CHAPTER IV

PARTIALLY ORDERED DISTRIBUTIVE SEMINEAR-FIELDS

In this chapter, we shall classify some complete ordered distributive seminear-field up to isomorphism.

Definition 4.1. A system $(K,+,\cdot,\leqslant)$ is called an ordered distributive seminear-field of zero type if $(K,+,\cdot)$ is a distributive seminear-field of zero type and $\leqslant$ is a total order on K satisfying the following properties:

(i) For any $x,y,z \in K$, $x \leqslant y$ implies that $x + z \leqslant y + z$ and $z + x \leqslant z + y$.

(ii) For any $x,y,z \in K$, $x \leqslant y$ and $z \geqslant 0$ imply that $xz \leqslant yz$ and $zx \leqslant zy$.

(iii) $0 < 1$.

Let K be an ordered distributive seminear-field of zero type. The positive part of K, denoted by D_K^+, is $\{x \in K \mid x > 0\}$ and the negative part of K, denoted by D_K^-, is $\{x \in K \mid x < 0\}$. Note that D_K^+ is nonempty and $K = D_K^+ \cup \{0\} \cup D_K^-$. The following statements hold:

(1) For any $x \in D_K^+$, $y \in D_K^-$, $xy, yx, y^{-1} \in D_K^-$.

(2) $D_K^+ \subseteq D_K^- x$ for all $x \in D_K^-$.

Proposition 4.2. Let K be an ordered distributive seminear-field of zero type. Then D_K^+ is an ordered distributive ratio seminear-ring.

Proof: Let $x, y \in D_K^+$. Then $x + y \geq 0 + y = y > 0$, so $x + y \in D_K^+$. If $y^{-1} < 0$ then $1 = yy^{-1} < y0 = 0$, a contradiction. Hence $y^{-1} > 0$. It follows that $xy^{-1} > 0$, so $xy^{-1} \in D_K^+$. Therefore D_K^+ is an ordered distributive ratio seminear-ring. #

The following theorem has been proven in [3], page 46, and the assumption that the addition is commutative was not used in the proof.

Theorem 4.3. Let K be an ordered distributive seminear-field of zero type such that $1 + 1 \neq 1$. Then the prime distributive seminear-field of K is order isomorphic to $(\mathbb{Q}_0^+, +, \cdot, \leq)$.

The following fact will be used to prove the next theorem: Let K be a complete ordered distributive seminear-field of zero type such that $1 + 1 \neq 1$. Assume that $D_K^- \neq \emptyset$. Then for each $x \in D_K^-$, $0 = \sup \{n^{-1}x \mid n \in \mathbb{Z}^+\}$ (see [3], page 48).

Theorem 4.4. Let $(K, +, \cdot, \leq)$ be a complete ordered distributive seminear-field of zero type such that $1 + 1 \neq 1$. Assume that K is not a skew field. Then $(K, +, \cdot, \leq)$ is order isomorphic to $(\mathbb{R}_0^+, +, \cdot, \leq)$.

Proof: Since K is complete, D_K^+ is complete. Hence by Proposition 4.2, Theorem 2.39 and 4.3, $(D_K^+, +, \cdot, \leq)$ is order isomorphic to $(\mathbb{R}^+,+,\cdot,\leqslant)$. Since K is not a skew field, by Proposition 1.43, $x + y \neq 0$ for all $x,y \in K\setminus\{0\}$. We shall show that $D_K^- = \emptyset$. Suppose not.

We shall show that there exist $x_o \in D_K^-$ and $y_o \in K$ such that $x_o + y_o > 0$. Suppose not. Then $x + y < 0$ for all $x \in D_K^-$, $y \in K$. Let $x \in D_K^-$. Then $x + x(x+x) < 0$, so we have that $D_K^+ \subseteq D_K^-(x+x(x+x))$ (see page 83). Hence $1 = t(x+x(x+x)) = tx + tx(x+x) > 0$ for some $t \in D_K^-$. Therefore $tx > 0$. Thus $1 + (x+x) = (tx)^{-1} \in D_K^+$. Since $(D_K^+,+,\cdot,\leqslant)$ is order isomorphic to $(\mathbb{R}^+,+,\cdot,\leqslant)$, $(1+1)(1+x) = (1+1)+(x+x) = 1 + (1+(x+x)) > 1$. But $x + (1+x) < 0$, so $(1+1)(1+x) = (1+x)+(1+x) = 1 + (x+(1+x)) \leqslant 1 + 0 = 1$, a contradiction. Hence there exist $x_o \in D_K^-$ and $y_o \in K$ such that $x_o + y_o > 0$.

