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APPENDIX A

GEOMETRIC CONSTRUCTION FOR LENZ VECTOR

We can easily make a simple geometric construction for this vector $\vec{U}$. First, let us recall Eq (3.2), the definition of this vector,

$$\vec{U} = \frac{1}{m_o e^2}(\vec{L} \times \vec{p}) + \frac{\vec{r}}{r}$$

Rewriting the above equation in the form below (the notation ^ is used to denote unit vector)

$$\vec{U} = \frac{1}{m_o e^2} \cdot \left[(m_o vr)(m_o v)\sin\alpha \right] \widehat{Lxp} + \hat{r} \qquad \text{(A.1)}$$

where α is the angle between vector $\vec{r}$ and $\vec{p}$

$$\vec{U} = - \frac{m_o v^2/2}{-e^2/r} \cdot 2\sin\alpha \cdot \widehat{Lxp} + \hat{r}$$

or

$$\vec{U} = - 2R \sin\alpha \cdot \widehat{Lxp} + \hat{r} \qquad \text{(A.2)}$$

$R = \frac{T}{V}$ be the ratio of the kinetic energy $(T = \frac{mv^2}{2})$ to the potential energy $(V = - \frac{e^2}{r})$

The construction is given now. For our example below we will take R to be $- \frac{1}{4}$.

Step 1 Draw circles around and through the nucleus S and electron E as shown in Fig. A.1 Mark off arc QS so it is double the arc RS on the circle around the electron and draw straight line through E and Q

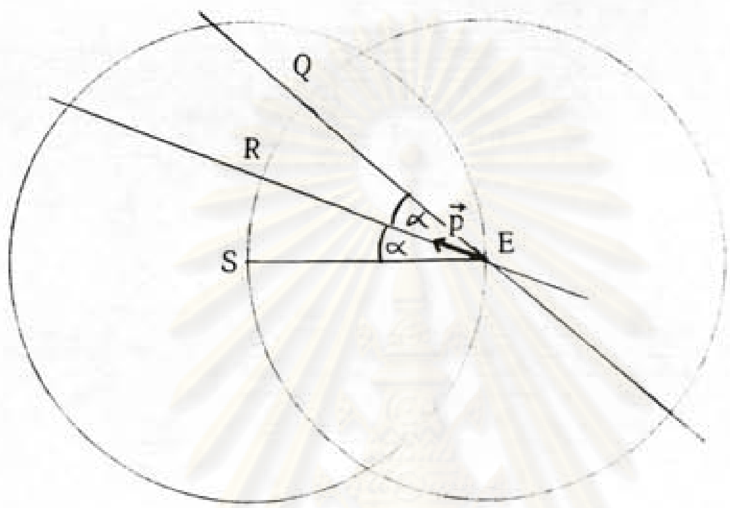

Fig. A.1 Doubling the angle between the momentum and the position vectors gives a line QE which must contain the orbit focus

Step 2 Copy the arc SRQ onto the arc ET of the circle around the nucleus. A line drawn throguh E and T should be perpendicular to $\vec{p}$ and will serve as the $R = \frac{T}{V}$ axis. Point E corresponds to $R = 0$ while T corresponds to $R = -1$, and the R scale is linear (see Fig. A.2)

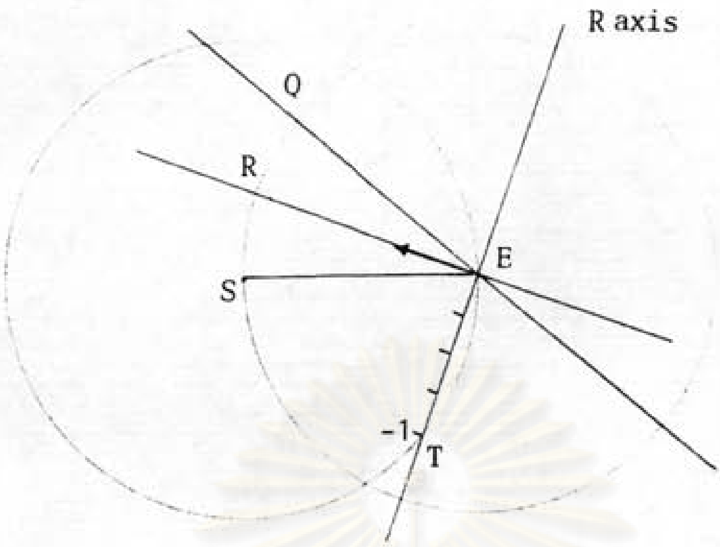

Fig. A.2 A line perpendicular to the momentum will serve as an energy scale, linear in the initial $R = \frac{T}{V}$

