CHAPTER II.

THEORETICAL BACKGROUND

Introduction

Many structures and structural systems whose response cannot be adequately described by linear theories are being encountered in significant structural engineering applications. The nonlinear interaction of pile and surrounding soil is an example of such problems. For the purpose of structural safety and economy in design, it is important to have accurate structural analysis methods that take into account nonlinear behaviour of the structures. Although nonlinear theories in continuum mechanics have been in existence for many years, practical applications of these theories have been limited by the complexity of the initial and boundary value problems. For most practical structural applications, nonlinear analyses can be performed only with the aid of numerical methods, e.g., finite element analysis. Applications of the finite element method to nonlinear problems in soil mechanics were found to produce accurate solutions (17,18,19). Therefore, in order to make a comprehensive study of the problem of nonlinear behaviour of sway piles under horizontal soil movements, a nonlinear elastic static finite element program should be employed for analyses.

In this chapter, the development of a nonlinear finite element program will be briefly revived and discussed. Then, the computation of pile curvatures will be described.

Finite Element Formulation for Large Deformation Elastic Static Analysis

An incremental finite element formulation for nonlinear elastic static analyses which account for large displacements and large strains can be derived using the concept of the incremental linear analyses with corrections to account for changes in geometry. The numerical solution of the continuum mechanics is carried out using isoparametric finite element discretization which has proved to be very effective in many applications.

1. An Incremental Nonlinear Formulation of Equations of Motion for Finite Deformation

A formulation of the equations of motion for finite deformation response can be developed using the concept of the incremental theory in connection with the priciple of virtual work. Kinematic variables are derived from the basic theories in continuum mechanics. The formulation presented here closely follows those appearing in references (20,21,22).

Conceptually, the formulation of the incremental nonlinear equations of motion requires that the path of deformation of a body be divided into a number of equilibrium states ${}^{o}\Omega$, ${}^{1}\Omega,\ldots,$ ${}^{n}\Omega$, ${}^{n+1}\Omega,\ldots,$ ${}^{f}\Omega$ where ${}^{o}\Omega$ and ${}^{f}\Omega$ are the initial and final states of the deformation, respectively, while ${}^{n}\Omega$ is an arbitrary intermediate state. It is assumed that all the state variables such as stresses, strains and displacements, together with the loading history, are known up to the $^{n}\Omega$ state and the state variables in the $^{n+1}\Omega$ state are required next. Then the equation of incremental virtual work between the state $^{n}\Omega$ and $^{n+1}\Omega$ is established to express the equilibrium of the body in the state $^{n+1}\Omega$. However, the configuration at $^{n+1}\Omega$ is unknown, and therefore all the state variables must be referred to a known or previously calculated equilibrium state. In principle, any one of the already calculated equilibrium states can be used. Basically, the state variables are referred to either the initial $^{o}\Omega$ state or the current equilibrium $^{n}\Omega$ state and the corresponding formulations are called Lagrangian formulation and Eulerian or moving co-ordinate or updated formulation, respectively. The stresses and strains in the first formulation are the second Piola-Kirchhoff stress and the Green-Lagrange strain, while those in the second formulation are the Cauchy stress and strain. The incremental process for the next required equilbrium state is typical and would be applied repetitively until the final state, $^{f}\Omega$, has been reached.

Consider the motion of the body as shown in Figure 2.1. There are three configulations in its path of deformation that are of interest, i.e.,

(a) the undeformed configuration C_o,

(b) the current deformed configuration C_1, and

(c) a second deformed configuration C_2

taken as a neighboring configuration to the current deformed configuration C_1.

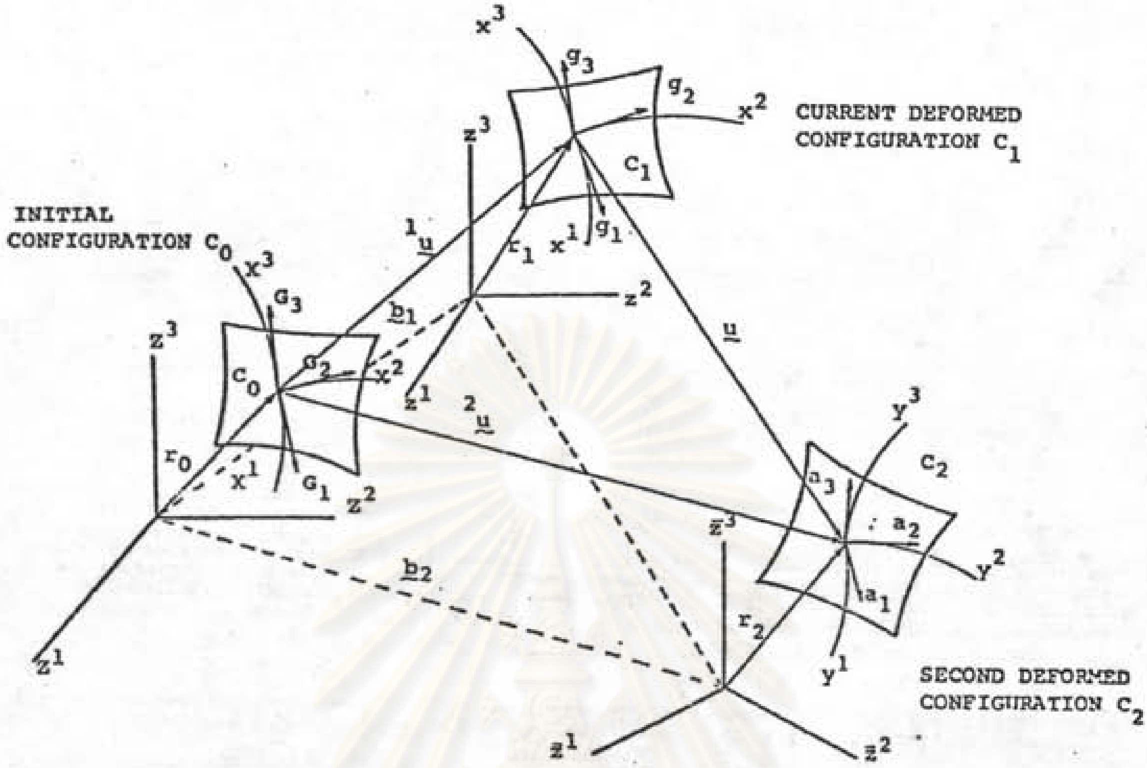

FIGURE 2.1 Deformation Path of a Body.

