เซตจุดตรึงของฟังก์ชันกึ่งไม่ขยายตัวบนปริภูมิ CAT(0)

นายอนิรุทธ ผลอ่อน

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FIXED-POINT SET OF A QUASI-NONEXPANSIVE MAP ON A CAT(0) SPACE

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อนิรุทธ ผลอ่อน: เซตจุดตรึงของฟังก์ชันกึ่งไม่ขยายตัวบนปริภูมิ CAT(0)(FIXED-POINT SET OF A QUASI-NONEXPANSIVE MAP ON A CAT(0) SPACE) อ.ที่ปรึกษา : ผศ.ดร. พิเชฐ ชาวหา, 23 หน้า ISBN 974-17-6656-4

ให้ (X, ρ) เป็น ปริภูมิ $CAT(0)$ และ $f \colon X \to X$ เป็นฟังก์ชัน ต่อเนื่อง กึ่งไม่ขยายตัว ในวิทยานิพนธ์นี้ เราจะแสดงว่า เซตจุดตรึงของ f เป็นเซต $\rho-$คอนเวกซ์และหดตัว และผลที่ตามมา จะได้ว่า ถ้า เซตจุดตรึงของ f เป็นเซตที่ไม่ว่าง แล้ว เซตจุดตรึงของ f เป็นเซตโทน หรือ เซตอนันต์

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Let (X, ρ) be a CAT(0) space and $f\colon X \to X$ a continuous quasi-nonexpansive map. In this thesis, we show that the fixed point set of f is $\rho-$convex and contractible. As a consequence, if the fixed point set of f is nonempty, it is either a singleton or an infinite.

Department **Mathematics**	Student's signature...............................
Field of study **Mathematics**	Advisor's signature..............................
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CONTENTS

CHAPTER I

INTRODUCTION

When we deal with a continuous self-map of a metric space, it is usual to discuss whether the function attains a fixed point. It is not true in general that every continuous function has a fixed point. So, it is a challenging problem in fixed point theory to find necessary and sufficient conditions on a space or a map for the existence of such a fixed point. The set of all fixed points of a map, called the fixed point set, is usually denoted by $F(f)$. It is well-known that if f is continuous and X is a Hausdorff space, then $F(f)$ is always closed but we do not exactly know about other properties such as convexity and connectivity. In general, the fixed point set of a continuous self-map of a metric space need not be convex or contractible.For example, if $\imath_{S^1}$ is the identity map and S^1 is the boundary of unit open ball D^2, then $F(\imath_{S^1}) = S^1$ which is not contractible but convex (with respect to the metric defined to be the length of the shortest arc joining any two points). Also, if $f(x) = x^2$ on $\mathbb{R}$, we have $F(f) = \{0, 1\}$ which is neither convex (with respect to the Euclidean metric) nor contractible . Therefore, it is natural to ask:

Under what circumstances, $F(f)$ is convex and contractible?

For an inner product space, we can show that the fixed point set $F(f)$ of a nonexpansive, or even a quasi-nonexpansive, map is both convex and contractible. Since every nonempty closed convex subset of a Hilbert space X is always a nonexpansive retract of X, it follows that the nonempty fixed point set of a nonexpansive map of a Hilbert space is a nonexpansive retract of X.

In this thesis, we consider a larger class of metric spaces, called CAT(0), which includes Hilbert spaces and a larger class of maps, called (continuous) quasi-nonexpansive, which includes nonexpansive maps. We show that the fixed point set of a quasi-nonexpansive map on a CAT(0) space is always convex and contractible. There are three chapters in this thesis. In chapter II, we recall some basic facts used throughout this work. In chapter III, we recall the definitions and some properties of a CAT(0) space and show that the fixed point set of a quasi-nonexpansive map on this space is convex and contractible. Also, we show that every nonempty closed subset of a CAT(0) space can be realized as the fixed point set of a continuous map.

CHAPTER II

PRELIMINARIES

In this chapter, we review some notations, terminologies, and fundamental facts that will be used throughout this work.

DEFINITION 2.1. *A* **metric** *on a set* $X \neq \emptyset$ *is a map* $\rho\colon X \times X \to \mathbb{R}$ *satisfying the following properties for all* $x, y, z \in X$:

i $\rho(x, y) \geq 0$ *; equality holds if and only if* $x = y$

ii $\rho(x, y) = \rho(y, x)$

iii $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$.

The pair (X, ρ) *is called a* **metric space.**

DEFINITION 2.2. *A sequence* (x_n) *in a topological space* X *is said to* **converge** *to a point* x *of* X *if for each neighborhood* U *of* x*, there exists a positive integer* N *such that* x_i *lies in* U *for all* $i \geq N$*. In this case,* x *is called a* **limit** *of the sequence* (x_n)*, and we write* $(x_n) \to x$.

LEMMA 2.3. *In a metric space, every convergent sequence has a unique limit.*

Proof. The proof can be found in *[1]*. □

THEOREM 2.4. *Let* (X, ρ) *be a metric space and* $f\colon X \to X$*. Then,* f *is continuous if and only if for every convergent sequence* $(x_n) \to x$*, we have* $(f(x_n)) \to f(x)$.

Proof. The proof can be found in *[1]*. □

DEFINITION 2.5. *Let* X *be a set and* $f\colon X \to X$ *a map. A point* x *in* X *is said to be a* **fixed point** *of* f *provided that* $f(x) = x$.

