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VARIOUS KINDS OF CENTRES OF SIMPLICES

Miss Somluck Outudee

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By Miss Somluck Outudee

Department Mathematics

Thesis Advisor Stephen New, Ph.D.

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สามเหลี่ยมในระนาบแบบยูคลิดมีจุดกลางหลายชนิด ตัวอย่างเช่น เซนทรอยด์ G ศูนย์กลางวงล้อม O ศูนย์กลางวงกลมแนบใน I จุดออร์โทเซนเตอร์ H และจุดคลีแวนซ์-เซนเตอร์ J ในงานวิจัยนี้เราหาจุดกลางเหล่านี้สำหรับซิมเพล็กซ์ในปริภูมิแบบยูลิด n มิติ และในปริภูมิทรงกลม n มิติ แต่ละจุดกลางเป็นจุดตัดของระนาบเกิน (หรือทรงกลมเกินใหญ่ในกรณีของปริภูมิทรงกลม) เราจะทำให้ทฤษฎีบทที่เกี่ยวข้องกับจุดกลางชนิดต่างๆของสามเหลี่ยมหลายทฤษฎีเป็นกรณีทั่วไปในมิติที่สูงขึ้น ตัวอย่างเช่น เราแสดงว่าสำหรับทุกๆซิมเพล็กซ์ในปริภูมิแบบยูคลิด จุดกลาง G O และ H อยู่บนเส้นตรงเดียวกัน และในทำนองเดียวกัน จุดกลาง G I และ J อยู่บนเส้นเดียวกัน ในที่สุดเราจะได้ลักษณะเฉพาะสำหรับจุดกลางบางชนิด

ภาควิชา คณิตศาสตร์ ลายมือชื่อนิสิต สมลักษณ์ อุตุดี

สาขาวิชา คณิตศาสตร์ ลายมือชื่ออาจารย์ที่ปรึกษา สตีเฟน นิว

ปีการศึกษา 2542 ลายมือชื่ออาจารย์ที่ปรึกษาร่วม -

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A triangle in the Euclidean plane has various kinds of centres. For example the **centroid** G, the **circumcentre** O, the **incentre** I, the **orthocentre** H, and the **cleavance centre** J. In this paper, we find higher dimensional analogous of these centres for simplices in Euclidean n-space and also in spherical n-space. Each centre is described as the point of intersection of certain hyperplanes (or great hyperspheres in the spherical case). Several of the theorems relating the various kinds of centres for triangles are generalized to higher dimensions, for example we show that the centres O, G and H are colinear and similarly, that the centres J, G and I are colinear for any simplex in Euclidean n-space. Finally, we obtain some new characterizations for several of these centres.

ภาควิชา คณิตศาสตร์	ลายมือชื่อนิสิต สมลักษณ์ อุตุดี
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Contents

INTRODUCTION

A triangle in the Euclidean plane has various kinds of centres. For example; the centroid G (the point of intersection of the medians), the circumcentre O (the point of intersection of the perpendicular bisectors, which is the centre of the circumcircle), the incentre I (the point of the intersection of the angle bisectors, and also the centre of the inscribed circle), the orthocentre H (which is the common point of the altitudes), the cleavance centre J (which is the intersection of the cleavers), and the nine-point centre (which is the centre of the circle passing through the midpoints of the sides, the midpoints of the lines joining the orthocentre to the vertices, and the feet of the altitudes).

In this paper, we find higher dimensional analogous of these centres for simplices in Euclidean n-space and also in spherical n-space. Each centre is described as the point of intersection of certain hyperplanes (or great hyperspheres in the spherical case). For example; the medial plane at a given edge of a simplex is the hyperplane which passes through the midpoint of an edge of the simplex and through the other vertices of the simplex. We show that the medial planes meet at a point G, called the centroid. The perpendicular bisector of a given edge is the hyperplane which passes through the midpoint of an edge of the simplex and is perpendicular to this edge. We show that the perpendicular bisectors meet at a point O, called the circumcentre. The altitudinal plane at a given edge of a simplex is the hyperplane which passes through the centroid of a face opposite the edge and is perpendicular to this edge. We show that the altitudinal planes meet at a point H. The incentre I is the point of intersection of the hyperplanes each of which is the interior angle bisector of two faces of the simplex. The cleavance centre J is the point of intersection of the hyperplanes each of which passes through the midpoint of an edge of the simplex and is parallel to the interior angle bisector of the two opposite faces.

Several of the theorems relating the various kinds of centres for triangles are generalized to higher dimensions, for example, we show that the centres O, G and H are colinear and similarly, that the centres I, G and J are colinear for any simplex in Euclidean n-space.

Moreover, we define the medial simplex of a given simplex in Euclidean n-space to be the simplex each of whose vertices is the centroid of an edge of the given simplex. We compare the centres of the given simplex with those of its medial simplex. For example, we shall generalize the nine-point centre theorem to higher dimensions.

Finally, we present characterization theorems of the centroid and the circumcentre of simplices in Euclidean n-space. The centroid of an n-simplex has the property that for any $(n+1)$-simplex, the lines from each vertex to the centroid of the opposite face all meet. We show that this property characterizes the centroid. The circumcentre of an n-simplex has the property that in any $(n+1)$-simplex, the lines through the circumcentre of each face and perpendicular to the face all meet. We show that this property characterizes the circumcentre.

CHAPTER I

VARIOUS KINDS OF CENTRES OF SIMPLICES IN $\mathbf{R}^N$

Simplices in $\mathbf{R}^N$

The following definitions and theorems may be found in [3].

Definition 1.1. A set $\{a_0, a_1, ..., a_n\}$ of points in $\mathbf{R}^N$ is said to be *affinely independent* if and only if $\sum_{i=0}^{n} t_i a_i = 0$ and $\sum_{i=0}^{n} t_i = 0$ imply that $t_i = 0$ for all i.

Theorem 1.2. *Let $\{a_0, a_1, ..., a_n\}$ be a set of points in $\mathbf{R}^N$. For each pair of indices, i, j let u_{ij} denote the vector $u_{ij} = a_j - a_i$. Then the following are equivalent.*

(1) *$\{a_0, a_1, ..., a_n\}$ is affinely independent*

(2) *$\{ u_{ik} \mid i \neq k \}$ is linearly independent for some fixed index k*

(3) *$\{ u_{ik} \mid i \neq k \}$ is linearly independent for any fixed index k*

(4) *There exists a unique n-plane which contains $\{a_0, a_1, ..., a_n\}$.*

When these conditions hold, the vector space $V = span\{ u_{ik} \mid i \neq k \}$ is independent of k, and the n-plane containing $\{a_0, a_1, ..., a_n\}$ is the plane $P = a_k + V$.

Definition 1.3. A *convex combination* of points $a_0, a_1, ..., a_n$ of $\mathbf{R}^N$ is a linear combination of the form $\sum_{i=0}^{n} t_i a_i$ such that $\sum_{i=0}^{n} t_i = 1$ and $t_i \geq 0$ for all i.

Theorem 1.4. *A subset S of $\mathbf{R}^N$ is convex if and only if S contains every convex combination of points of S.*

Definition 1.5. The *convex hull* of a subset S of $\mathbf{R}^N$ is defined to be the intersection $[S]$ of all the convex subsets of $\mathbf{R}^N$ containing S. If S is a finite set, say $S = \{a_0, a_1, \ldots, a_n\}$, then we shall write $[S]$ as $[a_0, a_1, \ldots, a_n]$.

Theorem 1.6. *$[S]$ is the set of all convex combinations of points of S.*

Definition 1.7. An *n-simplex* is the convex hull of a set of $n+1$ affinely independent points. A 1-simplex is called a *line segment*, a 2-simplex is called a *triangle* and a 3-simplex is called a *tetrahedron*.

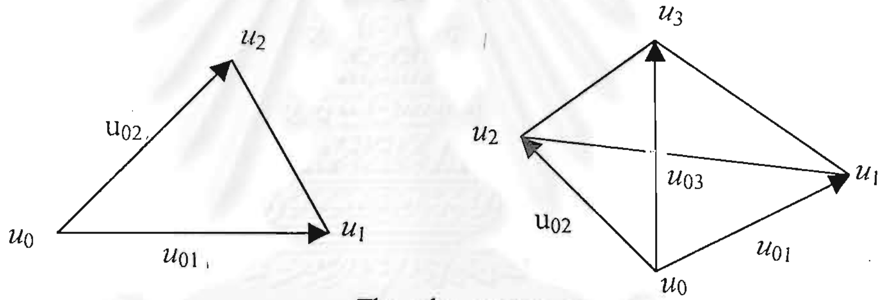
By theorem 1.4, the n-simplex $[a_0, a_1, \ldots, a_n]$ is the set

$$[a_0, a_1, \ldots, a_n] = \left\{ \sum_{i=0}^{n} t_i a_i \,\middle|\, \sum_{i=0}^{n} t_i = 1, t_i \geq 0 \text{ for all } i \right\}.$$

Notation. Given an n-simplex $T=[a_0, a_1, \ldots, a_n]$, we shall use the following notation :

We let u_{ij} or $u_{ij}(T)$ denote the edge vector

$$u_{ij} = a_j - a_i.$$



The edge vectors u_{0i}

We let V or $V(T)$ denote the vector space

$$V(T) = \text{span}\{ u_{ki} \mid i \neq k \}$$

for any fixed k and we let $\langle T \rangle$ denote the n-plane

$$\langle T \rangle = a_k + \text{span}\{ u_{ki} \mid i \neq k \}$$

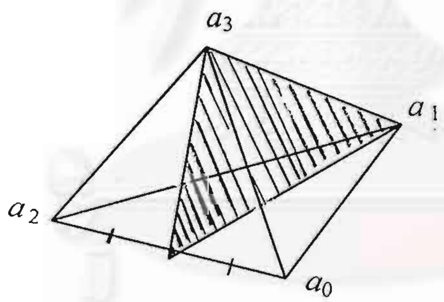
for any fixed k.

Definition 1.8. Let $T=[a_0, a_1, \ldots, a_n]$ be an n-simplex in $\mathbf{R}^N$. A k-face of T is a simplex of the form $[b_0, b_1, \ldots, b_k]$ where each $b_i \in \{a_0, a_1, \ldots, a_n\}$. A 0-face $[a_i]$ is caled a *vertex* of T and a 1-face $[a_i, a_j]$ is called an *edge* of T. For $k \in \{0, 1, \ldots, n\}$, we let T_k denote the $(n-1)$-face of T which is opposite the vertex a_k, that is $T_k = [a_0, a_1, \ldots, \hat{a}_k, \ldots, a_n]$, where $\hat{a}_k$ indicates omission of the vertex a_k. Similarly for $k \neq l$, we let T_{kl} denote the $(n-2)$-face of T opposite the edge $[a_k, a_l]$, that is $[a_0, a_1, \ldots, \hat{a}_k, \ldots, \hat{a}_l, \ldots, a_n]$.

The Centroid

In $\mathbf{R}^2$, it can be shown that the three medians of any triangle meet at a point G, called the *centroid* of the triangle. We would like to obtain an analogous result for n-simplices. We shall define the medial hyperplanes of an n-simplex, and we shall show that they have a unique point of intersection.

Definition 1.9. Given an n-simplex $T = [a_0, a_1, \ldots, a_n]$ in $\mathbf{R}^N$, and given $i \neq j$ with $0 \leq i, j \leq n$, let M_{ij} be the $(n-1)$-plane which passes through the midpoint of $[a_i, a_j]$ and through all the other vertices a_k, $k \neq i, j$. M_{ij} is called the *medial plane* of T at the edge $[a_i, a_j]$. Note that if T is a triangle, then its medial hyperplanes are in fact its medians.



The medial plane M_{02}

The medial plane M_{ij} is given by

$$M_{ij} = \frac{1}{2}(a_i + a_j) + \operatorname{span}\{u_k \mid k \neq i, j\}$$

where $u_k = \frac{1}{2}(a_i + a_j) - a_k$. Or, if we fix $k \neq i, j$, we can write M_{ij} as

$$M_{ij} = a_k + \operatorname{span}\{v_l \mid l \neq i, j\}$$

where $v_l = u_{kl} = a_l - a_k$ for $l \neq k$ and $v_k = \frac{1}{2}(a_i + a_j) - a_k$.

In other words $x \in M_{ij}$ iff $x - \frac{1}{2}(a_i + a_j) \in \text{span}\{u_k \mid k \neq i,j\}$

iff $x - a_k \in \text{span}\{v_l \mid l \neq i, j\}$.

Lemma 1.10. *Let $T = [a_0, a_1,\ldots, a_n]$ be an n-simplex in $\mathbf{R}^N$. Then $a_i, a_j \notin M_{ij}$ for all $i \neq j$.*

Proof. Suppose that $a_i \in M_{ij}$. Let $k \in \{0, 1,\ldots, n\}\backslash\{i,j\}$. Then $a_i - a_k \in \text{span}\{v_l \mid l \neq i,j\}$ where $v_l = a_l - a_k$ where $l \neq k$ and $v_k = \frac{1}{2}(a_i + a_j) - a_k$. Thus there exist $\beta, \alpha_l \in \mathbf{R}$ where $l \neq i,j,k$ such that

$$a_i - a_k = \sum_{l \neq i,j,k} \alpha_l (a_l - a_k) + \beta\left(\frac{1}{2}(a_i + a_j) - a_k\right).$$

So $0 = \sum_{l \neq i,j,k} \alpha_l (a_l - a_k) + \frac{\beta}{2}(a_j - a_k) + (\frac{\beta}{2} - 1)(a_i - a_k)$. Since $\{a_l - a_k \mid l \neq k\}$ is linearly independent, $\beta = 0 = \beta - 1$, this is a contradiction. Hence $a_i \notin M_{ij}$. #

Theorem 1.11. *The medial planes M_{ij} of any n-simplex $T = [a_0, a_1,\ldots, a_n]$ have a unique point of intersection G, called the **centroid** (or the **barrycentre**) of T. It is given by $(n+1)G = \sum_{i=0}^{n} a_i$.*

Proof. Existence. Let $i,j \in \{0, 1,\ldots, n\}, i \neq j$. Let G be the point in $\mathbf{R}^N$ which is defined by $G = \frac{1}{n+1}\sum_{i=0}^{n} a_i$. Fix $k \neq i, j$. Since

$$G - a_k = \frac{1}{n+1}\sum_{l \neq k}(a_l - a_k) = \sum_{l \neq i,j,k} \frac{1}{n+1}(a_l - a_k) + \frac{2}{n+1}\left(\frac{a_i + a_j}{2} - a_k\right)$$

which lies in $\text{span}\{v_l \mid l \neq i, j\}$ where $v_l = a_l - a_k$ and $v_k = \frac{1}{2}(a_i + a_j) - a_k$, we have $G \in M_{ij}$ for all $i \neq j$.