Let $A = \{y \in K \mid x_o + y > 0\}$. Then $A \neq \emptyset$. Since for each $y \in A$, $0 < x_o + y \leqslant 0 + y = y$, 0 is a lower bound of A. Since K is complete, inf(A) exists, say z. Then $z \geqslant 0$.

<u>Case 1</u>: $x_o + z > 0$. Then $z > 0$. Since $(D_K^+,+,\cdot,\leqslant)$ is order isomorphic to $(\mathbb{R}^+,+,\cdot,\leqslant)$, $r < x_o + z \leqslant 0 + z = z$ for some $r \in D_K^+$. Thus $z = t + r$ for some $t \in D_K^+$. If $x_o + t < 0$ then $x_o + z = x_o + t + r \leqslant 0 + r = r < x_o + z$, a contradiction. Hence $x_o + t > 0$. Therefore $t \in A$, so $t \geqslant z$. But $t,r \in D_K^+$, so $z = t + r > t \geqslant z$, a contradiction.

<u>Case 2</u>: $x_o + z < 0$. We shall show that there exists an $a_o \in D_K^+$ such that $x_o + z + a_o < 0$. Suppose not. Then $x_o + z + a > 0$ for all $a \in D_K^+$. Claim that $y + a > 0$ for all $y \in D_K^-$, $a \in D_K^+$. Let $y \in D_K^-$. Then $0 = \sup\{n^{-1}y \mid n \in \mathbb{Z}^+\}$ (see page 84). Since $x_o + z < 0$, there

exists an $n \in Z^+$ such that $x_o + z < n^{-1}y$. Then for each $a \in D_K^+$,

$0 < x_o + z + a \leqslant n^{-1}y + a = n^{-1}(y+na)$. Since $n^{-1}a \in D_K^+$ for all $a \in D_K^+$,

$0 < n^{-1}(y+n(n^{-1}a)) = n^{-1}(y+a)$ for all $a \in D_K^+$. Therefore $0 < y + a$

for all $a \in D_K^+$. Hence we have the claim.

Let $w \in D_K^-$. By the claim, $w^{-1} + 1 > 0$ ....(*)

Suppose that $1 + w < 0$. Then $w + (1+w) < 0$. Also, $(1+1)(w+1) =$

$(w+1)+(w+1) = (w+(1+w)) + 1 \leqslant 0 + 1 = 1$. Since $w + w < 0$, by the claim,

we have that $(w+w) + 1 > 0$. Since $(D_K^+,+,\cdot,\leqslant)$ is order isomorphic to

$(\mathbb{R}^+,+,\cdot,\leqslant)$, $1 < ((w+w)+1) + 1 = (1+1)w + (1+1) = (1+1)(w+1)$, a

contradiction. Hence $1 + w > 0$. Therefore $w^{-1} + 1 = w^{-1}(1+w) < 0$

which contradicts (*). Hence there exists an $a_o \in D_K^+$. Such that

$x_o + z + a_o < 0$. Since $z \geqslant 0$, $z + a_o > z$. But $z = \inf(A)$, so there

exists a $y \in A$ such that $y < z + a_o$. Thus $x_o + z + a_o \geqslant x_o + y > 0$, a

contradiction.

Therefore $D_K^- = \emptyset$. Thus $K = D_K^+ \cup \{0\}$. Hence $(K,+,\cdot,\leqslant)$ is

order isomorphic to $(\mathbb{R}_0^+,+,\cdot,\leqslant)$. # .

<u>Theorem 4.5</u>. Let $(K,+,\cdot,\leqslant)$ be a complete ordered distributive seminear-field of zero type. Then $(D_K^+,+,\cdot,\leqslant)$ is not order isomorphic to the following:

(1) $(\mathbb{R}^+,\min,\cdot,\leqslant)$.

