# CHAPTER V

# MINIMAL QUASI-HYPERIDEALS

The purpose of this chapter is to study minimal quasi-hyperideals of hyperrings in order to generalize Proposition 1.15 – Proposition 1.18.

**Theorem 5.1.** *A nonzero quasi-hyperideal $Q$ of a hyperring $A$ is a minimal quasi-hyperideal if and only if $(x)_q = Q$ for all $x \in Q\backslash\{0\}$.*

*Proof.* Let $Q$ be a nonzero quasi-hyperideal of a hyperring $A$. Suppose that $Q$ is a minimal quasi-hyperideal and let $x \in Q\backslash\{0\}$. Since $(x)_q$ is a nonzero quasi-hyperideal of $A$ contained in $Q$, by the minimality of $Q$, $(x)_q = Q$.

Conversely, assume that $(x)_q = Q$ for all $x \in Q\backslash\{0\}$. Let $Q'$ be a nonzero quasi-hyperideal of $A$ contained in $Q$. Then there exists a nonzero element in $Q'$, say $y$, so $(y)_q = Q$. Then $Q = (y)_q \subseteq Q'$ since $y \in Q'$. Hence $Q = Q'$. Therefore $Q$ is a minimal quasi-hyperideal of $A$. □

We obtain Proposition 1.15 as an immediate consequence of Theorem 5.1.

**Corollary 5.2.** *A nonzero quasi-ideal $Q$ of a ring $A$ is a minimal quasi-ideal of $A$ if and only if $(x)_q = Q$ for all $x \in Q\backslash\{0\}$.*

There is a relation among minimal quasi-hyperideals, minimal left hyperideals and minimal right hyperideals of a hyperring as follows:

**Theorem 5.3.** *The intersection of a minimal left hyperideal $L$ and a minimal right hyperideal $R$ of a hyperring $A$ is either $\{0\}$ or a minimal quasi-hyperideal of $A$.*

*Proof.* We have that $Q = L \cap R$ is a quasi-hyperideal of $A$. Assume that $Q \neq \{0\}$. We shall show that $Q$ is minimal. Suppose that there exists a quasi-hyperideal $Q'$ of $A$ such that $\{0\} \neq Q' \subsetneqq Q$. Then $Q' \subsetneqq L$. Since $< AQ' >$ is a left hyperideal of $A$ contained in $L$ and $L$ is a minimal left hyperideal, it follows that $< AQ' >= \{0\}$ or $< AQ' >= L$. If $< AQ' >= \{0\}$, then $Q'$ is a left hyperideal of $A$ such that $\{0\} \neq Q' \subsetneqq L$ which contradicts the minimality of $L$. Then $< AQ' >= L$. Similarly, one can show that $< Q'A >= R$. Hence $Q = L \cap R = < AQ' > \cap < Q'A > \subseteq Q'$, which contradicts that $Q' \subsetneqq Q$. Therefore $Q$ is a minimal quasi-hyperideal of $A$. □

We then have Proposition 1.16 as a corollary of Theorem 5.3

**Corollary 5.4.** *If $L$ and $R$ are a minimal left ideal and a minimal right ideal of a ring $A$, respectively, then either $L \cap R = \{0\}$ or $L \cap R$ is a minimal quasi-ideal of $A$.*

Necessary conditions and a partial converse for a quasi-hyperideal of a hyperring $A$ to be minimal are as follows:

**Theorem 5.5.** *Let $A$ be a hyperring.*

(i) *A minimal quasi-hyperideal $Q$ of $A$ is either a zero subhyperring or a division subhyperring. In the second case, $Q = eAe (= eA \cap Ae)$ where $e$ is the identity of $Q$.*

(ii) *If a quasi-hyperideal $Q$ of $A$ is a division hyperring, then $Q$ is a minimal quasi-hyperideal of $A$.*