Uniqueness. Let P(m) be the statement "the intersection of the medial planes M_{0j} with $j = 1, 2, \ldots, m$ is the $(n-m)$-plane in $\mathbf{R}^N$ which passes through $\frac{1}{m+1}(\sum_{j=0}^{m} a_j)$ and the other vertices $a_{m+1}, a_{m+2}, \ldots, a_n$". We claim that P($m$) is true for $m = 1, 2, \ldots, n$. We will prove this by induction. It is clear that P(1) is true. Suppose that P(k) is true, that is, $\bigcap_{j=1}^{k} M_{0j}$ is the $(n-k)$-plane in $\mathbf{R}^N$ which passes through $\frac{1}{k+1}(\sum_{j=0}^{k} a_j)$ and the other vertices $a_{k+1}, a_{k+2}, \ldots, a_n$. Since $a_{k+1} \in \bigcap_{j=1}^{k} M_{0j}$ but $a_{k+1} \notin M_{0,k+1}$, $\dim(\bigcap_{j=1}^{k+1} M_{0j}) < n-k$. By the definition of $M_{0,k+1}$ and the induction hypothesis, we have that $a_{k+2}, a_{k+3}, \ldots, a_n \in \bigcap_{j=1}^{k+1} M_{0j}$.

Let $l \in \{1, 2, \ldots, k+1\}$. Then

$$\begin{aligned}\frac{1}{k+2}\left(\sum_{j=0}^{k+1} a_j\right) - a_{k+2} &= \sum_{j=0}^{k+1} \frac{1}{k+2}(a_j - a_{k+2}) \\ &= \frac{2}{k+2}\left(\frac{1}{2}(a_0 + a_l) - a_{k+2}\right) + \sum_{j \neq 0, l} \frac{1}{k+2}(a_j - a_{k+2}) \\ &\in \operatorname{span}\{ v_j \mid j \neq 0, l \}\end{aligned}$$

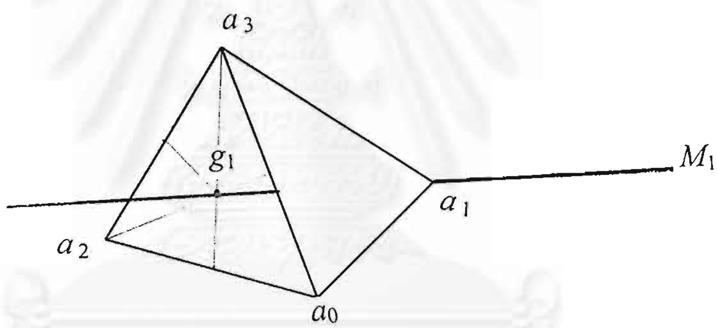
where $v_j = a_j - a_{k+2}$ for $j \neq k+2, 0, l$ and $v_{k+2} = \frac{1}{2}(a_0 + a_l) - a_{k+2}$. This shows that $\frac{1}{k+2}\left(\sum_{j=0}^{k+1} a_j\right) \in M_{0l}$. Since $l \in \{1, 2, \ldots, k+1\}$ was arbitrary, we have that $\frac{1}{k+2}\left(\sum_{j=0}^{k+1} a_j\right) \in \bigcap_{j=1}^{k+1} M_{0j}$.

We claim that $\left\{ \frac{1}{k+2}\left(\sum_{j=0}^{k+1} a_j\right) - a_{k+2}, a_l - a_{k+2} \mid l = k+3, \ldots, n \right\}$ is linearly independent. Let $\alpha, \alpha_{k+3}, \alpha_{k+4}, \ldots, \alpha_n \in \mathbf{R}$ be such that

$$\alpha\left(\frac{1}{k+2}\left(\sum_{j=0}^{k+1} a_j\right) - a_{k+2}\right) + \sum_{j=k+3}^{n} \alpha_j (a_j - a_{k+2}) = 0.$$ Then

$$\sum_{j=0}^{k+1} \frac{\alpha}{k+2}(a_j - a_{k+2}) + \sum_{j=k+3}^{n} \alpha_j (a_j - a_{k+2}) = 0.$$ Since $\{ a_j - a_{k+2} \mid j = 0, 1, \ldots, k+1, k+3, \ldots, n \}$ is linearly independent, $\alpha = 0 = \alpha_j$ for all $j \in \{k+3, k+4, \ldots, n\}$. Thus we have the claim and P(k+1) is true. So P(m) is true for $m = 1, 2, \ldots, n$. Hence G is the unique point of intersection of the medial planes M_{0j} where $j \neq 0$ and hence also of all the medial planes M_{ij} where $j \neq i$. #

Definition 1.12. Given an n-simplex $T = [a_0, a_1, \ldots, a_n]$ in $\mathbf{R}^N$, let M_i be the line through a_i and the centroid $g_i = \frac{1}{n}(\sum_{k \neq i} a_k)$ of the opposite face T_i. M_i is called the *medial line* of T at a_i in $\mathbf{R}^N$.



The medial line M_1

Note that the equation of M_i is

$$x = a_i + (g_i - a_i)\, t \qquad , t \in \mathbf{R}.$$

Since $g_i = \frac{1}{n}(\sum_{k \neq i} a_k)$, the equation of M_i can be written as

$$x = a_i + \left[\frac{1}{n}\left(\sum_{k \neq i} a_k\right) - a_i\right] t \qquad , t \in \mathbf{R}$$

or

$$x = a_i + \left[\sum_{k \neq i} a_k - n a_i\right] s \qquad , s \in \mathbf{R}$$

or

$$x = (1 - ns) a_i + \left(\sum_{k \neq i} a_k\right) s \qquad , s \in \mathbf{R}.$$

Theorem 1.13. *Given an n-simplex* $T=[a_0, a_1, \ldots, a_n]$ *in* $\mathbf{R}^N$, *the medial lines* M_i *meet at the centroid* G *of* T.

Proof. Let $i, j \in \{0, 1, \ldots, n\}, i \neq j$. First we shall show that $M_i \neq M_j$. By definition, we have $a_j \in M_j$. Suppose that, we also have $a_j \in M_i$. Since $a_j \in M_i$, there exists some $t \in \mathbf{R}$ such that $a_j = a_i + (g_i - a_i)\, t$. Then

$$\begin{aligned} a_j - a_i &= (g_i - a_i)t \\ &= \left(\frac{1}{n}\sum_{k\neq i} a_k - a_i\right)t \\ &= \sum_{k\neq i}\frac{t}{n}(a_k - a_i) \\ &= \sum_{k\neq i,j}\left[\frac{t}{n}(a_k - a_i)\right] + \frac{t}{n}(a_j - a_i). \end{aligned}$$

So $(1-\frac{t}{n})(a_j - a_i) = \sum_{k\neq i,j}\frac{t}{n}(a_k - a_i)$, and $a_j - a_i = \sum_{k\neq i,j}\frac{t}{n-t}(a_k - a_i)$. This is a contradiction since $\{a_k - a_i \mid k \neq i\}$ is linearly independent. Thus $a_j \notin M_i$ but $a_j \in M_j$ so $M_i \neq M_j$. So the intersection of M_i and M_j is the empty set or a singleton.

Next, let G be the point defined by $G = \frac{1}{n+1}\sum_{k=0}^{n} a_k$, fix $i \in \{0, 1, \ldots, n\}$.

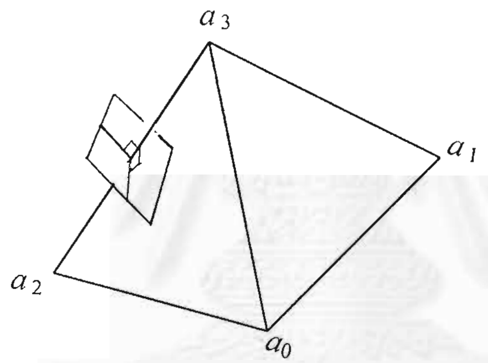
Since $G = \frac{1}{n+1}\sum_{k=0}^{n} a_k = \frac{1}{n+1}a_i + \frac{1}{n+1}\sum_{k\neq i} a_k = \left(1 - \frac{n}{n+1}\right)a_i + \frac{1}{n+1}\sum_{k\neq i} a_k$,

we have that $G = (1-t)a_i + \frac{t}{n}\sum_{k\neq i} a_k$ where $t = \frac{n}{n+1}$. So $G \in M_i$. Hence the medial lines M_i meet at the point G. #

The Circumcentre

In $\mathbf{R}^2$, the three perpendicular bisectors of the sides of any triangle meet at a point O, called the *circumcentre*, which is the centre of the circumscribed circle. We shall show that an analogous result holds for any n-simplex in $\mathbf{R}^N$.

Definition 1.14. Given an n-simplex $T = [a_0, a_1, \ldots, a_n]$ in $\mathbf{R}^N$, let P_{ij} be the $(n-1)$-plane in $\langle T \rangle$ which is perpendicular to the edge $[a_i, a_j]$ and which passes through the midpoint of $[a_i, a_j]$. P_{ij} is called the *perpendicular bisector* of the edge $[a_i, a_j]$ in $\langle T \rangle$.



The perpendicular bisector P_{23}

Note that

$$P_{ij} = \left\{ x \in \langle T \rangle \,\middle|\, \left\langle x - \frac{1}{2}(a_i + a_j), a_i - a_j \right\rangle = 0 \right\}$$

$$= \left\{ x \in \langle T \rangle \,\middle|\, \langle 2x, a_i - a_j \rangle = \langle a_i + a_j, a_i - a_j \rangle \right\}$$

or

$$P_{ij} = \left\{ x \in \langle T \rangle \,\middle|\, \left\langle x - a_j + \frac{1}{2}(a_j - a_i), a_i - a_j \right\rangle = 0 \right\}$$

$$= \left\{ x \in \langle T \rangle \,\middle|\, \langle x - a_j, a_i - a_j \rangle = \frac{1}{2}|a_j - a_i|^2 \right\}.$$

We shall show that the perpendicular bisectors P_{ij} meet at a point O, called the *circumcentre*, which is the centre of the circumscribed $(n-1)$-sphere.

Notation. Given an n-simplex $T = [a_0, a_1, \ldots, a_n]$, choose an orthonormal basis for the vector space $V(T)$ spanned by T. Set $u_{ji} = a_i - a_j$. For fixed k with $0 \le k \le n$, let $A_k(T)$ denote the $n \times n$ matrix whose rows are the vectors $u_{ki} = a_i - a_k$ with respect to the chosen basis.

Theorem 1.15. *Given an n-simplex* $T = [a_0, a_1, \ldots, a_n]$ *in* $\mathbf{R}^N$, *the perpendicular bisectors* P_{ij} *of the edges* $[a_i, a_j]$ *meet at a unique point O. If we fix k with* $0 \le k \le n$ *then O is given by*

$$O = a_k + \frac{1}{2} A^{-1} P$$

where $A = A_k(T)$ *and P is the* $n \times 1$ *matrix whose rows are* $|u_{ki}|^2$.

Proof. Let $T=[a_0, a_1, \ldots, a_n]$ be an n-simplex in $\mathbf{R}^N$, fix $k \in \{0, 1, \ldots, n\}$.

Existence. Let O be the point in $\mathbf{R}^N$ which is defined by $O = a_k + \frac{1}{2} A^{-1} P$ or $2A(O - a_k) = P$. Since A is the matrix with rows $a_i - a_k$ and P is the matrix whose rows $|u_{ki}|^2$, this means that $\langle a_i - a_k, 2(O - a_k) \rangle = \langle a_i - a_k, a_i - a_k \rangle$

$$\langle a_i - a_k, 2(O - a_k) - (a_i - a_k) \rangle = 0.$$

So $\langle a_i - a_k, 2O \rangle = \langle a_i - a_k, a_i + a_k \rangle$ for all i. For all $i, j \in \{0, 1, 2, \ldots, n\}$, $i \neq j$, we have

$$\begin{aligned}
\langle a_j - a_i, 2O \rangle &= \langle a_j - a_k + a_k - a_i, 2O \rangle \\
&= \langle a_j - a_k, 2O \rangle - \langle a_i - a_k, 2O \rangle \\
&= \langle a_j - a_k, a_j + a_k \rangle - \langle a_i - a_k, a_i + a_k \rangle \\
&= \langle a_j, a_j \rangle + \langle a_j, a_k \rangle - \langle a_k, a_j \rangle - \langle a_k, a_k \rangle - \\
&\quad \langle a_i, a_i \rangle - \langle a_i, a_k \rangle + \langle a_k, a_i \rangle + \langle a_k, a_k \rangle \\
&= \langle a_j, a_j \rangle - \langle a_i, a_i \rangle \\
&= \langle a_j - a_i, a_j + a_i \rangle.
\end{aligned}$$

This shows that O is a point of intersection of the perpendicular bisectors P_{ij}.

We shall show that O is the only point of intersection of the perpendicular bisectors P_{ij}.

Uniqueness. Suppose that $x \in P_{ij}$ for all $i, j \in \{0, 1, 2,..., n\}$, $i \neq j$. Then $\langle a_j - a_i, 2x \rangle = \langle a_j - a_i, a_j + a_i \rangle$ for all $i, j \in \{0, 1, 2,..., n\}$, $i \neq j$. and in particular, if we fix k then

$$\langle a_i - a_k, 2x \rangle = \langle a_i - a_k, a_i + a_k \rangle \quad \text{for all } i \neq k.$$

Thus $\langle a_i - a_k, 2(x - a_k) \rangle = \langle a_i - a_k, a_i - a_k \rangle$ for all $i \neq k$,

or equivalently, $2A(x - a_k) = P$. Since A is invertible, $x = a_k + \frac{1}{2}A^{-1}P = O$.

This shows that the point O is the unique point of intersection of the perpendicular bisectors P_{ij}. #

Theorem 1.16. *Given an n-simplex* $T=[a_0, a_1,..., a_n]$ *in* $\mathbf{R}^N$, *the perpendicular bisector* P_{ij} *of the edge* $[a_i, a_j]$ *is the set of all* $x \in \langle T \rangle$ *such that* $d_R(x, a_i) = d_R(x, a_j)$.