(2) $(\mathbb{R}^+,+_\ell,\cdot,\leqslant)$.

(3) $(\mathbb{R}^+,+_r,\cdot,\leqslant)$.

(4) $(\{2^n \mid n \in \mathbb{Z}\},\min,\cdot,\leqslant)$.

(5) $(\{2^n \mid n \in \mathbb{Z}\},+_\ell,\cdot,\leqslant)$.

(6) $(\{2^n \mid n \in \mathbb{Z}\},+_r,\cdot,\leqslant)$.

Proof: Suppose that there exists an order isomorphism $\varphi: (R^+,\min,\cdot,<)\to(D_K^f,+,\cdot,<)$. Then $0 < \varphi(1) < \varphi(2)$. But $\varphi(2) = \varphi(2) + 0 < \varphi(2) + \varphi(1) = \varphi(2+1) = \varphi(1)$, a contradiction. Hence $(D_K^f,+,\cdot,<)$ is not order isomorphic to $(R^+,\min,\cdot,<)$.

Similarly, we can show that $(D_K^+,+,\cdot,\leqslant)$ is not order isomorphic to (2)-(6). #

Remark 4.6. Let $(K,+,\cdot,\leqslant)$ be a complete ordered distributive seminear-field of zero type such that $1 + 1 = 1$. From Theorem 2.32 and 4.5, we have that $(D_K^+,+,\cdot,\leqslant)$ is order isomorphic to exactly one of the following:

(1) $(\{1\},+,\cdot,\leqslant)$.

(2) $(R^+,\max,\cdot,\leqslant)$.

(3) $(\{2^n \mid n \in \mathbb{Z}\},\max,\cdot,\leqslant)$.

Hence D_K^+ is additively commutative.

Remark 4.7. The following are examples of complete ordered distributive seminear-field of zero type:

(1) $(R,+_1,\cdot,\leqslant)$ where $x +_1 y = \begin{cases} x & \text{if } |y| < |x|, \\ y & \text{if } |x| \leqslant |y|. \end{cases}$

(2) $(R,+_2,\cdot,\leqslant)$ where $x +_2 y = \begin{cases} x & \text{if } |y| \leqslant |x|, \\ y & \text{if } |x| < |y|. \end{cases}$

(3) $(\{2^n \mid n \in \mathbb{Z}\} \cup \{0\} \cup \{-(2^n) \mid n \in \mathbb{Z}\},+_1,\cdot,\leqslant)$.

(4) $(\{2^n \mid n \in \mathbb{Z}\} \cup \{0\} \cup \{-(2^n) \mid n \in \mathbb{Z}\},+_2,\cdot,\leqslant)$

(5) $(\{-1,0,1\},+_1,\cdot,\leqslant)$.

(6) $(\{-1,0,1\},+_2,\cdot,\leqslant)$.

Proof: (1) For any $x,y,z \in \mathbb{R}\setminus\{0\}$,

$$(x+_1y)+_1 z = x +_1 (y+_1 z) = \begin{cases} x & \text{if } |y| < |x| \text{ and } |z| < |x|, \\ y & \text{if } |x| \leqslant |y| \text{ and } |z| < |y|, \\ z & \text{if } |x| \leqslant |z| \text{ and } |y| \leqslant |z|, \end{cases}$$

$$x(y +_1 z) = xy +_1 xz = \begin{cases} xy & \text{if } |z| < |y| \\ xz & \text{if } |y| \leqslant |z| \end{cases}$$

and

$$(x+_1y)z = xz +_1 yz = \begin{cases} xz & \text{if } |y| < |x|, \\ yz & \text{if } |x| \leqslant |y|. \end{cases}$$

Hence $+_1$ is associative and $\cdot$ is distributive over $+_1$ in $\mathbb{R}$.

To show that for any $x,y,z \in \mathbb{R}\setminus\{0\}$, $x \leqslant y$ implies that $x +_1 z \leqslant y +_1 z$ and $z +_1 x \leqslant z +_1 y$, let $x,y,z \in \mathbb{R}\setminus\{0\}$ be such that $x \leqslant y$.

