# CHAPTER V

## CONCLUSION AND DISCUSSIONS

The main purpose of this thesis is to establish the fundamental field equations by using the continuity equation. We first *assume* that there exists some conserved quantity, denoted by Q, in nature. The global conservation of Q implies directly that it is also conserved locally as well. Its local conservation is guaranteed by the continuity equation in three-space,

$$\frac{\partial \rho(\mathbf{x},t)}{\partial t} + \nabla . \mathbf{j}(\mathbf{x},t) = 0. \qquad (5.1)$$

The local conservation implies that Q must be a conserved scalar. As a result, the continuity equation should have *form invariance* under some transformation laws of the reference frames.

The requirement of form invariance (or covariance) of the continuity equation forces us to remind of the tensor equations which are form invariance under the so-called *tensor transformations*. After taking some considerations, we find that continuity equation Eq.(5.1) can actually be able to write in terms of tensor equation,



$$\partial_\nu J^\nu = 0. \qquad (5.2)$$



This tensor equation is valid in four-dimensional of space and time, or called for brevity as *four-space*, because both operator $\partial_\nu$ and function $J^\nu$ are composed of four components,

$$\partial_\nu \equiv (\partial_0, \partial_1, \partial_2, \partial_3) = [\partial/\partial(kt), \nabla], \qquad (5.3a)$$

and

$$J^\nu \equiv (J^0, J^1, J^2, J^3) = (k\rho, \mathbf{j}). \qquad (5.3b)$$

Notice that the coordinates involved in the operator $\partial_\nu$ are constituted both scalar part and vector part thus this operator is defined in four-space corresponding to what we have stated earlier. So, the operator $\partial_\nu$ and the function $J^\nu$, defined in Eq.(5.3), can be treated as vectors in four-space, called the *four-vectors.* The zeroth-component of both four-vectors is called the *scalar part* and the other three left components are elaborated the *vector part.* In order to make the vector part be compatible with the scalar part, we have to introduce the *universal constant velocity* denoted by k in the scalar part as shown in Eq.(5.3). The appearance of k is so strange because it was never mentioned in the realm of Galilean transformation of classical mechanics. Therefore, we conclude that the continuity equation Eq.(5.2) must be invariant under a new kind of transformation under which the existence of k is allowed. We try to formulate this new transformation laws by using only three basic postulates:

I. Postulate of the inertial motion of frames.
II. Postulate of the universality of k.
III. Postulate of the reciprocity of space.

We find that the first two postulates give us a new addition law for velocity in the x-direction,

$$u_x' = \frac{u_x - v}{1 - vu_x/k^2}, \qquad (5.4)$$



under which k is conserved, and the third assumption gives rise that k is also playing the role as the *limiting speed* in nature, no other particle speeds can exceed the universal speed k. Finally, from the above three postulates, we discover, in Chapter III, that the suitable tensor transformation laws for the four-vectors in the continuity equation Eq.(5.2), is a linear homogeneous coordinate transformations,

$$x' = \beta(x-vt), \tag{5.5a}$$

$$y' = y, \tag{5.5b}$$

$$z' = z, \tag{5.5c}$$

$$t' = \beta(t-vx/k^2), \tag{5.5d}$$

where $\beta = 1/\left[1-(v^2/k^2)\right]^{1/2}$ is the constant of the transformation. We call the set of the transformation laws in Eq.(5.5) *the inertial transformation.*

We find that if we replace k in Eq.(5.5) with c, the velocity of light in vacuum, we will obtain the well-known *Lorentz transformations* that was first developed by Einstein in his special theory of relativity. To assert the uniqueness of inertial transformation, we are convinced that there should be only one universal constant velocity exists in nature but what it be, at this stage, is dependent on the empirical facts. Note that, we will also call any quantity that transforms under inertial transformation, similar to the four-vectors $\partial_\nu$ and $J^\nu$, *the four-vector.* The roles of Lorentz transformations are also generalized from (1+1)-dimension, as we have done, to (1+n)-dimension where n is finite or infinite by Ungar (1992). This new formalism allows one to solve in the abstract Lorentz group previously poorly understood problems in the standard (1+3)-dimensional Lorentz group.

After we have known that the continuity equation could be written in terms of tensor equation in four-space, we then proceed to show that this fact is so useful in deriving the *fundamental field equations* in four-space. This idea is partly stimulated from the logical fact that Robert Mills has presented in his brilliant paper in 1989. In the paper, Mills has pointed out the astonishing fact, from gauge theories, that *for every true conservation law there is a complete theory of a gauge field for which the given conserved quantity is the source.* The only restriction is that the conservation law be associated with a continuous symmetry. We find that our conserved Q is satisfied with this constraint so it should be related to some real fields in nature. It is evident from the invariant form of continuity equation Eq.(5.2) that the source in four-space must be the entire four-vector $J^{\nu}$, not just its zero components.