Proof. Let $T=[a_0, a_1,..., a_n]$ be an n-simplex in $\mathbf{R}^N$. For any $x \in \langle T \rangle$,

$$\begin{aligned}
x \in P_{ij} &\text{ iff } \langle 2x - (a_i + a_j), a_i - a_j \rangle = 0 \\
&\text{ iff } \langle (x - a_i) + (x - a_j), a_i - a_j \rangle = 0 \\
&\text{ iff } \langle (x - a_i) + (x - a_j), (x - a_j) - (x - a_i) \rangle = 0 \\
&\text{ iff } \langle x - a_i, x - a_j \rangle - \langle x - a_i, x - a_i \rangle + \\
&\qquad \langle x - a_j, x - a_j \rangle - \langle x - a_j, x - a_i \rangle = 0 \\
&\text{ iff } \langle x - a_i, x - a_i \rangle = \langle x - a_j, x - a_j \rangle \\
&\text{ iff } |x - a_i|^2 = |x - a_j|^2 \\
&\text{ iff } d_R(x, a_i) = d_R(x, a_j).
\end{aligned}$$

Hence the perpendicular bisector of $[a_i, a_j]$ is the set of all x such that $d_R(x, a_i) = d_R(x, a_j)$. #

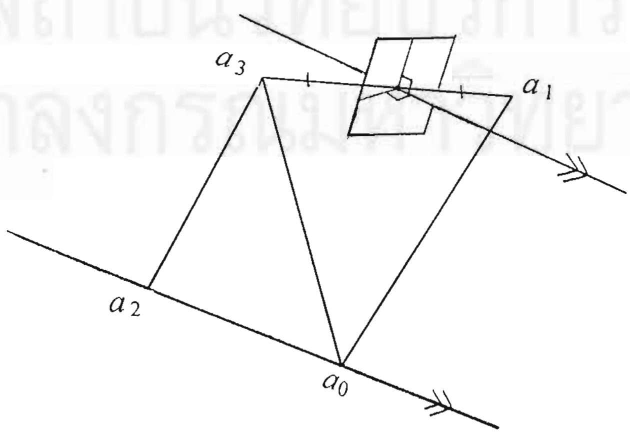
Definition 1.17. Combining the above two theorems, we see that the circumcentre O is the unique point with the property that $d_R(O, a_i) = d_R(O, a_j)$ for all i, j. We define the *circumradius* of T to be $R = d_R(O, a_k)$ for any fixed k.

Corollary 1.18. Given an n-simplex $T = [a_0, a_1, \dots, a_n]$, there is a unique $(n-1)$-sphere in $\langle T \rangle$, called the *circumscribed sphere* of T, which passes through each of the points a_i. It is the sphere with centre O and radius R.

The Orthocentre

In $\mathbf{R}^2$, it can be shown that the three altitudes of any triangle meet at a point H, called the *orthocentre* of the triangle. In $\mathbf{R}^N$, however the altitudes of an n-simplex do not always intersect, so we shall give an alternate definition for the orthocentre.

Definition 1.19. Given an n-simplex $T = [a_0, a_1, \dots, a_n]$ in $\mathbf{R}^N$, let A_{ij} denote the $(n-1)$-plane in $\langle T \rangle$ which is perpendicular to the edge $[a_i, a_j]$ and which passes through the centroid g_{ij} of the opposite $(n-2)$-face $T_{ij} = [a_0, a_1, \dots, \hat{a}_i, \dots, \hat{a}_j, \dots, a_n]$. We shall call A_{ij} the *altitudinal plane* of T at $[a_i, a_j]$. Note that if T is a triangle then the altitudinal planes are in fact its altitudes.



The altitudinal plane A_{02}

The altitudinal plane A_{ij} is given by

$$A_{ij} = \{x \in \langle T\rangle | \langle x - g_{ij}, a_j - a_i\rangle = 0$$

We shall show that these altitudinal planes A_{ij} have a unique point of intersection H, which we shall call it the *orthocentre* of T.

Theorem 1.20. *Given an n-simplex* $T = [a_0, a_1, ..., a_n]$ *in* $\mathbf{R}^N$, *the altitudinal planes* A_{ij} *have a unique point of intersection H. If we fixed k with* $0 \le k \le n$ *and let* $A = A_k(T)$ *then H is given by*

$$H = a_k + A^{-1}K$$

where K is the $n\times 1$ *matrix whose rows are* $\langle u_{ki}, g_{ik} - a_k\rangle$.

Proof. Let $T=[a_0, a_1, ..., a_n]$ be an n-simplex in $\mathbf{R}^N$, fix $k \in \{0, 1, ..., n\}$. **Existence.** Let H be the point in $\mathbf{R}^N$ which is defined by $H = a_k + A^{-1}K$. Then $A(H - a_k) = K$. Since A is the matrix whose rows are $a_i - a_k$ and K is the matrix whose rows are $\langle u_{ki}, g_{ik} - a_k\rangle$, this means that $\langle u_{ki}, H - a_k\rangle$ = $\langle u_{ki}, g_{ik} - a_k\rangle$ for all $i \ne k$. So $\langle u_{ki}, H - g_{ik}\rangle = 0$ for all $i \ne k$. This shows that H lies on each of the altitudinal planes A_{ik}. It remains to show that H lies on the altitudinal planes with $i, j \ne k$. We have

$$\begin{aligned} g_{ij} &= \frac{1}{n-1}\sum_{l \ne i,j} a_l \\ &= \frac{1}{n-1}\left[\left(\sum_{l \ne k,j} a_l\right) - a_i + a_k\right] \\ &= g_{kj} - \frac{1}{n-1}(a_i - a_k) \text{ for all } i, j, \end{aligned}$$

and so

$$\langle a_j - a_i, H - g_{ij}\rangle = \langle a_j - a_k, H - g_{ij}\rangle + \langle a_k - a_i, H - g_{ij}\rangle$$

$$= \left\langle a_j - a_k, H - g_{kj} + \frac{1}{n-1}(a_i - a_k) \right\rangle$$

$$- \left\langle a_i - a_k, H - g_{ki} + \frac{1}{n-1}(a_j - a_k) \right\rangle$$

$$= \langle a_j - a_k, H - g_{kj} \rangle + \left\langle a_j - a_k, \frac{1}{n-1}(a_i - a_k) \right\rangle$$

$$- \langle a_i - a_k, H - g_{ki} \rangle + \left\langle a_i - a_k, \frac{1}{n-1}(a_j - a_k) \right\rangle$$

$$= \left\langle a_j - a_k, \frac{1}{n-1}(a_i - a_k) \right\rangle - \left\langle a_i - a_k, \frac{1}{n-1}(a_j - a_k) \right\rangle$$

$$= \frac{1}{n-1}\langle a_j - a_k, a_i - a_k \rangle - \frac{1}{n-1}\langle a_i - a_k, a_j - a_k \rangle$$

$$= 0.$$

This shows that H is a point of intersection of all the altitudinal planes A_{ij}.

Uniqueness. We shall now show that H is the only point of intersection of the altitudinal planes $A_{ki}, i \neq k$ (hence also of all the altitudinal planes A_{ij}) . Suppose that $x \in A_{ki}$, for all $i \neq k$. Then $\langle a_i - a_k, x - g_{ik} \rangle = 0$ for all $i \neq k$. That is $\langle a_i - a_k, x - a_k \rangle = \langle a_i - a_k, g_{ik} - a_k \rangle$ for all $i \neq k$. Or equivalently, $A(x - a_k) = K$. Since A is invertible, $x = a_k + A^{-1}K = H$. This shows that the point H is the unique point of intersection of the altitudinal planes A_{ij}. #

In $\mathbf{R}^2$, the circumcentre O, the centroid G, and the orthocentre H of a triangle, all lie on a line called the *Euler line* of the triangle, and $H + 2O = 3G$. We shall show more generally that the points O, G and H of an n-simplex in $\mathbf{R}^N$ all lie on a line, also called the Euler line.

Theorem 1.21. *O, G and H are colinear and* $(n-1)H + 2O = (n+1)G$.

Proof. Let $A = A_0(T)$.

By Theorem 1.20, $H - a_0 = \begin{bmatrix} \langle a_1 - a_0, g_{01} - a_0 \rangle \\ \langle a_2 - a_0, g_{02} - a_0 \rangle \\ \vdots \\ \langle a_n - a_0, g_{0n} - a_0 \rangle \end{bmatrix}$

$$= A^{-1} \begin{bmatrix} \left\langle a_1 - a_0, \frac{1}{n-1} \sum_{i \neq 0,1} a_i - a_0 \right\rangle \\ \left\langle a_2 - a_0, \frac{1}{n-1} \sum_{i \neq 0,2} a_i - a_0 \right\rangle \\ \vdots \\ \left\langle a_n - a_0, \frac{1}{n-1} \sum_{i \neq 0,n} a_i - a_0 \right\rangle \end{bmatrix}$$

$$= \frac{1}{n-1} A^{-1} \begin{bmatrix} \left\langle a_1 - a_0, \sum_{i \neq 0,1} a_i - (n-1)a_0 \right\rangle \\ \left\langle a_2 - a_0, \sum_{i \neq 0,2} a_i - (n-1)a_0 \right\rangle \\ \vdots \\ \left\langle a_n - a_0, \sum_{i \neq 0,n} a_i - (n-1)a_0 \right\rangle \end{bmatrix}$$

$$= \frac{1}{n-1} A^{-1} \begin{bmatrix} \left\langle a_1 - a_0, \sum_{i=0}^{n} a_i - a_1 - na_0 \right\rangle \\ \left\langle a_2 - a_0, \sum_{i=0}^{n} a_i - a_2 - na_0 \right\rangle \\ \vdots \\ \left\langle a_n - a_0, \sum_{i=0}^{n} a_i - a_n - na_0 \right\rangle \end{bmatrix}.$$

By Theorem 1.15, $O - a_0 = \frac{1}{2} A^{-1} \begin{bmatrix} \langle a_1 - a_0, a_1 - a_0 \rangle \\ \langle a_2 - a_0, a_2 - a_0 \rangle \\ \vdots \\ \langle a_n - a_0, a_n - a_0 \rangle \end{bmatrix}$. Thus

$$(n-1)(H-a_0)+2(O-a_0) = A^{-1}\begin{bmatrix} \left\langle a_1-a_0,\ \sum_{i=0}^{n} a_i-(n+1)a_0 \right\rangle \\ \left\langle a_2-a_0,\ \sum_{i=0}^{n} a_i-(n+1)a_0 \right\rangle \\ \cdots \\ \left\langle a_n-a_0,\ \sum_{i=0}^{n} a_i-(n+1)a_0 \right\rangle \end{bmatrix}$$

$$= A^{-1}A\left(\sum_{i=0}^{n} a_i-(n+1)a_0\right)$$

$$= (n+1)(G-a_0)\,.$$

Therefore $(n-1)H+2O = (n+1)G$. #

The Incentre

In $\mathbf{R}^2$, the three angle bisectors of a triangle meet at a point I, called the *incentre*, which is the centre of the inscribed circle. We shall show that an analogous result holds for any n-simplex in $\mathbf{R}^N$.

Notation. Given an n-simplex $T=[a_0, a_1,\ldots, a_n]$ in $\mathbf{R}^N$, fix k with $0 \le k \le n$. Recall that $\langle T \rangle = a_k + V(T)$, where $V(T) = \text{span}\{u_{ki} \mid i \neq k\}$. Similarly, for $i \neq k$, we have $\langle T_i \rangle = a_k + V(T_i)$ where $V(T_i) = \text{span}\{u_{kj} \mid j \neq k, i\}$. There are two normal vectors $\pm m_i$ for the face T_i, that is there are two vectors $\pm m_i \in V(T)$ such that $|m_i| = 1$ and $\langle m_i, u_{kj} \rangle = 0$ for all $k, j \neq i$. Let m_i or $m_i(T)$ denote the *inward normal vector* for the face T_i, in other words m_i is the normal vector such that $\langle m_i, u_{ki} \rangle > 0$ for all $k \neq i$, then $-m_i$ is the *outward normal vector.*

Note that $\langle m_i, u_{ji} \rangle = \langle m_i, u_{ki} \rangle > 0$ *for all* $k, j \neq i$, since $\langle m_i, u_{ji} \rangle =$ $\langle m_i, a_i - a_j \rangle = \langle m_i, a_i - a_k + a_k - a_j \rangle = \langle m_i, u_{ki} \rangle - \langle m_i, u_{kj} \rangle = \langle m_i, u_{ki} \rangle$.

We define the angle between two $(n-1)$-faces of an n-simplex T in $\mathbf{R}^N$ as follows ;

Definition 1.22. Given an n-simplex $T=[a_0, a_1,\ldots, a_n]$ in $\mathbf{R}^N$, the *angle* between the two faces $\langle T_i \rangle$ and $\langle T_j \rangle$ is the angle $\theta(T_i, T_j) = \arccos|\langle m_i, m_j \rangle| \in (0, \frac{\pi}{2})$ where m_i and m_j are normal vectors of T_i and T_j, respectively.

Definition 1.23. Given an n-simplex $T=[a_0, a_1,\ldots, a_n]$ in $\mathbf{R}^N$, an *angle bisector* of T_i and T_j is an $(n-1)$-plane B in $\langle T \rangle$ which contains T_{ij} such that $\theta(B, T_i) = \theta(B, T_j)$.

Theorem 1.24. *Given an n-simplex* $T=[a_0, a_1, \ldots, a_n]$ *in* $\mathbf{R}^N$, *there are two angle bisectors of* T_i *and* T_j. *They are the two* $(n-1)$*-planes in* $\langle T \rangle$ *with orthogonal vectors* $m_i + m_j$ *and* $m_i - m_j$.

Proof. Let B be an angle bisector of T_i and T_j. Then B is an $(n-1)$-plane in $\langle T \rangle$ which contains T_{ij}, and $\theta(B, T_i) = \theta(B, T_j)$. Let b be a normal vector for B. We have that

$$\begin{aligned}\theta(B, T_i) = \theta(B, T_j) \quad &\text{iff} \quad \arccos|\langle b, m_i\rangle| = \arccos|\langle b, m_j\rangle| \\ &\text{iff} \quad |\langle b, m_i\rangle| = |\langle b, m_j\rangle| \\ &\text{iff} \quad \langle b, m_i\rangle = \langle b, m_j\rangle \text{ or } \langle b, m_i\rangle = -\langle b, m_j\rangle \\ &\text{iff} \quad \langle b, m_i - m_j\rangle = 0 \text{ or } \langle b, m_i + m_j\rangle = 0.\end{aligned}$$

Case 1. $\langle b, m_i - m_j\rangle = 0$

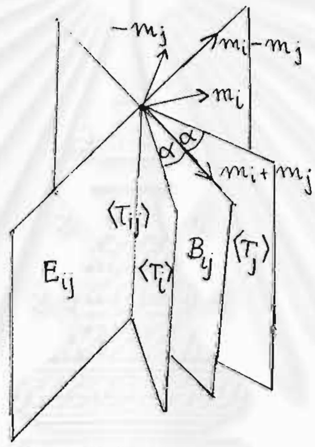
Since $\langle m_i + m_j, m_i - m_j\rangle = |m_i|^2 - |m_j|^2 = 0$, $m_i + m_j$ is orthogonal to $m_i - m_j$. Since m_i is a normal vector of T_i and $T_{ij} \subseteq T_i$, m_i is orthogonal to $\langle T_{ij}\rangle$. Similarly, m_j is orthogonal to $\langle T_{ij}\rangle$. So we have $\langle m_i + m_j, u_{kl}\rangle = 0 = \langle m_i - m_j, u_{kl}\rangle$ for all $k, l \neq i, j$. This show that $m_i + m_j$ and $m_i - m_j$ are both orthogonal to $\langle T_{ij}\rangle$. Since $m_i - m_j$ is an orthogonal vector of $\langle T_{ij}\rangle$, $\langle V(T_{ij}) \cup \{m_i - m_j\}\rangle$ is an $(n-1)$-dimensional vector subspace of $V(T)$, and $\{m_i + m_j\}$ is a basis for $\langle V(T_{ij}) \cup \{m_i - m_j\}\rangle^{\perp}$ in $V(T)$. So if b is a normal vector of T_{ij} and b is orthogonal to $m_i - m_j$, then $b \in \text{span}\{m_i + m_j\}$. Hence b is either equal to $\dfrac{m_i + m_j}{|m_i + m_j|}$ or to $-\dfrac{m_i + m_j}{|m_i + m_j|}$.