The state variables in configuration C_1 and C_2 are defined as follows:

$${}^{1}S_{IJ}\ ,\ {}^{1}E_{IJ}\ ,\ {}^{1}U_{I}\ ,\ {}^{1}t_{I}\ ,\ {}^{1}f_{I} \qquad \text{in } C_1$$

$${}^{2}S_{IJ}\ ,\ {}^{2}E_{IJ}\ ,\ {}^{2}U_{I}\ ,\ {}^{2}t_{I}\ ,\ {}^{2}f_{I} \qquad \text{in } C_2$$

where S_{IJ}, E_{IJ}, U_I, t_I and f_I are stresses, strains, displacements, surface tractions and body forces, respectively; a left superscript indicates the configuration of the body in which the quantity occurs.

The incremental decomposition of the state variables are given by

$$^{2}S_{IJ} = {}^{1}S_{IJ} + S_{IJ} \tag{2.1a}$$

$$^{2}E_{IJ} = {}^{1}E_{IJ} + E_{IJ} \tag{2.1b}$$

$$^{2}U_{I} = {}^{1}U_{I} + U_{I} \tag{2.1c}$$

where S_{IJ}, E_{IJ} and U_I are incremental stress, incremental strain and incremental displacement between C_1 and C_2.

There are three state variables describing the state of deformation that are stresses, strains and displacements. These state variables are related by the following natural relations:

(a) Kinematic Relations or Strain-Displacement Relations

(b) Equilibrium Equations

(c) Constitutive Relations

1.1 Kinematic Relations: The incremental strain E_{IJ} expressed in terms of displacements can be decomposed into a linear and nonlinear components as follow,

$$E_{IJ} = e_{IJ} + \eta_{IJ} \tag{2.2}$$

in which

$$2e_{IJ} = U_{I|J} + U_{J|I} + U_{K|I}\,{}^{1}U_{K|J} + {}^{1}U_{K|I}\,U_{K|J} \tag{2.3}$$

$$2\eta_{IJ} = U_{K|I}\,U_{K|J} \tag{2.4}$$

where a vertical bar indicates the covariant derivative in the undeformed configuration C_o.

1.2 Equilibrium Equations: An incremental nonlinear equations of motion describing the deformation of the body between the two neighboring configuration C_1 and C_2 will be derived from the principle of virtual work.

Consider the body in its deformed equilibrium configuration C_1. The virtual work of the external forces in moving through an infinitesimal virtual displacement δU_I from the current state is given by

$$\delta\,{}^1W_{ext} = \int_{{}^1A} \delta U_I\,{}^1t_I\,da + \int_{{}^1V} \delta U_I\,{}^1f_I\,dv \tag{2.5}$$

in which ${}^1A$ is the part of the surface area in C_1 which has prescribed surface tractions, and ${}^1V$ is the volume of the body in C_1. da and dv are the differential area and volume of the body in C_1, respectively.

The virtual work done by the internal forces in configuration C_1 in an arbitrary virtual displacement δU_I can be expressed as

$$\delta\,{}^1W_{int} = \int_{{}^oV} {}^1S_{IJ}\,\delta e_{IJ}\,dV \tag{2.6}$$

in which ${}^oV$ is the volume of the body in C_o and dV is the differential volume of the body in C_o.

The equation of virtual work in configuration C_1 can be obtained by equating equation (2.6) with equation (2.5), i.e.,

$$\int_{{}^{o}V} {}^{1}S_{IJ}\, \delta e_{IJ}\, dV = \int_{{}^{1}A} \delta U_I\, {}^{1}t_I\, da + \int_{{}^{1}V} \delta U_I\, {}^{1}f_I\, dv \tag{2.7}$$

In order to develop the incremental virtual work equation between configuration C_1 and C_2, the virtual work equation in the deformed equilibrium configuration C_2 has to be established. This can be done by following the same procedures similar to those for the deformed configuration C_1. That is

$$\delta^2 W_{ext} = \int_{{}^{2}A} \delta U_I\, {}^{2}t_I\, da + \int_{{}^{2}V} \delta U_I\, {}^{2}f_I\, dv \tag{2.8}$$

in which ${}^{2}A$ is the part of the surface area of the body in C_2 which has prescribed surface tractions, ${}^{2}V$ is the volume of the body in C_2, and da and dv are the differential area and volume of the body in C_2, respectively.

