabla_R n(R) \rangle \cdot dR$$

$$= \oint_C A_n(R) \cdot dR$$

$$= \oint_C A_n(R) \cdot \frac{dR}{dt} dt \tag{20}$$

This is essentially the same as the phase obtained by Berry. However the present derivation of Kuratsuji and Iida is quite different form Berry's and somewhat similar to Simon's (Simon, 1983) which is based upon the holonomy of vector bundles over R-space. Eq.(15) will be rewritten in term of Berry's phases

$$T_{nn}(C) = \exp\left[-\frac{i}{\hbar}\int_0^T E_n(\mathbf{R}_t)\,dt\right]\exp[i\gamma_n(C)] \tag{21}$$

and we can write the propagator in Eq.(9) as

$$K(T) = \sum_n \int \exp\left[-\frac{i}{\hbar}\int_0^T E_n(\mathbf{R}_t)\,dt\right]\exp[i\gamma_n(C)]\exp\left[\frac{i}{\hbar}S_0(C)\right]\prod_t d\mu(\mathbf{R}_t, \mathbf{P}_t) \tag{22}$$

Thus we arrive at the effective path integral associated with the adiabatic change of the external dynamical variable **R**,

$$K^{eff}(T) = \sum_n \int \exp\left[\frac{i}{\hbar}\left(S_n^{ad} + \hbar\,\gamma_n(C)\right)\right]\prod_t d\mu(\mathbf{R}_t, \mathbf{P}_t) \tag{23}$$

where

$$S_n^{ad} = S_0 - \int_0^T E_n(\mathbf{R}_t)\,dt$$

is the adiabatic action function. From (23) we get a natural explanation that the phase $\gamma_n(C)$ appears as a topological action function which is to be added to the usual dynamical action. We want the dynamics in **R**-space to correspond to a particle moving in an additional vector space $\mathbf{A}_n(\mathbf{R})$. Thus we expect to find a piece in the effective action S_n^{eff} in **R**-space which corresponds to the effective Lagrangian L_n^{eff} (Aitchison, 1987)

$$L_n^{eff} = \mathbf{A}_n(\mathbf{R})\cdot\frac{d\mathbf{R}}{dt}$$

Indeed, in that case

$$S_n^{eff}(T) = \int_0^T L_n^{eff}\,dt = \int_0^T \mathbf{A}_n(\mathbf{R}) \cdot \frac{d\mathbf{R}}{dt}\,dt$$

and

$$\exp\left[iS_n^{eff}(T)\right] = \exp\left[i\gamma_n(T)\right]$$

This result is just what Kuratsuji and Iida obtain.

Semiclassical Quantization Rule

In this section we will consider the effect of the topological phase. The most direct way for this is to examine the energy spectra. The energy spectra is rapidly estimated by the semiclassical quantization rule (Gutzwiller, 1972; Miller, 1975) which is derived base on the effective propagator Eq.(23). Consider the Fourier transform of $K^{eff}(T)$, where we restrict ourselves to a specific adiabatic level n. Firstly the semiclassical limit of $K^{eff}(T)$ is approximated by the method of stationary phase,

$$K^{sc}(T) \approx \sum_{P.O.} \exp\left[\frac{i}{\hbar} S^{ad}(C) + i\gamma(C) - i\frac{\pi}{2}\alpha(C)\right] \tag{24}$$

where $\alpha(C)$ denotes the so-called Keller-Maslov index and $\sum_{P.O.}$ indicated is the sum over periodic orbits. Next, taking the Fourier transform of effective propagator $K^{eff}(T)$

$$K(E) = i\int K^{eff}(T)\exp\left[\frac{i}{\hbar}ET\right] dT$$

$$K^{sc}(E) = i \sum_{P.O.} \int_0^\infty \exp\left[\frac{i}{\hbar} S^{ad}(C) + i\gamma(C) - i\frac{\pi}{2}\alpha(C)\right] \exp\left[\frac{i}{\hbar} ET\right] dT$$

$$= i \sum_{P.O.} \int_0^\infty \exp\left[\frac{i}{\hbar} S^{ad}(C) + \frac{i}{\hbar} ET + i\gamma(C) - i\frac{\pi}{2}\alpha(C)\right] dT . \qquad (25)$$

and evaluating the integral over T by the stationary phase, then we get the semiclassical propagator in energy form

$$K^{sc}(E) \approx \sum_{P.O.} \exp\left[\frac{i}{\hbar} S^{ad}(C) + \frac{i}{\hbar} ET + i\gamma(C) - i\frac{\pi}{2}\alpha(C)\right]$$

$$\approx \sum_{P.O.} \exp\left[\frac{i}{\hbar} W^{ad}(E) + i\gamma(C) - i\frac{\pi}{2}\alpha(C)\right] \qquad (26)$$

where $W^{ad}(E) = S^{ad} + ET$ (action integral) and T(E) is determined by the stationary phase condition

$$\frac{\partial}{\partial T}\left(S^{ad} + ET\right) = 0.$$

Here, we restrict ourselves to the case that there appear a finite number of isolated closed orbits for each value of the energy. For this case, a semiclassical quantization condition can be written down explicitly. Namely, taking account of the contribution from the multiple traversals of basic orbits, i.e., puting $W^{ad} \rightarrow m \cdot W^{ad}$, $\alpha \rightarrow m \cdot \alpha$ and $\gamma \rightarrow m \cdot \gamma$ for m-times traversals and summing over m, $K^{sc}(E)$ turns out to be

$$K^{sc}(E) \approx \sum_{P.O.} \exp\left[\frac{i}{\hbar} \widetilde{W}\right] \left(1 - \exp\left[\frac{i}{\hbar} \widetilde{W}\right]\right)^{-1} \qquad (27)$$

with

$$\widetilde{W}(E) = W^{ad}(E) + \hbar\gamma(C) - \frac{\hbar\alpha}{2\pi} .$$

From the pole of Eq.(27)

$$\exp\left[\frac{i}{\hbar}\widetilde{W}\right] = 1$$

so that

$$\frac{\widetilde{W}}{\hbar} = 2n\pi$$

where n is the integer. We find the following quantization rule.

$$W^{ad}(E) = \oint_C \mathbf{P}\, d\mathbf{R} = \left(n + \frac{\alpha}{4} - \frac{\gamma(C)}{2\pi}\right) 2\pi\hbar \qquad (28)$$

This gives the energy spectrum for the collective motion including the effect of the topological phase γ.

The relation between geometrical phase and magnetic monopole will be explained in the next chapter.