Case 2. $\langle b, m_i + m_j\rangle = 0$

The proof of this case is similar to of case 1. We have that b is equal to $\dfrac{m_i - m_j}{|m_i - m_j|}$ or to $-\dfrac{m_i - m_j}{|m_i - m_j|}$

Therefore the two angle bisectors of T_i and T_j are the planes through T_{ij} in $\langle T \rangle$ with orthogonal vectors $m_i + m_j$ or $m_i - m_j$. #

Definition 1.25. Given an n-simplex $T = [a_0, a_1, \ldots, a_n]$ in $\mathbf{R}^N$, the *internal angle bisector* B_{ij} of T_i and T_j is the angle bisector of the faces T_i and T_j with orthogonal vector $m_i - m_j$. The *external angle bisector* E_{ij} of T_i and T_j is the angle bisector of the faces T_i and T_j with orthogonal vector $m_i + m_j$.



The internal angle bisector B_{ij} and the external angle bisector E_{ij}

Note that if we fix $k \neq i, j$ then

$$B_{ij} = \{x \in \langle T \rangle | \langle x - a_k, m_i - m_j \rangle = 0$$

and $$E_{ij} = \{x \in \langle T \rangle | \langle x - a_k, m_i + m_j \rangle = 0$$

Remark 1.26. $\langle a_k, m_i - m_j \rangle = \langle a_l, m_i - m_j \rangle$ for all $k, l \neq i, j$ since $\langle a_l - a_k, m_i - m_j \rangle = \langle u_{kl}, m_i - m_j \rangle = \langle u_{kl}, m_i \rangle - \langle u_{kl}, m_j \rangle = 0$.

Notation. *Given an n-simplex* T=$[a_0, a_1, \ldots, a_n]$, fix an orthonormal basis for $V(T)$. Given k with $0 \leq k \leq n$, let $B = B_k(T)$ denote the matrix whose rows are the vectors $m_i - m_k$, $i \neq k$ with respect to the chosen basis.

Theorem 1.27. *Given an n-simplex* $T = [a_0, a_1, \ldots, a_n]$ *in* $\mathbf{R}^N$, *fix* $k \in \{0, 1, \ldots, n\}$. *Then the matrix* $B = B_k(T)$ *is invertible.*

Proof. Let $X = \{m_i - m_k \mid i \neq k\}$. We shall show that X is linearly independent. Suppose that $\sum_{i \neq k} \alpha_i (m_i - m_k) = 0$. We claim that $\sum_{i \neq k} \alpha_i = 0$, suppose not. Then $m_k = \dfrac{\sum_{i \neq k} \alpha_i m_i}{\sum_{i \neq k} \alpha_i}$. Set $t_i = \dfrac{\alpha_i}{\sum_{i \neq k} \alpha_i}$. Then $\sum_{i \neq k} t_i = 1$ and $m_k = \sum_{i \neq k} t_i m_i$. For each $i \neq k$, $\langle m_k, u_{ki} \rangle = \left\langle \sum_{i \neq k} t_i m_i, u_{ki} \right\rangle = t_i \langle m_i, u_{ki} \rangle$. Thus $t_i = \dfrac{\langle m_k, u_{ki} \rangle}{\langle m_i, u_{ki} \rangle} < 0$ which contradicts the fact that $\sum_{i \neq k} t_i = 1$. Hence $\sum_{i \neq k} \alpha_i = 0$.

Since $\sum_{i \neq k} \alpha_i (m_i - m_k) = 0$ and $\sum_{i \neq k} \alpha_i = 0$, we have $\sum_{i \neq k} \alpha_i m_i = 0$ and so for any $j \neq k$, $0 = \left\langle \sum_{i \neq k} \alpha_i m_i, u_{kj} \right\rangle = \alpha_j \langle m_j, u_{kj} \rangle$. Since $\langle m_j, u_{kj} \rangle \neq 0$, $\alpha_j = 0$ for each $j \neq k$. Hence X is linearly independent, so B is invertible. #

Theorem 1.28. *Given an n-simplex* $T=[a_0, a_1, \ldots, a_n]$ *in* $\mathbf{R}^N$, *the internal angle bisectors* B_{ij} *have a unique point of intersection I. If we fix k and let* $B = B_k(T)$ *then I is given by*

$$I = a_k + B^{-1}M$$

where M is an $n \times 1$ *matrix whose rows are* $\langle m_i - m_k, a_{l_i} - a_k \rangle$, *where* $l_i \neq i, k$.

Proof. Let $T=[a_0, a_1, \ldots, a_n]$ be an n-simplex in $\mathbf{R}^N$, fix $k \in \{0, 1, \ldots, n\}$.
Existence. Let I be the point in $\mathbf{R}^N$ which is defined by $I = a_k + B^{-1}M$.

Since B is an $n\times n$ matrix whose rows are $m_i - m_k$ and M is an $n\times n$ matrix whose rows are $\langle m_i - m_k, a_{l_i} - a_k \rangle$, this means that $\langle m_i - m_k, I - a_k \rangle$ = $\langle m_i - m_k, a_{l_i} - a_k \rangle$ for all $l_i \neq i, k$.

For all $i, j \in \{0, 1, 2, \ldots, n\}$, $i \neq j$, we have

$$\begin{aligned}\langle m_i - m_j, I - a_k \rangle &= \langle m_i - m_k + m_k - m_j, I - a_k \rangle \\ &= \langle m_j - m_k, I - a_k \rangle - \langle m_i - m_k, I - a_k \rangle \\ &= \langle m_j - m_k, a_{l_p} - a_k \rangle - \langle m_i - m_k, a_{l_p} - a_k \rangle \ , l_p \neq i, j, k \\ &= \langle m_j - m_i, a_{l_p} - a_k \rangle \qquad , l_p \neq i, j, k.\end{aligned}$$

This shows that I is a point of intersection of the angle bisectors B_{ij}.

Uniqueness. We shall now show that I is the only point of intersection of the angle bisectors B_{ik}, $i \neq k$ (hence also of all the angle bisector B_{ij}).

Suppose that $x \in B_{ik}$ for all $i \neq k$. Then $\langle m_i - m_k, x - a_{j_l} \rangle = 0$ for all $j_l \neq i, k$.

That is $\langle m_i - m_k, x - a_k + a_k - a_{j_l} \rangle = 0$ for all $j_l \neq i, k$, so

$\langle m_i - m_k, x - a_k \rangle = \langle m_i - m_k, a_{j_l} - a_k \rangle$ for all $j_l \neq i, k$. Or equivalently, $B(x - a_k) = M$. Since B is invertible, $x = a_k + B^{-1}M = I$. Hence the point I is the unique point of intersection of the angle bisectors B_{ij}. #

Theorem 1.29. *Given an n-simplex $T=[a_0, a_1,\ldots, a_n]$ in $\mathbf{R}^N$ and given any point $x \in \langle T \rangle$, we have $d_R(x, \langle T_i \rangle) = d_R(x, \langle T_j \rangle)$ if and only if x lies on one of the two angle bisectors B_{ij} and E_{ij} of T_i and T_j.*

Proof. Let $a \in \langle T_{ij} \rangle$. For each $x \in \langle T \rangle$, we have

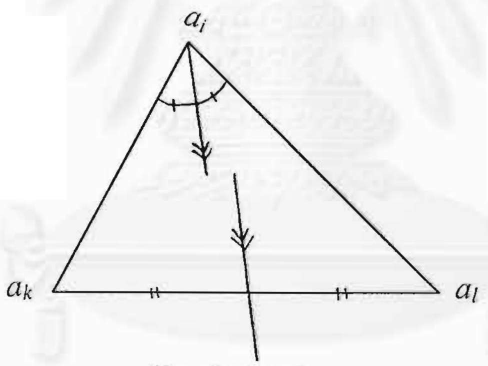
$$\begin{aligned} d_R(x, \langle T_i \rangle) = d_R(x, \langle T_j \rangle) \quad &\text{iff} \quad |\langle x - a, m_i \rangle| = |\langle x - a, m_j \rangle| \\ &\text{iff} \quad \langle x - a, m_i \rangle = \pm \langle x - a, m_j \rangle \\ &\text{iff} \quad \langle x - a, m_i \pm m_j \rangle = 0 \end{aligned}$$

iff x lies on one of the two angle bisectors of T_i and T_j. #

Definition 1.30. Since $I \in B_{ij}$ for all i, j, by Theorem 1.28, $d_R(I, \langle T_i \rangle) = d_R(I, \langle T_j \rangle)$ for all i, j. We define the *inradius* to be $r = d_R(I, \langle T_i \rangle)$, which is dependent of i. The $(n-1)$-sphere $S(I, r) = \{x \in \langle T \rangle \mid d_R(I, x) = r\}$ is called the *inscribed sphere* of T.

The Cleavance Centre

Definition 1.31. Given a triangle $[a_0, a_1, a_2]$ in $\mathbf{R}^N$, and given an index i, we define the *cleaver* from a_i to be the line L_i which is parallel to the angle bisector at a_i and which passes through the midpoint of $[a_k, a_l]$.



The cleaver L_2

Theorem 1.32. *Given a triangle* $[a_0, a_1, a_2]$ in $\mathbf{R}^N$, *the cleavers* L_i *meet at a point* J, *called the* ***cleavance centre****. This point lies on the line which passes through* I *and* G, *and is given by* $2J = 3G - I$.

Proof. Let b_i be the midpoint of the edge opposite the vertex a_i.

Case 1. If $I = G$, then each median bisects the respective angle, and the triangle is "isoceles three ways", that is, equilateral.

Case 2. If a triangle $[a_0, a_1, a_2]$ is not equilateral, then there is a unique line which passes through I and G. Consider a point J such that $(I-G) = 2(G-J)$. Since $(G-a_0) = -2(G-b_0)$, the edge $[a_0, I]$ is parallel to the edge $[b_0, J]$. Since $[a_0, I]$ is the angle bisector at a_0, $[b_0, J]$ is parallel to the angle bisector at a_0. Similarly, $[b_1, J]$ and $[b_2, J]$ are parallel to the angle bisector at a_1 and a_2, respectively. #

We would like to find a higher dimensional analog for the cleavance centre of any n-simplex in $\mathbf{R}^N$.

Definition 1.33. Given an n-simplex $T=[a_0, a_1,\ldots, a_n]$ in $\mathbf{R}^N$, let Q_{ij} denote the $(n-1)$-plane which passes through the midpoint of $[a_i, a_j]$ and is parallel to the internal angle bisector of T_i and T_j. Q_{ij} is called the *cleavance plane* of T at the edge $[a_i, a_j]$. Note that the cleavance planes of a triangle are its cleavers.

Note that $Q_{ij} = \left\{x \in \langle T\rangle \mid \left\langle x - \frac{1}{2}(a_i + a_j), m_i - m_j \right\rangle = 0\right\}$

$$= \left\{x \in \langle T\rangle \mid \left\langle x - a_j + \frac{1}{2}(a_j - a_i), m_i - m_j \right\rangle = 0\right\}$$

$$= \left\{x \in \langle T\rangle \mid \langle 2(x - a_j), m_i - m_j \rangle = \langle a_i - a_j, m_i - m_j \rangle\right\}.$$

We shall show that the cleavance planes Q_{ij} meet at a unique point J, called the *cleavance centre.*

Theorem 1.34. *Given an n-simplex* $T=[a_0, a_1,\ldots, a_n]$ *in* $\mathbf{R}^N$, *the cleavance planes* Q_{ij} *have a unique point of intersection J. If we fix* $k \in \{0, 1, 2,\ldots, n\}$ *and let* $B = B_k(T)$ *then*

$$J = a_k + \frac{1}{2}B^{-1}Q$$

where Q is the $n\times 1$ *matrix whose rows are* $\langle m_i - m_k, a_i - a_k \rangle$.

Proof. Let $T = [a_0, a_1, \ldots, a_n]$ be an n-simplex in $\mathbf{R}^N$, fix $k \in \{0, 1, 2, \ldots, n\}$.

Existence. Let J be the point in $\mathbf{R}^N$ which is defined by $J = a_k + \frac{1}{2}B^{-1}Q$. Then $2B(J - a_k) = Q$. Since B is the matrix whose rows are $m_i - m_k$ and Q is the matrix whose rows are $\langle m_i - m_k, a_i - a_k \rangle$, this means that

$$\langle m_i - m_k, 2(J - a_k) \rangle = \langle m_i - m_k, a_i - a_k \rangle \quad \text{for all } i \neq k.$$

So $$\langle m_i - m_k, 2J \rangle = \langle m_i - m_k, a_i + a_k \rangle \quad \text{for all } i \neq k.$$

This shows that J lies on each of the cleavance planes Q_{ki} for all $i \neq k$.

For all $i, j \in \{0, 1, 2, \ldots, n\}$, $i \neq j$, we have

$$\begin{aligned}
\langle m_j - m_i, 2J \rangle &= \langle m_j - m_k, 2J \rangle - \langle m_i - m_k, 2J \rangle \\
&= \langle m_j - m_k, a_j + a_k \rangle - \langle m_i - m_k, a_i + a_k \rangle \\
&= \langle m_j - m_k, a_j + a_k \rangle + \langle m_j, a_i - a_k \rangle - \langle m_k, a_i - a_k \rangle - \\
&\quad \langle m_i, a_j - a_k \rangle + \langle m_k, a_j - a_k \rangle - \langle m_i - m_k, a_i + a_k \rangle
\end{aligned}$$

(since $\langle m_j, u_{ki} \rangle = 0 = \langle m_i, u_{kj} \rangle$ and $\langle m_k, u_{ki} \rangle = \langle m_k, u_{kj} \rangle$)

$$\begin{aligned}
&= \langle m_j - m_k, a_j + a_k \rangle + \langle m_j - m_k, a_i - a_k \rangle - \\
&\quad \langle m_i - m_k, a_j - a_k \rangle - \langle m_i - m_k, a_i + a_k \rangle \\
&= \langle m_j - m_k, a_j + a_i \rangle - \langle m_i - m_k, a_i + a_j \rangle \\
&= \langle m_j - m_i, a_j + a_i \rangle.
\end{aligned}$$

This shows that J is a point of intersection of all the cleavance planes Q_{ij}.

Uniqueness. Suppose that $x \in Q_{ij}$ for all $i \neq j$.

Then $\langle m_j - m_i, 2J \rangle = \langle m_j - m_i, a_j + a_i \rangle$ for all $i \neq j$,

and in particular, for fixed k, $\langle m_i - m_k, 2J \rangle = \langle m_i - m_k, a_i + a_k \rangle$ for all $i \neq k$.