The virtual work done by the internal forces in configuration C_2 in an arbitrary virtual displacement δU_I can be expressed as

$$\delta^2 W_{int} = \int_{{}^{o}V} {}^{2}S_{IJ}\, \delta E_{IJ}\, dV \tag{2.9}$$

Therefore, equation of virtual work in configuration C_2 is obtained by equating equation (2.9) with equation (2.8), i.e.,

$$\int_{{}^{o}V} {}^{2}S_{IJ}\,\delta E_{IJ}\,dV = \int_{{}^{2}A} \delta U_{I}\,{}^{2}t_{I}\,d\bar{a} + \int_{{}^{2}V} \delta U_{I}\,{}^{2}f_{I}\,d\bar{v} \tag{2.10}$$

Since

$$ {}^{2}S_{IJ} = {}^{1}S_{IJ} + S_{IJ} \tag{2.1a}$$

$$ {}^{2}U_{I} = {}^{1}U_{I} + U_{I} \tag{2.1c}$$

$$ E_{IJ} = e_{IJ} + \eta_{IJ} \tag{2.2}$$

Substituting the above relations into equation (3.10), resulting in

$$\int_{{}^{o}V} [({}^{1}S_{IJ} + S_{IJ})\,(\delta e_{IJ} + \delta\eta_{IJ})]\,dV = \int_{{}^{2}A} \delta U_{I}\,{}^{2}t_{I}\,d\bar{a} + \int_{{}^{2}V} \delta U_{I}\,{}^{2}f_{I}\,d\bar{v} \tag{2.11}$$

The incremental virtual work between configuration C_1 and C_2 can be obtained by subtracting equation (2.7) from (2.11), i.e.,

$$\int_{{}^{o}V} [S_{IJ}(\delta e_{IJ} + \delta\eta_{IJ}) + {}^{1}S_{IJ}\,\delta\eta_{IJ}]\,dV$$

$$= \int_{{}^{2}A}\delta U_{I}\,{}^{2}t_{I}\,da + \int_{{}^{2}V}\delta U_{I}\,{}^{2}f_{I}\,dv - [\int_{{}^{1}A}\delta U_{I}\,{}^{1}t_{I}\,da + \int_{{}^{1}V}\delta U_{I}\,{}^{1}f_{I}\,dv] \tag{2.12}$$

Equation (2.12) can be interpreted to mean that the incremental virtual work of the body forces and surface tractions in the deformed equilibrium configuration C_1 and C_2 must equal the incremental virtual work of the state of stress in these configurations.

1.3 Constitutive Relation: The incremental equations of motion (2.12) are valid for any type of material irrespective of its constitutions. However, the application of these equations to physical nonlinear problems requires detailed knowledge of the material characterization, specifically the relationship between incremental stress and incremental strain.

For elastic materials, the incremental stress S_{IJ}, is linearly related to the incremental strain, E_{IJ}. That is

$$S_{IJ} = C_{IJMN} E_{IJ} \tag{2.13}$$

in which C_{IJMN} are the components of the constitutive matrix.

1.4 Incremental Nonlinear Equations of Motion with Equilibrium Corrections: The solution of the equations of motion (2.12) cannot be achieved directly since they are nonlinear in displacement increments. Therefore, it needs to be linearized for practical applications. However, the process of linearization must take account of three effects as follows:

(a) It is sufficient to assume the linear stress-strain relationship in the general form of equation (2.13).

(b) If the relationship (2.13) is substituted into equation (2.12) it would result in terms such as

$$C_{IJMN}\,(e_{MN}\,\delta\eta_{IJ} + \eta_{MN}\,\delta e_{IJ})$$

and $C_{IJMN}\,\eta_{MN}\,\delta\eta_{IJ}$

which are nonlinear in the incremental displacements. The linearization process requires that these terms be omitted.

(c) If the prescribed surface tractions are deformation dependent, the external virtual work integrals in configuration C_2 can be evaluated approximately.

Due to the above linearization and computational inaccuracies, the current deformed configuration C_1 may not be in complete equilibrium, thus resulting in residual work. That is,

$$\delta\,{}^{1}W_{res} = \int_{{}^{1}A} \delta U_I\,{}^{1}t_I\,da + \int_{{}^{1}V} \delta U_I\,{}^{1}f_I\,dv - \int_{{}^{o}V} {}^{1}S_{IJ}\,\delta e_{IJ}\,dV \qquad (2.14)$$

To prevent excessive departure of the solution from the true response, the corrective term (Equation 2.14) should be added to the right hand side of equation (2.12). Thus we have

$$\int_{{}^{o}V}[S_{IJ}(\delta e_{IJ}+\delta\eta_{IJ})+{}^{1}S_{IJ}\delta\eta_{IJ}]dV = \int_{{}^{2}A}\delta U_I\,{}^{2}t_I\,d\bar{a} + \int_{{}^{2}V}\delta U_I\,{}^{2}f_I\,d\bar{v} - \int_{{}^{o}V}{}^{1}S_{IJ}\delta e_{IJ}\,dV \qquad (2.15)$$

The incremental nonlinear equation (2.15) will be solved using the linearized form and applying the load in small steps together with an iterative process for equilibrium correction. Furthermore, it will be assumed that the components of the surface tractions are always known in the reference system and are defined per unit undeformed area and volume. Therefore, the integral expressions can be approximated by the following expression:

$$\int_{{}^{2}A} \delta U_i \, {}^{2}t_i \, d\bar{a} + \int_{{}^{2}V} \delta U_i \, {}^{2}f_i \, d\bar{v} \approx \int_{{}^{o}A} \delta U_i \, {}^{2}t_i \, dA + \int_{{}^{o}V} \delta U_i \, {}^{2}f_i \, dV \qquad (2.16)$$

The equations of motion suitable for use as a basis for discretization by the finite element method then take the form

$$\int_{{}^{o}V} [S_{ij}(\delta e_{ij} + \delta \eta_{ij}) + {}^{1}S_{ij}\delta \eta_{ij}] \, dV$$

$$= \int_{{}^{o}A} \delta U_i \, {}^{2}t_i \, dA + \int_{{}^{o}V} \delta U_i \, {}^{2}f_i \, dV - \int_{{}^{o}V} {}^{1}S_{ij}\delta e_{ij} \, dV \qquad (2.17)$$

2. Finite Element Formulation of Equations of Motion for Finite Deformation

2.1 Introduction: An important aspect of the finite element concept is that we can consider an individual, typical element to be completely isolated from the element assemblage and its behaviour can be studied independently of the behaviour of the other elements. Moreover, the connectivity of the discrete models is established independently of the linearity or nonlinearity of the system by a simple mapping.