DEFINITION 2.6. *Let X be a Hausdorff topological space and $f\colon X \to X$ a continuous map. The* **fixed point set** *of f, denoted by $F(f)$, is defined by*

$$F(f) = \{x \in X \mid f(x) = x\}.$$

The following lemma shows basic properties of notion above.

LEMMA 2.7. *Let (X, ρ) be a metric space and $f\colon X \to X$ a continuous map. Then the following statements hold :*

(1) *$F(f)$ is closed in X.*

(2) *$F(f)$ is invariant under f; i.e., $f(F(f)) \subseteq F(f)$.*

Proof. (1) If $F(f) = \emptyset$, then we are done. Assume that $F(f) \neq \emptyset$. Let (x_n) be a sequence in $F(f)$ converging to x. That is, $(x_n) \to x$. We will show that $x \in F(f)$. Since f is continuous, we have $f(x_n) \to f(x)$. Note that $f(x_n) = x_n$ for all n and $x_n \to x$ so $f(x_n)$ also converges to x. Since the limit is unique, we have $f(x) = x$ which forces $x \in F(f)$. This finishes the proof.

(2) Clear. $\square$

DEFINITION 2.8. *Let (X, ρ) be a metric space and $f\colon X \to X$ a map. Then f is called*

(1) *a* **contraction** *if there is a constant $\alpha \in [0, 1)$ such that for each $x, y \in X$,*

$$\rho(f(x), f(y)) \leq \alpha\rho(x, y);$$

(2) *a* **contractive** *if for each $x \neq y \in X$,*

$$\rho(f(x), f(y)) < \rho(x, y);$$

(3) *an* **isometry** *if for each $x, y \in X$,*

$$\rho(f(x), f(y)) = \rho(x, y);$$

(4) *a* **nonexpansive** *if for each* $x, y \in X$,

$$\rho(f(x), f(y)) \leq \rho(x, y);$$

(5) *a* **quasi-nonexpansive** *if for each* $x \in X$ *and for each fixed point* y *of* f

$$\rho(f(x), y) \leq \rho(x, y).$$

Remark 2.9. *From the definitions, we have* (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5).

Definition 2.10. *Let* (X, ρ) *be a metric space and let* A *be a nonempty subset of* X. *For each* $x \in X$, *we define the* **distance** *between* x *and* A *by*

$$\rho(x, A) = \inf\{\rho(x, a) \mid a \in A\}.$$

It is easy to show that for a fixed A, the function $\rho(x, A)$ is a continuous map of x. Moreover, for $x, y \in X$, one has the inequality

$$|\rho(x, A) - \rho(y, A)| \leq \rho(x, y).$$

Lemma 2.11. *Let* (X, ρ) *be a metric space and* A *a nonempty closed subset of* X. *Let* $x \in X$. *Then* $x \in A$ *if and only if* $\rho(x, A) = 0$.

Proof. It is clear that if $x \in A$ then $\rho(x, A) = 0$. For the converse, assume that $x \notin A$. Since A is closed, x is not a limit point of A. Then there is an open ball $B(x, \epsilon)$ of x such that

$$(B(x, \epsilon) - \{x\}) \cap A = \emptyset.$$

So, for each $a \in A$, we have $\rho(x, a) \geq \epsilon$. It follows that

$$\begin{aligned} \rho(x, A) &= \inf\{\rho(x, a) \mid a \in A\} \\ &\geq \epsilon \\ &> 0. \end{aligned}$$

This finishes the proof. $\square$

LEMMA 2.12. *(**The pasting lemma**). Let $X = A \cup B$, where A and B are closed in X. Let $f\colon A \to Y$ and $g\colon B \to Y$ be continuous. If $f(x) = g(x)$ for every $x \in A \cap B$, then the map $h\colon X \to Y$, defined by*

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B \end{cases}$$

is continuous.

Proof. The proof can be found in *[1]*. □

DEFINITION 2.13. *Let $f, g\colon X \to Y$. We say that f is **homotopic** to g if there exists a continuous function $H\colon X \times I \to Y$ such that*

- *H(-,0) = f,*
- *H(-,1) =g.*

*The map H is called a **homotopy** from f to g.*

DEFINITION 2.14. *A map $f\colon X \to Y$ is called **nullhomotopic** if f is homotopic to a constant map.*

DEFINITION 2.15. *A space X is called **contractible** if 1_X, the identity map on X, is nullhomotopic.*

CHAPTER III

FIXED POINT SET IN A CAT(0) SPACE

Let (X, ρ) be a metric space and $b \in \mathbb{R}$. A map $\phi\colon [0, b] \to X$ is a **geodesic path** of $[0, b]$ into X if

$$\rho(\phi(s), \phi(t)) = |s - t| \text{ for all } s, t \in [0, b].$$

The image of such a map will be called a **geodesic segment** or a **metric segment** from $\phi(0)$ to $\phi(b)$, denoted by $\langle \phi(0), \phi(b) \rangle$. When the geodesic from x to y is unique, its image will be denoted by $[x, y]$.

The space (X, ρ) is said to be **geodesic** if for distinct points $x, y \in X$, there is a geodesic segment from x to y. We say that X is **uniquely geodesic** if there is exactly one geodesic segment from x to y for any distinct points $x, y \in X$.

The followings are simple facts about a geodesic space X.

LEMMA 3.1. *Let $x, y \in X$ and $z, w \in \langle x, y \rangle$. Then*

(i) $0 \leq \rho(x, z) \leq \rho(x, y)$

(ii) *If $\rho(x, z) = \rho(x, w)$, then $z = w$.*

Proof. Let $\phi\colon [0, b] \to \langle x, y \rangle$ be a geodesic path such that $\phi(0) = x$ and $\phi(b) = y$ where $b \in \mathbb{R}$.

