CHAPTER III

FORMULATION OF INERTIAL TRANSFORMATION

In this chapter it is shown that the continuity equation is covariant under the new transformation called inertial transformation not under the classical Galilean transformation and the existence of universal constant velocity will be arised automatically, we are not necessary to postulate it at the outset.

The Form Invariance of Continuity Equation

The continuity equation is emerged directly as a consequence of the conservation law of nature. If some quantity in nature, may be denoted by Q, is conserved globally, this means it must be a conserved scalar which is not dependent on any coordinate systems. Thus, Q must be conserved in all frames of reference. The global conservation of Q implies directly that it should be conserved locally as well (Feynman,1965). This means at any point in spacetime the continuity equation,

$$\frac{\partial \rho}{\partial t} + \nabla . \mathbf{j} = 0 \qquad (3.1)$$

must be existed, where $\rho(\mathbf{x},t)$ is the *density* of Q and $\mathbf{j}(\mathbf{x},t)$ is the corresponding *current density vector* defined classically as $\mathbf{J} = \rho\mathbf{v}$, where $\mathbf{v}(\mathbf{x},t)$ is the typical velocity of Q at the considered point $(\mathbf{x},t)$ (Sokolnikoff and Redheffer, 1987).

The continuity equation Eq.(3.1) implies conservation of Q within a volume of space that can be made arbitrary small, which means that Q is locally conserved; processes that destroy Q at one point and create it at another are forbidden, even though they may conserve the total of Q. This is because global conservation of Q would require the propagation of instantaneous signals between distant points, which is inconsistent with the principle of relativity. This concept of a local conservation law will play a central role in the material to follow.

To preserve the global conservation of Q, there requires that it must be conserved in all frames of reference and, as a result, Eq.(3.1) should be covariant or have the same form in all reference frames. Following this idea, we suggest that the continuity equation Eq.(3.1) should be written in form of tensor equation,

$$\partial_\nu J^\nu = 0, \tag{3.2}$$

where we take the legitimate assumption that ∂_ν is the *covariant* partial derivative vector and J^ν is the *contravariant* current-density vector. The scalar product of these two vectors is ensured the covariant form of the continuity equation. Note that the convention that repeated Greek indices are summed from 0 to 3 is used. The two vectors each has four components, one for the scalar part and three for vector part,

$$\partial_\nu \equiv (\partial_0, \partial_1, \partial_2, \partial_3) \equiv (\partial_0, \nabla) \tag{3.3a}$$

$$J^\nu \equiv (J^0, J^1, J^2, J^3) \equiv (J^0, \mathbf{j}), \tag{3.3b}$$

then they are exhibited as vectors, or first-rank tensors in four-dimension of space and time, or *spacetime*. Thus, we may call ∂_ν the *derivative four-vector* and J^ν the *current-density four-vector.*

From the definitions in Eq.(3.3), if we set $\partial_0 \equiv \partial/\partial t$ and $J^0 \equiv \rho$, we find that Eq.(3.2) apparently becomes the continuity equation Eq.(3.1) but there still contains some logical inconsistencies in these definitions. This because we have convinced that all components of a four-vector must be equivalent so they must have the same unit. Therefore our definitions for ∂_0 and J^0 above are improper because they have different units from the others in the vector part. For the sake of consistency, we have to introduce the constant, denoted by k, which has unit of velocity, into the definitions of ∂_0 and J^0 by the way as follows:

$$\partial_0 \equiv \partial/\partial(kt) \quad ; \quad J^0 \equiv k\rho. \tag{3.4}$$

The continuity equation still satisfies with the new definitions in Eq.(3.4) but the confusion of the inconsistency is now disappeared. The constant k, whose dimensions are velocity, must be appeared in all reference frames, according with the transformation of the four-vectors ∂_0 and J^0, thus it must be *universal*, we then call it the *universal constant velocity*. No such velocity is allowed in the classical Galilean transformation (Serway, Moses, and Moyer, 1989) so that the continuity equation is not covariant under Galilean transformation. The proper transformation under which the form invariance of continuity equation holds will be developed in the next section.