In the section to follow, attention needs only be confined to a single element. Within the scope of this research, typical elements to be used for the discrete analysis is a family of the isoparametric finite element, and its element property matrices will be derived in detail for two dimensional analysis.

2.2 Discretization of Equations of Motion by Finite Element Method: Consider a single isoparametric element and introduce local approximation of the displacement field within the element by

$$ {}^{\alpha}U_{K}(x) = \phi^{m}(x)\ {}^{\alpha}q_{mK} \quad , \alpha = 1,2 \tag{2.18}$$

where ${}^{\alpha}U_{K}(x)$ are the components of the displacement of material coordinate x, in configuration C_{α}. $\phi^{m}(x)$ are the interpolation functions at node m. ${}^{\alpha}q_{mK}$ are the components of displacement at node m. The index m is assumed over all nodes of the element.

For the isoparametric finite element, the incremental displacement between configuration C_1 and C_2 and the coordinates of the material point are interpolated similarly using the above functions. That is

$$ U_{K}(x) = \phi^{m}(x)\ q_{mK} \tag{2.19}$$

$$ X_{K} = \phi^{m}(x)\ \hat{X}_{mK} \tag{2.20}$$

$$^{\alpha}X_K = \phi^m(x)\ {}^{\alpha}\hat{X}_{mK} \quad , \alpha = 1,2 \tag{2.21}$$

where q_{mK} are the components of the incremental displacement at node m. $\hat{X}_{mK}$ and ${}^{\alpha}\hat{X}_{mK}$ are the components of nodal coordinates in the configurations C_o and C_α, respectively.

Substituting the stress-strain relation (2.13) into the incremental equations of motion (2.17) yields

$$\int_{{}^oV} [\ C_{IJMN}\ e_{MN}\ \delta e_{IJ} + C_{IJMN}\ (\ e_{MN}\ \delta\eta_{IJ} + \eta_{MN}\ \delta e_{IJ}\ )$$

$$+ C_{IJMN}\ \eta_{MN}\ \delta\eta_{IJ} + {}^1S_{IJ}\ \delta\eta_{IJ}\ ]\ dV$$

$$= \int_{{}^oA} \delta U_I\ {}^2t_I\ dA + \int_{{}^oV} \delta U_I\ {}^2f_I\ dV - \int_{{}^oV} {}^1S_{IJ}\ \delta e_{IJ}\ dV \tag{2.22}$$

Thus, the following discretized equations of motion for a typical finite element are obtained

$$\delta q.[\{K_L({}^1q) + K_G({}^1S)\}q] = \delta q.[{}^2P - {}^1R] \tag{2.23}$$

where

$$\delta q.K_L.q = \int_{{}^oV} C_{IJMN}\ e_{MN}\ \delta e_{IJ}\ dV \tag{2.24a}$$

$$\delta q.K_G.q = \int_{^oV} {^1S_{IJ}}\, \delta\eta_{IJ}\, dV \qquad (2.24b)$$

$$\delta q.{^1R} = \int_{^oV} {^1S_{IJ}}\, \delta e_{IJ}\, dV \qquad (2.24c)$$

$$\delta q.{^2P} = \int_{^oV} \delta U_I\, {^2t_I}\, dA + \int_{^oV} \delta U_I\, {^2f_I}\, dV \qquad (2.24d)$$

Equation (2.23) represents a system of nonlinear equations in the unknown nodal displacement components, describing the incremental finite deformation of an element between the current deformed configuration C_1 and a neighboring deformed configuration C_2.

In the above arrays, K_L is the linear stiffness matrix, including initial displacement effect; K_G is the geometric stiffness matrix, a function of the initial stress ^1S; ^2P is the generalized nodal loads due to the body forces and conservative surface tractions; ^1R is the consistent nodal load vector in equilibrium with the state of stress in configuration C_1.

2.3 Two Dimensional Isoparametric Finite Element Matrices

2.3.1 Axisymmetric Quadrilateral Ring Element:

In this section, the element matrices for a general 8-node quadrilateral isoparametric element for axisymmetric analysis are given in detail. The plane stress and plane strain formulations can be implemented directly from this axisymmetric formulation. Geometry and transformations (mappings) of coordinates are given in Figure 2.2.

(a) Components of Stresses and Strains

For the plane stress formulation we have

$$ {}^{1}S_{13} = {}^{1}S_{23} = {}^{1}S_{33} = 0 $$
$$ S_{13} = S_{23} = S_{33} = 0 \qquad (2.25) $$

$$ {}^{1}E_{13} = {}^{1}E_{23} = 0 \text{ , } E_{13} = E_{23} = 0 $$

For the plane strain formulation we have

$$ {}^{1}S_{13} = {}^{1}S_{23} = 0 \text{ , } S_{13} = S_{23} = 0 $$

$$ {}^{1}E_{13} = {}^{1}E_{23} = {}^{1}E_{33} = 0 \qquad (2.26) $$

$$ E_{13} = E_{23} = E_{33} = 0 $$

Thus, for both plane stress and plane strain formulations the following stress and strain components can be considered.

$$ E_{IJ} = (E_{11}\ E_{22}\ 2E_{12}) \qquad (2.27a) $$

$$ S_{IJ} = (S_{11}\ S_{22}\ S_{12}) \qquad (3.27b) $$

For the axisymmetric formulation there are four strain and stress components to be considered, namely