(i) Let $t \in [0, b]$ be such that $\phi(t) = z$. Then $\rho(x, z) = \rho(\phi(0), \phi(t)) = |t - 0| = t \leq b = |b - 0| = \rho(x, y)$.

(ii) Let $t, s \in [0, b]$ be such that $\phi(t) = z$ and $\phi(s) = w$. Then

$$t = |0 - t| = \rho(\phi(0), \phi(t)) = \rho(x, z) = \rho(x, w) = \rho(\phi(0), \phi(s)) = |0 - s| = s.$$

It follows that t=s which yields $z = \phi(t) = \phi(s) = w$, completing the proof. $\square$

LEMMA 3.2. *Let* $x, y \in X$, *and* $\phi\colon [0,b] \to \langle x, y\rangle$ *a geodesic path such that* $\phi(0) = x$ *and* $\phi(b) = y$ *where* $b \in \mathbb{R}$. *Then for* $t \in [0,1]$,

$$\rho(\phi(0), \phi(tb)) = t\rho(\phi(0), \phi(b)), \textit{ and}$$

$$\rho(\phi(b), \phi(tb)) = (1-t)\rho(\phi(0), \phi(b)).$$

Proof. We see that

$$\begin{aligned}\rho(\phi(0), \phi(tb)) &= |0 - tb| \\ &= t|0 - b| \\ &= t\rho(\phi(0), \phi(b))\end{aligned}$$

Similarly, we can show that $\rho(\phi(b), \phi(tb) = (1-t)\rho(\phi(0), \phi(b))$. □

PROPOSITION 3.3. *Let* $x, y \in X$. *For each* $t \in [0,1]$, *there is a unique point* $z \in \langle x, y\rangle$ *such that*

$$\rho(x, z) = t\rho(x, y) \textit{ and } \rho(y, z) = (1-t)\rho(x, y).$$

Proof. Let $t \in [0,1]$. If $x = y$, then the conclusion is obvious by letting $z = x = y$. We assume that $x \neq y$. Let $\phi\colon [0,b] \to \langle x, y\rangle$ be a geodesic path such that $\phi(0) = x$ and $\phi(b) = y$ where $b \in \mathbb{R}$. Let $z = \phi(tb)$. Then by Lemma 3.2 we have $\rho(x, z) = t\rho(x, y)$ and $\rho(y, z) = (1-t)\rho(x, y)$. The uniqueness part follows from Lemma 3.1(ii). □

The unique point z in the above proposition will be denoted by $(1-t)x \oplus ty$; i.e., $\phi(tb) = (1-t)\phi(0) \oplus t\phi(b)$ for all $t \in [0,1]$. Also, for each $x \in X$ and $t \in [0,1]$, $(1-t)x \oplus tx = x$. It is obvious that if $z \in \langle x, y\rangle$ satisfies one of the conditions in Proposition 3.3, it also satisfies the other.

REMARK 3.4. *Let* $x, y \in X$ *be such that* $x \neq y$ *and* $s, t \in [0,1]$. *Then*

(i) $(1-t)x \oplus ty = (1-s)x \oplus sy$ *if and only if* $t = s$

(ii) $(1-t)x \oplus ty = ty \oplus (1-t)x$.

LEMMA 3.5. *Let $x, y \in X$ and $\phi\colon [0, b] \to \langle x, y\rangle$ be a geodesic path such that $\phi(0) = x$ and $\phi(b) = y$ where $b \in \mathbb{R}$. Then for $s, t \in [0, 1]$,*

$$\rho((1-t)x \oplus ty, (1-s)x \oplus sy) = |t-s|\rho(x, y).$$

Proof. Note that $\phi(tb) = (1-t)x \oplus ty$ and $\phi(sb) = (1-s)x \oplus sy$. Then

$$\begin{aligned}\rho((1-t)x \oplus ty, (1-s)x \oplus sy) &= \rho(\phi(tb), \phi(sb)) \\ &= |tb - sb| \\ &= |t-s||0-b| \\ &= |t-s|\rho(x, y).\end{aligned}$$

This finishes the proof. $\square$

LEMMA 3.6. *Let $x, y \in X$ be such that $x \neq y$. Then*

(i) $\langle x, y\rangle = \{(1-t)x \oplus ty \mid t \in [0, 1]\}$.

(ii) *the map $f\colon [0, 1] \to \langle x, y\rangle$, $t \mapsto (1-t)x \oplus ty$, is continuous and bijective.*

(iii) $\rho(x, y) = \rho(x, z) + \rho(z, y)$ *for all $z \in \langle x, y\rangle$.*

(iv) *if $z \neq w \in X$ are such that $\rho(x, y) \leq \rho(z, w)$, then there is a unique $v \in \langle z, w\rangle$ such that $\rho(z, v) = \rho(x, y)$.*

Proof. (i) ($\supseteq$) It is clear by definition.

($\subseteq$) Let $z \in \langle x, y\rangle$. By Lemma 3.1 we have $0 \leq \rho(x, z) \leq \rho(x, y)$. Let $t = \dfrac{\rho(x, z)}{\rho(x, y)} \in [0, 1]$. It follows that $\rho(x, z) = t\rho(x, y)$ and $\rho(y, z) = (1-t)\rho(x, y)$. Therefore, $z = (1-t)x \oplus ty$.