Step 3 Locate the given ratio on the R scale (for the example we take $R = -\frac{1}{4}$) and extend a line from the nucleus S through this point until it intersects the line QES' at S'. Point S' is the second focus of the conic section orbit tangent to vector $\vec{p}$. In Fig. A.3 . the appropriate ellipse has been drawn by using foci S and S'

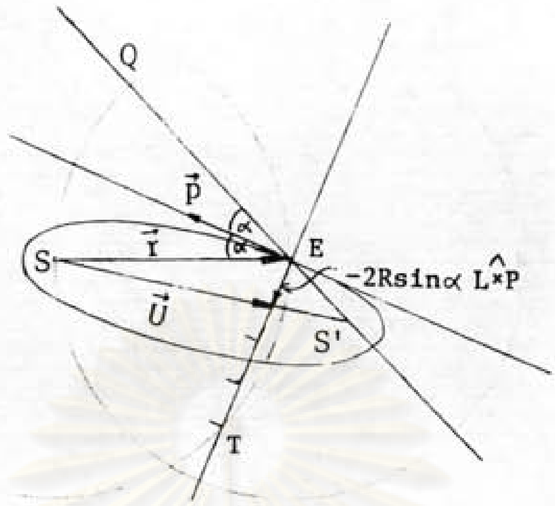

Fig. A.3 The desired energy $R = -\frac{1}{4}$ on the R scale points to the correct orbit focal point at S' on the focus locus

It is also shown in Fig. A.3 that $\vec{U}$ points along the line SS' which is the major axis of the elliptic orbit.

To show that the magnitude of this vector $\vec{U}$ is the eccentricity of the orbit, we take the dot product of this vector with the radius vector $\vec{r}$ as in Eq. (A.3). This then reduces to the following equation (A.4) of a conic section in polar coordinates, which is the general orbit equation

$$\vec{U}.\vec{r} = Ur\cos\theta = \frac{1}{m_o e^2}(\vec{L} \times \vec{p}.\vec{r}) + r \qquad (A.3)$$

$$= \frac{\vec{L}.\vec{L}}{m_o e^2} + r$$

$$\therefore \frac{1}{r} = \frac{m_o e^2}{L^2}(1 - U\cos\theta) \qquad (A.4)$$

APPENDIX B

PROVING THE CONSTANCY WITH TIME OF VECTOR MATRIX $\hat{\vec{U}}$

To show that the vector matrix $\hat{\vec{U}}$ is constant with time in the special case of a Coulomb field of force, we must start by setting up the rules of calculation for the radius-vector matrix $\hat{\vec{r}}$ and trying to evaluate the time-derivative of $\frac{\hat{\vec{r}}}{\hat{r}}$

let $\hat{x}, \hat{y}, \hat{z}$ be the matrices of the Cartesian coordinates of the electron which make up the components of the vector matrix $\hat{\vec{r}}$, and for the matrix $\hat{r}$ which represents the magnitude of the radius vector. obviously, they must satisfy the relation

$$\hat{r}^2 = \hat{x}^2 + \hat{y}^2 + \hat{z}^2 \tag{B.1}$$

From the quantum conditions (2.8), we have

$$\hat{p}_x\hat{x} - \hat{x}\hat{p}_x = \frac{h}{2\pi i}\mathbf{1} , \ \ldots \tag{B.2}$$

These rules can be extended by the following additional relations, on making use of the matrix $\hat{r}$. Firstly, $\hat{r}$ also commutes with $\hat{x}, \hat{y}, \hat{z}$. This can be written in the form of a vector equation :

$$\hat{r}\hat{\vec{r}} = \hat{\vec{r}}\hat{r} \tag{B.3}$$

Secondly, for any arbitrary rational function f of $\hat{r}, \hat{x}, \hat{y}, \hat{z}$, the relation

$$\hat{p}_x f - f\hat{p}_x = \frac{h}{2\pi i}\frac{\partial f}{\partial \hat{x}} , \ldots . \tag{B.4}$$

holds, and in particular for $f = \hat{r}$, we obtain

$$\hat{\vec{p}}\hat{r} - \hat{r}\hat{\vec{p}} = \frac{h}{2\pi i} \cdot \frac{\hat{\vec{r}}}{\hat{r}} \qquad \text{(B.5)}$$

Conversely, (B.4) follows generally from (B.2) and (B.5) for every function which can be expressed as a series of positive and negative powers of $\hat{x}$, $\hat{y}$, $\hat{z}$ and $\hat{r}$, as can easily be shown by induction. Relation (B.5) is also in accord with (B.1) For this reason, the existence of relations (B.3) and (B.4) constitutes a necessary requirement for the energy conservation law,

$$\frac{1}{2} m_o \hat{\vec{v}}^2 + F(\hat{x}, \hat{y}, \hat{z}, \hat{r}) = \hat{E} \qquad \text{(diagonal matrix)} \qquad \text{(B.6)}$$

together with the frequency condition, for any quantity $\hat{\phi}$,

$$\hat{E}\hat{\phi} - \hat{\phi}\hat{E} = \frac{h}{2\pi i}\hat{\dot{\phi}} \qquad \text{(B.7)}$$