The relation of source $J^{\nu}$ and field $F^{\mu\nu}$, which conventionally called *field equation in four-space*, are supposed to be

$$\partial_{\mu}F^{\mu\nu} = \alpha J^{\nu}; \tag{5.6}$$

where $\alpha$ is the scaling constant and the superposition of fields is also realized. To justify the continuity equation Eq.(5.2), we find that the tensor field $F^{\mu\nu}$ should be an antisymmetric tensor which, from sixteen, has left only six independent components. These six components can be exhibited, in three-space, in forms of two three-vector fields which called the *polar field* $\mathbf{P}(\mathbf{x},t)$, and the *axial field* $\mathbf{A}(\mathbf{x},t)$. They have components labeled as

$$\mathbf{P}(\mathbf{x},t) = (P_1, P_2, P_3) = (F^{10}, F^{20}, F^{30}), \tag{5.7a}$$

$$\mathbf{A}(\mathbf{x},t) = (A_1, A_2, A_3) = (-F^{23}, -F^{31}, -F^{12}). \tag{5.7b}$$

The field equations in three-space then become

(for $\nu = 0$)
$$\nabla\cdot\mathbf{P}(\mathbf{x},t) = \alpha k\rho(\mathbf{x},t), \tag{5.8a}$$

(for $\nu = 1,2,3$)
$$\nabla\times\mathbf{A}(\mathbf{x},t) - \frac{1}{k}\frac{\partial\mathbf{P}(\mathbf{x},t)}{\partial t} = \alpha\,\mathbf{j}(\mathbf{x},t), \tag{5.8b}$$

These two equations are, indeed, not enough to determine the characteristics of the fields, through we consider for the dual of the tensor field $F^{\mu\nu}$ defined by

$$^*F^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}, \qquad (5.9)$$

where $\varepsilon^{\mu\nu\alpha\beta}$ is the Levi-Civita symbol defined in Chapter II. The *dual field tensor* $^*F^{\mu\nu}$ contains precisely the same information as $F^{\mu\nu}$ but manifests in the different feature. The source of this dual field is called *dual* quantity, which in this thesis is denoted by $^*J^{\nu} \equiv (k^*\rho,\ ^*\mathbf{j})$. The relation of the dual field and its source in four-space is described as

$$\partial_\mu {}^*F^{\mu\nu} = \alpha {}^*J^{\nu}. \qquad (5.10)$$

The field equations for $^*F^{\mu\nu}$ and $^*J^{\nu}$ in three-space can be displayed as follows:

(for $\nu = 0$)
$$\nabla\cdot\mathbf{A}(\mathbf{x},t) = \alpha k^*\rho(\mathbf{x},t), \qquad (5.11a)$$

(for $\nu = 1,2,3$)
$$-\nabla\times\mathbf{P}(\mathbf{x},t) - \frac{1}{k}\frac{\partial\mathbf{A}(\mathbf{x},t)}{\partial t} = \alpha^*\mathbf{j}(\mathbf{x},t). \qquad (5.11b)$$

Finally, the complete forms of field equations in four-space are

$$\partial_\mu F^{\mu\nu} = \alpha J^{\nu}, \qquad (5.12a)$$
$$\partial_\mu {}^*F^{\mu\nu} = \alpha^*J^{\nu}. \qquad (5.12b)$$

These field equations will appear in three-space in the forms:

$$\nabla\cdot\mathbf{P}(\mathbf{x},t) = \alpha k\rho(\mathbf{x},t), \qquad (5.13a)$$
$$\nabla\times\mathbf{A}(\mathbf{x},t) - \frac{1}{k}\frac{\partial\mathbf{P}(\mathbf{x},t)}{\partial t} = \alpha\mathbf{j}(\mathbf{x},t), \qquad (5.13b)$$
$$\nabla\cdot\mathbf{A}(\mathbf{x},t) = \alpha k^*\rho(\mathbf{x},t), \qquad (5.13c)$$
$$-\nabla\times\mathbf{P}(\mathbf{x},t) - \frac{1}{k}\frac{\partial\mathbf{A}(\mathbf{x},t)}{\partial t} = \alpha^*\mathbf{j}(\mathbf{x},t). \qquad (5.13d)$$

In order to determine the physical meaning of k, we consider the special case, in the region where both $J^\nu$ and $^*J^\nu$ vanished, then Eq.(5.13) implies directly the existence of two wave fields in three-space,

$$\nabla^2\mathbf{P}(\mathbf{x},t) - \frac{1}{k^2}\frac{\partial^2\mathbf{P}(\mathbf{x},t)}{\partial t^2} = 0, \tag{5.14a}$$

$$\nabla^2\mathbf{A}(\mathbf{x},t) - \frac{1}{k^2}\frac{\partial^2\mathbf{A}(\mathbf{x},t)}{\partial t^2} = 0. \tag{5.14b}$$

In this case, k is regarded as the velocity of the propagation of waves **P** and **A** in empty three-space. Therefore, our universal constant velocity k, which first appeared as the logical addition, is determined to be the velocity of the wave fields **P** and **A** of which $J^\nu$ and $^*J^\nu$ are being the sources.