<u>Concepts of Inertial Transformation</u>

In this section we will determine the appropriate transformation under which the continuity equation Eq.(3.2) holds covariantly. We begin by considering the transformation of the derivative four-vector $\partial_\nu \equiv (\partial_0, \nabla)$ between the two frames S and S$'$ by using the rules of implicit differentiation,

$$\partial_{\nu'} \equiv \frac{\partial}{\partial X^{\nu'}} = \frac{\partial X^{\nu}}{\partial X^{\nu'}}\frac{\partial}{\partial X^{\nu}} = a^{\nu}{}_{\nu'}\,\partial_\nu \tag{3.5}$$

where $X^\nu \equiv (X^0, X^1, X^2, X^3) \equiv (X^0, \mathbf{x}) = (kt, \mathbf{x})$ is the coordinate vector in spacetime, and, for convenience, we will write

$$\frac{\partial X^{\nu}}{\partial X^{\nu'}} = a^{\nu}{}_{\nu'}, \quad \frac{\partial X^{\nu'}}{\partial X^{\nu}} = a^{\nu'}{}_{\nu}, \qquad (3.6)$$

and use a similar notation for other such derivatives. It is difficult to formulate the transformation laws directly from Eq.(3.5). We find that it is more suitable to obtain transformation laws from the transformation of differentials of coordinates dX^{ν}, where

$$dX^{\nu'} = \frac{\partial X^{\nu'}}{\partial X^{\nu}} dX^{\nu} = a^{\nu'}{}_{\nu} dX^{\nu}. \qquad (3.7)$$

It is evident from Eqs.(3.5) and (3.7) that the derivative four-vector ∂_{ν} can be calculated from the inverse of the transformation laws of the differentials dX^{ν}. Actually, we have already shown in Eq.(2.22) that the differentials of the coordinates, dX^{ν} is the simplest example of a contravariant vector in four-space. (It is for this reason that we conventionally write coordinates with superscripts.)

We then provoke the fundamental assumption that spacetime must be *homogeneous*, in that it has "everywhere and every time" the same properties. The homogeneity assumption implies that the transformation equations which furnish spacetime coordinates x' and t' as functions of x and t are linear (Berzi and Gorini, 1969). Under *linear* transformation,

$$X^{\nu'} = A^{\nu'}{}_{\nu} X^{\nu} + B^{\nu'} \qquad (3.8)$$

where ($A^{\nu'}{}_{\nu}$, $B^{\nu'} \equiv$ constant), the coordinate differences ΔX^{ν} transform like the coordinate differentials dX^{ν} and thus constitute a qualified four-vector called *displacement four-vector*. Because of this, the displacement four-vector can then serve to represent any contravariant four-vector. As a result, the coordinate vector

$X^{\nu} \equiv (kt, \mathbf{x})$ itself behaves as a four-vector only under linear homogeneous transformation ($B^{\nu'} = 0$) in order to conserve the common origin, if $X^{\nu} \equiv (0, \mathbf{0})$ then $X^{\nu'} \equiv (0, \mathbf{0})$, too.

The linearity of the proper transformation has an important physical consequence called *inertial motions*, the uniform motion with a constant velocity. From the requirement of linear homogeneous transformation, we can suggest that the relation between the pair components of coordinate four-vectors $X^{\nu'} \equiv (kt', \mathbf{x}')$ and $X^{\nu} \equiv (kt, \mathbf{x})$, for the special case that the two frames of reference S' and S are moving relatively along xx' axis so that only x-component is changed and the other spatial components are left undisturbed, could be written as

$$x' = A(v)x - B(v)t \tag{3.9a}$$

$$t' = C(v)t - D(v)x \tag{3.9b}$$

where A, B, C, and D are the constant functions of the unspecified parameter v and the minus sign has been introduced for further convenience. It is obviously seen from Eq.(3.9) that an object has an inertial motion, with velocity v, in some reference frame S if it is at rest in another equivalent frame S'. Inertial motions then are characterized by the same parameter as the transformation of the coordinate four-vector. Their general equation of motion, according to (3.9a), is

$$x = [B(v)/A(v)]t, \tag{3.10}$$

therefore the *inertial velocity* v equals B/A. Thus we can say that the inertial motions are obtained from rest by the proper transformation which will be referred to later as *inertial transformation* because it represents for the transformation of any inertial frame of reference. With an adequate change of notation, our inertial transformation equations Eq.(3.9) may be written

$$x' = E(v)(x-vt) \qquad (3.11a)$$

$$t' = E(v)\left[F(v)t-G(v)x\right] \qquad (3.11b)$$

depending on three unknown functions E, F, and G.