(ii) From (i) and Remark 3.4, we get that f is well-defined and bijective. It remains to show that f is continuous. Let $\phi\colon [0, b] \to \langle x, y\rangle$ be a geodesic path such that $\phi(0) = x$ and $\phi(b) = y$ where $b \in \mathbb{R}$. Then for all $t \in [0, 1]$, $f(t) = \phi(tb)$. Note that the map $g\colon [0, 1] \to [0, b]$, $t \mapsto tb$, is continuous. Since g and ϕ are continuous, so is $f = \phi \circ g$.

(iii) Let $z \in \langle x, y\rangle$. Then by (i) $z = (1-t)x \oplus ty$ for some $t \in [0,1]$. Thus $\rho(x,z) = t\rho(x,y)$ and $\rho(y,z) = (1-t)\rho(x,y)$ which imply $\rho(x,z) + \rho(z,y) = t\rho(x,y) + (1-t)\rho(x,y) = \rho(x,y)$.

(iv) Assume $z \neq w \in X$ are such that $\rho(x,y) \leq \rho(z,w)$. Let $t = \dfrac{\rho(x,y)}{\rho(z,w)}$. Then $\rho(x,y) = t\rho(z,w)$. Let $v = (1-t)z \oplus tw$ so that $\rho(z,v) = t\rho(z,w) = \rho(x,y)$. The uniqueness part follows from Lemma 3.1(ii). $\square$

Lemma 3.7. *Let X be a uniquely geodesic space. Let $x, y \in X$ be such that $x \neq y$. Define*

$$L(x,y) = \{\ z \in X \mid \rho(x,y) = \rho(x,z) + \rho(z,y)\ \}.$$

Then $[x,y] = L(x,y)$.

Proof. From Lemma 3.6, it is clear that $[x,y] \subseteq L(x,y)$.

Conversely, let $z \in L(x,y)$. Then $\rho(x,y) = \rho(x,z) + \rho(z,y)$. Let

$$\phi\colon [0, \rho(x,y)] \to [x,y]$$

$$\phi_1\colon [0, \rho(x,z)] \to [x,z]$$

$$\phi_2\colon [0, \rho(z,y)] \to [z,y]$$

be geodesic paths such that $\phi(0) = x, \phi(\rho(x,y)) = y$, $\phi_1(0) = x, \phi_1(\rho(x,z)) = z$, $\phi_2(0) = z$, and $\phi_2(\rho(z,y)) = y$. First, we show that $[x,z] \cap [z,y] = \{z\}$. Let $w \in [x,z] \cap [z,y]$. Then

$$\rho(x,w) + \rho(w,z) = \rho(x,z) \text{ and } \rho(z,w) + \rho(w,y) = \rho(z,y).$$

So we have

$$\begin{aligned}
\rho(x,y) &= \rho(x,z) + \rho(z,y) \\
&= \rho(x,w) + \rho(w,z) + \rho(z,w) + \rho(w,y) \\
&= \rho(x,w) + \rho(w,y) + 2\rho(z,w)
\end{aligned}$$

It follows that $\rho(z,w) = 0$ so $z = w$. Therefore $[x,z] \cap [z,y] = \{z\}$ as desired. By the pasting lemma, $\bar{\phi} = \phi_1 \cup \phi_2 \colon [0, \rho(x,y)] \to [x,y]$ is a geodesic path from x to y and $\bar{\phi}(0) = x$ and $\bar{\phi}(\rho(x,y)) = y$. Since X is uniquely geodesic, we have $\bar{\phi}([0,\rho(x,y)]) = \phi([0,\rho(x,y)]) = [x,y]$, so $z \in [x,y]$. This shows that $L(x,y) \subseteq [x,y]$ and hence $L(x,y) = [x,y]$. □

LEMMA 3.8. *Let X be a uniquely geodesic space, let $x, y \in X, x \neq y$ and f a quasi-nonexpansive map. If x and y are fixed points of f, then $f([x,y]) \subseteq [x,y]$. That is $[x,y]$ is invariant under f.*

Proof. Assume that x and y are fixed points of f. Let $z \in [x,y]$. Then $\rho(x,y) = \rho(x,z) + \rho(z,y)$. Since f is quasi-nonexpansive and x, y are its fixed points, we have

$$\begin{aligned} \rho(x,y) &\leq \rho(f(z),x) + \rho(f(z),y) \\ &\leq \rho(z,x) + \rho(z,y) \\ &= \rho(x,y). \end{aligned}$$

This forces $\rho(x,y) = \rho(x,f(z)) + \rho(f(z),y)$. By Lemma 3.7, it follows that $f(z) \in [x,y]$. Therefore, $f([x,y]) \subseteq [x,y]$. □

Next, we will define the CAT(0) space.

Let (X,ρ) be a geodesic space. A **geodesic triangle** $\Delta(x_1,x_2,x_3)$ in X consists of three geodesic segments $\langle x_1,x_2\rangle, \langle x_2,x_3\rangle, \langle x_1,x_3\rangle$ joining each pair of x_1,x_2,x_3. A **comparison triangle** for Δ in $\mathbb{R}^2$ is a triangle $\bar{\Delta}(\bar{x_1},\bar{x_2},\bar{x_3})$ such that $\rho(x_i,x_j) = |\bar{x_i} - \bar{x_j}|$ for $i,j \in \{1,2,3\}$. Let $x \in \Delta$ be a point on the geodesic $\langle x_i, x_j\rangle$. The **comparison point** of x in $\bar{\Delta}$ is a point $\bar{x} \in \langle \bar{x_i}, \bar{x_j}\rangle$ such that $\rho(x_i,x) = |\bar{x_i} - \bar{x}|$.