We therefore postulate the existence of a matrix $\hat{r}$ which satisfies relations (B.1) , (B.3), (B.5)

Next, we will evaluate the time-derivative of $\frac{\hat{\vec{r}}}{\hat{r}}$. For example, for the x-component we have, with the help of Eq. (B.7),

$$\frac{d}{dt}\left(\frac{\hat{x}}{\hat{r}}\right) = \frac{2\pi i}{h}\left(\hat{E}\frac{\hat{x}}{\hat{r}} - \frac{\hat{x}}{\hat{r}}\hat{E}\right)$$

$$= \frac{2\pi i}{h} \cdot \frac{1}{2m_o}\left\{(\hat{\vec{p}}.\hat{\vec{p}})\frac{\hat{x}}{\hat{r}} - \frac{\hat{x}}{\hat{r}}(\hat{\vec{p}}.\hat{\vec{p}})\right\}$$

$$= \frac{2\pi i}{h} \cdot \frac{1}{2m_o}\left\{\hat{\vec{p}}.\left(\hat{\vec{p}}\frac{\hat{x}}{\hat{r}} - \frac{\hat{x}}{\hat{r}}\hat{\vec{p}}\right) + \left(\hat{\vec{p}}\frac{\hat{x}}{\hat{r}} - \frac{\hat{x}}{\hat{r}}\hat{\vec{p}}\right).\hat{\vec{p}}\right\}$$

Now we make use of (B.4) to obtain

$$\frac{d}{dt}\left(\frac{\hat{x}}{\hat{r}}\right) = \frac{1}{2m_o}\left\{ \left(\hat{p}_x \frac{\hat{y}^2+\hat{z}^2}{\hat{r}^3}\right) - \hat{p}_y \frac{\hat{x}\hat{y}}{\hat{r}^3} - \hat{p}_z \frac{\hat{x}\hat{z}}{\hat{r}^3}\right)$$

$$+ \left(\frac{\hat{y}^2+\hat{z}^2}{\hat{r}^3}\hat{p}_x - \frac{\hat{x}\hat{y}}{\hat{r}^3}\hat{p}_y - \frac{\hat{x}\hat{z}}{\hat{r}^3}\hat{p}_z\right) \right\}$$

$$= \frac{1}{2m_o}\left\{ \left(\hat{L}_y \cdot \frac{\hat{z}}{\hat{r}^3} - \hat{L}_z \cdot \frac{\hat{y}}{\hat{r}^3}\right) + \left(\frac{\hat{z}}{\hat{r}^3}\cdot\hat{L}_y - \frac{\hat{y}}{\hat{r}^3}\cdot\hat{L}_z\right)\right\}$$

Thus, generally

$$\frac{d}{dt}\left(\frac{\hat{\vec{r}}}{\hat{r}}\right) = \frac{1}{2m_o}\left\{ \left(\hat{\vec{L}} \times \frac{\hat{\vec{r}}}{\hat{r}^3}\right) - \left(\frac{\hat{\vec{r}}}{\hat{r}^3} \times \hat{\vec{L}}\right)\right\} \qquad (B.8)$$

For the special case of a Coulomb field of force we set

$$F(\hat{x}, \hat{y}, \hat{z}, \hat{r}) = -\frac{e^2}{\hat{r}}$$

in (B.6) The equation of motion (2.10) derived from energy conservation with the aid of the quantum rules assume the same form here as in classical mechanics :

$$\dot{\hat{\vec{p}}} = m_o\ddot{\hat{\vec{r}}} = -\frac{e^2}{\hat{r}^3}\hat{\vec{r}} \qquad (B.9)$$

It then follows from (B.8) with the help of (B.9) that

$$\frac{d}{dt}\left(\frac{\hat{\vec{r}}}{\hat{r}}\right) = \frac{1}{2m_o}\left\{\left[\hat{\vec{L}} \times \left(-\frac{\dot{\hat{\vec{p}}}}{e^2}\right)\right] - \left[\left(-\frac{\dot{\hat{\vec{p}}}}{e^2}\right) \times \hat{\vec{L}}\right]\right\}$$

$$= \frac{1}{m_o e^2}\cdot\frac{1}{2}\frac{d}{dt}\left\{\hat{\vec{p}} \times \hat{\vec{L}} - \hat{\vec{L}} \times \hat{\vec{p}}\right\}$$

Hence,

$$\frac{d}{dt}\left[\frac{1}{m_o e^2} \cdot \frac{1}{2}\left\{(\hat{\vec{L}} \times \hat{\vec{p}}) - (\hat{\vec{p}} \times \hat{\vec{L}})\right\} + \frac{\hat{\vec{r}}}{\hat{r}} \right] = 0$$

The above relation shows that $\hat{\vec{U}}$ is constant with time.

