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GRADIENT ESTIMATE FOR HARMONIC FUNCTIONS ON POINCARE DISC

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ฟังก์ชันฮาร์มอนิกบนแผ่นวงกลมปวงกาเรเป็นฟังก์ชัน C^2 ที่สอดคล้องกับ $\Delta_g u = 0$ เกรเดียนต์ของ บนแผ่นวงกลมของปวงกาเรเป็นสนามเวกตอร์ที่นิยามโดย $\nabla_g u$ ในวิทยานิพนธ์นี้ $\left|\nabla_g u\right|$ ถูกประมาณโดยข้อมูลเชิงปริมาณของ u บนวงกลมจีออเดสิกรัศมี r

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The harmonic function on the Poincare disc is a C^2 function satisfying $\Delta_g u = 0$, where Δ_g is the Laplace-Beltrami operator. The gradient on the Poincare disc of u is $\nabla_g u$. In this work, $|\nabla_g u|$ is to be estimated in terms of given qualitative data of u on a geodesic ball of radius r.

Department **Mathematics** Student's signature...............................

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CONTENTS

CHAPTER I

INTRODUCTION

A harmonic function on a Riemannian manifold can be viewed as a geralization of the classical notion of a harmonic function on a Euclidean space.
A **harmonic function** on $\mathbb{R}^2$ is a C^2 function $u : \mathbb{R}^2 \to \mathbb{R}$ which satisfies

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

It follows that $|\nabla u(0)| \leq Cr^{-1} \sup\limits_{B(0,r)} |u|$ on the ball $B(0, r)$ where C is independent of u and r.

This gradient estimate has many applications. One of them is the Liouville theorem which says that there is no non-constant bounded harmonic function in $\mathbb{R}^2$. A harmonic function on an n-dimensional Riemannian manifold is the generalization of a harmonic function on $\mathbb{R}^n$ by using the following notion. A **harmonic function** on a smooth n-dimensional Riemannian manifold M equipped with a metric (g_{ij}) is a differentiable function $u : M \to \mathbb{R}$ such that locally

$$\Delta_g f = \frac{1}{\sqrt{g}} \sum_{j=1}^{n} \frac{\partial}{\partial x^j} \left((\sqrt{g} \sum_{j=1}^{n} g^{jk} \frac{\partial f}{\partial x^k}) \right) = 0,$$

where

$$\Delta_g = \frac{1}{\sqrt{g}} \sum_{j=1}^{n} \frac{\partial}{\partial x^j} \left((\sqrt{g} \sum_{j=1}^{n} g^{jk} \frac{\partial}{\partial x^k}) \right)$$

is the **Laplace-Beltrami operator** on the n-dimensional Riemannian manifold. In any smooth Riemannian manifold, we also have the notion of gradient which will be clarified in the next chapter. In 1975, S.T.Yau gave a generalization on a certain smooth Riemannian manifold as the following. For more details see [1].

Theorem 1.1. *(S.T.Yau) Let M be a smooth Riemannian manifold equipped with metric (g_{ij}) such that M has a non-negative Ricci curvature and x a fixed point in M. If $\Delta_g u = 0$ on $B_r(x)$, then*

$$\|\nabla u(x)\| \leq Cr^{-1} \|u\|_{\infty},$$

where $\|u\|_{\infty}$ is the supremum norm of the function u on $B_r(x)$ and C depends only on the dimension of M.

In fact, he also extended Theorem 1.1 to a Riemannian manifold with Ricci curvature bounded from below (see [1], [2]).

Theorem 1.2. *(S.T.Yau) Let M be a smooth Riemannian manifold equipped with metric (g_{ij}) such that $\mathrm{Ric}_M \geq -A$, where $A \geq 0$ is a non-negative constant and x a fixed point in M. If u is positive and satisfies $\Delta_g u = 0$ on $B_r(x)$, then*

$$\frac{\|\nabla_g u\|}{u} \leq C\left(\frac{1 + r\sqrt{A}}{r}\right),$$

on where $B_{\frac{r}{2}}(x)$ and C depends only on the dimension of M.

In our work, we will establish the spacial case of Theorem 1.2 by using more elementary method. In particular, we will obtain the gradient estimate the of the harmonic function u on the 2-dimensional Poincare disc $\mathbb{D}_P$. The

Poincare disc $\mathbb{D}_P$ is the set

$$\mathbb{D}_P = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$$

equipped with Poincare metric

$$(g_{ij}(x, y)) = \frac{4}{(1 - (x^2 + y^2))^2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The Laplace-Beltrami operator and the gradient can be computed as

$$\Delta_g = \frac{(1 - (x^2 + y^2))^2}{4} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

and

$$\nabla_g = \frac{(1 - (x^2 + y^2))^2}{4} \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right),$$

respectively.

For the motivation, we consider the case which our 2-dimensional Riemannian manifold is just a 2-dimensional Euclidean space $B_1(0) \subseteq \mathbb{R}^2$.

Define the cutoff function $\eta(x) = 1 - |x|^2$, so that $|\nabla \eta| \leq 2$ and $\Delta \eta = -4$. We compute that

$$\begin{aligned} \Delta(\eta^2 |\nabla u|^2) &\geq -4 |\nabla u|^2 - 16\eta |\nabla u| |\mathrm{Hess}_u| + 2\eta^2 |\mathrm{Hess}_u|^2 \\ &\geq -(40) |\nabla u|^2, \end{aligned}$$

where the last inequality used the A.M.-G.M. inequality $16ab \leq 2a^2 + 32b^2$. In particular, the function $w = 20u^2 + \eta^2 |\nabla u|^2$ is superharmonic on $B_1(0)$ (i.e.,$\Delta w \geq 0$) By the maximum principle, the maximum of w occurs on the boundary so that

$$|\nabla u|^2 (0) \leq w(0) \leq \max_{\partial B_1(0)} w = 20 \max_{\partial B_1(0)} u.$$

Therefore, we easily have the desired gradient estimate of harmonic function.

In general, the difficulties arise from the fact that it is not easy to find the cutoff function η in the Poincare disc as above. Thus, we cannot mimic this proof in the Poincare disc. Our main result is a simpler proof of S.T.Yau's gradient estimate on the Poincare disc. Our approach is to use the normal coordinate to simplify the expressions of Δ_g and ∇_g. In the process we apply the Bochner-Weitzenbock identity to get a certain equality. Finally, we are able to construct a cutoff function and use the maximum principle to obtain the gradient estimate as following.

Theorem 1.3. *Let u be a positive harmonic function on the Poincare disc and $B_r(x)$ is a geodesic ball. Then*

$$\frac{\|\nabla_g u\|}{u} \leq C'\left(\frac{1+r}{r}\right) \quad on \quad B_{\frac{r}{2}}(x)$$

where C' is a constant independent of x and r.

CHAPTER II

PRELIMINARIES

2.1 ABSTRACT SURFACES

The concept of an embedded surface in $\mathbb{R}^3$ is concrete. It can be viewed as a set of $(x, y, z) \subseteq \mathbb{R}^3$, where z represents the real valued function $f(x, y)$, such as a surface of a paraboloid $z = x^2 + y^2 + c$. To further study the geometry of surfaces in $\mathbb{R}^3$, we need the concept of the coordinate chart which will enable us to define more general surfaces. For more details, see [3], [4], [5].

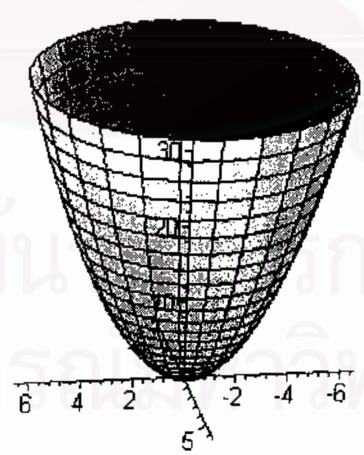

Figure 2.1: paraboloid

Definition 2.1. A smooth **abstract surface** (or smooth **two-dimensional manifold**) is a set M equipped with a collection of one-to-one functions called **coordinate charts** or patches

$$\mathcal{A} = \{x_\alpha : (U_\alpha \subseteq \mathbb{R}^2) \to M \; ; \; \alpha \in A\}$$

such that

(1) U_α is an open subset of $\mathbb{R}^2$,

(2) $\bigcup_\alpha x_\alpha(U_\alpha) = M$,

(3) If α and β are in A and $x_\alpha(U_\alpha) \cap x_\beta(U_\beta) = V_{\alpha\beta} \neq \varnothing$, then the composite

$$x_\alpha^{-1} \circ x_\beta : x_\beta^{-1}(V_{\alpha\beta}) \to x_\alpha^{-1}(V_{\alpha\beta})$$

is a smooth mapping (called a **transition function**) between open sets of $\mathbb{R}^2$. The collection $\mathcal{A}$ generates a maximal set called an **atlas** of charts on M. That is, if $x : U \to M$ is another chart such that $x_\alpha^{-1} \circ x$ and $x^{-1} \circ x_\alpha$ are smooth for all $\alpha \in A$, then x is in the collection generated by $\mathcal{A}$. The atlas generated by $\mathcal{A}$ is called a **differentiable structure** on M.