It is not necessary to introduce potentials, but they can be introduced for convenience. If the dual quantity $^*J^\nu$ is vanished, tensor field equations in four-space become

$$\partial_\mu F^{\mu\nu} = J^\nu, \tag{5.15a}$$

$$\partial_\mu {}^*F^{\mu\nu} = 0. \tag{5.15b}$$

From Eq.(5.15), we can define $F^{\mu\nu}$ in terms of a new four-vector called *potential four-vector*, denoted by $U^\mu \equiv (\Phi, \mathbf{u})$, where

$$F^{\mu\nu} = \partial^\mu U^\nu - \partial^\nu U^\mu. \tag{5.16}$$

In general, *gauge transformations* can be made on $U^\mu$,

$$U^{\mu\prime} = U^\mu - \partial^\mu \chi, \tag{5.17}$$

where $\chi(x^{\mu})$ is the independent function of $x^{\mu}$. Field equations in four-space, Eq.(5.15), are left invariantly under gauge transformation, this because

$$F^{\mu'\nu'} = \left[\partial^{\mu}U^{\nu} - \partial^{\mu}\partial^{\nu}\chi\right] - \left[\partial^{\nu}U^{\mu} - \partial^{\nu}\partial^{\mu}\chi\right] = \left[\partial^{\mu}U^{\nu} - \partial^{\nu}U^{\mu}\right] = F^{\mu\nu}.$$

In classical electrodynamics, the vector and scalar potentials were first introduced as a convenient mathematical aid for calculating the fields. But in quantum mechanics, the potentials have some significant aspects because they provide an important role in causing the so-called *Aharonov-Bohm effect* (Aharonov and Bohm, 1959). It is implied from these effects that the potentials can, in certain cases, be considered as physically effective, even when there are no fields acting on the charged particles. These counterintuitive effects play an important roles in the theory of electromagnetic interactions, in solid-state physics and possibly in the development of new microelectronics devices (Imry and Webb, 1989).

The most excellent example of our studies, and indeed the only one that we find now, is the approach to electromagnetic fields. The classical Maxwell equations are easily obtained if we replace the electric charge density $\rho_e$ and the electric current density $\mathbf{j}_e$ in Eq.(5.13) and use $\alpha = 4\pi/k$ for Gaussian unit, we find that



$$\nabla\cdot\mathbf{P}(\mathbf{x},t) = 4\pi\rho_e(\mathbf{x},t), \qquad (5.18a)$$

$$\nabla\times\mathbf{A}(\mathbf{x},t) - \frac{1}{k}\frac{\partial\mathbf{P}(\mathbf{x},t)}{\partial t} = \frac{4\pi}{k}\mathbf{j}_e(\mathbf{x},t), \qquad (5.18b)$$

$$\nabla\cdot\mathbf{A}(\mathbf{x},t) = 0, \qquad (5.18c)$$

$$\nabla\times\mathbf{P}(\mathbf{x},t) + \frac{1}{k}\frac{\partial\mathbf{A}(\mathbf{x},t)}{\partial t} = 0, \qquad (5.18d)$$

where the *magnetic charge* $J^{\nu}{}_{m}$ is disappeared. Eq.(5.18) will exactly be represented the complete set of *Maxwell equations* in three-space if we replace in the definitions of the electric field $\mathbf{E} \equiv \mathbf{P}$, the magnetic field $\mathbf{B} \equiv \mathbf{A}$, and k=c is the velocity of the electromagnetic waves in vacuum. It is unfortunately that we can not discover any candid conserved quantities that can be related to fields as we have proposed. The magnetic charges may exist in nature, as we predict, but no compelling experiment to confirm is found except that the only one performed by Cabrera (Cabrera, 1981).

In classical mechanics, we are convinced to believe that the gravitational field is produced by the conserved quantity called *mass*, so the gravitational equations in three-space should be appeared similarly to the fundamental field equations we have developed in Eq.(5.13). As a result, for static case, the gravitational field should be presented similarly to the Coulomb's law shown in Eq.(4.43), or,

$$\mathbf{g}(\mathbf{x}) = \int_{V} \frac{\rho_g(\mathbf{x}')(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} d^3x',$$

where $\mathbf{g}(\mathbf{x})$ is the static gravitational field and $\rho_g$ is the mass density. However, there is evident that the freely falling frames in gravitational field can also be considered as the inertial frames because no net force acts on the particles. Then, the results of local experiments in free fall are consistence with inertial frames in special theory of relativity. This statement is generally known as the strong equivalence in the general theory of relativity developed by Einstein (Kenyon,1990). Implication from this principle, in a freely falling laboratory, we feel no gravitational field at all. By transforming into a frame of reference which is freely falling, we completely replace gravity by accelerated frames of reference. Thus, Einstein's fundamental insight was to abolish gravity altogether and replace it by appropriate transforms between accelerated frames of reference. This idea is so different from our approach which is involved only the transformation between inertial frames of reference. Therefore, in accelerated frames, the continuity equation described in Eq.(5.2) should be modified in order to take account for the gravitational field.

The another important role of sources is the *interaction* with its field, in this thesis, we do not develop such item because it needs more assumptions than usual and it is not in the scope of our interests. The example of this logical process to obtain the Lorentz force law was studied by Zeleny (1991).