APPENDIX C

PROVING THE COMMUTATION RELATION IN EQ. (3.15b)

To prove the correction of the commutation relation in Eq. (3.15b)

$$\left[\hat{L}_\alpha \, , \; \hat{U}_\beta\right] \; = \; -\frac{h}{2\pi i} \, \epsilon_{\alpha\beta\gamma} \, \hat{U}_\gamma \; , \tag{C.1}$$

we will use the commutation rules from Eq. (3.10) ;

$$\left[\hat{L}_\alpha \, , \; \hat{p}_\beta\right] \; = \; -\frac{h}{2\pi i} \, \epsilon_{\alpha\beta\gamma} \, \hat{p}_\gamma \tag{C.2}$$

and from (3.12) ;

$$\left[\hat{L}_\alpha \, , \; \hat{L}_\beta\right] \; = \; -\frac{h}{2\pi i} \, \epsilon_{\alpha\beta\gamma} \, \hat{L}_\gamma \tag{C.3}$$

Now we can sufficiently verify the correction of the commutation relation (C.1) by proving only one fragment of this relation. Suppose we choose

$$\left[\hat{L}_x \, , \hat{U}_y\right] \; = \; -\frac{h}{2\pi i} \, \hat{U}_z$$

for our proof,

Noting that $\hat{U}_y \; = \; \frac{1}{2m_o e^2}\left\{(\hat{L}_z\hat{p}_x-\hat{L}_x\hat{p}_z) - (\hat{p}_z\hat{L}_x - \hat{p}_x\hat{L}_z)\right\} + \frac{\hat{y}}{\hat{r}}$,

we thus have,

$$\hat{L}_x\hat{U}_y - \hat{U}_y\hat{L}_x \; = \frac{1}{2m_o e^2}\left[\left\{\hat{L}_x\hat{L}_z\hat{p}_x-\hat{L}_x\hat{L}_x\hat{p}_z-\hat{L}_x\hat{p}_z\hat{L}_x + \hat{L}_x\hat{p}_x\hat{L}_z - \hat{L}_z\hat{p}_x\hat{L}_x+ \hat{L}_x\hat{p}_z\hat{L}_x + \hat{p}_z\hat{L}_x\hat{L}_x-\hat{p}_x\hat{L}_z\hat{L}_x\right\} + \frac{(\hat{L}_x\hat{y}-\hat{y}\hat{L}_x)}{\hat{r}}\right]$$

$$= \frac{1}{2m_o e^2}\Big[(\hat{L}_x\hat{p}_z\hat{L}_x-\hat{L}_x\hat{L}_x\hat{p}_z)-(\hat{L}_z\hat{p}_x\hat{L}_x-\hat{L}_x\hat{L}_z\hat{p}_x)$$

$$-(\hat{p}_x\hat{L}_z\hat{L}_x-\hat{L}_x\hat{p}_x\hat{L}_z)+(\hat{p}_z\hat{L}_x\hat{L}_x-\hat{L}_x\hat{p}_z\hat{L}_x)+\frac{(-\frac{h}{2\pi i})\hat{z}}{\hat{r}}\Big]$$

$$= \frac{1}{2m_o e^2}\Big[\hat{L}_x(\hat{p}_z\hat{L}_x-\hat{L}_x\hat{p}_z)-(\hat{L}_z\hat{L}_x-\hat{L}_x\hat{L}_z)\hat{p}_x$$

$$-\hat{p}_x(\hat{L}_z\hat{L}_x-\hat{L}_x\hat{L}_z)+(\hat{p}_z\hat{L}_x-\hat{L}_x\hat{p}_z)\hat{L}_x+\frac{(-\frac{h}{2\pi i})\hat{z}}{\hat{r}}\Big]$$