Thus $\langle m_i - m_k, 2(x - a_k) \rangle = \langle m_i - m_k, a_i - a_k \rangle$ for all $i \neq k$,

or equivalently, $2B(x - a_k) = Q$. Since B is invertible, $x = a_k + \frac{1}{2}B^{-1}Q = J$.

This shows that the point J is the unique point of intersection of the cleavance planes Q_{ij}. #

Theorem 1.35. *The centres G, I, and J of any n-simplex in* $\mathbf{R}^N$ *are colinear and* $(n-1)I + 2J = (n+1)G$.

Proof. Let $B = B_0(\mathrm{T})$.

By Theorem 1.28, $I - a_0 = B^{-1}\begin{bmatrix} \langle m_1 - m_0, a_{j_1} - a_0\rangle \\ \langle m_2 - m_0, a_{j_2} - a_0\rangle \\ \vdots \\ \langle m_n - m_0, a_{j_n} - a_0\rangle \end{bmatrix}$ where $j_i \neq 0, i$. Since

$\langle m_i - m_0, a_j\rangle = \langle m_i - m_0, a_l\rangle$ for all $j, l \neq i, 0$, we have

$(n-1)\langle m_i - m_0, a_{j_i}\rangle = \left\langle m_i - m_0, \sum\limits_{k\neq 0,i} a_k\right\rangle$.

So $(n-1)(I - a_0) = (n-1)B^{-1}\begin{bmatrix} \langle m_1 - m_0, a_{j_1} - a_0\rangle \\ \langle m_2 - m_0, a_{j_2} - a_0\rangle \\ \vdots \\ \langle m_n - m_0, a_{j_n} - a_0\rangle \end{bmatrix}$

$$= B^{-1}\begin{bmatrix} \left\langle m_1 - m_0, \sum\limits_{i\neq 0,1} a_i - (n-1)a_0\right\rangle \\ \left\langle m_2 - m_0, \sum\limits_{i\neq 0,2} a_i - (n-1)a_0\right\rangle \\ \vdots \\ \left\langle m_n - m_0, \sum\limits_{i\neq 0,n} a_i - (n-1)a_0\right\rangle \end{bmatrix}$$

$$= B^{-1}\begin{bmatrix} \left\langle m_1 - m_0, \sum\limits_{i=0}^{n} a_i - a_1 - na_0\right\rangle \\ \left\langle m_2 - m_0, \sum\limits_{i=0}^{n} a_i - a_2 - na_0\right\rangle \\ \vdots \\ \left\langle m_n - m_0, \sum\limits_{i=0}^{n} a_i - a_n - na_0\right\rangle \end{bmatrix}.$$

By Theorem 1.34, $2(J - a_0) = B^{-1}\begin{bmatrix} \langle m_1 - m_0, a_1 - a_0 \rangle \\ \langle m_2 - m_0, a_2 - a_0 \rangle \\ \vdots \\ \langle m_n - m_0, a_n - a_0 \rangle \end{bmatrix}$.

Thus $(n-1)(I - a_0) + 2(J - a_0) = B^{-1}\begin{bmatrix} \left\langle m_1 - m_0, \sum_{i=0}^{n} a_i - (n+1)a_0 \right\rangle \\ \left\langle m_2 - m_0, \sum_{i=0}^{n} a_i - (n+1)a_0 \right\rangle \\ \vdots \\ \left\langle m_n - m_0, \sum_{i=0}^{n} a_i - (n+1)a_0 \right\rangle \end{bmatrix}$

$$= B^{-1}B\left(\sum_{i=0}^{n} a_i - (n+1)a_0 \right) = (n+1)(G - a_0)$$

Hence $(n-1)I + 2J = (n+1)G$. #

CHAPTER II
MEDIAL SIMPLICES

Given an n-simplex $T=[a_0, a_1, \ldots, a_n]$ in $\mathbf{R}^N$, let g_i be the centroid of the face T_i. We have that $T'=[g_0, g_1, \ldots, g_n]$ is an n-simplex in $\mathbf{R}^N$, since

$v_{0i} = g_i - g_0 = \frac{1}{n}\left(\sum_{k\neq i} a_k - \sum_{k\neq 0} a_k\right) = \frac{1}{n}(a_0 - a_i)$ and since $\{ u_{0i} \mid u_i = a_i - a_0, i \neq 0 \}$

is linearly independent, and so $\{ v_{0i} \mid v_{0i} = g_i - g_0, i \neq 0 \}$ is linearly independent. The n-simplex T' is called the *medial simplex* of T in $\mathbf{R}^N$.

Note that $\langle T \rangle = \langle T' \rangle$.

Notation. Given $a \in \mathbf{R}^N$ and $k \in \mathbf{R}$ with $k \neq 0$, let $H_{a,k}$ denote the homothety centred at a of ratio k. Recall that $H_{a,k}$ is given by

$$H_{a,k}(p) = a + k(p - a)$$

for all $p \in \mathbf{R}^N$. The homothety maps any k-plane through the point a to itself.

Theorem 2.1. *Let $T=[a_0, a_1, \ldots, a_n]$ be an n-simplex in $\mathbf{R}^N$ with medial simplex $T'=[g_0, g_1, \ldots, g_n]$. Let $H_{g,-n} : \langle T \rangle \to \langle T \rangle$ denote the homothety*

$$H_{g,-n}(x) = g - 2(x - g).$$

Then

(1) $H_{g,-n}(g_i) = a_i$ *for all* $i \in \{0, 1, \ldots, n\}$

(2) $H_{g,-n}(T') = T$.

Proof. (1) Let $i \in \{0, 1, \ldots, n\}$. Since $a_i - G = a_i - \frac{1}{n+1}\sum_{j=0}^{n} a_j =$
$\frac{1}{n+1}\left(na_i - \sum_{j\neq i} a_j\right)$ and $g_i - G = \frac{1}{n}\sum_{j\neq i} a_j - \frac{1}{n+1}\sum_{j=0}^{n} a_j =$
$\frac{1}{n(n+1)}\left((n+1)\sum_{j\neq i} a_j - n\sum_{j=0}^{n} a_j\right) = \frac{1}{n(n+1)}\left(\sum_{j\neq i} a_j - na_i\right)$, we have
$a_i - G = -n(g_i - G)$. Hence $H_{g,-n}(g_i) = a_i$.

(2) ($\subseteq$) Let $x \in T'$. Then $x = \sum_{i=0}^{n} t_i g_i$ for some $t_0, t_1, \ldots, t_n \in \mathbf{R}$ such that $t_i \geq 0$ for all i and $\sum_{i=0}^{n} t_i = 1$. Since

$$\begin{aligned} H_{g,-n}(x) &= G - n(x - G) \\ &= \frac{1}{n} \sum_{i=0}^{n} a_i - n\left(\sum_{i=0}^{n} t_i g_i - \frac{1}{n+1} \sum_{i=0}^{n} a_i \right) \\ &= \sum_{i=0}^{n} a_i - n \sum_{i=0}^{n} t_i g_i \\ &= \sum_{i=0}^{n} a_i - \sum_{i=0}^{n} t_i \left(\sum_{j \neq i} a_j \right) \\ &= \sum_{i=0}^{n} \left(1 - \sum_{j \neq i} t_j \right) a_i \\ &= \sum_{i=0}^{n} t_i a_i \in T, \end{aligned}$$

we have $H_{g,-n}(T') \subseteq T$.

($\supseteq$) Let $x \in T$. Then $x = \sum_{i=0}^{n} t_i a_i$ for some $t_0, t_1, \ldots, t_n \in \mathbf{R}$ with $t_i \geq 0$ for all i and $\sum_{i=0}^{n} t_i = 1$. Since $H_{g,-n}\left(\sum_{i=0}^{n} t_i g_i \right) = \sum_{i=0}^{n} t_i a_i = x$ and $\sum_{i=0}^{n} t_i g_i \in T'$, we have $x \in H_{g,-n}(T')$. Thus $H_{g,-n}(T') \supseteq T$.

Hence $H_{g,-n}(T') = T$. #

We shall compare the centres of a given n-simplex with those of its medial simplex.

Theorem 2.2. *The n-simplex $T = [a_0, a_1, \ldots, a_n]$ in $\mathbf{R}^N$ and its medial simplex T' have the same centroid.*

Proof. Let G' be the centroid of the medial n-simplex T'. Then

$$G' = \frac{1}{n+1}\sum_{i=0}^{n} g_i$$

$$= \frac{1}{n+1}\left(\sum_{i=0}^{n}\frac{1}{n}\left(\sum_{k\neq i} a_k\right)\right)$$

$$= \frac{1}{n(n+1)}\left(\sum_{i=0}^{n}(n+1)G - a_i\right)$$

$$= \frac{(n+1)G}{n} - \frac{1}{n(n+1)}\sum_{i=0}^{n} a_i$$

$$= \frac{(n+1)}{n}G - \frac{G}{n}$$

$$= G.$$

Hence T and T' have the same centroid. #

Theorem 2.3. *Let* $T=[a_0, a_1, \ldots, a_n]$ *be an n-simplex in* $\mathbf{R}^N$ *and* O' *the circumcentre of the medial n-simplex* T'*. Then*

$$nO' = (n+1)G - O.$$

Proof. Let $T=[a_0, a_1, \ldots, a_n]$ be an n-simplex in $\mathbf{R}^N$ and O' the circumcentre of the medial n-simplex T'. By theorem 1.15, we have $O' = g_0 + \frac{1}{2}(A')^{-1}P'$ where A is the $n\times n$ matrix whose rows are $g_i - g_0$, $i \neq 0$ with respect to the chosen orthonormal basis for $V(T)$ and P' is the $n\times 1$ matrix whose rows are $\langle g_i - g_0, g_i - g_0\rangle$. Since $g_i - g_0 = -\frac{1}{n}(a_i - a_0)$ for all $i \neq 0$, $A' = -\frac{1}{n}A$ where A is the matrix whose rows are $a_i - a_0$. Since $\langle g_i - g_0, g_i - g_0\rangle = \frac{1}{n^2}\langle a_i - a_0, a_i - a_0\rangle$ for all $i \neq 0$, $P' = \frac{1}{n^2}P$ where P is the matrix whose rows are $\langle a_i - a_0, a_i - a_0\rangle$. Thus

$$O' = g_0 + \frac{1}{2}(-nA^{-1})(\frac{1}{n^2}P')$$

$$= \frac{1}{n}\sum_{i\neq 0} a_i - \frac{1}{2n}A^{-1}P$$

Then $$nO' = \sum_{i\neq 0} a_i - \frac{1}{2}A^{-1}P$$

$$= (n+1)G - a_0 - (O - a_0) \quad (\text{since } (n+1)G = \sum_{i=0}^{n} a_i)$$

$$= (n+1)G - O$$

Hence $nO' = (n+1)G - O$. #

Remark 2.4. (1) $O' = \dfrac{(n+1)G - O}{n}$. By Theorem 1.21, $(n-1)H + 2O = (n+1)G$, so $O' = \dfrac{(n+1)G + (n-1)H}{2n}$ or $O' = \dfrac{(n-1)H + O}{n}$.

(2) let H' be the orthocentre of T'. Since $(n-1)H' + 2O' = (n+1)G$, we have $$H' = \frac{(n-2)(n+1)G + 2O}{n(n-1)},$$

or $$H' = \frac{(n+1)G - H}{n},$$

or $$H' = \frac{(n-2)H + 2O}{n}.$$

(3) If T is a triangle, then O' is the midpoint of the edge $[O, H]$ and $H' = O$.

(4) O, G, H, O' and H' all lie on the Euler line.

Theorem 2.5. *Let $T = [a_0, a_1, ..., a_n]$ be an n-simplex in $\mathbf{R}^N$, let $S(O, R)$ be the circumscribed sphere of T and let $S(O', R')$ be the circumscribed sphere of T'. Then $R' = \frac{1}{n}R$.*

Proof. $$R' = |O' - g_i|$$

$$= \left|\left(\frac{(n-1)H + (n+1)G}{2n}\right) - \left(\frac{1}{n}\sum_{k\neq i} a_k\right)\right|$$

$$= \left| \left(\frac{(n-1)H + (n+1)G}{2n} \right) - \left(\frac{(n+1)G - a_i}{n} \right) \right|$$

$$= \left| \frac{(n-1)H - (n+1)G + 2a_i}{2n} \right|$$

$$= \left| \frac{O - a_i}{n} \right|$$

$$= \frac{1}{n} R'. \ \#$$

The nine-point theorem says that for any triangle, the midpoints of the sides, the feet of the altitudes and the midpoints of the segments joining the vertices to the orthocentre all lie on a circle which is called the *nine-point circle* of the triangle. The centre N of this circle lies midway between the orthocentre and the circumcentre. Note that N is the circumcentre of the medial triangle.

We shall generalize the nine-point theorem to higher dimension.

Theorem 2.6. (*The* $3(n+1)$*-point Theorem*)

Given an n-simplex $T = [a_0, a_1, \ldots, a_n]$ *in* $\mathbf{R}^N$, *let* g_i *be the centroid of the face* T_i, *let* h_i *the point which lies* $(1/n)^{th}$ *of the way from* H *to* a_i *and let* k_i *the point of intersection of* $\langle T_i \rangle$ *with the line* H_i *which passes through* h_i *and which is perpendicular to* $\langle T_i \rangle$. *Then the points* g_i, h_i *and* k_i *for all* $i \in \{0, 1, \ldots, n\}$ *all lie on the circumscribed sphere* $S(O', R')$ *of* T'.

Proof. Let $T = [a_0, a_1, \ldots, a_n]$ be an n-simplex in $\mathbf{R}^N$, let g_i be the centroid of the face T_i. The circumscribed sphere $S(O', R')$ of the medial simplex $T' = [g_0, g_1, \ldots, g_n]$ passes through g_i for all $i \in \{0, 1, \ldots, n\}$. Let h_i the point which lies $(1/n)^{th}$ of the way from H to a_i, that is

$h_i = a_i + \frac{n-1}{n}(H - a_i) = \frac{(n-1)H + a_i}{n}$. Since $g_i = \frac{(n+1)G - a_i}{n}$,

$\frac{g_i + h_i}{2} = \frac{(n-1)H + (n+1)G}{2n} = O'$. So $[g_i, h_i]$ is a diameter of $S(O', R')$. Thus h_i lies on $S(O', R')$ for all $i \in \{0, 1, \ldots, n\}$.

Next, we will show that $k_i \in S(O', R')$. Since $[h_i, k_i]$ is perpendicular to $\langle T_i \rangle$ and $[g_i, k_i] \subseteq \langle T_i \rangle$, $[h_i, k_i]$ is perpendicular to $[g_i, k_i]$. Thus the angle k_i in the triangle $[g_i, h_i, k_i]$ is $\pi/2$. Since $[g_i, h_i]$ is a diameter of the circumscribed sphere $S(O', R')$, we have $k_i \in S(O', R')$. Hence the points g_i, h_i and k_i where $i \in \{0, 1, \ldots, n\}$ all lie on the circumscribed sphere $S(O', R')$ of T'. #

Given an n-simplex $T=[a_0, a_1, \ldots, a_n]$ in $\mathbf{R}^N$, let $i \in \{0, 1, \ldots, n\}$. Since $\langle m_i, v_{kj} \rangle = \langle m_i, g_j - g_k \rangle = -\frac{1}{n}\langle m_i, a_j - a_k \rangle = -\frac{1}{n}\langle m_i, u_{kj} \rangle$, we see that the inward normal vector m_i of the face T_i is equal to the outward normal vector m_i' of the face T_i' of the medial simplex.