<u>Formulation of Inertial Transformation</u>

We have already known that the inertial transformations Eq.(3.11) is valid for two inertial frames of reference, moving with relative velocity v along xx' axis. At this point we will determine for the appropriate three constants E, F, and G. From the previous studies in the last two sections, it requires basically that the transformation laws in Eq.(3.11) must be satisfied two initial conditions:

I. The condition of inertial motion. If the particle is at rest in frame S' then it will have velocity v in frame S,

II. The condition of universal velocity. If the particle has the velocity k (-k) in frame S' then it will also have velocity k (-k) in frame S.

We find that Eq.(3.11) can give rise directly the additional law for velocities of inertial transformation,

$$u_x' = (u_x-v)/(F-u_xG) \qquad (3.12)$$

Then we make use of the second condition to obtain the following two equations:

$$k(F-kG) = (k-v) \qquad (3.13a)$$

$$k(F+kG) = (k+v). \qquad (3.13b)$$

From these equations we can readily prove that the parameters F(v) and G(v) should be defined by

$$F(v) = 1, \; G(v) = v/k^2. \qquad (3.14)$$

Then Eq.(3.11) becomes

$$x' = E(v)(x-vt) \qquad (3.15a)$$

$$t' = E(v)\left[t-vx/k^2\right] \qquad (3.15b)$$

and the additional transformation for velocity Eq.(3.10) becomes

$$u_x' = \frac{(u_x-v)}{(1-vu_x/k^2)} \qquad (3.16)$$

Indeed, it is more fundamental to obtain the transformation laws for coordinates by deriving first the additional law for velocities, Eq.(3.16) directly from the above two postulates, I and II, but the inertial transformation must be also proposed as an initial condition. After having the additional law, we can easily obtain the transformation laws for coordinates in four-space by the integration process. The complete treatment of this idea is shown in Appendix B.

To determine the only unsolved function E(v), we have to introduce another postulate of the transformation, the postulate of symmetry:

III. The inertial transformation is *symmetric* with respect to the inertial frames S and S′.

This assumption is arisen from the particular fact that if we put $u_x=0$ in Eq.(3.16), we find that $u_x'=-v$, in other words, S travels with constant velocity $-v$ relative to S'. Then the transformation equations Eq.(3.15) can be written as

$$x = E(-v)(x'+vt') \tag{3.17a}$$

$$t = E(-v)\left[t'+vx'/k^2\right] \tag{3.17b}$$

If we insert x and t from Eq.(3.15) into Eq.(3.17), we find that

$$E(v)E(-v) = 1/(1-v^2/k^2) \tag{3.18}$$

Then, the unknown function E(v) may be shown to be even function of the parameter v, or $E(v) = E(-v)$ as follows. We assume that space (not spacetime) is isotropic. The *isotropy* of space means space is nondirectional, so that both orientations of the space axis are physically equivalent. We state the isotropy principle by asserting that if two frames S and S' are connected by a transformation laws Eq.(3.15), then the two frames S^* and S'^* obtained from the preceding ones by inverting the direction of the x axis are connected by a transformation of the same type. Therefore

$$x'^* = E(v^*)(x^*-v^*t^*) \tag{3.19a}$$

$$t'^* = E(v^*)\left[t^*-v^*x^*/k^2\right] \tag{3.19b}$$

where v^* is the velocity of S'^* relative to S^*. However, if we apply $x^*=-x$, $t^*=t$, $x'^*=-x'$, and $t'^*=t'$ in Eq.(3.19), we will have the equations

$$-x' = E(v^*)(-x-v^*t) \tag{3.20a}$$

$$t' = E(v^*)\left[t+v^*x/k^2\right], \tag{3.20b}$$

By comparing Eq.(3.20) with Eq.(3.15), we obtain immediately that $E(v^*) = E(v)$ and $v^* = -v$, then, as a result, $E(v^*) = E(v)$ as required. Such natural result, expressing the relative velocity of the *reversed* reference frames as the opposite of the relative velocity of the initial frames, might have been taking for granted.