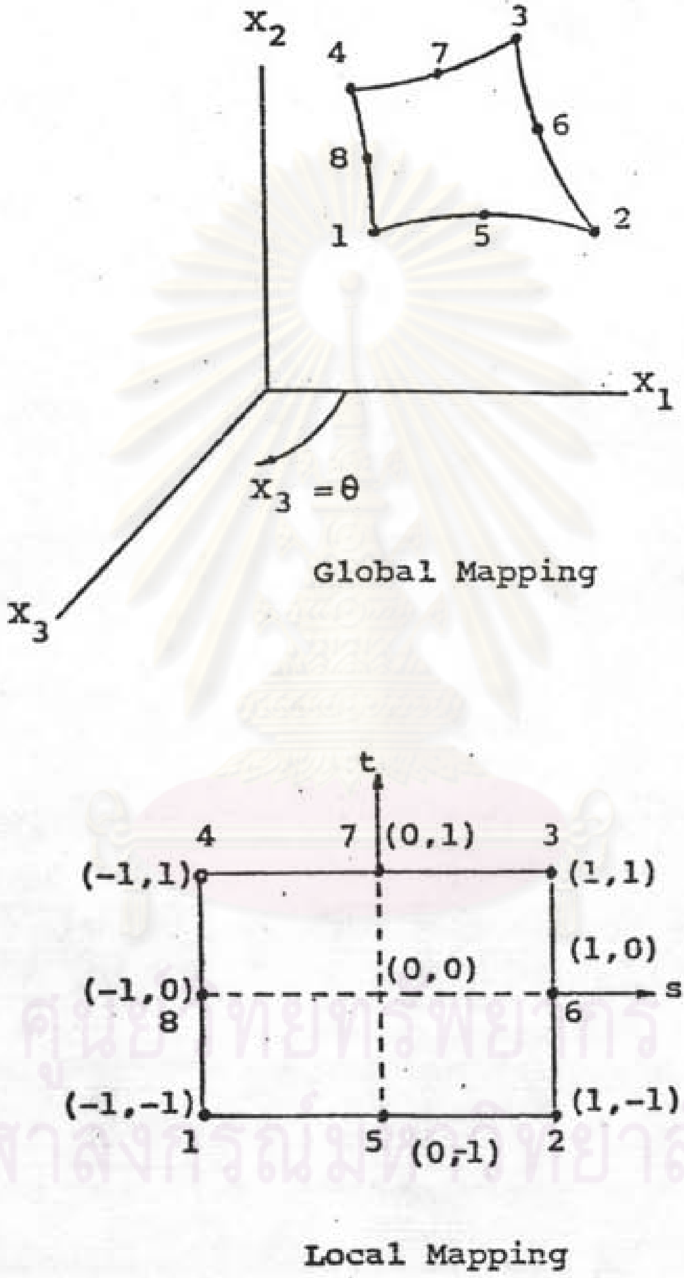

FIGURE 2.2 Two dimensional Isoparamatric Finite Element

$$E_{IJ} = (E_{11} \; E_{22} \; 2E_{12} \; E_{33}) \tag{2.28a}$$

$$S_{IJ} = (S_{11} \; S_{22} \; S_{12} \; S_{33}) \tag{2.28b}$$

(b) <u>Interpolation Functions</u>

For an 8-node quadrilateral element as shown in Figure 2.2, the interpolation functions for the corner nodes written in terms of the natural coordinates (s,t) are given by

$$\phi^m(s,t) = \tfrac{1}{4}(1+s.s_m)(1+t.t_m) - \tfrac{1}{4}(1+s.s_m)(1-t^2)$$

$$- \tfrac{1}{4}(1-s^2)(1+t.t_m) \tag{2.29}$$

in which m = 1,...,4 and $(s_m,t_m) = +1$.

For the midside nodes

$$\phi^m(s,t) = \tfrac{1}{2}(1-s^2)(1+t.t_m) \;,\; s_m = 0 \tag{2.30a}$$

$$\phi^m(s,t) = \tfrac{1}{2}(1+s.s_m)(1-t^2) \;,\; t_m = 0 \tag{2.30b}$$

in which m = 5,...,8.

In a matrix form we have

$$\phi = \left\{ \begin{array}{l} \frac{1}{4}(1-s)(1-t) - \frac{1}{4}(1-s)(1-t^2) - \frac{1}{4}(1-s^2)(1-t) \\ \frac{1}{4}(1+s)(1-t) - \frac{1}{4}(1+s)(1-t^2) - \frac{1}{4}(1-s^2)(1-t) \\ \frac{1}{4}(1+s)(1+t) - \frac{1}{4}(1+s)(1-t^2) - \frac{1}{4}(1-s^2)(1+t) \\ \frac{1}{4}(1-s)(1+t) - \frac{1}{4}(1-s)(1-t^2) - \frac{1}{4}(1-s^2)(1+t) \\ \frac{1}{2}(1-s^2)(1-t) \\ \frac{1}{2}(1+s)(1-t^2) \\ \frac{1}{2}(1-s^2)(1+t) \\ \frac{1}{2}(1-s)(1-t^2) \end{array} \right\} \tag{2.31}$$

(c) <u>Strain-Displacement Transformation</u>

Decomposition of the incremental strain into a linear and nonlinear components are given by equation (2.2) as follows:

$$\left\{ \begin{array}{c} E_{11} \\ E_{22} \\ 2E_{12} \\ E_{33} \end{array} \right\} = \left\{ \begin{array}{c} e_{11} \\ e_{22} \\ 2e_{12} \\ e_{33} \end{array} \right\} + \left\{ \begin{array}{c} \eta_{11} \\ \eta_{22} \\ 2\eta_{12} \\ \eta_{33} \end{array} \right\} \tag{2.32}$$