Examples

(1) The simplest surface is $\mathbb{R}^2$ with the identity chart.

(2) Let $\mathbb{R}P^2$ denote the set of lines through the origin in $\mathbb{R}^3$. A set of algebraic coordinates may be defined for $\mathbb{R}P^2$ by taking the equivalence classes of 3-tuples, $(x, y, z) \sim (rx, ry, rz)$ whenever $r \neq 0$. In each equivalence class there are two representatives satisfying $x^2+y^2+z^2 = 1$. If we take a coordinate chart for S^2 such that $x_\alpha(U_\alpha) \cap -x_\alpha(U_\alpha) = \varnothing$, then this defines a coordinate

chart on $\mathbb{R}P^2$ by identifying lines with their representatives on S^2. The surface $\mathbb{R}P^2$ with the atlas generated by these charts is called the real projective plane.

(3) The unit sphere S^2 is a smooth surface. One method to cover S^2 with coordinate patches is to use the six coordinate patches

$$x^1, x^2, y^1, y^2, z^1, z^2 : \mathbb{D}^2 \longrightarrow S^2$$

where

$$\mathbb{D}^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$$

given by

$$x^1(\begin{pmatrix} s \\ t \end{pmatrix}) = \begin{pmatrix} \sqrt{1-s^2-t^2} \\ s \\ t \end{pmatrix}, \quad x^2(\begin{pmatrix} s \\ t \end{pmatrix}) = \begin{pmatrix} -\sqrt{1-s^2-t^2} \\ s \\ t \end{pmatrix},$$

$$y^1(\begin{pmatrix} s \\ t \end{pmatrix}) = \begin{pmatrix} s \\ \sqrt{1-s^2-t^2} \\ t \end{pmatrix} \quad y^2(\begin{pmatrix} s \\ t \end{pmatrix}) = \begin{pmatrix} s \\ -\sqrt{1-s^2-t^2} \\ t \end{pmatrix},$$

$$z^1(\begin{pmatrix} s \\ t \end{pmatrix}) = \begin{pmatrix} s \\ t \\ \sqrt{1-s^2-t^2} \end{pmatrix} \quad z^2(\begin{pmatrix} s \\ t \end{pmatrix}) = \begin{pmatrix} s \\ t \\ -\sqrt{1-s^2-t^2} \end{pmatrix}.$$

Each of these coordinate patches covers an open hemisphere (see [6]).

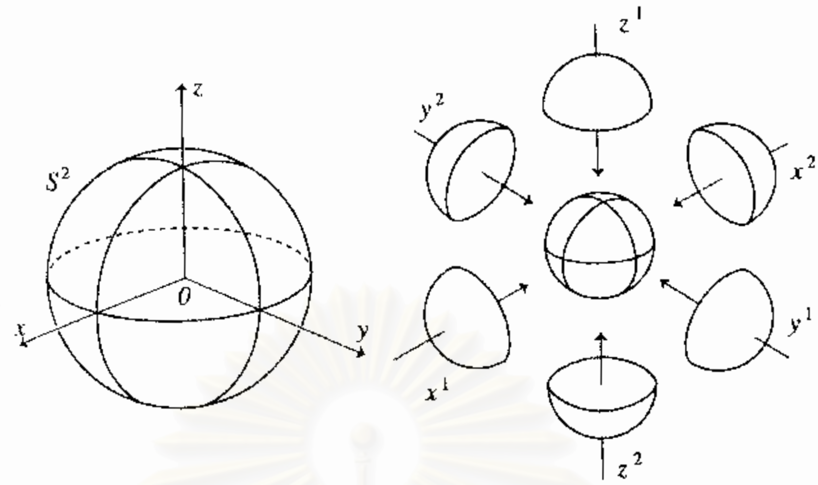

Figure 2.2: coordinate charts of sphere

Definition 2.2. Given two surfaces S and S', a function $\phi : S \to S'$ is **differentiable** at a point $p \in S$ if, for any coordinate charts $x_\alpha : (U_\alpha \subset \mathbb{R}^2) \to S$ around p and $y_\beta : (V_\beta \subset \mathbb{R}^2) \to S'$ around $f(p)$, the composite $y_\beta^{-1} \circ \phi \circ x_\alpha$ has continuous partial derivatives of all orders. A function $\phi : S \to S'$ is **differentiable** if it is differentiable at every point $p \in S$. A function $\phi : S \to S'$ is a **diffeomorphism** if ϕ is differentiable, one-to-one, and onto, and has a differentiable inverse function.

Definition 2.3. Let S be a smooth abstract surface. Given a curve $\lambda : (-\epsilon, \epsilon) \to S$, for some $\epsilon > 0$, through a point $p = \lambda(0)$ in S, define the **tangent vector** to λ at $t = 0$ as the linear mapping

$$\lambda'(0) : C^\infty(p) \to \mathbb{R}, \quad \lambda'(0)(f) = \frac{d}{dt}(f \circ \lambda(t))\Big|_{t=0}.$$

The collection of all such linear mappings for all smooth curves through p is denoted by $T_p(S)$, the **tangent space** of S at p.

Definition 2.4. A **Riemannian metric** on an abstract surface S is a choice of positive-definite inner product $\langle\ ,\ \rangle_p$ on each tangent space, $T_p(S)$ for $p \in S$, such that the choice varies smoothly from point to point.

In detail, we require $\langle\ ,\ \rangle_p$ to satisfy, for X, Y and Z in $T_p(S)$, and $r \in \mathbb{R}$,

(1) $\langle rX + Y,\ Z\rangle_p = r\,\langle X,\ Z\rangle_p + \langle Y,\ Z\rangle_p$.

(2) $\langle X,\ Y\rangle_p = \langle Y,\ X\rangle_p$.

(3) $\langle X,\ X\rangle_p \geq 0$ *and* $\langle X,\ X\rangle_p = 0$ if and only only if $X = 0$.

For a particular coordinate chart $x : (U \subset \mathbb{R}^2) \longrightarrow S$, the inner product may be represented by a symmetric matrix of smooth functions $(g_{ij}(x^1, x^2))$. If $X = \sum_i a^i \frac{\partial}{\partial x^i}$ and $Y = \sum_i b^i \frac{\partial}{\partial x^i}$ are tangent vectors at $p = x(u^1, u^2)$, then

$$\langle X, Y\rangle_p = \begin{bmatrix} a^1 & a^2 \end{bmatrix} (g_{ij}(u^1, u^2)) \begin{bmatrix} b^1 \\ b^2 \end{bmatrix}.$$

Independence of the metric on the choice of a coordinate chart requires that the functions g_{ij} form a $\binom{0}{2}$-tensor. We call $(g_{ij}) = \begin{bmatrix} E & F \\ F & G \end{bmatrix}$ the **metric tensor**, or the **first fundamental form** of S.

Definition 2.5. A differentiable **vector field** X on a smooth surface S is an association $p \longmapsto X_p \in T_pS$, for each $p \in S$, such that in every coordinate chart $x : (U_\alpha \subseteq \mathbb{R}^2) \to S$ with coordinates x^1, x^2, the coefficients $\xi^i : U \to \mathbb{R}$ in the representation (valid at a point)

$$X_p = \sum_{i=1}^{n} \xi^i(p) \frac{\partial}{\partial x^i}\Big|_p$$

are differentiable functions.