DEFINITION 3.9. *We say that a geodesic metric space (X,ρ) is a* **CAT(0) space** *if each geodesic triangle Δ in X satisfies the following property:*

Let $\bar{\Delta} \subset \mathbb{R}^2$ be a comparison triangle for Δ. Then for $x, y \in \Delta$ and their comparison points $\bar{x}, \bar{y} \in \bar{\Delta}$, we have

$$\rho(x, y) \leq |\bar{x} - \bar{y}|.$$

Example 3.10. *The following spaces are CAT(0):*

(i) *Euclidean space, $\mathbb{R}^n$*

(ii) *Hyperbolic spaces, $\mathbb{H}^n$.*

(iii) *$\mathbb{R}$-Trees.*

Proof. The proof can be founded in *[2]*. $\square$

Theorem 3.11. *A CAT(0) space is a uniquely geodesic space.*

Proof. Let (X, ρ) be a CAT(0) space, $x \neq y \in X$ and $\phi_1, \phi_2 \colon [0, \rho(x, y)] \to [x, y]$ geodesic paths from x to y such that $\phi_1(0) = x = \phi_2(0)$ and $\phi_1(\rho(x, y)) = y = \phi_2(\rho(x, y))$. We will show that $\phi_1([0, \rho(x, y)]) = \phi_2([0, \rho(x, y)])$. Let $p \in \phi_1([0, \rho(x, y)])$ and $q \in \phi_2([0, \rho(x, y)])$ be such that $\rho(x, p) = \rho(x, q)$. Then $\Delta(x, p, y)$ is a geodesic triangle in X. Let $\bar{\Delta}(\bar{x}, \bar{p}, \bar{y})$ be a comparison triangle for Δ in $\mathbb{R}^2$. Note that $\rho(x, p) = |\bar{x} - \bar{p}|$ and $\rho(x, q) = |\bar{x} - \bar{q}|$ so $|\bar{x} - \bar{p}| = |\bar{x} - \bar{q}|$. Since $|\bar{y} - \bar{p}| = |\bar{y} - \bar{q}|$, it follows that $\bar{p} = \bar{q}$. Since $\rho(p, q) \leq |\bar{p} - \bar{q}| = 0$, we obtain $p = q$. Therefore, $\phi_1([0, \rho(x, y)]) = \phi_2([0, \rho(x, y)])$. $\square$

Lemma 3.12. *Let (X, ρ) be a CAT(0) space. Then*

$$\rho((1-t)x \oplus ty, (1-t)x \oplus tz) \leq t\rho(y, z)$$

for all $x, y, z \in X$ and $t \in [0, 1]$.

Proof. Let $x, y, z \in X$ and $t \in [0,1]$. Let $\Delta(x, y, z)$ be a geodesic triangle in X and $\bar{\Delta}(\bar{x}, \bar{y}, \bar{z})$ a comparison triangle for Δ in $\mathbb{R}^2$. Since X is CAT(0), we have

$$\begin{aligned}\rho((1-t)x \oplus ty, (1-t)x \oplus tz) &\le |(1-t)\bar{x} + t\bar{y} - (1-t)\bar{x} - t\bar{z}| \\ &= t|\bar{y} - \bar{z}| \\ &= t\rho(y, z).\end{aligned}$$

□

Definition 3.13. *Let X be a geodesic space. A set $K \subseteq X$ is called $\rho-$* **convex** *if for all $x, y \in K$, there is a geodesic path $\phi\colon [0, b] \to \langle x, y\rangle$ such that $\langle x, y\rangle \subseteq K$.*

Theorem 3.14. *Let X be a CAT(0) space and $K \subseteq X$ a nonempty $\rho-$ convex subset of X. Then K is contractible.*

Proof. Let X be a CAT(0) space and $K \subseteq X$ a nonempty $\rho-$convex subset of X. We will show that K is contractible. First, fix $x_0 \in K$. Define $H\colon K \times I \to K$ by

$$H(x, t) = (1-t)x \oplus tx_0 \text{ for all } x \in K, t \in I.$$

We will show that H is continuous. Let $(x, t) \in K \times I$. Given $\epsilon > 0$. Let $\delta = \frac{\epsilon}{\rho(x_0, x)+1} > 0$. Let $(y, s) \in B_d((x, t), \delta)$, where

$$d((x, t), (y, s)) = \max\{\rho(x, y), |t - s|\}$$

is a metric on $K \times I$. By Lemma 3.5 and Lemma 3.12, we have

$$\begin{aligned}\rho(H(x, t), H(y, s)) &= \rho((1-t)x \oplus tx_0, (1-s)y \oplus sx_0) \\ &\le \rho((1-t)x \oplus tx_0, (1-s)x \oplus sx_0) \\ &\quad + \rho((1-s)x \oplus sx_0, (1-s)y \oplus sx_0) \\ &\le |t - s|\rho(x, x_0) + \rho(x, y)\end{aligned}$$

$$< \delta\rho(x_0, x) + \delta$$

$$= \delta(\rho(x_0, x) + 1)$$

$$= \epsilon$$

Thus H is continuous. Also,

$$H(x, 0) = 1x \oplus 0x_0 = x \text{ and}$$

$$H(x, 1) = 0x \oplus 1x_0 = x_0.$$

It follows that K is contractible. This completes the proof. □

THEOREM 3.15. *Let X be a CAT(0) space and f a quasi-nonexpansive map. Then $F(f)$ is $\rho-$convex, and hence contractible. In a word, if $F(f)$ is nonempty , then $F(f)$ is either a singleton or an infinite.*