Let B_{ij}' be the internal angle bisector of the two faces T_i' and T_j'. Then B_{ij}' is the $(n-1)$-plane in $\langle T' \rangle$ which contains T_{ij}' and which has orthogonal vector $m_i' - m_j'$. Since $m_i' = -m_i$ for all i, $m_i - m_j$ is an orthogonal vector of B_{ij}'.

Theorem 2.7. *Let $T=[a_0, a_1, \ldots, a_n]$ be an n-simplex in $\mathbf{R}^N$ and let J' be the cleavance centre of its medial simplex T'. Then*

$$nJ' = (n+1)G - J.$$

Proof. Let $T=[a_0, a_1, \ldots, a_n]$ be an n-simplex in $\mathbf{R}^N$ and let J' be the cleavance centre of its medial simplex T'. By Theorem 1.34, we have $J' = g_0 + \frac{1}{2}(B')^{-1}Q'$ where B' is the $n \times n$ matrix whose rows are $m_i - m_0$ with respect to the orthonormal basis for $V(T)$ and Q' is the $n \times 1$ matrix whose rows are $\langle m_i - m_0, g_i - g_0 \rangle$. Since $g_i - g_0 = \frac{1}{n}(a_0 - a_i)$ for all $i \neq 0$, $Q' = -\frac{1}{n}Q$ where Q is the matrix whose rows are $\langle m_i - m_0, a_i - a_0 \rangle$. Thus

$$\begin{aligned} J' &= g_0 - \frac{1}{2n}B^{-1}Q \\ &= \frac{1}{n}\left(\sum_{i \neq 0} a_i\right) - \frac{1}{n}(J - a_0) \\ &= \frac{1}{n}\big((n+1)G - a_0\big) - \frac{1}{n}(J - a_0) \end{aligned}$$

Thus

$$\begin{aligned} nJ' &= \sum_{i \neq 0} a_i - (J - a_0) \\ &= (n+1)G - J. \ \# \end{aligned}$$

Remark 2.8. (1) We have $J' = \dfrac{(n+1)G - J}{n}$. By theorem 1.35, $(n-1)I + 2J = (n+1)G$, so $J' = \dfrac{(n+1)G + (n-1)I}{2n}$, or $J' = \dfrac{(n-1)I + J}{n}$.

(2) Since $(n-1)I' + 2J' = (n+1)G$, We have that

$$I' = \frac{(n-2)(n+1)G + 2J}{n(n-1)},$$

or

$$I' = \frac{(n+1)G - I}{n},$$

or

$$I' = \frac{(n-2)I + 2J}{n}.$$

(3) The centres G, I, J, I' and J' of the n-simplex all lie on a line.

(4) If T is a triangle, then $I' = J$.

CHAPTER III

VARIOUS KINDS OF CENTRES OF SIMPLICES IN $\mathbf{S}^n$

Definition 3.1. The *n-sphere* in $\mathbf{R}^{n+1}$, $\mathbf{S}^n$, is the set of all points u in $\mathbf{R}^{n+1}$ such that $|u| = 1$,

$$\mathbf{S}^n = \{ u \in \mathbf{R}^{n+1} \mid |u| = 1 \}.$$

Given $u \in \mathbf{S}^n$, the point $-u$ is called the *antipodal point* of u.

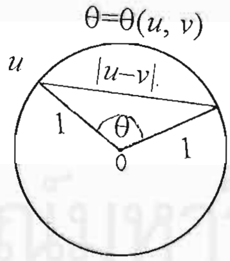
Definition 3.2. Given two points u and v in $\mathbf{S}^n$, we define the *distance* between u and v in $\mathbf{S}^n$ to be equal to the angle between the vectors u and v in $\mathbf{R}^{n+1}$, that is $d_S(u, v) = \theta(u, v) = \arccos\langle u, v\rangle$.

Note that $d_S(u, v) \leq \pi$ with equality if and only if v is the antipodal point of u and $d_S(u, v) + d_S(u, -v) = \pi$.

Theorem 3.3. *The distance $d_S(u, v)$ between two point u and v in $\mathbf{S}^n$ determines and is determined by the distance $|u - v|$ between u and v in $\mathbf{R}^{n+1}$.*

Proof. We have $0 \leq d_S(u, v) \leq \pi$ and $0 \leq |u - v| \leq 2$. By the law of cosines,

$$|u - v|^2 = 2 - 2\cos\theta(u, v).$$



Hence we see that

$$|u - v| = \sqrt{2 - 2\cos d_S(u, v)}$$

and

$$d_S(u, v) = \arccos\left(1 - \frac{1}{2}|u - v|^2\right). \ \#$$

Definition 3.4. We define a *k-sphere* in $\mathbf{S}^n$ to be any set of the form, $SP = P \cap \mathbf{S}^n$ for some $(k+1)$-plane P in $\mathbf{R}^{n+1}$ with $d_R(0, P) \leq 1$. For $k = n-1$, SP is called a *hypersphere* in $\mathbf{S}^n$.

If P is a $(k+1)$-dimensional vector subspace of $\mathbf{R}^{n+1}$, then SP is called the *great k-sphere* in $\mathbf{S}^n$ and if $k = n-1$, SP is called a *great hypersphere* in $\mathbf{S}^n$, and if $k = 1$, SP is called a *great circle* in $\mathbf{S}^n$.

Remark 3.5. If SP is a great hypersphere in $\mathbf{S}^n$, then we can write

$$SP = \{ u \in \mathbf{S}^n \mid \langle u, m \rangle = 0 \}$$

for some vector m in $\mathbf{S}^n$. We say that SP is the *great hypersphere with pole m* and we denote it by SP_m.

Notice that $SP_m = SP_l$ if and only if $m = \pm l$. The points m and $-m$ are called the *poles* of the great hypersphere SP_m. Two great hyperspheres are said to be *perpendicular* in $\mathbf{S}^n$ if their poles are perpendicular in $\mathbf{R}^{n+1}$, (or equivalently, if their hyperspaces are perpendicular in $\mathbf{R}^{n+1}$). More generally, a great k-sphere SP and a great l-sphere SQ are said to be *perpendicular* when the $(k+1)$-plane P and the $(l+1)$-plane Q are perpendicular in $\mathbf{R}^{n+1}$.

Definition 3.6. Given $u_1, u_2, \ldots, u_{k+1}$ in $\mathbf{S}^n$, we define

$$(u_1, u_2, \ldots, u_{k+1}) = \left\{ \sum_{i=1}^{k+1} t_i u_i \in \mathbf{S}^n \,\middle|\, t_i \geq 0 \right\}.$$

If $\{ u_1, u_2, \ldots, u_{k+1} \}$ is linearly independent in $\mathbf{R}^{n+1}$ then $(u_1, u_2, \ldots, u_{k+1})$ is called a *k-simplex* in $\mathbf{S}^n$.

A 1-simplex is called an *arc* and a 2-simplex is called a *spherical triangle*. An l-face of the simplex $S = (u_1, u_2, \ldots, u_{k+1})$ is an l-simplex of the form $(v_1, v_2, \ldots, v_{l+1})$ with each $v_i \in \{u_1, u_2, \ldots, u_{k+1}\}$.

A 0-face (u_i) is called a *vertex*, and a 1-face (u_i, u_j) is called an *edge*. Given an index i, the $(n-1)$-face of S which is opposite the vertex u_i is the $(k-1)$-simplex $S_i = (u_1, u_2, \ldots, \hat{u}_i, \ldots, u_{k+1})$.

Given a k-simplex $S = (u_1, u_2, \ldots, u_{k+1})$ in $\mathbf{S}^n$, we write $[0, S]$ for the $(k+1)$-simplex in $\mathbf{R}^{n+1}$ given by $[0, S] = [0, u_1, u_2, \ldots, u_{k+1}]$.

Definition 3.7. Given u_i, u_j in $\mathbf{S}^n$, the *midpoint* of the edge (u_i, u_j) in $\mathbf{S}^n$ is the point $\dfrac{u_i+u_j}{|u_i+u_j|}$.

Note that $d_S\left(u_i, \dfrac{u_i+u_j}{|u_i+u_j|}\right) = d_S\left(u_j, \dfrac{u_i+u_j}{|u_i+u_j|}\right)$ since

$$\begin{aligned} d_S\left(u_i, \frac{u_i+u_j}{|u_i+u_j|}\right) &= \arccos\left\langle u_i, \frac{u_i+u_j}{|u_i+u_j|}\right\rangle \\ &= \arccos\left(\frac{1}{|u_i+u_j|}\langle u_i, u_i+u_j\rangle\right) \\ &= \arccos\left(\frac{1}{|u_i+u_j|}\left(\langle u_i, u_i\rangle + \langle u_i, u_j\rangle\right)\right) \\ &= \arccos\left(\frac{1}{|u_i+u_j|}\left(1 + \langle u_i, u_j\rangle\right)\right) \end{aligned}$$

and similarly

$$d_S\left(u_j, \frac{u_i+u_j}{|u_i+u_j|}\right) = \arccos\left(\frac{1}{|u_i+u_j|}\left(1 + \langle u_i, u_j\rangle\right)\right).$$

The Centroid

Definition 3.8. Given an n-simplex $S=(u_1, u_2, \ldots, u_{n+1})$ in $\mathbf{S}^n$, let SM_{ij} be the great hypersphere in $\mathbf{S}^n$ which passes through the midpoint of the edge (u_i, u_j) and through the points u_k where $k \neq i, j$. We call SM_{ij} the *medial great hypersphere* of S in $\mathbf{S}^n$.

Lemma 3.9. *Given an n-simplex* $S=(u_1, u_2, \ldots, u_{n+1})$ *in* $\mathbf{S}^n$, *let* M_{ij} *be the medial hyperspace of the simplex* $[0, S]$ *in* $\mathbf{R}^{n+1}$ *and let* SM_{ij} *be the medial great hypersphere of* S *in* $\mathbf{S}^n$. *Then* $SM_{ij} = M_{ij} \cap \mathbf{S}^n$.

Proof. Let M_{ij} be the medial hyperplane of the $(n+1)$-simplex $[0, S]$ *in* $\mathbf{R}^{n+1}$. By definition, M_{ij} is the hyperspace which passes through the point $\frac{u_i+u_j}{2}$ and the points u_k with $k \neq i, j$. Since $0 \in M_{ij}$ and $\frac{u_i+u_j}{2} \in M_{ij}$, we also have $\frac{u_i+u_j}{|u_i+u_j|} \in M_{ij}$. So M_{ij} is the hyperspace which passes through the midpoint of the arc (u_i, u_j) and through each u_k with $k \neq i, j$. Hence $SM_{ij} = M_{ij} \cap \mathbf{S}^n$. #

Lemma 3.10. *Given an n-simplex* $S=(u_1, u_2, \ldots, u_{n+1})$ *in* $\mathbf{S}^n$ *and given* $i \neq j$, *let* SM_{ij} *be the medial great hypersphere of* S *in* $\mathbf{S}^n$. *The intersection of the hypersphere* SM_{ij} *with* $i, j \neq 0$ *and* $i \neq j$ *is the pair of points* $\pm\frac{G}{|G|}$ *where G is the centroid of the simplex* $[0, S]$.

Proof. Let G be the centroid of the $(n+1)$-simplex $[0, u_1, u_2, \ldots, u_{n+1}]$ in $\mathbf{R}^{n+1}$. Then $G = \dfrac{1}{n+2}\left(\sum_{i=1}^{n+1} u_i\right)$ and G lies on the line which passes through 0 and $\sum_{i=1}^{n+1} u_i$. Thus $\bigcap_{i \neq j} SM_{ij} = \bigcap_{i \neq j} (M_{ij} \cap \mathbf{S}^n)$

$$= (\text{the line through } 0 \text{ and } \sum_{i=1}^{n+1} u_i) \cap \mathbf{S}^n$$

$$= (\text{the line through } 0 \text{ and } G) \cap \mathbf{S}^n$$

$$= \{\text{the two points } \pm \frac{G}{|G|}\}$$

Hence the medial great hyperspheres SM_{ij} meet at the two points $\pm \dfrac{G}{|G|}$. #

Remark 3.11. Since $G = \dfrac{1}{n+2} \sum_{i=1}^{n+1} u_i$, it is clear that $\dfrac{G}{|G|} \in S$ and $-\dfrac{G}{|G|} \notin S$.

We call the point $\dfrac{G}{|G|}$ the *centroid* of S and denote it by G_S.

The Circumcentre

Definition 3.12. Given an n-simplex $S = (u_1, u_2, \ldots, u_{n+1})$ in $\mathbf{S}^n$, let SP_{ij} be the great hypersphere in $\mathbf{S}^n$ which is perpendicular to the edge (u_i, u_j) and which passes through the midpoint of the edge (u_i, u_j). SP_{ij} is called the *perpendicular bisector* of (u_i, u_j) in $\mathbf{S}^n$.