Definition 2.6. Let X, Y be (differentiable) vector fields on a smooth surface S, and let $f : S \to \mathbb{R}$ be a differentiable function. Through the relation

$$[X, Y](f) := X(Y(f)) - Y(X(f))$$

we define a vector field $[X, Y]$, which is referred to as the **Lie bracket** of X, Y (also called the **Lie derivative** $\mathcal{L}_X Y$) of Y in the direction X). At a point $p \in S$ we have $[X, Y]_p(f) = X_p(Yf) - Y_p(Xf)$.

Definition 2.7. A **Riemannian connection** ∇ on a Riemannian manifold $(S, \langle , \rangle)$ is a map

$$(X, Y) \longmapsto \nabla_X Y,$$

which associates to two given diffentiable vector fields X, Y a third differentiable vector field $\nabla_X Y$, such that the following conditions are satisfied: ($f : S \to \mathbb{R}$ denotes a differentiable function):

(i) $\nabla_{X_1+X_2} Y = \nabla_{X_1} Y + \nabla_{X_2} Y$;

(ii) $\nabla_{fX} Y = f \cdot \nabla_X Y$;

(iii) $\nabla_X (Y_1 + Y_2) = \nabla_X Y_1 + \nabla_X Y_2$;

(iv) $\nabla_X (fY) = f \cdot \nabla_X Y + (X(f)) \cdot Y$;

(v) $X(\langle Y, Z \rangle) = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle$;

(vi) $\nabla_X Y - \nabla_Y X - [X, Y] = 0$.

Remark: Instead of connection one also speaks of a **covariant derivative** and instead of Riemannian connection, one also uses the term **Levi-Civita connection**.

The metric tensor provides a full set of geometric multi-index quantities defined at each point of S:

$g = \det(g_{ij})$;

$(g^{ij}) = (g_{ij})^{-1}$; that is, $g_{ij}g^{ik} = g^{ik}g_{ij} = \delta_j^k$;

The Christoffel symbols of the first kind: $\Gamma_{ij,k} = \left\langle \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j},\, \frac{\partial}{\partial x^k} \right\rangle_p = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{kj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k} \right)$;

The Christoffel symbols of the second kind: $\Gamma_{ij}^k = \sum_{l=1}^2 g^{kl}\Gamma_{ij,l}$ (as well as $\Gamma_{ij,k} = \sum_{l=1}^2 g_{lk}\Gamma_{ij}^l$.

By using notations above, the Remannian connection can be expressed explicitly in the following. Let $X = \sum_{i=1}^2 \xi^i(x^1, x^2)\frac{\partial}{\partial x^i}$ and $Y = \sum_{j=1}^2 \eta^j(x^1, x^2)\frac{\partial}{\partial x^j}$ be vector fields. In order to determine $\nabla_X Y$, it is sufficient to know the quantities $\left\langle \nabla_X Y,\, \frac{\partial}{\partial x^k} \right\rangle_p$ for all k. From the definition of the connection, we get the equation

$$\begin{aligned}
\nabla_X Y &= \sum_{i=1}^2 \xi^i \nabla_{\frac{\partial}{\partial x^i}} Y \\
&= \sum_{i=1}^2 \xi^i \sum_{j=1}^2 \nabla_{\frac{\partial}{\partial x^i}} \left(\eta^j \frac{\partial}{\partial x^j} \right) \\
&= \sum_{i,j=1}^2 \xi^i \left(\frac{\partial \eta^j}{\partial x^i}\frac{\partial}{\partial x^j} + \eta^j \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} \right),
\end{aligned}$$

and consequently

$$\begin{aligned}
\left\langle \nabla_X Y,\, \frac{\partial}{\partial x^k} \right\rangle_p &= \sum_{i,j=1}^2 \xi^i \left(\frac{\partial \eta^j}{\partial x^i} g_{jk} + \eta^j \left\langle \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j},\, \frac{\partial}{\partial x^k} \right\rangle_p \right) \\
&= \sum_{i,j=1}^2 \xi^i \left(\frac{\partial \eta^j}{\partial x^i} g_{jk} + \eta^j \Gamma_{ij,k} \right).
\end{aligned}$$

Remark: The first fundamental form (g_{ij}) uniquely determines the Christoffel symbols and thus also the covariant derivative through the equation

$$\nabla_X Y = \sum_{i,j,k=1}^{2} \xi^i \left(\frac{\partial \eta^k}{\partial x^i} + \eta^j \Gamma^k_{ij} \right) \frac{\partial}{\partial x^k},$$

where$X = \sum_{i=1}^{2} \xi^i(x^1, x^2) \frac{\partial}{\partial x^i}$ and $Y = \sum_{j=1}^{2} \eta^j(x^1, x^2) \frac{\partial}{\partial x^j}$. For more details see [7].

Observe that if we can choose some suitable coordinates so that all Christoffel symbols vanish, the form of covariant derivative is the same as directional derivative in the Euclidean space. Practically, working on the form of directional derivative in the Euclidean space enables us arrange terms of variables in many situations easier. We will find such coordinates by using the notion of **geodesic** and **exponential map**.

The Riemannian metric determines the Riemannian connection, and this in turn determines the notion of parallelness in the same way that the covariant derivative in the Euclidean space.

Definition 2.8. 1. A vector field Y is said to be **parallel**, if $\nabla_X Y = 0$ for every X.

2. A vector field Y along a (regular) curve c is said to be **parallel along the curve** c, if $\nabla_{c'} Y = 0$ (this is independent of the parametrization).

3. A non-constant curve c is called a **geodesic**, if $\nabla_{c'} c' = \lambda c'$ for some scalar function λ. This is equivalent to the equation $\nabla_{c'} c' = 0$, provided c is parametrized by arc length.

Physical interpretation: If we view $c(t)$ as the motion of a mass particle, then the expression $\nabla_{c'}c' = c''$ is just the acceleration vector in the Euclidean space. The motions free of acceleration (the lines) are characterized by the vanishing of this expression. Similarly, on the surface the expression $\nabla_{c'}c'$ is the vector of acceleration on the surface, i.e.,the tangential component of the acceleration. In this sense the geodesics are the motions on the surface which are free of acceleration (meaning without consideration of the forces which act perpendicular to the surface). On the surface of the sphere, the geodesics are precisely the great circles. The consideration of $\nabla_{c'}c'$ requires only the knowledge of the first fundamental form. From this it is clear that geodesics are quantities of the intrinsic geometry of a surface.

Definition 2.9. (Exponential mapping)

For a fixed point $p \in M$ let $c_V^{(p)}$ denote the uniquely determined geodesic through p which is parametrized by arc length in the direction of a unit vector V. In some neighborhood U of $0 \in T_pM$, the following mapping is well defined:

$$T_pM \supseteq U \ni (p, tV) \longmapsto c_V^{(p)}(t).$$

Here the parameters are chosen is such a way that $(p, 0) \mapsto p$. This map is called the **exponential map** at the point p, and it is denoted by $\exp_p : U \longmapsto M$. For variable points p one can define a mapping exp in a similar manner by the formula $\exp(q, tV) = \exp_q(tV) = c_V^q(t)$.

Lemma 2.10. *The exponential mapping* $\exp_p$, *restricted to a certain neighborhood* U *of the origin neighborhood* U *of the origin in* T_pM, *is a diffeomorphism*

$$\exp_p : U \longrightarrow \exp_p(U).$$

The inverse mapping $\exp_p^{-1}$ *thus defines a chart at* p. *The corresponding coordinates are call* ***normal coordinates*** *or* ***Riemannian normal coordinates***.

Examples

1. In $\mathbb{R}^2$ the exponential mapping is nothing but the canonical identification of the tangent space $T_p\mathbb{R}^2$ with $\mathbb{R}^2$ itself, where the origin of the tangent space is mapped to the point p. More precisely, $\exp_p(tV) = p + tV$.

2. For the unit sphere S^2 with south pole $p = (0, 0, -1)$, the exponential mapping can be expressed in the following manner using polar coordinates, writing a tangent vector as $r\cos\phi\frac{\partial}{\partial x} + r\sin\phi\frac{\partial}{\partial y}$, the exponential mapping can be written as

$$\exp_p(r, \phi) = \left(\cos\phi\cos(r - \frac{\pi}{2}), \sin\phi\cos(r - \frac{\pi}{2}), \sin(r - \frac{\pi}{2})\right).$$

The circle $r = \frac{\pi}{2}$ in the tangent plane gets mapped to the equator, while the circle $r = \pi$ maps to the north pole. At this point the exponential mapping degenerates.