Proof. Let X be a CAT(0) space and f a quasi-nonexpansive map. Let $x, y \in F(f)$ and $z \in [x, y]$. We will show that $f(z) = z$. Since $z \in [x, y]$, it follows that $\rho(x, y) = \rho(x, z) + \rho(z, y)$. Since $[x, y]$ is invariant under f, $f(z) \in [x, y]$ which implies that $\rho(x, y) = \rho(x, f(z)) + \rho(f(z), y)$. Since f is quasi-nonexpansive, we have

$$\rho(x, z) = \rho(x, f(z)) \text{ and } \rho(z, y) = \rho(f(z), y).$$

By Lemma 3.1(ii), we have $z = f(z)$ which implies that z is a fixed point. Consequently, $[x, y] \subseteq F(f)$. Hence, $F(f)$ is $\rho-$convex and hence contractible. This completes the proof. □

It is necessary that a map f be quasi-nonexpansive. For example, consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = x^3 \text{ for all } x \in \mathbb{R}.$$

It is obvious that f is not quasi-nonexpansive and $F(f) = \{0, 1, -1\}$ is not $\rho-$convex and contractible.

The followings are examples that assert the theorem above.

EXAMPLE 3.16. *Let* $X = D^2$*. Define* $f\colon X \to X$ *by*

$$f(z) = \bar{z}.$$

It is clear that X *is a CAT(0) space and* f *is a quasi-nonexpansive. Also,* $F(f) = [-1, 1]$ *which is convex and contractible.*

EXAMPLE 3.17. *Let* $X = \mathbb{C}$*. Define* $f\colon X \to X$ *by*

$$f(z) = |z|.$$

It is clear that X *is a CAT(0) space and* f *is a quasi-nonexpansive map. Also,* $F(f) = [0, \infty)$*. Then we see that* $F(f)$ *is convex and contractible.*

EXAMPLE 3.18. *Let* $X = D^n$*, the unit disk in* $\mathbb{R}^n$*. Fix* $i \in \{1, 2, \cdots, n\}$ *and let* e_i *be the* i^{th} *standard basis element of* $\mathbb{R}^n$*. We define* $f\colon X \to X$ *by*

$$f(x) = \frac{x + (1 - \|x\|)e_i}{2}.$$

Clearly, f *is quasi-nonexpansive. We can easily see that* $F(f) = \{\frac{e_i}{2}\}$*. Therefore, it follows that* $F(f)$ *is convex and contractible.*

EXAMPLE 3.19. *Let* X *be the real Hilbert space* $\mathbb{R}^2$ *under the usual Euclidean inner product. If* $x = (a, b) \in X$ *we define* $x^{\perp} \in X$ *to be* $(b, -a)$*. Trivially, we have* $\|x^{\perp}\| = \|x\|$ *for all* $x, y \in X$*. We take our closed and bounded convex set* K *to be the closed unit ball in* X *and put* $K_1 = \{x \in X : \|x\| \leq \frac{1}{2}\}$ *and* $K_2 = \{x \in X : \frac{1}{2} \leq \|x\| \leq 1\}$*. We define the map* $f\colon K \to K$ *as follows :*

$$f(x) = \begin{cases} \frac{1}{4}(x + x^{\perp}), & \text{if } x \in K_1 \\ \frac{1}{4}(\frac{x}{\|x\|} - x + x^{\perp}), & \text{if } x \in K_2. \end{cases}$$

We notice that, for $x \in K_1 \cap K_2$, *the two possible expressions for* $f(x)$ *coincide and that* f *is continuous on both* K_1 *and* K_2. *Hence* f *is continuous on* K. *We now show that* f *is quasi-nonexpansive. Note that* $F(f) = \{(0,0)\}$ *which is convex and contractible. To show that* f *is quasi-nonexpansive, let* $x \in K$. *If* $x \in K_1$, *we have*

$$\|f(x) - 0\| = \|f(x)\| = \|\frac{1}{4}(x + x^{\perp})\| \leq \frac{1}{4}(\|x\| + \|x^{\perp}\|) = \frac{2}{4}\|x\| \leq \|x - 0\|.$$

If $x \in K_2$, *we have*

$$\begin{aligned} \|f(x) - 0\| &= \|f(x)\| \\ &= \|\frac{1}{4}(\frac{x}{\|x\|} - x + x^{\perp})\| \\ &\leq \frac{1}{4}(\frac{1}{\|x\|}\|x\| + \|x\| + \|x^{\perp}\|) \\ &\leq \frac{1}{4}(2\|x\| + \|x\| + \|x\|) \\ &= \|x - 0\|. \end{aligned}$$

It follows that f *is quasi-nonexpansive as desired.*

EXAMPLE 3.20. *Let* $f\colon \mathbb{R} \to \mathbb{R}$ *be defined by*

$$f(x) = \frac{1}{2}\ln(1 + e^x) \text{ for all } x \in \mathbb{R}.$$

We will show that f *is quasi-nonexpansive and* $F(f)$ *is convex and contractible. Notice that for all* $x \in \mathbb{R}$, *if* $f(x) = x$, *then* $\frac{1}{2}\ln(1 + e^x) = x$ *so* $\ln(1 + e^x) = 2x$. *By computing, we have* $x = \ln(\frac{1+\sqrt{5}}{2})$. *Therefore,* $F(f) = \{\ln(\frac{1+\sqrt{5}}{2})\}$ *which is convex and contractible. It remains to show that* f *is quasi-nonexpansive. First, we claim that for all* $x, y \in \mathbb{R}$ *with* $y \leq x$,