Case 1: $|x| \leqslant |z|$ and $|y| \leqslant |z|$. Then $x +_1 z = z = y +_1 z$.

Case 2: $|z| < |x|$ and $|z| < |y|$. Then $x +_1 z = x \leqslant y = y +_1 z$.

Case 3: $|x| \leqslant |z| < |y|$. Since $x \leqslant y$, $y > 0$. This implies that $x +_1 z = z < y = y +_1 z$.

Case 4: $|y| \leqslant |z| < |x|$. Since $x \leqslant y$, $x < 0$. This implies that $x +_1 z = x < z = y +_1 z$.

Similarly, $z +_1 x \leqslant z +_1 y$.

Therefore $(\mathbb{R},+_1,\cdot,\leqslant)$ is a complete ordered distributive seminear-field of zero type.

The proof of (2)-(6) are similar to the proof of (1). #

Let K be an ordered distributive seminear-field of zero type. For each $x \in K \setminus \{0\}$, let $LI_K^-(x) = LI_K(x) \cap D_K^-$ and $RI_K^-(x) = RI_K(x) \cap D_K^-$. Then the following statements hold:

(1) If $x \in LI_K^-(1)$ then $\{y \in D_K^- \mid y \geq x\} \subseteq LI_K^-(1)$.

(2) If $x \in RI_K^-(1)$ then $\{y \in D_K^- \mid y \geq x\} \subseteq RI_K^-(1)$.

(3) If $x \in LI_K^-(1)$ then $x^{-1} \notin RI_K^-(1)$.

Note that from (1) and (2), we have that $RI_K^-(1) \subseteq LI_K^-(1)$ or $LI_K^-(1) \subseteq RI_K^-(1)$.

We shall now classify all complete ordered distributive seminear-field of zero type such that $1 + 1 = 1$. First, we shall need a lemma.

<u>Lemma 4.8</u>. Let K be an ordered distributive seminear-field of zero type such that $x + y = \max\{x,y\}$ for all $x,y \in D_K^+$. Assume that $D_K^- \neq \emptyset$. Then the following statements hold:

(1) $1 + x = 1$ or $1 + x = x$ for all $x \in D_K^-$.

(2) $x + 1 = 1$ or $x + 1 = x$ for all $x \in D_K^-$.

(3) If $|D_K^+| > 1$ then $LI_K^-(1)$ and $RI_K^-(1)$ are nonempty proper subsets of D_K^-.

(4) If $LI_K^-(1) = RI_K^-(1)$ then K is additively commutative.

(5) If there exists an $a_o \in LI_K^-(1) \setminus RI_K^-(1)$ such that $a_o^2 = 1$ then $D_K^- = a_o D_K^+$.

Proof: (1) Let $x \in D_K^-$. Since $1 + 1 = 1$, by Remark 1.30 and Proposition 1.43, $1 + x \neq 0$. If $1 + x > 0$ then $1 = 1 + 0 \leqslant 1 + (1+x) = (1+1) + x = 1 + x \leqslant 1 + 0 = 1$, so $1 + x = 1$. If $1 + x < 0$ then $x = 0 + x \leqslant 1 + x = 1 + (x+x) = (1+x) + x \leqslant 0 + x = x$, so $1 + x = x$.

(2) The proof is similar to the proof of (1).

(3) Assume that $|D_K^+| > 1$. We shall show that $\emptyset \subset LI_K^-(1) \subset D_K^-$. Suppose not.

Case 1: $LI_K^-(1) = \emptyset$. Then for each $x \in D_K^-$, $x^{-1} \notin LI_K^-(1)$, by (2), we get that $x^{-1} + 1 = x^{-1}$ for all $x \in D_K^-$. Hence $1 + x = 1$ for all $x \in D_K^-$ which implies that $D_K^- = RI_K^-(1)$. Let $x \in D_K^+$ and $y \in D_K^-$ be such that $x > 1$. Since $x + y = \max\{x,y\}$ for all $x,y \in D_K^+$, $(1+y) + x = 1 + x = x$. Since $xy^{-1} \in D_K^-$, $1 + (y+x) = 1 + (1+xy^{-1})y = 1 + y = 1$, a contradiction.