Theorem 3.13. *Given an n-simplex $S = (u_1, u_2, \ldots, u_{n+1})$ in $\mathbf{S}^n$, let P_{ij} be the perpendicular bisector of the edge (u_i, u_j) of $[0, S]$ in $\mathbf{R}^{n+1}$ and let SP_{ij} be the perpendicular bisector of the edge (u_i, u_j) in $\mathbf{S}^n$. Then $SP_{ij} = P_{ij} \cap \mathbf{S}^n$.*

Proof. By definition, SP_{ij} is the hypersphere in $\mathbf{S}^n$ which is perpendicular to the edge (u_i, u_j) and passes through the point $\dfrac{u_i + u_j}{|u_i + u_j|}$. Thus SP_{ij} is the intersection with $\mathbf{S}^n$ of the hyperspace P_{ij} in $\mathbf{R}^{n+1}$ which is perpendicular to span $\{u_i, u_j\}$ and contains span$\left\{\dfrac{u_i + u_j}{|u_i + u_j|}\right\}$. Since span$\{u_i, u_j\}$=span$\{u_i + u_j, u_i - u_j\}$ and span$\left\{\dfrac{u_i + u_j}{|u_i + u_j|}\right\}$ = span$\{u_i + u_j\}$, P_{ij} is the hyperspace in $\mathbf{R}^{n+1}$ which is perpendicular to span$\{u_i - u_j\}$ and passes through $\dfrac{u_i + u_j}{2}$. Thus P_{ij} is the perpendicular bisector of $[u_i, u_j]$ of the simplex $[0, S]$ in $\mathbf{R}^{n+1}$ and SP_{ij} = $P_{ij} \cap \mathbf{S}^n$. #

Theorem 3.14. *Let u_i and u_j be two distinct points in $\mathbf{S}^n$. Then the perpendicular bisector of $[u_i, u_j]$ is the set of all point u in $\mathbf{S}^n$ such that $d_S(u, u_i) = d_S(u, u_j)$.*

Proof. Given $u_i, u_j \in \mathbf{S}^n$, let $u \in \mathbf{S}^n$. Then

$u \in SP_{ij}$	iff	$u \in P_{ij}$	by theorem 2.16
	iff	$d_R(u, u_i) = d_R(u, u_j)$	by lemma 1.19
	iff	$d_S(u, u_i) = d_S(u, u_j)$. #	

Lemma 3.15. *Given an n-simplex $S = (u_1, u_2, \ldots, u_{n+1})$ in $\mathbf{S}^n$, the intersection of the perpendicular bisectors P_{ij} in $\mathbf{R}^{n+1}$ where $i \neq j$ is the line which passes through 0 and O where O is the circumcentre of the simplex $[0, S]$ in $\mathbf{R}^{n+1}$.*

Proof. It is clear that the circumcentre O of the simplex $[0, S]$ lies on each P_{ij}. Since $d_R(0, u_i) = 1 = d_R(0, u_j)$, by the Theorem 1.16, 0 lies on each P_{ij}. Since $d_R(0, 0) = 0$ but $d_R(0, u_i) = 1$, we have that $0 \notin P_{0i}$. So $0 \neq O$. Since the intersection of the perpendicular bisectors P_{ij} is a line, it must be the line which passes through 0 and O. #

Theorem 3.16. *Given an n-simplex* $S = (u_1, u_2, \ldots, u_{n+1})$ *in* $\mathbf{S}^n$, *the perpendicular bisectors* SP_{ij} *of the edges* (u_i, u_j) *in* $\mathbf{S}^n$ *meet at the two points* $\pm O_S$, *where* $O_S = \dfrac{O}{|O|}$ *and* O *is the circumcentre of the simplex* $[0, S]$ *in* $\mathbf{R}^{n+1}$.

Proof. Let O be the circumcentre of the $(n+1)$-simplex $[0, S]$ in $\mathbf{R}^{n+1}$. Since

$$\bigcap_{i \neq j} SP_{ij} = \bigcap_{i \neq j} (P_{ij} \cap \mathbf{S}^n)$$

$$= (\text{ the line which passes through } 0 \text{ and } O\,) \cap \mathbf{S}^n$$

$$= \{ \text{ the point } \pm \frac{O}{|O|} \} ,$$

we see that the perpendicular bisectors SP_{ij} meet at the two point $\pm O_S$. #

Definition 3.17. (1) Since $O_S \in SP_{ij}$, by Theorem 3.14, $d_S(O_S, u_i) = d_S(O_S, u_j)$ for all i, j with $i \neq j$.

(2) Since
$$\langle u_i, O_S \rangle = \left\langle u_i, \frac{O}{|O|} \right\rangle$$

$$= \frac{1}{2|O|} \langle u_i, 2O \rangle$$

$$= \frac{1}{2|O|} \langle u_i, u_i \rangle \qquad (\text{ since } O \in P_{0i}\,)$$

$$= \frac{1}{2|O|} > 0 ,$$

we have $d_S(O_S, u_i) = \arccos\langle u_i, O_S \rangle < \dfrac{\pi}{2}$. Since $d_S(O_S, u_i) + d_S(-O_S, u_i) = \pi$, so $d_S(-O_S, u_i) > \dfrac{\pi}{2}$. We let $R_S = d_S(O_S, u_i)$ for any i. The sphere in $\mathbf{S}^n$

$$\mathrm{S}(O_S, R_S) = \{ u \in \mathbf{S}^n \mid d_S(O_S, u) = R_S \}$$

$$= \quad \{ u \in \mathbf{S}^n \mid d_S(-O_S, u_i) = \pi - R_S \}$$

is called the *circumscribed sphere* of S, O_S is called the *circumcentre* of S and R_S is called the *circumradius* of S.

The Incentre

Definition 3.18. Let SP and SQ be two great hyperspheres in $\mathbf{S}^n$, say $SP = P \cap \mathbf{S}^n$ and $SQ = Q \cap \mathbf{S}^n$ for some hyperspaces P and Q in $\mathbf{R}^{n+1}$. The *angle* between SP and SQ in $\mathbf{S}^n$, denoted by $\theta_S(SP, SQ)$, is defined to be equal to the angle between P and Q in $\mathbf{R}^{n+1}$, that is $\theta(P, Q)$.

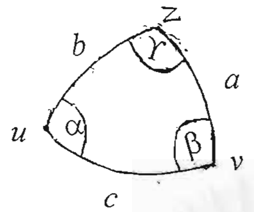
Definition 3.19. Given a point u and a great hypersphere SP in $\mathbf{S}^n$, the *distance* between u and SP, denoted by $d_S(u, SP)$, is given by

$$d_S(u, SP) = \inf\{ d_S(u, v) \mid v \in SP \}.$$

Theorem 3.20. *Given a point u and a great hypersphere SP in $\mathbf{S}^n$, say $SP = P \cap \mathbf{S}^n$ for some hyperspace P in $\mathbf{R}^{n+1}$ the distance $d_S(u, SP)$ between u and SP in $\mathbf{S}^n$ determines and is determined by the distance $d_R(u, P)$ between the point u and the hyperspace P in $\mathbf{R}^{n+1}$.*

Proof. Let w be the point on P which is nearest to u. Then $d(u, P) = |u - w|$ and the vector $u - w$ is an orthogonal vector of P (unless $u = w$ in which case $d_S(u, SP) = d_R(u, P) = 0$). Let $M = \text{span}\{u, w\} = \text{span}\{w, u - w\}$. Then M is perpendicular to P and $M \cap P = \text{span}\{w\}$. Let $SM = M \cap \mathbf{S}^n$. Then $SM \cap SP = (M \cap P) \cap \mathbf{S}^n$ which consists of two points, namely $\dfrac{w}{|w|}$ and $-\dfrac{w}{|w|}$ (unless $w = 0$, in which case $d_S(u, SP) = \dfrac{\pi}{2}$ and $d_R(u, P) = 1$). Let v be the point on $SM \cap SP$ which is nearest to u in $\mathbf{S}^n$.

let z be any point of SP with $z \neq w$. We consider the spherical triangle (u, v, z) with angles α at u, β at v, and γ at z and sides of spherical length a opposite u, b opposite v and c opposite z.



By the spherical law of cosines for angles, we have that

$$\cos\alpha = -\cos\beta \cos\gamma + \sin\beta \sin\gamma \cos a,$$

$$\cos\beta = -\cos\alpha \cos\gamma + \sin\alpha \sin\gamma \cos b,$$

and $$\cos\gamma = -\cos\alpha \cos\beta + \sin\alpha \sin\beta \cos c.$$

Since $\operatorname{span}\{u, v\} = \operatorname{span}\{u, w\} = M$ which is perpendicular to P and $\operatorname{span}\{v, z\} \subseteq P$, we see that $\operatorname{span}\{u, v\}$ is perpendicular to $\{v, z\}$. Thus the angle β is equal to $\pi/2$. So we have that

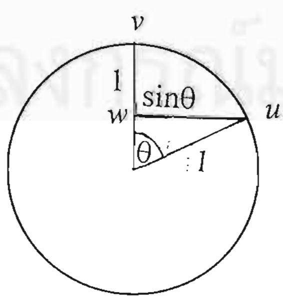
$$\cos a = \frac{\cos\alpha}{\sin\gamma},$$

$$\cos b = \frac{\cos\alpha \cos\gamma}{\sin\alpha \sin\gamma},$$

and $$\cos c = \frac{\cos\gamma}{\sin\alpha}.$$

Hence $\cos b = \cos a \cos c$ and so $b > c$. This show that v is the point on SP which is nearest to u. Hence $d_S(u, SP) = d_S(u, v) = \theta(u, v)$.

We consider the triangle $0wu$.



We have $d_R(u, P) = |\, u - w \,| = \sin\theta(u, v) = \sin d_S(u, v) = \sin d_S(u, SP)$ and so $d_S(u, SP) = \arcsin d(u, P)$. #

Definition 3.21. Let SP and SQ be great hyperspheres in $\mathbf{S}^n$, say $SP = P \cap \mathbf{S}^n$ and $SQ = Q \cap \mathbf{S}^n$ for some hyperspaces P, Q in $\mathbf{R}^{n+1}$, and say $SP \cap SQ = SL = L \cap \mathbf{S}^n$ where L is an $(n-1)$-dimensional vector subspace of $\mathbf{R}^{n+1}$. An *angle bisector* SB of SP and SQ at SL is a great hypersphere in $\mathbf{S}^n$ which contains SL such that $\theta_S(SB, SP) = \theta_S(SB, SQ)$.

Equivalently, an *angle bisector* SB of SP and SQ at SL is the intersection of $\mathbf{S}^n$ with a hyperspace B in $\mathbf{R}^{n+1}$ which contains L such that $\theta(B, P) = \theta(B, Q)$.

In other words, an *angle bisector* SB of SP and SQ at SL is the intersection of $\mathbf{S}^n$ with an angle bisector B of P and Q at L.

Lemma 3.22. *Let SP and SQ be great hyperspheres in $\mathbf{S}^n$ with $SP \cap SQ = SL$, and let $u \in \mathbf{S}^n$. Then $d_S(u, SP) = d_S(u, SQ)$ if and only if u lies on one of the two angle bisectors of SP and SQ at SL.*

Proof. Say $SP = P \cap \mathbf{S}^n$ and $SQ = Q \cap \mathbf{S}^n$, where P and Q are hyperspaces in $\mathbf{R}^{n+1}$. Let $u \in \mathbf{S}^n$. Then

$d_S(u, SP) = d_S(u, SQ)$ iff $d(u, P) = d(u, Q)$ by theorem 2.23

iff u lies on the angle bisector of P and Q at L by theorem 1.32

iff u lies on one of the angle bisector of SP and SQ at SL. #

Definition 3.23. Given an n-simplex $S = (u_1, u_2, \ldots, u_{n+1})$ in $\mathbf{S}^n$, the *inward pole* of the face S_i is the pole m_i such that $\langle m_i, u_i \rangle > 0$.

Note that m_i is the inward normal vector for the $(n-1)$-face $[0, S]_i$ which is opposite the vertex u_i of the simplex $[0, S]$ in $\mathbf{R}^{n+1}$.

Definition 3.24. Given an n-simplex $S = (u_1, u_2, \ldots, u_{n+1})$ in $\mathbf{S}^n$, let m_i and m_j be the inward poles of S_i and S_j, respectively. The *internal angle bisector* SB_{ij} of S_i and S_j is the great hypersphere which passes through u_k, where $k \neq i, j$ with the pole $\dfrac{m_i - m_j}{|m_i - m_j|}$.

Equivalently, SB_{ij} is the intersection of $\mathbf{S}^n$ with the hyperspace which passes through u_k where $k \neq i, j$ with the orthogonal vector $m_i - m_j$.

In other words, SB_{ij} is the intersection of $\mathbf{S}^n$ with the internal angle bisector B_{ij} of the faces $[0, S]_i$ and $[0, S]_j$ of the simplex $[0, S]$ in $\mathbf{R}^{n+1}$.

Lemma 3.25. *Given an n-simplex* $S = (u_1, u_2, \ldots, u_{n+1})$ *in* $\mathbf{S}^n$, *the intersection of the internal angle bisectors* B_{ij} *of the simplex* $[0, S]$ *in* $\mathbf{R}^{n+1}$ *is the line through* 0 *and* I *where* I *is the incentre of* $[0, S]$.

Proof. It is clear that the incentre I of the simplex $[0, S]$ lies on each B_{ij}. By definition of B_{ij}, 0 lies on each B_{ij}. Since $d_R(0, \langle S \rangle) = |\langle u_i, m_i \rangle| \neq 0$ but $d_R(0, F_i) = 0$ since $0 \in F_i$ where F_i is the plane spanned by $[0, S]_i$ of the simplex $[0, S]$. We have that $0 \notin B_{0i}$. So $0 \neq I$. Since the intersection of the internal angle bisectors B_{ij} is the line, it must be the line which passes through 0 and I. #

Thoerem 2.27. *Given an n-simplex* $S = (u_1, u_2, \ldots, u_{n+1})$ *in* $\mathbf{S}^n$, *the internal angle bisectors* SB_{ij} *meet at the two points* $\pm I_S$ *where* $I_S = \dfrac{I}{|I|}$ *and* I *is the incentre of the* $(n+1)$*-simplex* $[0, S]$ *in* $\mathbf{R}^{n+1}$.

Proof. Let I be the incentre of the simplex $[0, S]$ in $\mathbf{R}^{n+1}$. Then

$$\bigcap_{i \neq j} SB_{ij} = \bigcap_{i \neq j} (B_{ij} \cap \mathbf{S}^n)$$

$$= (\text{the line through } 0 \text{ and } I) \cap \mathbf{S}^n$$

$$= \{\text{two points } \pm \frac{I}{|I|}\}.$$

Hence the internal angle bisectors SB_{ij} meet at the two points $\pm I_S$. #

CHAPTER IV

CHARACTERIZATION THEOREMS

Throughout this chapter, we shall use the following notations;

$$T = \text{ the } n\text{-simplex } [a_0, a_1, \ldots, a_n]$$

$$[T, a_{n+1}] = \text{ the } (n+1)\text{-simplex } [a_0, a_1, \ldots, a_n, a_{n+1}]$$

$$T_i = \text{ the } (n-1)\text{-simplex } [a_0, a_1, \ldots, \hat{a}_i, \ldots, a_n]$$

$$R^k = \text{ the } (n-k)\text{-simplex } [a_k, a_{k+1}, \ldots, a_n]$$

$$[T, a_{n+1}]_0 = \text{ the } n\text{-simplex } [a_1, a_2, \ldots, a_n, a_{n+1}].$$

In chapter I, we showed that for any simplex T in $\mathbf{R}^N$, the centroid of T is unique. We can let g be the function from the collection of simplices in $\mathbf{R}^N$ to $\mathbf{R}^N$ which is defined by $g(T) = G$ where G is the centroid of T.

The centroid of an n-simplex has the property that for any $(n+1)$-simplex, the lines from each vertex to the centroid of the opposite face all meet. We shall show that this property characterizes the centroid.

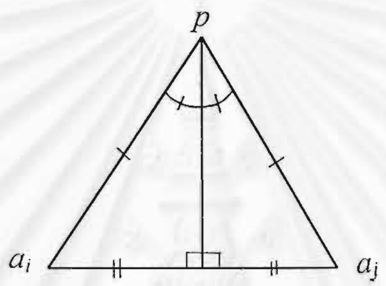
We begin with the definitions of a regular simplex and a regular function.