Lemma 2.11. *(Normal coordinates) Let* $X_1,\ X_2$ *be an orthonormal basis in* T_pM *and let*

$$\exp_p : U \longrightarrow \exp_p(U)$$

be the diffeomorphism of the previous lemma, defined on an open neighborhood of the origin $U \subset T_pM$. The associated coordinates are the normal coordinates, and we denote by ∂_i the elements of a basis of these coordinates on M, so that in particular $\partial_i|_p = \left(D\exp_p\right)|_0(X_i)$. Then all Christoffel symbols vanish for these coordinates at the point p.

Definition 2.12. The **gradient** of f with respect to a metric g, written **grad f** is the vector determined by the relation $\langle grad\, f, X\rangle_p := \nabla f(X) = \nabla_X f$.

In local coordinates, the components f^i of the gradient result from the components $f_j = \frac{\partial f}{\partial x^j}$ by raising indices, $f^i = \sum_{j=1}^{2} f_j g^{ji}$. In standard chart of Euclidean space there is no noticeable difference between f^i and f_i, but in polar coordinates with $(g_{ij}(r,\theta)) = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix}$ and $(g^{ij}(r,\theta)) = \begin{bmatrix} 1 & 0 \\ 0 & r^{-2} \end{bmatrix}$, one has $f_r = \frac{\partial f}{\partial r}, f_\theta = \frac{\partial f}{\partial \theta}$ and similarly $f^r = f_r g^{rr}, f^\theta = f_\theta g^{\theta\theta} = f_\theta r^{-2}$.

Definition 2.13. The second covariant derivative of f is given by $\nabla^2 f = \nabla\nabla f$. Explicitly,

$$\nabla^2 f(X,Y) := (\nabla_X \nabla f)(Y) := (\nabla_X \nabla f(Y)) - \nabla f(\nabla_X Y) = (\nabla_X \nabla_Y) f - (\nabla_X Y)(f).$$

$\nabla^2 f$ is also referred to as the Hess_f or the **Hessian** of f.

for a local coordinate (x^1, x^2), we have $\nabla^2 f(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = \nabla_i \nabla_j f := \nabla_i f_j := \frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma^k_{ij} f_k$. Since all Christoffel symbols vanish in standard chart of Euclidean space,

$$(\mathrm{Hess}_f)_{ij}(x,y)) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}$$

Definition 2.14. Let M be a 2-dimensional smooth Riemannian manifold equipped with metric (g_{ij}) and f a differentiable function. **Laplace-Beltrami operator** of f is defined by

$$\Delta_g f = \frac{1}{\sqrt{g}} \sum_{j=1}^{2} \frac{\partial}{\partial x^j} \left(\sqrt{g} \sum_{j=1}^{2} g^{jk} \frac{\partial f}{\partial x^k} \right),$$

where $(g^{ij}) = (g_{ij})^{-1}$.

2.2 Curvature

In general, there are two types of curvature: **extrinsic curvature** and **intrinsic curvature**. The **extrinsic curvature** of curves in two and three-space was the first type of curvature to be studied historically, culminating in the Frenet formulas, which describe a space curve entirely in terms of its curvature, torsion, and the initial starting point and direction.

After the curvature of two and three-dimensional curves was studied, attention turned to the curvature of surfaces in $\mathbb{R}^3$. The main objects to study are mean curvature and Gaussian curvature. Mean curvature was the most important for applications at the time and was the most studied, but Gauss was the first to recognize the importance of the Gaussian curvature.

A flat plane can be wrapped around a cone or cylinder without stretching or tearing; we say that the plane can be **developed** on the cone or cylinder. In fact, any one of these three surfaces can be developed on any one of others. More generally, we say that a surface S_1 can be developed on another surface S_2 if there is a map $f : S_1 \to S_2$ that preserves distances. A distance-preserving map is called an **isometry**.

Under what considerations can one surface be developed on another? Gauss took up this question in his long paper **General Investigations of curved surfaces**(1827) and determined that the surfaces must have the same Gaussian curvature at corresponding points. The crucial step in Gauss proof is a formula that expresses the curvature function of a surface entirely in terms of the metric tensor and its derivatives. Because Gaussian curvature is **intrinsic**, it is detectable to two-dimensional inhabitants of the surface, whereas mean curvature is not detectable to someone who can't study the three-dimensional space surrounding the surface on which he resides.

Riemann and many others generalized the concept of curvature to higher dimensional underlying space such as sectional curvature, scalar curvature, the Riemann tensor, Ricci curvature.

Gaussian curvature is an intrinsic property of a space independent of the coordinate system (see [7]). Let $x : U \to S$ be a coordinate chart. The Gaussian curvature can be given entirely in terms of the first fundamental form

$$(g_{ij}) = \begin{bmatrix} E & F \\ F & G \end{bmatrix},$$

and the metric discriminant

$$g \equiv EG - F^2.$$

In terms of tthe first fundamental form, Gaussian curvature can be computed as

$$
\begin{aligned}
K =&\frac{1}{2g}\left[2\frac{\partial^2 F}{\partial x\partial y}-\frac{\partial^2 E}{\partial y^2}-\frac{\partial^2 G}{\partial x^2}\right]-\frac{G}{4g^2}\left[\frac{\partial E}{\partial x}\left(2\frac{\partial F}{\partial y}-\frac{\partial G}{\partial x}\right)-\left(\frac{\partial E}{\partial y}\right)^2\right]\\
&+\frac{F}{4g^2}\left[\frac{\partial E}{\partial x}\frac{\partial G}{\partial y}-2\frac{\partial E}{\partial y}\frac{\partial G}{\partial x}+\left(2\frac{\partial F}{\partial x}-\frac{\partial E}{\partial y}\right)\left(2\frac{\partial F}{\partial y}-\frac{\partial G}{\partial x}\right)\right]\\
&-\frac{E}{4g^2}\left[\frac{\partial G}{\partial y}\left(2\frac{\partial F}{\partial x}-\frac{\partial E}{\partial y}\right)-\left(\frac{\partial G}{\partial x}\right)^2\right].
\end{aligned}
$$

In particular, if $F = 0$ (see [8]) the Gaussian curvature takes the form

$$K = -\frac{1}{2\sqrt{EG}}\left(\frac{\partial}{\partial y}\left(\frac{E_y}{\sqrt{EG}}\right)+\frac{\partial}{\partial x}\left(\frac{E_x}{\sqrt{EG}}\right)\right).$$

2.3 The Poincare Disc

One of the most familiar examples of a surface whose geometry is usually defined intrinsically by a metric rather than by an embedding in space is the Poincare disc D. It arises in the study of classical non-Euclidean geometry, which follows all the axioms of Euclid except for the famous parallel postulate: given a line and a point not on that line, there exists precisely one line through the point that is parallel to the line. In non-Euclidean geometry this postulate is modified in two different ways, leading to two different geometries: Either assume that there is no parallel (Elliptic geometry) or assume that there are many parallels (Hyperbolic geometry).

Poincare(1854 - 1912) devised a model of hyperbolic 2-dimensional space, a conformal image of the hyperbolic plane with distance invariant under inversion. In one of his popular and philosophical writings, Science and Hypothesis 1901, he wrote of his model as an imaginary universe occupying the interior of a disc in the Euclidean plane. The inhabitants are seen by us, as observers of their world, to shrink as they approach the infinitely distant horizon, the boundary of the disc. They do not register the effect as their ruler shrinks with them. They think that they live in a normal non-Euclidean space, we see them in a Euclidean space with their dimensions behaving strangely.

Poincare disc is the set $\mathbb{D}_P = \{(x, y) \in \mathbb{R}^2 | \ x^2 + y^2 < 1\}$ equipped with Poincare metric

$$(g_{ij}(x,y)) = \begin{bmatrix} \frac{4}{(1-(x^2+y^2))^2} & 0 \\ 0 & \frac{4}{(1-(x^2+y^2))^2} \end{bmatrix}.$$

The geodesics in the model $\mathbb{D}_P$ play the role of straight lines in the geometry. Once we defined the metric in $\mathbb{D}_P$, we can show that the geodesics are the circular arcs orthogonal to the unit disc, combined with the diameters of the circle (which can be viewed as circular arcs with infinite radius). In Figure 2.3, l, m, n are examples of geodesics in the Poicare disc.