Case 2: $LI_K^-(1) = D_K^-$. If there exists an $x \in RI_K^-(1)$ then $1 + x = 1$, so $x^{-1} + 1 = x^{-1}$, hence $x^{-1} \notin LI_K^-(1)$, a contradiction. Hence $RI_K^-(1) = \emptyset$. Let $x \in D_K^+$ and $y \in D_K^-$ be such that $x < 1$. Since $yx^{-1} \in D_K^-$, $1 + (y+x) = 1 + (yx^{-1}+1)x = 1 + x = 1$. By (1), $(1+y) + x = y + x = (yx^{-1}+1)x = x$, a contradiction.

Hence $\emptyset \subset LI_K^-(1) \subset D_K^-$. Similarly, we can show that $\emptyset \subset RI_K^-(1) \subset D_K^-$.

(4) Assume that $LI_K^-(1) = RI_K^-(1)$. Let $x \in D_K^-$. By (1), $1 + x = 1$ or $1 + x = x$. If $1 + x = 1$ then $x \in RI_K^-(1) = LI_K^-(1)$, so

$x + 1 = 1$. Assume that $1 + x = x$. Then $x \notin RI_K^-(1)$, so $x \notin LI_K^-(1)$. By (2), $x + 1 = x$. This shows that $x + 1 = 1 + x$ for all $x \in D_K^-$. Let $x,y \in K \setminus \{0\}$.

Case 1: $x,y \in D_K^+$. Then $x + y = y + x = \max\{x,y\}$.

Case 2: $x,y \in D_K^-$. Without loss of generality, assume that $x \leqslant y$. Then $x = x + x \leqslant x + y \leqslant x + 0 = x$, so $x + y = x$. Similarly, $y + x = x$. Hence $x + y = y + x$.

Case 3: $x \in D_K^+$ and $y \in D_K^-$ or $x \in D_K^-$ and $y \in D_K^+$. Then $xy^{-1} \in D_K^-$. Hence $xy^{-1} + 1 = 1 + xy^{-1}$, so $x + y = y + x$.

Therefore K is additively commutative.

(5) Assume that there exists an $a_o \in LI_K^-(1) \setminus RI_K^-(1)$ such that $a_o^2 = 1$. Then $a_o + 1 = 1$ and $1 + a_o = a_o$ by (1). Let $x \in D_K^-$. Suppose that $a_o x \in D_K^-$. Then $a_o x = (1+a_o)x = x + a_o x \leqslant x + 0 = x = (a_o+1)x = a_o x + x \leqslant a_o x + 0 = a_o x$. Therefore $a_o x = x$, so $a_o = 1$, a contradiction. Hence $a_o x \in D_K^+$. Thus $x = a_o(a_o x) \in a_o D_K^+$. This shows that $D_K^- \subseteq a_o D_K^+$. Since $a_o D_K^+ \subseteq D_K^-$, $D_K^- = a_o D_K^+$. #

Theorem 4.9. Let $(K,+,\cdot,\leqslant)$ be a complete ordered distributive seminear-field of zero type. Assume that $(D_K^+,+,\cdot,\leqslant)$ is order isomorphic to $(\mathbb{R}^+,\max,\cdot,\leqslant)$. Then exactly one of the following statements hold:

(1) K is additively commutative.

(2) $(K,+,\cdot,\leqslant)$ is order isomorphic to exactly one of the following:

2.1) $(\mathbb{R},+_1,\cdot,\leqslant)$.

2.2) $(\mathbb{R},+_2,\cdot,\leqslant)$.

<u>Proof</u>: By assumption, $x + y = \max\{x,y\}$ for all $x,y \in D_K^+$. If $D_K^- = \emptyset$ then $K = D_K^+ \cup \{0\}$ which is additively commutative. Assume that $D_K^- \neq \emptyset$. By Lemma 4.8(3), $LI_K^-(1)$ and $RI_K^-(1)$ are nonempty proper subsets of D_K^-. If $LI_K^-(1) = RI_K^-(1)$ then K is additively commutative by Lemma 4.8(4). Assume that $LI_K^-(1) \neq RI_K^-(1)$. By the note on page 89, $RI_K^-(1) \subset LI_K^-(1)$ or $LI_K^-(1) \subset RI_K^-(1)$.