An explicit relation between the nonlinear strains and nodal displacements is not needed, as will be evident when the evaluation of the geometric stiffness is considered in the next section. The relation between linear strains and nodal displacements from equation (2.3) can be written in terms of deformation gradients as

$$2e_{IJ} = (\delta_{KJ} + {}^1U_{K|J})\, U_{K|I} + (\delta_{KI} + {}^1U_{K|I})\, U_{K|J} \qquad (2.33a)$$

That is

$$e = {}^1F\, U_\partial \qquad (2.33b)$$

In matrix form we have

$${}^1F = \begin{bmatrix} (1 + \frac{\partial {}^1U_1}{\partial X_1}) & 0 & \frac{\partial {}^1U_2}{\partial X_1} & 0 & 0 \\ 0 & \frac{\partial {}^1U_1}{\partial X_2} & 0 & (1 + \frac{\partial {}^1U_2}{\partial X_2}) & 0 \\ \frac{\partial {}^1U_1}{\partial X_2} & (1 + \frac{\partial {}^1U_1}{\partial X_1}) & (1 + \frac{\partial {}^1U_2}{\partial X_2}) & \frac{\partial {}^1U_2}{\partial X_1} & 0 \\ 0 & 0 & 0 & 0 & (1 + \frac{{}^1U_1}{X_1}) \end{bmatrix} \qquad (2.34)$$

and $$U^T_\partial = \left(\frac{\partial U_1}{\partial X_1}\ \frac{\partial U_1}{\partial X_2}\ \frac{\partial U_2}{\partial X_1}\ \frac{\partial U_2}{\partial X_2}\ \frac{U_1}{X_1} \right) \qquad (2.35)$$

where the superscript T denotes the transpose of a vector or a matrix.

The displacement gradients $U_{I|J}$ and ${}^1U_{I|J}$ are related to the nodal displacements through the local approximations of the displacement field, equations (2.18 and 2.19). That is

$$U_o = N.q \tag{2.36a}$$

or

$$\begin{Bmatrix} \frac{\partial U_1}{\partial X_1} \\ \frac{\partial U_1}{\partial X_2} \\ \frac{\partial U_2}{\partial X_1} \\ \frac{\partial U_2}{\partial X_2} \\ \frac{U_1}{X_1} \end{Bmatrix} = \begin{bmatrix} \phi^T_{,X_1} & 0 \\ \phi^T_{,X_2} & 0 \\ 0 & \phi^T_{,X_1} \\ 0 & \phi^T_{,X_2} \\ \frac{\phi^T}{X_1} & 0 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} \tag{2.36b}$$

and $${}^1U_o = N.{}^1q \tag{2.37}$$

where $${}^1U_o^{\,T} = \left(\frac{\partial {}^1U_1}{\partial X_1} \; \frac{\partial {}^1U_1}{\partial X_2} \; \frac{\partial {}^1U_2}{\partial X_1} \; \frac{\partial U^1{}_2}{\partial X_2} \; \frac{{}^1U_1}{X_1} \right) \tag{2.38}$$

is required to construct the matrix of deformation gradient ${}^{1}F$. Commas in equation (2.36b) indicate "partial derivative with respect to".

Finally, the linear strain-displacement transformation matrix B can be obtained from equations (2.33 and 2.36) as follows:

$$e \;=\; {}^{1}F.N.q \;=\; B.q \tag{2.39}$$

(d) <u>Jacobian Transformation</u>

In the above matrix N, to calculate the derivative of the interpolation function $\phi^{m}(s,t)$ with respect to the material coordinates X_1 and X_2, a Jacobian transformation is needed to relate derivatives with respect to the local (s,t) coordinates to those with respect to the material coordinates. Consider a chain rule of differentiation as follows:

$$\begin{Bmatrix} \dfrac{\partial}{\partial s} \\ \\ \dfrac{\partial}{\partial t} \end{Bmatrix} = \begin{bmatrix} X_{1,s} & X_{2,s} \\ \\ X_{1,t} & X_{2,t} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial}{\partial X_1} \\ \\ \dfrac{\partial}{\partial X_2} \end{Bmatrix} \tag{2.40}$$

Inversely,

$$\left\{ \begin{matrix} \dfrac{\partial}{\partial X_1} \\ \\ \dfrac{\partial}{\partial X_2} \end{matrix} \right\} = \frac{1}{\det J} \begin{bmatrix} X_{2,t} & -X_{2,s} \\ \\ -X_{1,t} & X_{1,s} \end{bmatrix} \left\{ \begin{matrix} \dfrac{\partial}{\partial s} \\ \\ \dfrac{\partial}{\partial t} \end{matrix} \right\} \qquad (2.41)$$

$$\text{where } \det J = X_{1,s} X_{2,t} - X_{1,t} X_{2,s} \qquad (2.42)$$

According to equation (2.20) we can write

$$X_{1,s} = \phi^T_{,s} \hat{X}_1 \quad , \quad X_{1,t} = \phi^T_{,t} \hat{X}_1 \qquad (2.43a)$$

$$X_{2,s} = \phi^T_{,s} \hat{X}_2 \quad , \quad X_{2,t} = \phi^T_{,t} \hat{X}_2 \qquad (2.43b)$$

Substituting equation (2.43) into equation (2.42) gives

$$\det J = \hat{X}^T_1 \phi_{,s} \phi^T_{,t} X_2 - X^T_1 \phi_{,t} \phi^T_{,s} \hat{X}_2 \qquad (2.44a)$$

$$= \hat{X}^T_1 [\phi_{,s} \phi^T_{,t} - \phi_{,t} \phi^T_{,s}] \hat{X}_2 \qquad (2.44b)$$

$$= \hat{X}^T_1 \cdot P \cdot \hat{X}_2 \text{ or } = -\hat{X}^T_2 P \hat{X}_1 \qquad (2.44c)$$

$$\text{where } P = \phi_{,s} \phi^T_{,t} - \phi_{,t} \phi^T_{,s} = -P^T \quad \text{(anti-symmetric)} \qquad (2.45)$$