From these results, we immediately find that, from Eq.(3.18),

$$E(v) = 1/(1-v^2/k^2)^{1/2} \tag{3.21}$$

and the complete form of inertial transformation Eq.(3.15) becomes

$$x' = E(v)(x-vt) \tag{3.22a}$$

$$t' = E(v)\left[t-vx/k^2\right] \tag{3.22b}$$

and its reverse transformation defined in Eq.(3.17) becomes

$$x = E(v)(x'+vt') \tag{3.23a}$$

$$t = E(v)\left[t'+vx'/k^2\right] \tag{3.23b}$$

where $E(v)$ is the proper constant defined by Eq.(3.21) and the two undisturbed spatial components are transformed as $y' = y$ and $z' = z$. By collecting our results, we have found the *standard inertial transformation,*

$$x' = \beta(x-vt), \qquad y' = y, \qquad z' = z.$$
$$t' = \beta\left[t-vx/k^2\right], \tag{3.24a}$$

and its reversed transformations,

$$x = \beta(x'+vt'), \qquad y = y', \qquad z = z'.$$
$$t = \beta[t'+vx'/k^2] \tag{3.24b}$$

where $\beta \equiv E(v)$ is a costant function of v, $\beta = 1/(1-v^2/k^2)^{1/2}$.

It follows from Eq.(3.21) that the universal constant k plays the role of a *limiting* universal speed, which though unique, is as yet arbitrary, and need not be identified with the speed of light. The nature of the inertial formulas dictate that no particle may exceed this limiting speed without leading to imaginary value for the transformation coefficient $E(v) \equiv \beta$. We emphasize, however, that this limiting universal speed was *not* assumed in deriving the formulas, but follows as a consequence from them. The possibility that $k = \infty$, with which Eq.(3.24) becomes the classical Galilean transformation:

$$x' = (x-vt), \quad y' = y, \quad z' = z, \quad t' = t,$$

could not be ruled out on the basis of theory alone, but there is an abundance of experimental evidence which points to the fact that k is infinite. The existence of signals which travel with a finite invariant velocity leads us to eliminate the Galilean transformation. If the speed of light c is being the universal velocity, as Einstein has postulated, then our inertial transformations in Eq.(3.24) become the Lorentz transformation Eq.(1.22). Experiments actually show that the form of dependence of mass on velocity follow closely that derived from the Lorentz transformation, and that the limiting velocity k is indistinguishable from the speed of light c to within present experimental limits of accuracy. That the limiting universal velocity k is finite is an experimental facts.

To fulfill the requirement that the continuity equation must have invariant form, we find that, after taking a long logical process, it is covariant under the linear homogeneous equations called the *inertial* transformation for inertial frames of which we have established from only three fundamental postulates. This transformation also provides the crucial recognition that space and time are not absolute, a concept which is foreign to Newtonian mechanics.

The transformation may have a more generalized form if we omit some of the three postulates above. For example, Sewjathan has developed a fundamental rederivation of special relativity based on the c-invariance postulate but neglect other sufficient assumptions. He, finally, established the more generalized form of Lorentz transformation valid for all real values of particle's velocity (Sewjathan,1984). Lee and Kalotas have derived the Lorentz transformation by invoking the principle of reciprocity alone. They found that there must be a universal limiting speed in nature as we mention above (Lee and Kalotas,1975). The precise explanation of the significances of reciprocal principle was illuminated in the papers by Gorine and Zecca (Gorini and Zecca,1970) and Berzi and Gorini (Berzi and Gorini,1969).

Four-vectors

The inertial transformation Eq.(3.24a) describes the transformation of the contravariant coordinate four-vector $X^{\nu} \equiv (kt, \mathbf{x})$ from one inertial frame to another and its inverse Eq.(3.24b) for the transformation of covariant derivative four-vector ∂_{ν}. In three-dimensions we call $\mathbf{x}$ a *vector* and speak of x^1, x^2, x^3 as the components of a vector. We designate by the same name any three physical quantities that transform under rotations in the same way as the components of $\mathbf{x}$. It is natural therefore to anticipate that there are numerous physical quantities that transform under inertial transformation in the same manner as the four-vectors X^{ν}

and ∂_ν. By analogy we call them also being *four-vectors.* The concept of four-vectors involves closely with coordinate transformations. Each four-vector **A** can alternately be described by its *contravariant* or *covariant* components, A^ν or A_ν. The two kinds are distinguished by their transformation laws. The contravariant vector **A** = A^ν with four components A^0, A^1, A^2 , A^3 are transformed according to the rule,

$$A^{\nu'} = (\partial X^{\nu'}/\partial X^{\nu})A^{\nu} = a^{\nu'}{}_{\nu}A^{\nu}, \qquad (3.25a)$$

and the covariant vector **A** = A_ν with four components A_0, A_1, A_2, A_3 are transformed according to the rule,

$$A_{\nu'} = (\partial X^{\nu}/\partial X^{\nu'})A_{\nu} = a^{\nu}{}_{\nu'}A_{\nu}. \qquad (3.25b)$$

Our defined current-density four-vector is an example of contravariant four-vector under inertial transformation.