If then follows from (C.2) and (C.3) that

$$\left[\hat{L}_x, \hat{U}_y\right] = \frac{1}{2m_o e^2}\Big[\hat{L}_x(-\frac{h}{2\pi i}\hat{p}_y)-(-\frac{h}{2\pi i}\hat{L}_y)\hat{p}_x$$

$$-\hat{p}_x(-\frac{h}{2\pi i}\hat{L}_y)+(-\frac{h}{2\pi i}\hat{p}_y)\hat{L}_x+\frac{(-\frac{h}{2\pi i})\hat{z}}{\hat{r}}\Big]$$

$$= (-\frac{h}{2\pi i})\frac{1}{2m_o e^2}\Big[\hat{L}_x\hat{p}_y-\hat{L}_y\hat{p}_x-\hat{p}_x\hat{L}_y+\hat{p}_y\hat{L}_x+\frac{\hat{z}}{\hat{r}}\Big]$$

$$= -\frac{h}{2\pi i}\hat{U}_z$$

In a similar manner we find that

$$\left[\hat{L}_y, \hat{U}_z\right] = -\frac{h}{2\pi i}\hat{U}_x$$

$$\left[\hat{L}_z, \hat{U}_x\right] = -\frac{h}{2\pi i}\hat{U}_y$$

APPENDIX D

PROVING THE EQUATION (3.15c)

To prove the correction of the equation (3.15c)

$$\hat{\vec{U}} \times \hat{\vec{U}} = \frac{h}{2\pi i} \cdot \frac{2}{m_o e^4} \hat{E}\hat{\vec{L}} \qquad (D.1)$$

Taking the cross product $\hat{\vec{U}} \times \hat{\vec{U}}$ in (3.14)

$$\hat{\vec{U}} \times \hat{\vec{U}} = \left[\frac{1}{m_o e^2}(\hat{\vec{L}}x\hat{\vec{p}}) + \frac{\hat{\vec{r}}}{\hat{r}}\right] \times \left[\frac{1}{m_o e^2}(\hat{\vec{L}}x\hat{\vec{p}}) + \frac{\hat{\vec{r}}}{\hat{r}}\right]$$

$$= \frac{1}{m_o^2 e^4}(\hat{\vec{L}}x\hat{\vec{p}})x(\hat{\vec{L}}x\hat{\vec{p}}) + \frac{1}{m_o e^2}\left[\left\{\frac{\hat{\vec{r}}}{\hat{r}}\, x(\hat{\vec{L}}x\hat{\vec{p}})\right\} + \left\{(\hat{\vec{L}}x\hat{\vec{p}}) \times \frac{\hat{\vec{r}}}{\hat{r}}\right\}\right] \qquad (D.2)$$

With the help of the following formula from vector algebra,

$$(\vec{A}x\vec{B}) \times (\vec{A}x\vec{C}) = (\vec{A}.\vec{B}x\vec{C})\vec{A}$$

the term $(\hat{\vec{L}}x\hat{\vec{p}}) \times (\hat{\vec{L}}x\hat{\vec{p}})$ in (D.2) becomes

$$(\hat{\vec{L}}x\hat{\vec{p}}) \times (\hat{\vec{L}}x\hat{\vec{p}}) = (\hat{\vec{L}}.\hat{\vec{p}}x\hat{\vec{p}})\hat{\vec{L}}$$

$$= \left[\hat{L}_x\hat{P}_y\hat{P}_z + \hat{L}_y\hat{P}_z\hat{P}_x + \hat{L}_z\hat{P}_x\hat{P}_y - \hat{L}_x\hat{P}_z\hat{P}_y - \hat{L}_y\hat{P}_x\hat{P}_z - \hat{L}_z\hat{P}_y\hat{P}_x\right] \cdot \hat{\vec{L}}$$

$$(\hat{\vec{L}}x\hat{\vec{p}}) \times (\hat{\vec{L}}x\hat{\vec{p}}) = \left[(\hat{L}_z\hat{P}_y-\hat{L}_y\hat{P}_z)\hat{P}_x + (\hat{L}_x\hat{P}_z-\hat{L}_z\hat{P}_x)\hat{P}_y + (\hat{L}_y\hat{P}_x-\hat{L}_x\hat{P}_y)\hat{P}_z\right]\hat{\vec{L}}$$

We make use of (3.10) to obtain

$$(\hat{\vec{L}}x\hat{\vec{p}}) \times (\hat{\vec{L}}x\hat{\vec{p}}) = \left[(\frac{h}{2\pi i}\hat{p}_x)\hat{p}_x + (\frac{h}{2\pi i}\hat{p}_y)\hat{p}_y + (\frac{h}{2\pi i}\hat{p}_z)\hat{p}_z\right]\hat{\vec{L}}$$

$$= \frac{h}{2\pi i}(\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2)\hat{\vec{L}}$$

$$= \frac{h}{2\pi i}\hat{\vec{p}}^2\hat{\vec{L}} \qquad (D.3)$$

Next, considering the term $\hat{\vec{r}} \times (\hat{\vec{L}}x\hat{\vec{p}}) + (\hat{\vec{L}}x\hat{\vec{p}})x\hat{\vec{r}}$ in (D.2) and by using the vector formula,

$$\vec{A} \times (\vec{B}x\vec{C}) = (\vec{A}.\vec{c})\vec{B} - (\vec{A}.\vec{B})\vec{C}$$

$$(\vec{A}x\vec{B}) \times \vec{C} = (\vec{A}.\vec{C})\vec{C} - (\vec{B}.\vec{C})\vec{A}$$