Definition 4.1. Let T be an n-simplex in $\mathbf{R}^N$. T is *regular* if there exists $d \in \mathbf{R}$ such that $|a_i - a_k| = d$ for all i, k with $i \neq k$.

Theorem 4.2. *If T is a regular simplex, then* $G = O = I = H = J$.

Proof. We claim that $M_{ij} = P_{ij}$. Since $d_R(a_k, a_i) = d_R(a_k, a_j)$ for all $k \neq i, j$, P_{ij} is the $(n-1)$-plane in $\langle T \rangle$ which passes through the midpoint of $[a_i, a_j]$ and the other vertices a_k, $k \neq i, j$ so $M_{ij} = P_{ij}$. Hence $O = G$ and $H = O = G$.

Next, we shall show that $P_{ij} = B_{ij}$. Fix i and j. Since $d_R(a_k, a_i) = d_R(a_k, a_j)$ for all $k \neq i, j$, we have $a_k \in B_{ij}$ for all $k \neq i, j$. Given $x \in \left\langle T_{ij} \right\rangle$, let B be the plane in $\langle T \rangle$ which passes through x and which is perpendicular to $\langle T_{ij} \rangle$. Then $\theta(T_i, T_j)$ is equal to the angle at x of the triangle $[a_i, x, a_j]$. For all $y \in \langle T_{ij} \rangle$, we have that $d_R(y, a_i) = d_R(y, a_j)$. Let p be the point in $\langle T_{ij} \rangle$ such that $d_R(p, a_i) = d_R(p, a_j) = \inf\{ d_R(y, a_i) \mid y \in \langle T_{ij} \rangle\}$ The plane which passes through p, a_i and a_j is perpendicular to $\langle T_{ij} \rangle$ and $\theta(T_i, T_j)$ is equal to the angle at p of the triangle $[a_i, p, a_j]$. We consider the triangle $[a_i, p, a_j]$.



Then the midpoint of $[a_i, a_j]$ lies on the angle bisector B_{ij}. Thus $P_{ij} = B_{ij}$. Hence $I = O = G = J = H$. #

Definition 4.3. An *n-regular function* is a function f: { n-simplices in $\mathbf{R}^N$}$\to\mathbf{R}^N$ such that $f(T) \in \langle T \rangle$ for every n-simplex T and $f(T) = g(T)$ for every regular n-simplex T.

Remark 4.4. If f: { n-simplices in $\mathbf{R}^N$}$\to\mathbf{R}^N$ is invariant under isometries, then f must be n-regular.

Definition 4.5. Given an $(n+1)$-simplex $T = [a_0, a_1, \ldots, a_n, a_{n+1}]$, given an n-regular function f, and given an index i, let $M_i(f, T)$ denote the line through the point a_i and the point $f(T_i)$.

Theorem 4.6. *If f is an n-regular function with the property that for any $(n+1)$-simplex T the lines $M_i(f, T)$ meet, then $f = g$.*

Proof. Let f be an n-regular function with the stated property. Let P(m) be the statement "If $T = [a_0, a_1, \ldots, a_n]$, is an n-simplex such that the $(n-m)$-simplex $R^m = [a_m, a_{m+1}, \ldots, a_n]$ is regular, then $f(T)$ lies on the m-plane in $\langle T \rangle$ which passes through the point $g(T_m)$ and the points a_j with $j \in \{0, 1, \ldots, m-1\}$". We claim that P($m$) is true for $m = 0, 1, \ldots, n-1$. We will prove this by induction.

For $m = 0$, we have $R^m = T$. If R^m is a regular n-simplex, then T is regular and $f(T) = g(T)$ since f is n-regular.

Next, suppose that P(k) is true. Let T be an n-simplex such that R^{k+1} is regular. Choose a point a_{n+1} such that $a_{n+1} \notin \langle T \rangle$ and such that the $(n-k)$-simplex $[R^{k+1}, a_{n+1}] = [a_m, a_{m+1}, \ldots, a_n, a_{n+1}]$ is regular .

Since $[T, a_{n+1}]_0$ is an n-simplex such that $[R^{k+1}, a_{n+1}]$ is regular, by the induction hypothesis, $f([T, a_{n+1}]_0)$ lies on the k-plane P in $\langle [T, a_{n+1}]_0 \rangle$ which passes through the point $g([R^{k+1}, a_{n+1}])$ and the points a_j with $j \in \{1, 2, \ldots, k\}$.

If $a_0 \in P$, then there exist constants α_j with $j \in \{1, 2, \ldots, k\}$ such that $a_0 - a_1 = \sum_{j=2}^{k} \alpha_j (a_j - a_1) + \alpha_1 \left(g([R^{k+1}, a_{n+1}]) - a_1\right)$. So $\sum_{j=2}^{k} \alpha_j (a_j - a_1) +$ $\frac{\alpha_1}{n-k+1} \left(\sum_{j=k+1}^{n+1} (a_j - a_1) \right) - (a_0 - a_1) = 0$. This is a contradiction since $\{ a_j - a_1 \mid j \neq 1 \}$ is linearly independent. Hence $a_0 \notin P$.

Let p be the point of intersection of the lines $M_i = M_i(f, [T, a_{n+1}])$.

Since $p \in M_0$, p lies on the line which passes through a_0 and the point $f([T, a_{n+1}]_0)$. Since $f([T, a_{n+1}]_0)$ lies on the k-plane P and since $a_0 \notin P$, we have that p lies on the $(k+1)$-plane Q in $\langle [T, a_{n+1}] \rangle$ which contains P and which passes through a_0. Note that Q passes through the point $g([R^{k+1}, a_{n+1}])$ and the points a_j with $j \in \{0, 1, \ldots, k\}$.

If $a_{n+1} \in Q$, then there exist constants α_j with $j \in \{0, 1, \ldots, k\}$ such that $a_{n+1} - a_0 = \sum_{j=1}^{k} \alpha_j (a_j - a_0) + \alpha_0 \left(g([R^{k+1}, a_{n+1}]) - a_0\right) = \sum_{j=1}^{k} \alpha_j (a_j - a_0) +$

$$\frac{\alpha_0}{n-k+1}\left(\sum_{j=k+1}^{n+1}(a_j-a_0)\right). \text{ So } \sum_{j=1}^{k}\alpha_j(a_j-a_0)+$$

$$\frac{\alpha_0}{n-k+1}\left(\sum_{j=k+1}^{n}(a_j-a_0)\right)+\left(\frac{\alpha_0}{n-k+1}-1\right)(a_{n+1}-a_0).$$ Since $\{a_j-a_0 \mid j\neq 0\}$

is linearly independent, $\frac{\alpha_0}{n-k+1}=0=\frac{\alpha_0}{n-k+1}-1$. This is a contradiction, so $a_{n+1}\notin Q$.

Also, p lies on the line which passes through $f(T)$ and a_{n+1}, or in other words $f(T)$ is on the line which passes through a_{n+1} and p. Since p lies on the $(k+1)$-plane Q and since $a_{n+1}\notin Q$, $f(T)$ lies on the $(k+2)$-plane K which contains Q and which passes through $g(R^{k+1})$ and a_{n+1}. Since $g([R^{k+1}, a_{n+1}])$ lies on the line which passes through a_{n+1}, we see that K is the $(k+2)$-plane which passes through the point $g(R^{k+1})$ and the points a_j with $j\in\{0, 1, \ldots, k\}\cup\{n+1\}$.

Since $f(T)\in K\cap\langle T\rangle$ and since $\dim(V(K)\cap V(T)) = \dim V(K) + \dim V(T) - \dim(V(K)+V(T)) = (k+2)+n-(n+1) = k+1$, we see that $f(T)$ lies on the $(k+1)$-plane which passes through the point $g(R^{k+1})$ and the points a_j with $j\in\{0, 1, \ldots, k\}$. Thus $P(m)$ is true for $m = 0, 1, \ldots, n-1$.

In particular, $P(n-1)$ holds. Since $[a_{n-1}, a_n]$ is automatically regular, $f(T)$ lies on the $(n-1)$-plane in $\langle T\rangle$ through the point $g([a_{n-1}, a_n])$,which is the midpoint of $[a_{n-1}, a_n]$, and through all the other vertices a_k with $k<n-1$. In other words,

$f(T)$ lies on the medial plane $M_{n-1,n}$. A similar argument shows that $f(T)$ lies on all the other medial planes also. Hence $f(T)=g(T)$ so $f=g$. #

In chapter I, we showed that for any simplex T in R^N, the circumcentre of T is unique. We can let o be the function from the collection of simplices in R^N to R^N which is defined by $o(T) = O$ where O is the circumcentre of T.

The circumcentre of an n-simplex has the property that in any $(n+1)$-simplex, the lines through the circumcentre of each face and perpendicular to the face all meet. We shall show that this property characterizes the circumcentre.

Definition 4.7. Given an $(n+1)$-simplex $T = [a_0, a_1, \ldots, a_n, a_{n+1}]$, given an n-regular function f, and given an index i, let $P_i(f, T)$ denote the line which passes through the point $f(T_i)$ and is perpendicular to T_i.

Theorem 4.8. *If f is an n-regular function with the property that for any $(n+1)$-simplex T the lines $P_i(f, T)$ meet, then $f = o$.*

Proof. Let f be an n-regular function with the stated property.

Let P(m) be the statement "If $T = [a_0, a_1, \ldots, a_n]$, is an n-simplex such that the $(n-m)$-simplex $R^m = [a_m, a_{m+1}, \ldots, a_n]$ is regular, then $f(T)$ lies on the m-plane in $\langle T \rangle$ which passes through the point $g(R^m)$ and which is perpendicular to $\langle R^m \rangle$". We claim that P(m) is true for $m = 0, 1, \ldots, n-1$. We will prove this by induction.

For $m = 0$, we have $R^m = T$. If R^m is a regular n-simplex, then T is regular and $f(T) = g(T)$ since f is n-regular.

Next, suppose that P(k) is true. Let T be an n-simplex such that R^{k+1} is regular. Choose a point a_{n+1} such that $a_{n+1} \notin \langle T \rangle$, the $(n-k)$-simplex $[R^{k+1}, a_{n+1}]$ $= [a_m, a_{m+1}, \ldots, a_n, a_{n+1}]$ is regular and $a_{n+1} - g(R^{k+1})$ is perpendicular to $\langle T \rangle$. Since $[T, a_{n+1}]_0$ is an n-simplex such that $[R^{k+1}, a_{n+1}]$ is regular, by the induction hypothesis, $f([T, a_{n+1}]_0)$ lies on the k-plane K in $\langle [T, a_{n+1}]_0 \rangle$ which is perpendicular to $\langle [R^{k+1}, a_{n+1}] \rangle$ and passes through $g([R^{k+1}, a_{n+1}])$.

Let p be the point of intersection of the lines $P_i = P_i(f, [T, a_{n+1}])$.

Since $p \in P_0$, p lies on the line which passes through $f([T, a_{n+1}]_0)$ and is perpendicular to $\langle [T, a_{n+1}]_0 \rangle$. Since $f([T, a_{n+1}]_0)$ lies in K, p lies on the

$(k+1)$-plane Q in $\langle [T, a_{n+1}] \rangle$ which contains K and is perpendicular to $\langle [T, a_{n+1}]_0 \rangle$. Note that Q is the $(k+1)$-plane in $\langle [T, a_{n+1}] \rangle$ which passes through $g([R^{k+1}, a_{n+1}])$ and is perpendicular to $\langle [R^{k+1}, a_{n+1}] \rangle$.

Also, p lies on the line P_{n+1} which passes through $f(T)$ and is perpendicular to $\langle T \rangle$ in $\langle [T, a_{n+1}] \rangle$, or in other words, $f(T)$ lies on the line which passes through p and is perpendicular to $\langle T \rangle$ in $\langle [T, a_{n+1}] \rangle$. Since p lies on the plane Q, $f(T)$ lies on the $(k+2)$-plane R which contains Q and which is perpendicular to $\langle T \rangle$. Also, $f(T)$ lies on $\langle T \rangle$. Thus $f(T) \in M$, where $M = R \cap \langle T \rangle$.

Since R is the plane which contains Q and is perpendicular to $\langle T \rangle$, $M = \pi R$, where π is the orthogonal projection from R to $\langle T \rangle$. Since $g([R^{k+1}, a_{n+1}]) \in [a_{n+1}, g(R^{k+1})]$, $g([R^{k+1}, a_{n+1}]) = g(R^{k+1}) + t\{ a_{n+1} - g(R^{k+1}) \}$ for some $t \in \mathbf{R}$. Since $a_{n+1} - g(R^{k+1})$ is perpendicular to $\langle T \rangle$, we have $g(R^{k+1}) = \pi(g([R^{k+1}, a_{n+1}])) \in \pi R = M$.

Let V be the $(n+1)$-dimensional vector space $V = V([T, a_{n+1}])$.

Since Q is the $(k+1)$-plane which passes through $g([R^{k+1}, a_{n+1}])$ and is perpendicular to $\langle [R^{k+1}, a_{n+1}] \rangle$, for each $q \in Q$, we can write $q = g([R^{k+1}, a_{n+1}]) + v$ where $v \in V$ is a vector which is perpendicular to $\langle [R^{k+1}, a_{n+1}] \rangle$. Since R is the plane which contains Q and which is perpendicular to $\langle T \rangle$, $R = Q + \text{span}\{u\}$ where $u \in V$ is a normal vector for $\langle T \rangle$. Thus for each $r \in R$, we can write $r = q + tu$ for some $q \in Q$ and $t \in \mathbf{R}$. So $r = g([R^{k+1}, a_{n+1}]) + v + tu$ for some $t \in \mathbf{R}$. Since tu is perpendicular to $\langle T \rangle$, $\pi(r) = \pi(g([R^{k+1}, a_{n+1}])+v)$. Since $\pi(g([R^{k+1}, a_{n+1}]) = g(R^{k+1})$ and since v is perpendicular to $\langle R^{k+1} \rangle$, we see that $\pi(r) = g(R^{k+1}) + w$ for some vector $w \in$

$V(T)$ which is perpendicular to $V(R^{k+1})$. Therefore for $r \in R$, $\pi(r)$ lies on the $(k+1)$-plane in $\langle T \rangle$ which passes through $g(R^{k+1})$ and which is perpendicular to $\langle R^{k+1} \rangle$. Thus property P(k+1) holds, and by induction P(m) holds for $m = 0, 1, \ldots, n-1$.

Finally, given any n-simplex T, the 2-simplex $R^{n-1} = [a_{n-1}, a_n]$ is regular, and so by property P($n-1$) we know that $f(T)$ lies on the $(n-1)$-plane which passes through the point $g([a_{n-1}, a_n])$ and which is perpendicular to $[a_{n-1}, a_n]$. In other words $f(T)$ lies on the perpendicular bisector $P_{n-1,n}$. A similar argument shows that $f(T)$ lies on all the other perpendicular bisectors, so we must have $f(T) = o(T)$. #

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Vita

Somluck Outudee was born in August 18, 1975. She received a bachelor degree in Mathematics from the Department of Mathematics, Faculty of Science, Chiangmai University in 1996.