Therefore, $\phi^T_{,X_1}$ and $\phi^T_{,X_2}$ can be calculated from equation (2.41) as follows:

$$\phi_{,X_1} = \frac{1}{\det J} [X_{2,t} \, \phi^T_{,s} - X_{2,s} \, \phi^T_{,t}] \tag{2.46}$$

Substituting for $X_{2,t}$ and $X_{2,s}$ from equation (2.43b) yields

$$\phi_{,X_1} = \frac{1}{\det J} [\hat{X}^T_2 \, \phi_{,t} \, \phi^T_{,s} - \hat{X}^T_2 \, \phi_{,s} \, \phi^T_{,t}] \tag{2.47a}$$

$$= \frac{1}{\det J} \hat{X}^T_2 [\phi_{,t} \, \phi^T_{,s} - \phi_{,s} \, \phi^T_{,t}] \tag{2.47b}$$

$$= - \frac{1}{\det J} \hat{X}^T_2 \cdot P \tag{2.47c}$$

Similarly, we have

$$\phi_{,X_2} = \frac{1}{\det J} \hat{X}^T_1 \cdot P \tag{2.48}$$

Also, the differential volume dV is given by

$$dV = (\det J) \, X_1 \, dX_3 \, ds \, dt \tag{2.49}$$

2.3.2 Evaluation of Element Matrices: The element matrices required to solve the discrete finite element equations of motion (2.23) can be obtained by evaluating the virtual work integrals (2.24) using the Gaussian quadrature formulas for numerical integrations.

(a) Linear Element Stiffness Matrix

The linear element stiffness matrix is given by the integral (2.24a) as

$$K_L = \int_{{}^{\circ}V} B^T . C . B \, dV \qquad (2.50)$$

For the purpose of numerical integration, it is written in the s and t system as

$$K_L = \int_{-1}^{1} \int_{-1}^{1} . B^T . C . B \det J.X_1.ds\,dt \qquad (2.51)$$

The direct application of one dimensional numerical integration formula yields

$$K_L = \sum_{j} \sum_{k} W_j W_k \, [B^T(s_j,t_k).C(s_j,t_k).B(s_j,t_k)] \det J(s_j,t_k).X_1(s_j,t_k) \qquad (2.52)$$

(b) Geometric Stiffness Matrix

The geometric stiffness matrix can be obtained by evaluating the integral (2.24b) as follows:

$$\delta q.K_G.q = \int_{{}^{\circ}V} {}^{1}S_{ij} \, \delta\eta_{ij} \, dV \qquad (2.53)$$

where the nonlinear incremental strain η_{ij} is given by

$$\begin{Bmatrix} \eta_{11} \\ \eta_{22} \\ 2\eta_{12} \\ \eta_{33} \end{Bmatrix} = \begin{Bmatrix} \frac{1}{2}\left[\left(\frac{\partial U_1}{\partial X_1} \right)^2 + \left(\frac{\partial U_2}{\partial X_1} \right)^2 \right] \\ \frac{1}{2}\left[\left(\frac{\partial U_1}{\partial X_2} \right)^2 + \left(\frac{\partial U_2}{\partial X_2} \right)^2 \right] \\ \left[\frac{\partial U_1}{\partial X_1} \frac{\partial U_1}{\partial X_2} + \frac{\partial U_2}{\partial X_1} \frac{\partial U_2}{\partial X_2} \right] \\ \frac{1}{2}\left(\frac{U_1}{X_1} \right)^2 \end{Bmatrix} \qquad (2.54)$$

To put η_{IJ} into a symmetric form with respect to ${}^1S_{IJ}$, we can write

$$2\,{}^1S_{IJ}\eta_{IJ} = \left(\frac{\partial X_1}{\partial X_1} \frac{\partial X_2}{\partial X_2} \right) \begin{bmatrix} {}^1S_{11} & {}^1S_{12} \\ {}^1S_{21} & {}^1S_{22} \end{bmatrix} \begin{Bmatrix} \frac{\partial U_1}{\partial X_1} \\ \frac{\partial U_1}{\partial X_2} \end{Bmatrix}$$

$$+ \left(\frac{\partial U_2}{\partial X_1} \frac{\partial U_2}{\partial X_2} \right) \begin{bmatrix} {}^1S_{11} & {}^1S_{12} \\ {}^1S_{21} & {}^1S_{22} \end{bmatrix} \begin{Bmatrix} \frac{\partial U_2}{\partial X_1} \\ \frac{\partial U_2}{\partial X_2} \end{Bmatrix}$$

$$+ \left(\frac{U_1}{X_1} \right) {}^1S_{33} \left(\frac{U_1}{X_1} \right) \qquad (2.55a)$$

or $$2\,{}^1S_{IJ}\eta_{IJ} = U^T_{\partial} \,.\, {}^1\hat{S} \,.\, U_{\partial} \qquad (2.55b)$$

in which U_{∂} is given by equation (2.36) and ${}^1\hat{S}$ is given by

$$ {}^1\hat{S} = \begin{bmatrix} {}^1S_{11} & {}^1S_{12} & 0 & 0 & 0 \\ {}^1S_{21} & {}^1S_{22} & 0 & 0 & 0 \\ 0 & 0 & {}^1S_{11} & {}^1S_{12} & 0 \\ 0 & 0 & {}^1S_{21} & {}^1S_{22} & 0 \\ 0 & 0 & 0 & 0 & {}^1S_{33} \end{bmatrix} \qquad (2.56)$$