We then have,

$$\frac{\hat{\vec{r}}}{\hat{r}} \times (\hat{\vec{L}}x\hat{\vec{p}}) + (\hat{\vec{L}}x\hat{\vec{p}}) \times \frac{\hat{\vec{r}}}{\hat{r}} = (\frac{\vec{r}}{\hat{r}} . \hat{\vec{p}})\hat{\vec{L}} - (\frac{\hat{\vec{r}}}{\hat{r}} . \hat{\vec{L}})\hat{\vec{p}} + (\hat{\vec{L}}. \frac{\hat{\vec{r}}}{\hat{r}})\hat{\vec{p}} - (\hat{\vec{p}}. \frac{\hat{\vec{r}}}{\hat{r}})\hat{\vec{L}}$$

$$= \left[(\frac{\hat{\vec{r}}}{\hat{r}} .\hat{\vec{p}}) - (\hat{\vec{p}}. \frac{\hat{\vec{r}}}{\hat{r}})\right]\hat{\vec{L}} \qquad (D.4)$$

From (B.4) we obtain

$$\frac{\hat{\vec{r}}}{\hat{r}} \times (\hat{\vec{L}}x\hat{\vec{p}}) + (\hat{\vec{L}}x\hat{\vec{p}}) \times \frac{\hat{\vec{r}}}{\hat{r}} = \left[- \frac{h}{2\pi i} \operatorname{div}(\frac{\hat{\vec{r}}}{\hat{r}})\right]\hat{\vec{L}}$$

$$= (- \frac{h}{2\pi i} \cdot \frac{2}{\hat{r}})\hat{\vec{L}} \qquad (D.5)$$

Finally, substituting (D.3) and (D.5) into (D.2),

$$\hat{\vec{U}} \times \hat{\vec{U}} = \frac{1}{m_o^2e^4} \cdot \frac{h}{2\pi i} \hat{\vec{p}}^2\hat{\vec{L}} + \frac{1}{m_oe^2}(- \frac{h}{2\pi i} \frac{2}{\hat{r}})\hat{\vec{L}}$$

$$= \frac{h}{2\pi i} \cdot \frac{2}{m_oe^4} (\frac{\hat{\vec{p}}^2}{2m_o} - \frac{e^2}{\hat{r}})\hat{\vec{L}}$$

$$= \frac{h}{2\pi i} \cdot \frac{2}{m_o e^4} \hat{E}\hat{\vec{L}}$$

APPENDIX E

PROVING THE EQUATION (3.15d)

To prove the correction of the equation (3.15d)

$$1 - \hat{\vec{U}}^2 = -\frac{2}{m_o e^4}\hat{E}(\hat{\vec{L}}^2 + \frac{h^2}{4\pi^2}\mathbb{1}) \qquad (E.1)$$

Squaring (3.14) , one obtains

$$\hat{\vec{U}}.\hat{\vec{U}} = \left[\frac{1}{m_o e^2}(\hat{\vec{L}}x\hat{\vec{p}}) + \frac{\hat{\vec{r}}}{\hat{r}}\right]\cdot\left[\frac{1}{m_o e^2}(\hat{\vec{L}}x\hat{\vec{p}}) + \frac{\hat{\vec{r}}}{\hat{r}}\right]$$

$$= \frac{1}{m_o^2 e^4}(\hat{\vec{L}}x\hat{\vec{p}}).(\hat{\vec{L}}x\hat{\vec{p}}) + \frac{1}{m_o e^2\hat{r}}\left[\hat{\vec{r}}.(\hat{\vec{L}}x\hat{\vec{p}}) + (\hat{\vec{L}}x\hat{\vec{p}}).\hat{\vec{r}}\right] + 1$$

$$\therefore\ 1-\hat{\vec{U}}^2 = -\frac{1}{m_o^2 e^4}\left|\hat{\vec{L}}x\hat{\vec{p}}\right|^2 - \frac{1}{m_o e^2\hat{r}}\ \hat{\vec{r}}.(\hat{\vec{L}}x\hat{\vec{p}}) + (\hat{\vec{L}}x\hat{\vec{p}}).\hat{\vec{r}} \qquad (E.2)$$