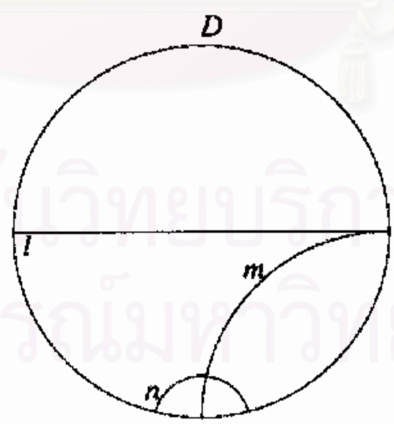

Figure 2.3: geodesics in the Poincare disc

In some situation, it is easier to deal with the geodesic polar coordinates around an arbitrary (but fixed) point (see [9]), the metric of the Poincare metric has the following form:

$$(g_{ij}(r,\theta)) = \begin{bmatrix} 1 & 0 \\ 0 & \sinh^2 r \end{bmatrix}.$$

Here r denotes the geodesic distance from a fixed point.

Now we can compute the Gaussian curvature of the Poincare disc.

Since

$$(g_{ij}(x,y)) = \begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} \dfrac{4}{(1-(x^2+y^2))^2} & 0 \\ 0 & \dfrac{4}{(1-(x^2+y^2))^2} \end{bmatrix}$$

and $F = 0$, we get

$$\begin{aligned} K &= -\frac{1}{2\sqrt{EG}}\left(\frac{\partial}{\partial y}\left(\frac{E_y}{\sqrt{EG}}\right) + \frac{\partial}{\partial x}\left(\frac{G_x}{\sqrt{EG}}\right)\right) \\ &= -(1-x^2-y^2)^2\left(\frac{8y^2}{(1-x^2-y^2)^2} + \frac{8}{1-x^2-y^2} + \frac{8x^2}{(1-x^2-y^2)^2}\right)\Big/ 8 \\ &= -1. \end{aligned}$$

Therefore, Poincare disc is a space of constant negative curvature.

2.4 The Bochner formula

On a Riemannian manifold M equipped with metric (g_{ij}), a very useful Bochner formula (see [10]) asserts that for any function f on M

$$\frac{1}{2}\Delta_g \left\|\nabla_g f\right\|^2 = \left\|\mathrm{Hess}_f\right\|^2 + \langle \Delta_g \nabla_g f, \nabla_g f\rangle + \mathrm{Ric}_M \langle \nabla f, \nabla f\rangle .$$

For the 2-dimensional case

Ric_M is just the Gaussian curvature K.

Two special cases of this formula are particularly useful. When u is a distance function, that is, when $\|\nabla_g u\| = 1$, then the above formula reduces to

$$0 = \|\mathrm{Hess}_u\|^2 + \langle \Delta_g \nabla_g u, \nabla_g u \rangle + \mathrm{Ric}_M \,.$$

This is the so-called **Ricatti equation**. The other useful special case of the Bochner formula is when u is a harmonic function. In this case, the Bochner formula reduces to

$$\frac{1}{2}\Delta_g \|\nabla_g u\|^2 = \|\mathrm{Hess}_u\|^2 + \mathrm{Ric}_M \langle \nabla u, \nabla u \rangle \,.$$

2.5 The Maximum Principle

Let Ω be, as usual, an open subset of $\mathbb{R}^2$. In this paragraph, we consider linear elliptic differential operators (see [11]) of the form

$$Lf(x) = \sum_{i,j=1}^{2} a^{ij}(x) \frac{\partial^2 f}{\partial x^i \partial x^j} + \sum_{i=1}^{2} b^i(x) \frac{\partial f}{\partial x^i}$$

which fulfill the following conditions:

(i) $a^{ij}(x) = a^{ji}(x)$ for all i, j, x,

(ii) uniform ellipticity: there are constants $0 < \lambda \le \mu < \infty$ with

$$\lambda |\xi|^2 \le \sum_{i,j=1}^{2} a^{ij}(x) \xi^i \xi^j \le \mu |\xi|^2$$

for all $x \in \Omega, \xi \in \mathbb{R}^2$,

(iii) there is a constant K such that $|b^i(x)| \le K$ for all $x \in \Omega, i \in 1, 2$.

Theorem 2.15. *Let Ω be bounded and let $f \in C^2(\Omega) \bigcap C^0(\overline{\Omega})$ satisfy $Lf \geq 0$ in Ω. Then f assumes its maximum on $\partial\Omega$, i.e.*

$$\sup_{x \in \Omega} f(x) = \max_{x \in \partial\Omega} f(x). \tag{1}$$

If $Lf \leq 0$, then the corresponding statement holds for the minimum.

Proof. We first consider the case that $Lf > 0$ in Ω. We claim that in this case, f cannot have a maximum in the interior of Ω. Namely, at an interior maximum x_0,

$$Df(x_0) = 0,$$

and

$$D^2 f(x_0) = \left((\frac{\partial^2 f}{\partial x^i \partial x^j}(x_0) \right)_{i,j=1,2}$$

is negative semi-definite. As the matrix $A = (a^{ij}(x_0))$ is, by assumption, positive definite,

$$Lf(x) = \sum_{i,j=1}^{2} a^{ij}(x) \frac{\partial^2 f}{\partial x^i \partial x^j}(x_0) = \mathrm{tr}(A \cdot D^2 f(x_0)) \leq 0$$

(tr denotes the trace of a matrix), contradicting the assumption $Lf(x_0) > 0$. Thus, in this case f cannot have maximum in the interior of Ω. We now consider for $\alpha =$ constant.

$$Le^{\alpha x^1} = (\alpha^2 a^{11}(x) + \alpha b^1(x)) e^{\alpha x^1} \leq (\lambda \alpha^2 - K\alpha) e^{\alpha x^1},$$

by (ii) and (iii). So, for sufficiently large α

$$Le^{\alpha x^1} > 0. \tag{2}$$

We now fix an α which satisfies (2). Then, for every $\epsilon > 0$

$$L(f(x) + \epsilon e^{\alpha x^1}) > 0.$$

Therefore by what has already been shown

$$\sup_{x \in \Omega}(f(x) + \epsilon e^{\alpha x^1}) = \sup_{x \in \Omega}(f(x) + \epsilon e^{\alpha x^1}).$$

Now (1) follows by letting $\epsilon \to 0$. $\square$

Definition 2.16. $f \in C^2(\Omega)$ is called a **subsolution** of $Lf = 0$ if $Lf \leq 0$, and a **supersolution** of $Lf = 0$, if $Lf \geq 0$ in Ω. A subsolution, respectively supersolution, of $\triangle f = 0$ in Ω is called **subhamonic** and **superharmonic**, respectively.

CHAPTER III

GRADIENT ESTIMATE

A further application of the maximum principle is a gradient estimate for solutions of the Laplace's equation in the Euclidean space. Gradient estimates have played a key role in both geometry and PDE. These are probably the most fundamental apriori estimates for elliptic and parabolic equations, leading to various results such as Liouville theorem.

A typical example for linear equations is the well-known gradient estimate of S.T. Yau for harmonic functions:

Theorem 3.1. *(S.T. Yau) Let M be a smooth Riemannian manifold equipped with metric (g_{ij}) such that $\mathrm{Ric}_M \geq -A$, where $A \geq 0$ is a non-negative constant and x a fixed point in M. If $\Delta_g u = 0$ on $B_r(x)$, then*

$$\frac{\|\nabla_g u\|}{u} \leq C\left(\frac{1 + r\sqrt{A}}{r}\right),$$

on $B_{\frac{r}{2}}(x)$ where C depends only on the dimension of M.

To give something of the flavor, we will use the maximum principle to prove the previous theorem on the Euclidean unit ball $B_1(0) \subset \mathbb{R}^2$. For more details, see [10].

Proof. (for $B_1(0) \subset \mathbb{R}^n$.) For convenient we let $u_1 = \frac{\partial}{\partial x}$ and $u_2 = \frac{\partial}{\partial y}$.
Define the cutoff function $\eta(x, y) = 1 - (x^2 + y^2)$, so that $\nabla\eta = (-2x, -2y)$

and $\Delta\eta = -4$.