*Proof.* (i) Suppose that $Q$ is a minimal quasi-hyperideal of $A$ which is not a zero hyperring. Then there exist $a, b \in Q \setminus \{0\}$ such that $ab \neq 0$ and so $Q^2 \neq \{0\}$. Since $0 \neq ab \in Ab \cap aA \subseteq < AQ > \cap < QA > \subseteq Q$, $Ab \cap aA$ is a nonzero quasi-hyperideal of $A$ contained in $Q$. But $Q$ is a minimal quasi-hyperideal of $A$, so we have $Q = Ab \cap aA$. Then there exist $r, s, t, u \in A$ such that

$$a = rb = as \quad \text{and} \quad b = tb = au.$$

Then $tba = ba = bas$, that is, $ba \in Aba \cap baA$. Since $Q = Ab \cap aA$, we have $Q \subseteq Ab$ and $Q \subseteq aA$ and so $Q^2 \subseteq AbaA$. But $Q^2 \neq \{0\}$, so $ba \neq 0$. Because $0 \neq ba = tba = bas \in Aba \cap baA \subseteq < AQ > \cap < QA > \subseteq Q$ and $Aba \cap baA$ is a quasi-hyperideal of $A$, we deduce that $Q = Aba \cap baA$. Thus there are $v, w, x, y \in A$ such that

$$a = vba = baw \quad \text{and} \quad b = xba = bay.$$

Consequently, the element $vbay$ is of the form

$$ay = vbay = v(bay) = vb. \tag{1}$$

It then follows that $ay \neq 0$ (since $0 \neq b = bay$) and $ay = vb \in Ab \cap aA = Q$. From (1), we have that $(ay)(ay) = (vb)(ay) = vbay = ay$. Let $e = ay \in Q$. Then $e \neq 0$ and $e^2 = e$, so we have $0 \neq e \in eA \cap Ae$. It is clear that $eA \cap Ae = eAe$. Thus $eA \cap Ae = eAe$ is a nonzero quasi-hyperideal of $A$ contained in $Q$, so by the minimality of $Q$, $Q = eA \cap Ae = eAe$. Consequently, $e$ is the identity of $(Q\backslash\{0\}, \cdot)$.

To show that every nonzero element in $Q$ has a left inverse element in $Q$, let $z \in A$ be such that $eze \in Q\backslash\{0\}$. By Lemma 2.3(iii), $eAe(eze)$ is a subhyperring of $A$. We have that $< (eAe(eze))A > \cap < A(eAe(eze)) > \subseteq < eA > \cap < Ae(eze) > = eA \cap Ae(eze) \subseteq eAe(eze)$, thus $eAe(eze)$ is a quasi-hyperideal of $A$. Since $0 \neq eze = (eee)eze \in eAe(eze) = Q(eze) \subseteq Q$ and $eAe(eze)$ is a quasi-hyperideal of $A$, by the minimality of $Q$, $eAe(eze) = Q$. Then $e = ez'e(eze)$ for some $z' \in A$, that is, $ez'e$ is a left inverse element of $eze$. Consequently, $(Q\backslash\{0\}, \cdot)$ is a group. Therefore $Q$ is a division subhyperring of $A$.

(ii) Let $Q'$ be a quasi-hyperideal of $A$ such that $\{0\} \neq Q' \subseteq Q$. Then $< Q'Q > \cap < QQ' > \subseteq < Q'A > \cap < AQ' > \subseteq Q'$, so $Q'$ is a quasi-hyperideal of $Q$. Since $Q$ is a division hyperring, by Theorem 2.5, $Q' = Q$. This shows that $Q$ is a minimal quasi-hyperideal of $A$. □

The following consequence is Proposition 1.17.