The term $\left|\hat{\vec{L}}x\hat{\vec{p}}\right|^2$ in (E.2) becomes

$$\left|\hat{\vec{L}}x\hat{\vec{p}}\right|^2 = (\hat{L}_y\hat{p}_z-\hat{L}_z\hat{p}_y)^2+(\hat{L}_z\hat{p}_x-\hat{L}_x\hat{p}_z)^2+(\hat{L}_x\hat{p}_y-\hat{L}_y\hat{p}_x)^2$$

$$= \hat{L}_y^2\hat{p}_z^2-2\hat{L}_y\hat{p}_z\hat{L}_z\hat{p}_y+\hat{L}_z^2\hat{p}_y^2+\hat{L}_z^2\hat{p}_x^2-2\hat{L}_z\hat{p}_x\hat{L}_x\hat{p}_z+\hat{L}_x^2\hat{p}_z^2+\hat{L}_x^2\hat{p}_y^2-2\hat{L}_x\hat{p}_y\hat{L}_y\hat{p}_x+\hat{L}_y^2\hat{p}_x^2$$

With the help of Eq. (3.10) ,we get

$$\left|\hat{\vec{L}}x\hat{\vec{p}}\right|^2 = \hat{p}_x^2\hat{L}_x^2+\hat{p}_x^2\hat{L}_y^2+\hat{p}_x^2\hat{L}_z^2+\hat{p}_y^2\hat{L}_x^2+\hat{p}_y^2\hat{L}_y^2+\hat{p}_z^2\hat{L}_z^2+\hat{p}_z^2\hat{L}_x^2+\hat{p}_z^2\hat{L}_y^2+\hat{p}_z^2\hat{L}_z^2$$

$$+ \hat{p}_x^2 \cdot \frac{h^2}{4\pi^2}\mathbb{1} + \hat{p}_y^2 \cdot \frac{h^2}{4\pi^2}\mathbb{1} + \hat{p}_z^2 \cdot \frac{h^2}{4\pi^2}\mathbb{1}$$

$$= (\hat{p}_x^2+\hat{p}_y^2+\hat{p}_z^2)\left[(\hat{L}_x^2+\hat{L}_y^2+\hat{L}_z^2) + \frac{h^2}{4\pi^2}\mathbb{1}\right]$$

$$\left|\hat{\vec{L}}x\hat{\vec{p}}\right|^2 = \hat{\vec{p}}^2(\hat{\vec{L}}^2 + \frac{h^2}{4\pi^2}\mathbb{1}) \qquad \text{(E.3)}$$

To determine the term $\hat{\vec{r}}.(\hat{\vec{L}}x\hat{\vec{p}}) + (\hat{\vec{L}}x\hat{\vec{p}}).\hat{\vec{r}}$ in (E.2), we will use the formula from vector algebra,

$$\vec{A}.(\vec{B}x\vec{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} \qquad \text{(E.4)}$$

Hence,

$$\hat{\vec{r}}.(\hat{\vec{L}}x\hat{\vec{p}})+(\hat{\vec{L}}x\hat{\vec{p}}).\hat{\vec{r}} = (\hat{x}\hat{L}_y\hat{p}_z+\hat{y}\hat{L}_z\hat{p}_x+\hat{z}\hat{L}_x\hat{p}_y-\hat{z}\hat{L}_y\hat{p}_x-\hat{y}\hat{L}_x\hat{p}_z-\hat{x}\hat{L}_z\hat{p}_y)$$

$$+(\hat{L}_y\hat{p}_z\hat{x}+\hat{L}_z\hat{p}_x\hat{y}+\hat{L}_x\hat{p}_y\hat{z}-\hat{L}_y\hat{p}_x\hat{z}-\hat{L}_x\hat{p}_z\hat{y}-\hat{L}_z\hat{p}_y\hat{x})$$

$$= \Big\{\hat{x}(\hat{z}\hat{p}_x-\hat{x}\hat{p}_z)\hat{p}_z+\hat{y}(\hat{x}\hat{p}_y-\hat{y}\hat{p}_x)\hat{p}_x+\hat{z}(\hat{y}\hat{p}_z-\hat{z}\hat{p}_y)\hat{p}_y-\hat{z}(\hat{z}\hat{p}_x-\hat{x}\hat{p}_z)\hat{p}_x-\hat{y}(\hat{y}\hat{p}_z-\hat{z}\hat{p}_y)\hat{p}_z$$

$$-\hat{x}(\hat{x}\hat{p}_y-\hat{y}\hat{p}_x)\hat{p}_y\Big\} + \Big\{(\hat{z}\hat{p}_x-\hat{x}\hat{p}_z)\hat{p}_z\hat{x}+(\hat{x}\hat{p}_y-\hat{y}\hat{p}_x)\hat{p}_x\hat{y}+(\hat{y}\hat{p}_z-\hat{z}\hat{p}_y)\hat{p}_y\hat{z}$$