Thus, $|\nabla\eta| = \sqrt{4(x^2+y^2)} \leq 2$.

Note that

$$\begin{aligned}
\Delta|\nabla u|^2 &= \Delta\left[(u_1)^2 + (u_2)^2\right] \\
&= \Delta(u_1)^2 + \Delta(u_2)^2 \\
&= 2[(u_{11})^2 + u_1u_{111} + (u_{12})^2 + u_1u_{122} \\
&\quad + (u_{21})^2 + u_2u_{221} + (u_{22})^2 + u_2u_{222}] \\
&= 2[(u_{11})^2 + 2(u_{12})^2 + (u_{22})^2 \\
&\quad + u_1(u_{11}+u_{22})_1 + u_2(u_{11}+u_{22})_2] \\
&= 2[(u_{11})^2 + 2(u_{12})^2 + (u_{22})^2 + 0 + 0] \\
&= 2\left|\mathrm{Hess}_u\right|^2.
\end{aligned}$$

Since $\Delta fg = f\Delta g + g\Delta f + \nabla f \cdot \nabla g$, we have

$$\begin{aligned}
\Delta\left(\eta^2|\nabla u|^2\right) &= \eta^2\Delta|\nabla u|^2 + |\nabla u|^2\Delta\eta^2 + 2\nabla\eta^2\cdot|\nabla u|^2 \\
&= \eta^2\left(2\left|\mathrm{Hess}_u\right|^2\right) + |\nabla u|^2\left(2\eta\Delta\eta + 2\nabla\eta\cdot\nabla\eta\right) + 4\eta\nabla\eta\cdot\nabla|\nabla u|^2 \\
&\geq 2\eta^2\left|\mathrm{Hess}_u\right|^2 - 8|\nabla u|^2 - 8\eta\left|\nabla|\nabla u|^2\right| \qquad (1)
\end{aligned}$$

$$\left(2\eta\Delta\eta + 2|\nabla\eta|^2 \geq 2\eta\Delta\eta \geq -8, \quad |\nabla\eta| \leq 2\right).$$

Next, we consider $\left|\nabla |\nabla u|^2\right|^2$,

$$\begin{aligned}\left|\nabla |\nabla u|^2\right| &= (2u_2u_{21} + 2u_1u_{11})^2 + (2u_2u_{22} + 2u_1u_{12})^2 \\ &= 4[u_2^2u_{21} + 2u_2u_1u_{21}u_{11} + u_1^2u_{11}^2 + u_2^2u_{22}^2 + 2u_2u_{22}u_1u_{12} + u_1^2u_{12}^2] \\ &= 4[u_2^2(u_{21}^2u_{22}^2) + u_1^2(u_{12}^2 + u_{11}^2) + 2u_2u_1u_{21}(u_{11} + u_{22})] \\ &= 4[u_2^2(u_{21}^2 + u_{22}^2) + u_1^2(u_{12}^2 + u_{11}^2) + 0] \\ &\leq 4(u_1^2 + u_2^2)(u_{11}^2 + 2u_{12}^2 + u_{22}^2).\end{aligned}$$

Thus,

$$\left|\nabla |\nabla u|^2\right| \leq 2 |\nabla u| |\text{Hess}_u| . \tag{2}$$

Substitute (2) in (1), we obtain that

$$\begin{aligned}\Delta\left(\eta^2 |\nabla u|^2\right) &\geq 2\eta^2 |\text{Hess}_u|^2 - 8 |\nabla u|^2 - 16\eta |\nabla u| |\text{Hess}_u| \\ &= -8 |\nabla u|^2 + \left(2\eta^2 |\text{Hess}_u|^2 - 16\eta |\nabla u| |\text{Hess}_u|\right) \\ &\geq -8 |\nabla u|^2 + \left(2\eta^2 |\text{Hess}_u|^2 - \left(2\eta^2 |\text{Hess}_u|^2 + 32 |\nabla u|^2\right)\right) \\ &(By \quad A.M. - G.M. \;\; inequality, \quad 16ab \leq 2a^2 + 32b^2) \\ &\geq 40 |\nabla u|^2 .\end{aligned}$$

Let $w = 20u^2 + \eta^2 |\nabla u|^2$, we get

$$\Delta w = 20(2u\Delta u + 2 |\nabla u|^2) + \Delta(\eta^2 |\nabla u|^2) \geq 40 |\nabla u|^2 - 40 |\nabla u|^2 = 0.$$

Hence, w is superharmonic. By the maximum principle, the maximum of w occurs on the boundary so that

$$|\nabla u|^2 \leq w(0) \leq 20 \max_{\partial B_1(0)} u^2.$$

Therefore, $|\nabla u| \leq \sqrt{20}\, \|u\|_\infty$ on $B_1(0)$.

Next we rescale u on $B_r(0)$.

Define $\tilde{u} : B_1(0) \to \mathbb{R}$ by

$$\tilde{u}(x,y) = u(rx, ry), \quad for \quad (x,y) \in B_1(0).$$

Then we get

$$\begin{aligned}\nabla \tilde{u}(x,y) &= \nabla\left(u(rx,ry)\right)\\ &= r\nabla u(rx,ry).\end{aligned}$$

Thus,

$$|r\nabla u(0)| = |\nabla \tilde{u}(0)| \leq \max_{\partial B_1(0)} |\tilde{u}| \leq \max_{\partial B_r(0)} |u|\,.$$

Hence we obtain the desired gradient estimate. $\square$

In general, the difficulties arise from the fact that we cannot find the cut off function η in the Poincare disc as above. Since we only work on the Poincare disc which is a 2-dimensional Riemannian manifold, we are enable to give a simpler proof of Theorem 3.1. The proof of the gradient estimate in the Poincare disc can be briefly explained as the followings:

Step 1. Use normal coordinate and Bochner formula to compute that for every point $p \in \mathbb{D}_P$

$$\|\nabla_g u(p)\|\, \Delta_g(\|\nabla_g u(p)\|) + \|\nabla_g u(p)\|^2 = \|\nabla_g(\|\nabla_g u(p)\|)\|^2\,.$$

Step 2. Define $\phi = \dfrac{|\nabla_g u|}{u} \neq 0.$

use the formula in step 2 show that $\Delta_g \phi \geq -\phi + \phi^3$.

Step 3. Use geodesic polar coordinate to verify

$$\Delta_g d(x) = \coth(d(x)).$$

Step 4.

$$\Delta_g d^2(x) \leq C(1 + d(x)),$$

where C is independent of x_0 and x

Step 5. Define cutoff function on a geodesic ball $B_r(x)$

by $\eta(y) = (r^2 - d(y)^2)$,

where d(y) is a distant function from a point x.

Step 6. Define

$$F(y) = (r^2 - d(y)^2)\phi(y) = (r^2 - d(y)^2)\frac{\|\nabla_g u(y)\|}{u(y)},$$

and use maximum principle to get the gradient estimate.

Now we are give the details of those steps in the sequence of lemmas.

Lemma 3.2. *Let u be a harmonic function on the Poincare disc equipped with metric (g_{ij}). Then for every point $p \in \mathbb{D}_P$*

$$\|\nabla_g u(p)\| \, \Delta_g(\|\nabla_g u(p)\|) + \|\nabla_g u(p)\|^2 = \|\nabla_g(\|\nabla_g u(p)\|)\|^2 .$$

Proof. Let $\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\}|_p$ be an orthonormal basis in a normal coordinate of T_pM, u_i and u_{ij} denote $\frac{\partial u}{\partial x^i}$ and $\frac{\partial^2 u}{\partial x^i \partial x^j}$, respectively, for $1 \leq i, j \leq 2$.