The above definitions are general. The specific of the spacetime is defined by the invariant interval ds^2, which can be verified directly from the transformation laws Eq.(3.24a),

$$\begin{aligned} ds^2 &= k^2dt^2 - dx^2 - dy^2 - dz^2, \\ &= (dX^0)^2 - (dX^1)^2 - (dX^2)^2 - (dX^3)^2. \end{aligned} \qquad (3.26)$$

This norm or metric is a special case of the general differential length element of the (pseudo-) Reimannian space,

$$ds^2 = g_{\mu\nu}dX^{\mu}dX^{\nu}. \qquad (3.27)$$

where $g_{\mu\nu} = g_{\nu\mu}$ is called the *metric tensor.* For *flat* spacetime, the metric tensor is diagonal,

$$g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1). \qquad (3.28)$$

As we have shown in the previous chapter, the covariant coordinate four-vector X_ν can be obtained from the contravariant X^ν by contraction with $g_{\mu\nu}$, that is,

$$X_\nu = g_{\mu\nu}X^\nu \qquad (3.29a)$$

and its inverse,

$$X^\nu = g^{\mu\nu}X_\nu, \qquad (3.29b)$$

where, for flat spacetime, $g_{\mu\nu} = g^{\mu\nu}$.

With the metric tensor $g_{\mu\nu}$, Eq.(3.28), it follows that if a contravariant four-vector has components A^0, A^1, A^2, A^3, its covariant partner has components, $A_0 = A^0$, $A_1 = -A^1$, $A_2 = -A^2$, $A_3 = -A^3$. We write this concisely as

$$A^\nu = (A^0, \mathbf{a}), \quad A_\nu = (A^0, -\mathbf{a}), \qquad (3.30)$$

where the three-vector **a** has components A^1, A^2, A^3. The scalar product of two four-vectors is

$$\mathbf{B}\cdot\mathbf{A} = B^\nu A_\nu = (B^0A^0 - \mathbf{b}\cdot\mathbf{a}). \qquad (3.31)$$

Consider now the transformation of derivative four vectors Eq.(3.5), it shows that differentiation with respect to a contravariant component of the coordinate vector transforms as the component of a covariant vector operator. From Eq.(3.29a) it follows that differentiation with respect to a covariant component gives a contravariant vector operator. We therefore employ the notations,

$$\partial^\nu \equiv \partial/\partial X_\nu = (\partial/\partial X^0, -\nabla)$$
$$\partial_\nu \equiv \partial/\partial X^\nu = (\partial/\partial X^0, \nabla) \qquad (3.32)$$

The four-divergence of a four-vector is the invariant,

$$\partial_\nu A^\nu = \partial^\nu A_\nu = (\partial A^0/\partial X^0) + \nabla\cdot\mathbf{a} \tag{3.33}$$

an equation familiar in form from continuity equation, Eq.(3.1), of density and current density of Q. The four-dimensional Laplacian operator is defined to be the invariant contraction,

$$\Box \equiv \partial_\nu \partial^\nu = \left[\partial^2/\partial(kt)^2\right] - \nabla^2. \tag{3.34}$$

This is, of course, just the operator of the wave equation in vacuum.

Note that, We can easily verify that Eq.(3.26) imply the very fundamental identity

$$R^2 = x'^2 - (kt')^2 = x^2 - (kt)^2 \tag{3.35}$$

where R^2 is the squared distance in spacetime. This constraint is analogous to the invariance of the squared distance, $r^2=x^2+y^2+z^2$, under rotations and translations of the orthogonal axes in Euclidean three-space. Actually, Eq.(3.35) can be written more precisely as

$$R^2 = x'^2 + (ikt')^2 = x^2 + (ikt)^2 \tag{3.36}$$

From this relation, the components x and kt can be written in polar form as

$$x = R\cos\theta, \qquad ikt = R\sin\theta, \tag{3.37}$$

where $\tan\theta = ik/v$. From Eq.(3.37), we obtain the following identity,

$$(x+kt) = R(\cos\theta - i\sin\theta) = Re^{-i\theta}. \qquad (3.38)$$

Finally, we find that

$$(x+kt)(x-kt) = x^2 - (kt)^2 = (Re^{-i\theta})(Re^{i\theta}) = R^2,$$

the same result as shown in Eq.(3.35).