$$-(\hat{z}\hat{p}_x-\hat{x}\hat{p}_z)\hat{p}_x\hat{z}-(\hat{y}\hat{p}_z-\hat{z}\hat{p}_y)\hat{p}_z\hat{y}-(\hat{x}\hat{p}_y-\hat{y}\hat{p}_x)\hat{p}_y\hat{x}\Big\}$$

$$= -2\left\{\hat{y}\hat{y}\hat{p}_z\hat{p}_z + \hat{z}\hat{z}\hat{p}_y\hat{p}_y + \hat{z}\hat{z}\hat{p}_x\hat{p}_x + \hat{x}\hat{x}\hat{p}_z\hat{p}_z + \hat{x}\hat{x}\hat{p}_y\hat{p}_y + \hat{y}\hat{y}\hat{p}_x\hat{p}_x\right\}$$

$$+\left\{\hat{z}\hat{p}_z\hat{p}_y\hat{y} + \hat{z}\hat{p}_z\hat{y}\hat{p}_y + \hat{z}\hat{p}_z\hat{x}\hat{p}_x + \hat{z}\hat{p}_z\hat{p}_x\hat{x} + \hat{y}\hat{p}_y\hat{x}\hat{p}_x + \hat{y}\hat{p}_y\hat{p}_x\hat{x}\right.$$

$$\left. + \hat{x}\hat{p}_x\hat{y}\hat{p}_y + \hat{x}\hat{p}_x\hat{p}_y\hat{y} + \hat{x}\hat{p}_x\hat{z}\hat{p}_z + \hat{x}\hat{p}_x\hat{p}_z\hat{z} + \hat{y}\hat{p}_y\hat{p}_z\hat{z} + \hat{y}\hat{p}_y\hat{z}\hat{p}_z\right\} \quad (E.5)$$

With the help of Eq.(2.8), (E.5) becomes

$$\hat{\vec{r}}.(\hat{\vec{L}}x\hat{\vec{p}}) + (\hat{\vec{L}}x\hat{\vec{p}}).\hat{\vec{r}} = -2\left\{\hat{y}\hat{y}\hat{p}_z\hat{p}_z + \hat{z}\hat{z}\hat{p}_y\hat{p}_y - \hat{z}\hat{p}_y\hat{y}\hat{p}_z - \hat{y}\hat{p}_z\hat{z}\hat{p}_y + \hat{z}\hat{z}\hat{p}_x\hat{p}_x\right.$$

$$+ \hat{x}\hat{x}\hat{p}_z\hat{p}_z - \hat{x}\hat{p}_z\hat{z}\hat{p}_x - \hat{z}\hat{p}_x\hat{x}\hat{p}_z + \hat{x}\hat{x}\hat{p}_y\hat{p}_y + \hat{y}\hat{y}\hat{p}_x\hat{p}_x$$

$$\left. - \hat{y}\hat{p}_x\hat{x}\hat{p}_y - \hat{x}\hat{p}_y\hat{y}\hat{p}_x + \frac{h^2}{4\pi^2}\mathbb{1}\right\}$$

$$\hat{\vec{r}}.(\hat{\vec{L}}x\hat{\vec{p}})+(\hat{\vec{L}}x\hat{\vec{p}}).\hat{\vec{r}} = -2\left[(\hat{y}\hat{p}_z-\hat{z}\hat{p}_y)(\hat{y}\hat{p}_z-\hat{z}\hat{p}_y)+(\hat{z}\hat{p}_x-\hat{x}\hat{p}_z)(\hat{z}\hat{p}_x-\hat{x}\hat{p}_z)\right.$$

$$\left.+(\hat{x}\hat{p}_y-\hat{y}\hat{p}_x)(\hat{x}\hat{p}_y-\hat{y}\hat{p}_x) + \frac{h^2}{4\pi^2}\mathbb{1}\right]$$

$$= -2\left[\hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 + \frac{h^2}{4\pi^2}\mathbb{1}\right] \quad (E.6)$$

Substituting (E.3) and (E.6) into (E.2)

$$1-\hat{\vec{U}}^2 = = -\frac{1}{m_o^2e^4}\hat{\vec{p}}^2(\hat{\vec{L}}^2 + \frac{h^2}{4\pi^2}\mathbb{1}) + \frac{2}{m_oe^2\hat{r}}(\hat{\vec{L}}^2 + \frac{h^2}{4\pi^2}\mathbb{1})$$

$$= -\frac{2}{m_oe^4}\left(\frac{p^2}{2m_o} - \frac{e^2}{\hat{r}}\right)\left[\hat{\vec{L}}^2 + \frac{h^2}{4\pi^2}\mathbb{1}\right]$$

$$1-\hat{\vec{U}}^2 = -\frac{2}{m_oe^4}\hat{E}(\hat{\vec{L}}^2 + \frac{h^2}{4\pi^2}\mathbb{1}) \quad (E.7)$$

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