Using the Bochner formula we have at a point p

$$\begin{aligned}
\frac{\Delta_g(\|\nabla_g u\|^2)}{2} &= \|Hess_u\|^2 + \langle \nabla_g(\Delta_g u), \nabla_g u\rangle - \langle \nabla_g u, \nabla_g u\rangle \\
&= \sum_{i,j=1}^{2} u_{ij}^2 + 0 - \|\nabla_g u\|^2 .
\end{aligned}$$

Thus,

$$\sum_{i,j=1}^{2} u_{ij}^2 - \|\nabla_g u\|^2 . \tag{1}$$

We can choose the normal coordinates at p so that

$$u_1(p) = \|\nabla_g u(p)\| , u_2(p) = 0.$$

Then

$$\begin{aligned}
\frac{\partial}{\partial x^j} \|\nabla_g u\| &= \frac{\partial}{\partial x^j}(\sqrt{u_1^2 + u_2^2}) \\
&= \frac{u_1 u_{1j} + u_1 u_{2j}}{\sqrt{u_1^2 + u_2^2}} \\
&= \frac{u_1 u_{1j} + u_2 u_{2j}}{u_1} \\
&= u_{1j}, \quad \text{for} \quad j = 1, 2.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|\nabla_g(\|\nabla_g u\|)\|^2 &= \left\| u_{11}\frac{\partial}{\partial x^1} + u_{12}\frac{\partial}{\partial u^2} \right\|^2 \\
&= u_{11}^2 + u_{12}^2. \qquad (2)
\end{aligned}$$

We have

$$\Delta_g(\|\nabla_g u\|^2) = 2\|\nabla_g u\| \, \Delta_g(\|\nabla_g u\|) + 2\|\nabla_g(\|\nabla_g u\|)\|^2 . \tag{3}$$

Substitute (2) and (3), we have

$$\sum_{i,j=1}^{2} u_{ij}^2 - \|\nabla_g u\|^2 = \|\nabla_g u\| \, \Delta_g(\|\nabla_g u\|) + u_{11}^2 + u_{12}^2.$$

Therefore

$$\|\nabla_g u\| \, \Delta_g \|\nabla_g u\| + \|\nabla_g u\|^2 = \sum_{i,j=1}^{2} u_{ij}^2 - u_{11}^2 - u_{12}^2$$

$$= u_{11}^2 + u_{12}^2 + u_{21}^2 + u_{22}^2 - u_{11}^2 - u_{12}^2$$

$$= u_{21}^2 + u_{22}^2.$$

Since $\Delta_g u = u_{11} + u_{22} = 0, \quad u_{11}^2 = u_{22}^2$.

Using this and (2), we get from the above equality that

$$\|\nabla_g u\| \, \Delta_g(\|\nabla_g u\|) + \|\nabla_g u\|^2 = \|\nabla_g(\|\nabla_g u\|)\|^2 .$$

$\square$

Lemma 3.3. *Let u be a positive harmonic function on the Poincare disc and $\phi = \dfrac{|\nabla_g u|}{u} \neq 0$. Then $\Delta_g \phi \geq -\phi + \phi^3$.*

Proof. Note that

$$\nabla_g \phi = \frac{\nabla_g \|\nabla u\|}{u} - \frac{\|\nabla u\| \, \nabla_g u}{u^2}. \tag{1}$$

Since $\|\nabla_g u\| = \phi u$, at any point where $\nabla_g u \neq 0$ we have by Lemma 3.2

$$\begin{aligned}
\Delta_g(\|\nabla_g u\|) &= \Delta_g(\phi u) \\
&= u\Delta_g \phi + \phi \Delta_g u + 2 \langle \nabla_g \phi, \nabla_g u \rangle \\
&= u\Delta_g \phi + 2 \langle \nabla_g \phi, \nabla_g u \rangle \\
\Delta_g \phi &= \frac{\Delta_g(\|\nabla_g u\|)}{u} - \frac{2 \langle \nabla_g \phi, \nabla_g u \rangle}{u} \\
&= \frac{\|\nabla_g u\| \, \Delta_g(\|\nabla_g u\|)}{\|\nabla_g u\| \, u} - \frac{2 \langle \nabla_g \phi, \nabla_g u \rangle}{u} \\
&\geq \frac{(\|\nabla_g(\|\nabla u\|)\|^2 - \|\nabla_g u\|^2)}{\|\nabla_g u\| \, u} - \frac{2 \langle \nabla_g \phi, \nabla_g u \rangle}{u} \\
&= \frac{\|\nabla_g(\|\nabla_g u\|)\|^2}{\|\nabla_g u\| \, u} - \phi - \frac{2 \langle \nabla_g \phi, \nabla_g u \rangle}{u}
\end{aligned}$$

If follows from (1) that

$$
\begin{aligned}
\frac{2\left\langle \nabla_g \phi, \nabla_g u\right\rangle}{u} &= \frac{2\left\langle \dfrac{\nabla_g \left\|\nabla_g u\right\|}{u} - \dfrac{\left\|\nabla_g u\right\| \nabla_g u}{u^2}, \nabla_g u\right\rangle}{u} \\
&= \frac{2\left\langle \nabla_g (\left\|\nabla_g u\right\|), \nabla_g u\right\rangle}{u^2} - \frac{2\left\|\nabla_g u\right\|^3}{u^3} \\
&\leq \frac{2\left\|\nabla_g (\left\|\nabla_g u\right\|)\right\| \left\|\nabla_g u\right\|}{u^2} - 2\phi^3. \qquad (2)
\end{aligned}
$$

Condsider

$$
\begin{aligned}
\frac{2\left\|\nabla_g (\left\|\nabla_g u\right\|)\right\| \left\|\nabla_g u\right\|}{u^2} &= \frac{2\left\|\nabla_g (\left\|\nabla_g u\right\|)\right\|}{(\left\|\nabla_g u\right\| u)^{\frac{1}{2}}} \frac{\left\|\nabla_g u\right\|^{\frac{3}{2}}}{u^{\frac{3}{2}}} \\
&= 2\left(\frac{\left\|\nabla_g (\left\|\nabla_g u\right\|)\right\|^2}{\left\|\nabla_g u\right\| u}\right)^{\frac{1}{2}} \left(\frac{\left\|\nabla_g u\right\|^3}{u^3}\right)^{\frac{1}{2}} \\
&\leq \frac{\left\|\nabla_g (\left\|\nabla_g u\right\|)\right\|^2}{\left\|\nabla_g u\right\| u} + \frac{\left\|\nabla_g u\right\|^3}{u^3} \quad (\textit{by} \quad A.M - GM \quad \textit{inequality}) \\
&= \frac{\left\|\nabla_g (\left\|\nabla_g u\right\|)\right\|^2}{\left\|\nabla_g u\right\| u} + \phi^3. \qquad (3)
\end{aligned}
$$

Using (2) and (3) we get that $\Delta_g \phi \geq -\phi + \phi^3$. □

Lemma 3.4. *Let x_0 be a fix point on the 2-dimensional Poincare disc and $d(x)$ a geodesic distance function from x_0 to x.*

Then

$$\Delta_g \left(d(x)\right) = \coth(d(x))$$

Proof. On the 2-dimensional Poincare disc, the metric in the geodesic polar coordinate (r, θ) is in the from

$$(g_{ij}) = \begin{bmatrix} 1 & 0 \\ 0 & \sinh^2 r \end{bmatrix}.$$

Note that $r(x) = d(x)$.

Since

$$\Delta_g f = \frac{1}{\sqrt{g}} \sum_{j=1}^{2} \frac{\partial}{\partial x^j} \left((\sqrt{g} \sum_{j=1}^{2} g^{jk} \frac{\partial f}{\partial x^k}) \right),$$

$$\begin{aligned}
\Delta_g d(x) &= \frac{1}{\sinh r} \left[\frac{\partial}{\partial r} \left(\sinh r \frac{\partial d}{\partial r} \right) + \frac{\partial}{\partial \theta} \left(\sinh r \cdot \sinh^{-2} r \frac{\partial d}{\partial \theta} \right) \right] \\
&= \frac{1}{\sinh r} \cdot \cosh d \quad + \quad 0 \\
&= \coth d(x).
\end{aligned}$$

□

Lemma 3.5. *Let x_0 be a point in the Poincare disc and $d(x)$ a geodesic distance function from x_0 to x.*

Then

$$\Delta_g d^2(x) \leq C(1 + d(x)),$$

where C is independent of x_0 and x.

Proof. From the Lemma 3.4, on the Poincare disc we have

$$\Delta_g d(x) = \coth(d(x)).$$

It follows that

$$\begin{aligned}
\Delta_g d^2 &= 2d\Delta_g d + 2 \langle \nabla_g d, \nabla_g d \rangle \\
&= 2d\Delta_g d + 2 \|\nabla_g d\|^2 \\
&= 2d\Delta_g d + 2 \quad (\|\nabla_g d\| = 1) \\
&= 2d \coth(d) + 2.
\end{aligned}$$

Note that $\lim_{d\to 0} d\coth(d) = 1$ and $\lim_{d\to\infty} \coth(d) = 1$.