$$\frac{1 + e^x}{1 + e^y} \leq \frac{e^x}{e^y}.$$

Let $x, y \in \mathbb{R}$ *be such that* $y \leq x$. *Then* $e^y \leq e^x$. *Thus,*

$$e^y(1 + e^x) = e^y + e^{x+y} \leq e^x + e^{x+y} = e^x(1 + e^y).$$

It follows that

$$\frac{1+e^x}{1+e^y} \leq \frac{e^x}{e^y}.$$

Next, let $y \in \mathbb{R}$ *and* $x = \ln(\frac{1+\sqrt{5}}{2})$ *the fixed point of* f. *Then*

$$\begin{aligned}
|f(y) - x| &= |\frac{1}{2}\ln(1+e^y) - \frac{1}{2}\ln(1+e^x)| \\
&= \frac{1}{2}|\ln(1+e^y) - \ln(1+e^x)| \\
&= \frac{1}{2}|\ln(\frac{1+e^x}{1+e^y})| \\
&\leq \frac{1}{2}|\ln(\frac{e^x}{e^y})| \\
&\leq \frac{1}{2}|x-y| \\
&\leq |y-x|.
\end{aligned}$$

Therefore, f *is quasi-nonexpansive.*

Remark 3.21. *There is no quasi-nonexpansive map on* D^2 *such that* $F(f) = S^1$.

Remark 3.22. *Every polynomial* $f\colon \mathbb{C} \to \mathbb{C}$ *of degree* n *where* $n \geq 2$ *is not a quasi-nonexpansive map.*

Theorem 3.23. *Every CAT(0) space is contractible.*

Proof. Let X be a CAT(0) space. The identity map $\imath_X$ on X is quasi-nonexpansive and $F(\imath_X) = X$. By Theorem 3.6, $F(\imath_X)$ is contractible, so X is also contractible. □

From Theorem 3.23, we obtain that every noncontractible space is not a CAT(0) space. Thus, for example, S^1 and $\mathbb{R}^2 \setminus \{0\}$ are not CAT(0) spaces.

Definition 3.24. *A map* $f\colon X \to X$ *is said to be* **eventually quasi-nonexpansive** *if there exists an integer* $N \in \mathbb{N}$ *such that for each* $n \geq N$, f^n *is quasi-nonexpansive.*

Example 3.25. *Every nonexpansive map is eventually quasi-nonexpansive.*

EXAMPLE 3.26. *Let* $f\colon \mathbb{R} \to \mathbb{R}$ *be defined by*

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ -2x & \text{if } x \geq 0 \end{cases}$$

We see that for all $n \geq 2$, $f^n(x) = 0$ *is clearly quasi-nonexpansive so* f *is eventually quasi-nonexpansive.*

EXAMPLE 3.27. *Let* $g\colon \mathbb{R} \to \mathbb{R}$ *be defined by*

$$g(x) = \begin{cases} -2x-1 & \text{if } x < -1 \\ -x & \text{if } -1 \leq x < 0 \\ x & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x > 1. \end{cases}$$

We see that

$$g^n(x) = \begin{cases} 1 & \text{if } x < -1 \\ -x & \text{if } -1 \leq x < 0 \\ x & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x > 1 \end{cases}$$

for all $n \geq 2$. *It is obvious that* g^n *is quasi-nonexpansive for all* $n \geq 2$. *So* g *is eventually quasi-nonexpansive.*

Next, we will show the interesting fact for the fixed point of an eventually quasi-nonexpansive map.

THEOREM 3.28. *Let* (X, ρ) *be a CAT(0) space. If* $f\colon X \to X$ *is an eventually quasi-nonexpansive map, then* $F(f)$ *is* $\rho-$ *convex.*

Proof. Since f is eventually quasi-nonexpansive, there is $N \in \mathbb{N}$ such that f^n is quasi-nonexpansive for all $n \geq N$. First, claim that $F(f) = F(f^N) \cap F(f^{N+1})$. It is clear that $F(f) \subseteq F(f^N) \cap F(f^{N+1})$ since for each $x \in F(f)$ we have $f(x) = x$ so $f^N(x) = x$ and $f^{N+1}(x) = x$. Conversely, let $x \in F(f^N) \cap F(f^{N+1})$. Then $f^N(x) = x$ and $f^{N+1}(x) = x$ so

$$x = f^{N+1}(x) = f(f^N(x)) = f(x)$$

which implies that $x \in F(f)$. Therefore, the claim is proved. Next, we will show that $F(f)$ is $\rho-$convex. Let $x, y \in F(f)$. Then $x, y \in F(f^N)$ and $x, y \in F(f^{N+1})$. Since f^N and f^{N+1} are quasi-nonexpansive so by Theorem 3.15, we have $[x, y] \subseteq F(f^N)$ and $[x, y] \subseteq F(f^{N+1})$. This yields

$$[x, y] \subseteq F(f^N) \cap F(f^{N+1}) = F(f).$$

Therefore, $F(f)$ is $\rho-$convex. □

THEOREM 3.29. *Let X be a CAT(0) space. If $f \colon X \to X$ is an eventually quasi-nonexpansive map, then $F(f)$ is contractible.*

Proof. It follows directly from Theorem 3.14. □

EXAMPLE 3.30. *Let $X = \mathbb{R}$ and f, g be maps in Example 3.26 and Example 3.27, respectively. Note that both f and g are not quasi-nonexpansive but eventually quasi-nonexpansive. Moreover, we see that $F(f) = \{0\}$ and $F(g) = [0, 1]$. It is clear that $F(f)$ and $F(g)$ are convex and contractible.*

Next, we will show that every nonempty closed subset K of a CAT(0) space X is the fixed point set of a continuous self map T on X.