<u>Case 1</u>: $RI_K^-(1) \subset LI_K^-(1)$. Let $a \in D_K^- \setminus LI_K^-(1)$. Then $a \notin RI_K^-(1)$. By Lemma 4.8(1), $1 + a = a$, so $a^{-1} + 1 = 1$. For each $x \in LI_K^-(1)$, $1 = x + 1 = x(a^{-1}+1) + 1 = (xa^{-1}+x) + 1 = xa^{-1} + (x+1) = xa^{-1} + 1$, so $a = x + a \leqslant x + 0 = x$ for all $x \in LI_K^-(1)$. Hence a is a lower bound of $LI_K^-(1)$. Let $a_o = \inf(LI_K^-(1))$. Claim that $LI_K^-(1) \setminus RI_K^-(1) = \{a_o\}$ Suppose not. Then there exists an $x \in LI_K^-(1) \setminus RI_K^-(1)$ such that $x \neq a_o$. Then $x > a_o$, so there exists a $y \in LI_K^-(1)$ such that $x > y$. Therefore $(y+1) + x = 1 + x = x > y = y + 0 \geqslant y + x = y + (1+x)$, a contradiction. Hence we have the claim. By the claim and Lemma 4.8(1), $a_o^{-1} \in LI_K^-(1) \setminus RI_K^-(1)$, hence $a_o^{-1} = a_o$. Therefore $a_o^2 = 1$. By Lemma 4.8(5), $D_K^- = a_o D_K^+$.

To show that $x + 1 = 1 + x$ for all $x \in D_K^- \setminus \{a_o\}$, let $x \in D_K^- \setminus \{a_o\}$. If $x < a_o$ then $x \notin LI_K^-(1)$, so $x \notin RI_K^-(1)$, hence $x + 1 = 1 + x = x$ by Lemma 4.8(1) and (2). Assume that $x > a_o$.

Then $x \in LI_K^-(1)$ (see page 89). By the claim, $x \in RI_K^-(1)$. Hence $x + 1 = 1 + x = 1$.

To show that $a_o x = x a_o$ for all $x \in K \setminus \{0\}$, let $x \in K \setminus \{0\}$. Since $a_o + 1 \neq 1 + a_o$, $x^{-1} a_o x + 1 \neq 1 + x^{-1} a_o x$. If $x \in D_K^+$ then $x^{-1} a_o x \in D_K^-$, so $x^{-1} a_o x = a_o$, hence $a_o x = x a_o$, so we are done. Assume that $x \in D_K^-$. Then $x = a_o d$ for some $d \in D_K^+$. It follows that $x^{-1} a_o x = (d^{-1} a_o^{-1}) a_o (a_o d) = d^{-1} a_o d \in D_K^-$. Hence $x^{-1} a_o x = a_o$, so $a_o x = x a_o$.

For simplicity, we shall assume that $D_K^+ = \mathbb{R}^+$. Define $f: (K,+,\cdot,\leqslant) \to (\mathbb{R},+_1,\cdot,\leqslant)$ as follows: For $x \in K$,

$$f(x) = \begin{cases} x & \text{if } x \in D_K^+ \cup \{0\}, \\ -d & \text{if } x = a_o d \text{ for some } d \in D_K^+. \end{cases}$$

It is easy to see that f is a bijection.

To show that f is a homomorphism, let $x,y \in K \setminus \{0\}$.

<u>Case I</u>: $x,y \in D_K^+$. Then $f(x+y) = x + y = f(x) +_1 f(y)$ and $f(xy) = xy = f(x)f(y)$.

<u>Case II</u>: $x,y \in D_K^-$. Then $x = a_o d$ and $y = a_o r$ for some $d,r \in D_K^+$. Also, $x + y = a_o(d+r)$ and $xy = (a_o d)(a_o r) = a_o(d a_o)r = a_o(a_o d)r = dr$. Hence $f(x+y) = -(d+r) = -\max\{d,r\} = -(d +_1 r) = (-d) +_1 (-r) = f(x) +_1 f(y)$ and $f(xy) = dr = (-d)(-r) = f(x)f(y)$.