Let $m = \max_{0\le d\le 1} d\coth(d)$ and $M = \max_{1\le d<\infty} \coth(d)$.

Then for $0 \le d \le 1$,

$$\begin{aligned}\Delta_g d^2 &= 2d\coth(d) + 2\\ &\le 2m + 2\\ &\le (2m+2)(1+d),\end{aligned}$$

and for $0 \le d \le \infty$,

$$\begin{aligned}\Delta_g d^2 &= 2d\coth(d) + 2\\ &\le 2dM + 2\\ &\le (2M+2)(1+d),\end{aligned}$$

Let $C = \max\{2m+2, 2M+2\}$. Then

$$\Delta_g d^2(x) \le C(1 + d(x)),$$

where C is independent from x_0 and x. $\square$

Lemma 3.6. *Let F be the function on geodesic ball $B_r(x)$ defined by*

$$F(y) = (r^2 - d(y)^2)\phi(y) = (r^2 - d(y)^2)\frac{\|\nabla_g u(y)\|}{u(y)},$$

where u is a positive harmonic function and $d(y)$ the distance function from x on the Poincare disc.

Then

$$F(y)^2 - C_1(1+r)^2 r^2 \le 0,$$

where C_1 is independent of x and y.

Proof. Since $F|_{\partial B_r(x)} = 0$, if $\nabla_g u \not\equiv 0$ then F must achieve its maximum at some interior point $x_0 \in B_r(x)$.

Note that the Laplace-Beltrami operator and the gradient of F are

$$\Delta_g F = \frac{(1-(x^2+y^2))^2}{4}\left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}\right)$$

and

$$\nabla_g F = \frac{(1-(x^2+y^2))^2}{4}\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}\right), \text{respectively}.$$

Thus,

$$\nabla_g F(x_0) = 0. \tag{1}$$

By the Maximum principle, we get

$$\Delta_g F(x_0) \leq 0. \tag{2}$$

By (1) and (2) we have at x_0

$$\frac{\nabla_g d^2}{r^2 - d^2} = \frac{\nabla_g \phi}{\phi}, \tag{3}$$

$$\frac{-\Delta_g d^2}{r^2 - d^2} + \frac{\Delta_g \phi}{\phi} - \frac{2\left\langle \nabla_g d^2, \nabla_g \phi \right\rangle}{(r^2 - d^2)\phi} \leq 0. \tag{4}$$

It follows that

$$\frac{\Delta_g \phi}{\phi} - \frac{\Delta_g d^2}{r^2 - d^2} - \frac{2|\nabla_g d^2|^2}{(r^2 - d^2)^2} \leq 0. \tag{5}$$

Notice that $|\nabla_g d^2| = 2d|\nabla_g d| = 2d$ and by Lemma 3.5 we get

$$\Delta_g d^2 \leq C_0(1 + d),$$

where C_0 is a constant. Using these and Lemma 3.3 in (5) we obtain

$$\begin{aligned}0 &\geq \frac{\Delta_g \phi}{\phi} - \frac{C_0(1+d)}{r^2-d^2} - \frac{8d^2}{(r^2-d^2)^2} \\ &\geq -1+\phi^2 - \frac{C_0(1+d)}{r^2-d^2} - \frac{8d^2}{(r^2-d^2)^2}.\end{aligned}$$

Note that by (3) we have

$$\frac{\langle \nabla_g \phi, \nabla_g u\rangle}{\phi u} = \frac{2d \langle \nabla_g d, \nabla_g u\rangle}{(r^2-d^2)u} \leq \frac{2d\phi}{(r^2-d^2)}.$$

Substituting this into the previous inequality it and use the fact that $F = (r^2-d^2)\phi$ we get

$$\begin{aligned}0 &\geq F^2 - C_0(1+d)(r^2-d^2) - 8d^2 - (r^2-d^2)^2 \\ &\geq F^2 - C_0(1+r)r^2 - 8r^2 - r^4 \\ &\geq F^2 - C_1(1+r)^2 r^2,\end{aligned}$$

where C_1 are constant independent from x and y. □

Finally, we can prove our main result which is the gradient estimate on the Poincare disc as followings:

Theorem 3.7. *Let u be a positive harmonic function on the Poincare disc and $B_r(x)$ is a geodesic ball . Then*

$$\frac{\|\nabla_g u\|}{u} \leq C' \left(\frac{1+r}{r}\right) \quad on \quad B_{\frac{r}{2}}(x)$$

where C' is a constant independent of x and r.

Proof. From Lemma 3.6, we have at x_0,

$$\begin{aligned}0 &\geq F^2 - C_1(1+r)^2 r^2 \\ &= F^2 + BF + C\end{aligned}$$

where $B = 0,\ C = -C_1(1+r)^2r^2$.

Consider the solutions of the equation

$$X^2 + BX + C = 0,$$

we get $X = \dfrac{-B + \sqrt{B^2 - 4C}}{2}$.

Since $0 \geq F^2 + BF + C$, we have $F \leq \dfrac{-B + \sqrt{B^2 - 4C}}{2}$.

Thus,

$$\begin{aligned} F(x_0) &\leq \frac{-B + \sqrt{B^2 - 4C}}{2} \\ &= 0 + \sqrt{0 + 4C_1(1+r)^2r^2} \\ &\leq C_2 r(1+r), \quad \text{where} \quad C_2 = 2\sqrt{C_1}. \end{aligned}$$

Therefore,

$$\begin{aligned} \sup_{B_r(x)} F &= F(x_0) \\ &\leq C_2 r(1+r). \end{aligned}$$

Since in $B_{\frac{r}{2}(x)}$

$$\begin{aligned} F(y) &= (r^2 - d^2)\phi(y) \\ &= (r^2 - d^2)\frac{\|\nabla_g u\|}{u} \\ &\geq \left(r^2 - (\frac{r}{2})^2\right)\frac{\|\nabla_g u\|}{u} \\ &= \frac{3r^2}{4}\frac{\|\nabla_g u\|}{u}, \end{aligned}$$

we have that

$$\begin{aligned}
\frac{3r^2}{4} \sup_{B_{\frac{r}{2}}(x)} \frac{\|\nabla_g u\|}{u} &= \sup_{B_{\frac{r}{2}}(x)} \frac{3r^2}{4} \frac{\|\nabla_g u\|}{u} \\
&\le \sup_{B_{\frac{r}{2}}(x)} F \\
&\le F(x_0) \\
&\le C' r(1+r), \quad \text{where} \quad C' = \frac{4}{3r^2} C_2.
\end{aligned}$$

Thus $\sup\limits_{B_{\frac{r}{2}}(x)} \dfrac{\|\nabla_g u\|}{u} \le C' \left(\dfrac{1+r}{r} \right)$. □

REFERENCES

[1] Yau, S. T. Harmonic function on complete Riemannian manifolds. Comm. Pure Appl. Math. 28(1975): 201-208.

[2] Nihonyanagi, T. Geometric of Poisson equations on complete Riemannian manifolds[Online]. Available from: www.st.hirosaki-u.ac.jp/mathsci/Graduate[2004, January 15]

[3] Garmo, M. P. Riemannian geometry. Boston: Birkhauser, 1992.

[4] Boothby, W. M. An introduction to differentiable manifolds and Riemannian geometry. New York: Academic Press, INC, 1975.

[5] Kuhnel, W. Diffential geometry: curve-surface-manifold. Rhode Island: American Mathematical Society, 2002.

[6] Bloch,E. D. A first course in geometric topological and differential geometry. Boston: Birkhauser, 1996.

[7] Callahan, J. J. The geometry of spacetime: an introduction to space and general relativity. New York: Springer-Verlag, 2000.

[8] McCleary, J. Geometry from a differentiable viewpoint. New York: Cambridge University Press, 1997.

[9] Candel, A. Eigenvalue estimates for minimal surfaces in hyperbolic space[Online]. 2004. Available from: http://www.csun.edu/ac53971/research[2005, March 6]

[10] Colding, T. H., and Minicozzi II, W. P. An excursion into geometric analysis[Online]. 2003. Available from: http://arXiv.math.DG/0309021/[2004, May 21]

[11] Jost, J. Post modern analysis. Heidelberg: Springer-Verlag, 2003.

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