**Corollary 5.6.** *Let $Q$ be a quasi-ideal of a ring $A$.*

(i) *If $Q$ is a minimal quasi-ideal of $A$, then $Q$ is either a zero ring or a division subring of $A$. In the second case, $Q = eAe = Ae \cap eA$ where $e$ is the identity of $Q$.*

(ii) *If $Q$ is a division subring of $A$, then $Q$ is a minimal quasi-ideal of $A$.*

Next, a necessary and sufficient condition for a quasi-hyperideal of a hyperring to be minimal in terms of principal left hyperideals and right hyperideals is given as follows:

**Theorem 5.7.** *A quasi-hyperideal $Q$ of $A$ is minimal if and only if for any elements $x, y \in Q \backslash \{0\}$,*

$$(x)_l = (y)_l \quad \textit{and} \quad (x)_r = (y)_r.$$

*Proof.* Assume that $Q$ is a minimal quasi-hyperideal of $A$. Let $x, y \in Q\backslash\{0\}$. Then $(x)_l \cap Q$ is a quasi-hyperideal of $A$ containing $x \neq 0$ and $(x)_l \cap Q \subseteq Q$. By the minimality of $Q$, we have that $Q = (x)_l \cap Q$. This implies that $Q \subseteq (x)_l$, so $y \in (x)_l$. Hence $(y)_l \subseteq (x)_l$. By a similar argument, we obtain $(x)_l \subseteq (y)_l$ so that $(x)_l = (y)_l$. Dually, we can show that $(x)_r = (y)_r$.

Conversely, assume that $(x)_l = (y)_l$ and $(x)_r = (y)_r$ for all $x, y \in Q\backslash\{0\}$. To show that $Q$ is a minimal quasi-hyperideal of $A$, let $Q'$ be a nonzero quasi-hyperideal of $A$ contained in $Q$.

**Case 1**: $< AQ' > \cap Q = \{0\}$. Let $y \in Q'\backslash\{0\}$. Then for any $x \in Q\backslash\{0\}$,

$(x)_l = (y)_l$, so $x \in (y)_l = \mathbb{Z}y + Ay$. Thus $x \in c + ry$ for some $c \in \mathbb{Z}y$ and $r \in A$. By the reversibility of $(A, +)$, $ry \in x - c \subseteq Q$, so that $ry \in < AQ' > \cap Q = \{0\}$, that is, $ry = 0$. Then $x \in c + 0 = \{c\} \subseteq \mathbb{Z}y \subseteq Q'$. Hence $x \in Q'$, that is, $Q \subseteq Q'$.

**Case 2**: $< Q'A > \cap Q = \{0\}$. Dually to Case 1, one can prove that $Q \subseteq Q'$.

**Case 3**: $< AQ' > \cap Q \neq \{0\}$ and $< Q'A > \cap Q \neq \{0\}$. Let $q \in (< AQ' > \cap Q) \backslash \{0\}$ and $p \in (< Q'A > \cap Q) \backslash \{0\}$. Let $x \in Q \backslash \{0\}$. Then $(x)_l = (q)_l$ and $(x)_r = (p)_r$, so $x \in (q)_l$ and $x \in (p)_r$. Thus $x \in (q)_l = \mathbb{Z}q + Aq \subseteq \mathbb{Z} < AQ' > +A < AQ' >$. By Proposition 1.29 and Lemma 2.3(ii), $\mathbb{Z} < AQ' > +A < AQ' > \subseteq < AQ' > + < AQ' > = < AQ' >$. Also, $x \in (p)_r = \mathbb{Z}p + pA \subseteq \mathbb{Z} < Q'A > + < Q'A > A \subseteq < Q'A > + < Q'A > = < Q'A >$ (Proposition 1.29 and Lemma 2.3(ii)). Hence $x \in < AQ' > \cap < Q'A > \subseteq Q'$, that is, $Q \subseteq Q'$.

In any cases, we obtain $Q = Q'$. Therefore $Q$ is a minimal quasi-hyperideal of $A$. □

Proposition 1.18 is an immediate consequence of the above theorem.

**Corollary 5.8.** *A quasi-ideal $Q$ of a ring $A$ is a minimal quasi-ideal of $A$ if and only if for any two nonzero elements $x, y$ in $Q$,*

$$(x)_l = (y)_l \quad \textit{and} \quad (x)_r = (y)_r.$$