Substituting for U_{∂} from equation (2.36) into equation (2.55) yields

$$ {}^1S_{IJ}\,\eta_{IJ} = \tfrac{1}{2}\, q^T\, [N]^T.[{}^1\hat{S}].[N].q \qquad (2.57)$$

and by taking the variation with respect to nonlinear strain as

$$ {}^1S_{IJ}\delta\eta_{IJ} = \delta q^T\, [N]^T.[{}^1\hat{S}].[N].q \qquad (2.58)$$

Therefore, the geometric stiffness matrix is given in a symmetric form by

$$ K_G = \int_{{}^oV} [N]^T.[{}^1\hat{S}].[N]\, dV \qquad (2.59a)$$

$$\text{or}\quad K_G = \int_{-1}^{1}\int_{-1}^{1} [N]^T.[{}^1\hat{S}].[N] \det J.X_1 \, ds\, dt \tag{2.59b}$$

numerically

$$K_G = \sum_j \sum_k W_j W_k \, [N^T(s_j,t_k).{}^1\hat{S}(s_j,t_k).N(s_j,t_k)] \det J(s_j,t_k).X_1(s_j,t_k) \tag{2.60}$$

(c) Equivalent Nodal Load Vector ${}^1R$ for Equilibrium Correction

This is given by evaluating in the integral (2.24c) as follows:

$$\delta q^T.{}^1R = \int_{{}^oV} {}^1S_{ij}\, \delta e_{ij}\, dV \tag{2.61}$$

That is

$$ {}^1R = \int_{{}^oV} B^T.{}^1S\, dV \tag{2.62}$$

where

$$({}^1S) = ({}^1S_{11} \quad {}^1S_{22} \quad {}^1S_{12} \quad {}^1S_{33}) \tag{2.63}$$

In the (s,t) system we then have

$$^{1}R = \int_{-1}^{1} \int_{-1}^{1} B^{T} \,.\, ^{1}S \;\det J \,.\, X_1 \; ds\, dt \qquad (2.64)$$

or numerically

$$^{1}R = \sum_{j} \sum_{k} W_j W_k \, [B^{T}(s_j,t_k).\, ^{1}S(s_j,t_k)] \det J(s_j,t_k).X_1(s_j,t_k) \qquad (2.65)$$

(d) <u>Consistent Nodal Load Vector ^{2}P</u>

Consistent nodal load vector, ^{2}P, is obtained from the evaluation of the body forces and surface tractions by the conventional finite element method. That is

$$^{2}P = \int_{^{o}A} [\hat{N}]^{T}.\{^{2}t\}.dA + \int_{^{o}V} [N^{*}]^{T}.\{^{2}f\}.dV \qquad (2.66)$$

in which $[\hat{N}]$ is the displacement transformation which relates displacements of the loaded surface to the nodal displacements; { t} is the vector of surface tractions; $[N^{*}]$ is the matrix of interpolation functions given by equations (2.19,2.30); and $\{^{2}f\}$ is the vector of body forces. The intregrals in the equation (2.66) are calculated numerically.

3. <u>Direct Stiffness Assembling Process</u>

The global form of the discrete finite element formulation of the equations of motion for the entire assemblage of elements are achieved through the standard finite element direct stiffness assembling process. That is

$$\delta q.[\{\overline{K}_L + \overline{K}_G\}q] = \delta q.[{}^2\overline{P} - {}^1\overline{R}] \tag{2.67}$$

where

$$\overline{K}_L = \sum_{e=1}^{M} K_L^{(e)} \tag{2.68a}$$

$$\overline{K}_G = \sum_{e=1}^{M} K_G^{(e)} \tag{2.68b}$$

$${}^2\overline{P} = \sum_{e=1}^{M} {}^2P^{(e)} \tag{2.68c}$$

$${}^1\overline{R} = \sum_{e=1}^{M} {}^1R^{(e)} \tag{2.68d}$$

in which $\sum_{e=1}^{M}$ is the summation in direct stiffness assembling process

and M is the number of elements.

4. Numerical Methods for the Solutions of Nonlinear Equations

In the actual analysis of nonlinear response of a structural system it would lead to systems of nonlinear equations in the unknown nodal displacement components. However, in a majority of nonlinear analyses the linearized forms of the finite element formulation, such as the one given by equation (2.67), are used in connection with various numerical techniques. Each of these solution techniques seeks to trace the nonlinear load-deformation path of the structure by solving systems of nonlinear equation by one or a combination of the following techniques:

(a) Incremental loading procedures

(b) Iterative procedures

(c) Minimization procedures

A system of n nonlinear equations in n unknown incremental displacement components is obtained using minimization procedure as follows:

$$(\overline{K}_L + \overline{K}_G).q = {}^2\overline{P} - {}^1\overline{R} \qquad (2.69a)$$

or simply,

$$\overline{K} \,.\, q = \overline{P} \qquad (2.69b)$$

where $\overline{K}$ = the nonlinear (global) stiffness matrix

q = the incremental nodal displacement vector

P = the generalized force vector

In the incremental loading procedure, the load is applied in increments which are sufficiently small so that during each increment the response of the body is linear. At the end of each increment, a new updated stiffness relation is obtained and another increment of load is applied. By continuing this process, the response of the body is generated as a sequence of linear steps. Naturally, some error will inevitably enter each step of the process. To improve the accuracy of the incremental loading method, Newton-Raphson iterations with equilibrium corrections are applied at the end of each load increment. Details of the incremental loading method and the Newton-Raphson iterative scheme can be found in references (21,22).

Computation of Pile Curvatures

In the nonlinear elastic static analysis by the present finite element computer program, the incremental displacements in each equilibrium configuration of the body are obtained. By updating the coordinates of the nodal points, the deformed configuration can then be obtained. Finally, curvature distribution along the pile can be calculated using the Lagrange polynomial. The concept of Lagrange polynomial can be found in many texts concerning finite element and numerical analyses (23,24).