THEOREM 3.31. *Let X be a CAT(0) space and K a nonempty closed subset of X. Then there exists a continuous map $T \colon X \to X$ such that $F(T) = K$.*

Proof. Let K be a nonempty closed subset of X. First note that for each $x \in X$, we have $\rho(x, K) < \infty$. So we let $k_x = \frac{\rho(x,K)}{1+\rho(x,K)}$. Note that for each $x, y \in X$,

$$\begin{aligned}
|k_x - k_y| &= |\frac{\rho(x,K)}{1+\rho(x,K)} - \frac{\rho(y,K)}{1+\rho(y,K)}| \\
&= |1 - \frac{1}{1+\rho(x,K)} - 1 + \frac{1}{1+\rho(y,K)}| \\
&= |\frac{1}{1+\rho(y,K)} - \frac{1}{1+\rho(x,K)}| \\
&= |\frac{\rho(x,K) - \rho(y,K)}{(1+\rho(y,K))(1+\rho(x,K))}| \\
&\leq |\rho(x,K) - \rho(y,K)|.
\end{aligned}$$

Now, fix $x_0 \in K$ and define $T\colon X \to X$ by

$$T(x) = (1-k_x)x \oplus k_x x_0 \text{ for all } x \in X.$$

We first show that T is continuous on X. Let $x \in X$. Let $\epsilon > 0$. Choose $\delta = \frac{\epsilon}{\rho(x,x_0)+1}$. Let $y \in B_\rho(x, \delta)$. By Lemma 3.5, Lemma 3.12 and the above note, we have

$$\begin{aligned}
\rho(T(x), T(y)) &= \rho((1-k_x)x \oplus k_x x_0, (1-k_y)y \oplus k_y x_0) \\
&\leq \rho((1-k_x)x \oplus k_x x_0, (1-k_y)x \oplus k_y x_0) \\
&\quad + \rho((1-k_y)x \oplus k_y x_0, (1-k_y)y \oplus k_y x_0) \\
&\leq |k_x - k_y|\rho(x, x_0) + \rho(x, y) \\
&\leq |\rho(x,K) - \rho(y,K)|\rho(x_0, x) + \rho(x, y) \\
&\leq \rho(x, y)(\rho(x_0, x) + 1) \\
&< \delta(\rho(x_0, x) + 1) \\
&= \epsilon.
\end{aligned}$$

This shows that T is continuous.

Also, we see that $T(x) = x$ if and only if $(1-k_x)x \oplus k_x x_0 = x$ if and only if $k_x = 0$ if and only if $\frac{\rho(x,K)}{1+\rho(x,K)} = 0$ if and only if $\rho(x,K) = 0$ if and only if $x \in K$. Thus, $F(f) = K$, completing the proof. $\square$

The following is an example of a continuous map and closed set in Theorem 3.31.

EXAMPLE 3.32. *Let $X = [0,1]$ and $K = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ a closed subset of X. By Theorem 3.31, define $T \colon X \to X$ by*

$$T(x) = \frac{x}{1+\rho(x,K)} \text{ for all } x \in X.$$

For each $n \in \mathbb{N} \cup \{0\}$, let $T_0(0) = 0$ and $T_n \colon [\frac{1}{n+1}, \frac{1}{n}] \to [\frac{1}{n+1}, \frac{1}{n}]$ be defined by

$$T_n(x) = \begin{cases} \frac{x}{1+\frac{1}{n}-x} & ; \frac{2n+1}{2n(n+1)} \leq x \leq \frac{1}{n} \\ \frac{x}{1-\frac{1}{n+1}+x} & ; \frac{1}{n+1} \leq x \leq \frac{2n+1}{2n(n+1)}. \end{cases}$$

We see that $F(T_0) = \{0\}$ and $F(T_n) = \{\frac{1}{n+1}, \frac{1}{n}\}$. It follows that $T = \bigcup_{n=0}^{\infty} T_n$ and $F(T) = \bigcup_{n=0}^{\infty} F(T_n)$. It is obvious that T is continuous and $F(T) = K$.

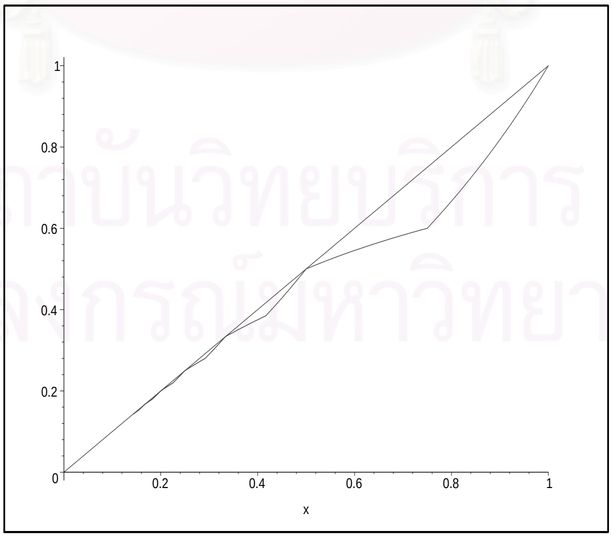

FIGURE 1. $f(x) = x, T(x) = \frac{x}{1+\rho(x,E)}$

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