<u>Case III</u>: $x \in D_K^+$ and $y \in D_K^-$. Then $y = a_o d$ for some $d \in D_K^+$. Also, $xy = x(a_o d) = a_o(xd)$. Hence $f(xy) = -(xd) = x(-d) = f(x)f(y)$ and

$$f(x) +_1 f(y) = x +_1 (-d) = \begin{cases} x & \text{if } d < x, \\ -d & \text{if } x \leqslant d. \end{cases}$$

If $d < x$ then $a_o x = a_o(x+d) = a_o x + a_o d < 0 + a_o d = y$, so $a_o < yx^{-1}$ which implies that $x + y = (1 + yx^{-1})x = x$. Similarly, if $x \leqslant d$ then $yx^{-1} \leqslant a_o$ which implies that $x + y = (1+yx^{-1})x = (yx^{-1})x = y = a_o d$.

Hence

$$f(x+y) = \begin{cases} x & \text{if } d < x, \\ -d & \text{if } x \leqslant d. \end{cases}$$

Therefore $f(x+y) = f(x) +_1 f(y)$.

<u>Case IV</u>: $x \in D_K^-$ and $y \in D_K^+$. This proof is similar to the proof of Case III.

Hence f is a homomorphism.

To show that f is isotone, let $x,y \in K\setminus\{0\}$ be such that $x < y$. It is clear that if $0 < x < y$ or $x < 0 < y$ then $f(x) < f(y)$. Assume that $x < y < 0$. Then $x = a_o d$ and $y = a_o r$ for some $d,r \in D_K^+$. If $d < r$ then $y = a_o r = a_o(d+r) = a_o d + a_o r \leqslant a_o d + 0 = x$, a contradiction. Hence $r < d$, so $f(x) = -d < -r = f(y)$. Therefore f is isotone.

Therefore f is an order isomorphism.

<u>Case 2</u>: $LI_K^-(1) \subset RI_K^-(1)$. This proof is similar to the proof of Case 1 and shows that $(K,+,\cdot,\leqslant)$ is order isomorphic to $(\mathbb{R},+_2,\cdot,\leqslant)$

Finally, we shall show that 2.1) are not order isomorphic to 2.2), suppose that there exists an order isomorphism $f: (\mathbb{R},+_1,\cdot,\leqslant) \to (\mathbb{R},+_2,\cdot,\leqslant)$. Since $f(-1) = f(1 +_1 (-1)) = f(1) +_2 f(-1) =$

$1 +_2 f(-1)$, $1 < |f(-1)|$. Since $1 = f(1) = f(-1 +_1 1) = f(-1) +_2 f(1) = f(-1) +_2 1$, $|f(-1)| < 1$, a contradiction. #

Theorem 4.10. Let $(K,+,\cdot,\leqslant)$ be a complete ordered distributive seminear-field of zero type. Assume that $(D_K^+,+,\cdot,\leqslant)$ is order isomorphic to $(\{2^n \mid n \in Z\},\max,\cdot,\leqslant)$. Then exactly one of the following statements holds:

(1) K is additively commutative.

(2) $(K,+,\cdot,\leqslant)$ is order isomorphic to exactly one of the following:

2.1) $(\{2^n \mid n \in \mathbb{Z}\} \cup \{0\} \cup \{-(2^n) \mid n \in \mathbb{Z}\},+_1,\cdot,\leqslant)$.

2.2) $(\{2^n \mid n \in \mathbb{Z}\} \cup \{0\} \cup \{-(2^n) \mid n \in \mathbb{Z}\},+_2,\cdot,\leqslant)$.

Proof: The proof is the same as Theorem 4.9. #

Theorem 4.11. Let $(K,+,\cdot,\leqslant)$ be a complete ordered distributive seminear-field of zero type. Assume that $(D_K^+,+,\cdot,\leqslant)$ is order isomorphic to $(\{1\},+,\cdot,\leqslant)$. Then exactly one of the following statements holds:

(1) K is additively commutative.

(2) $(K,+,\cdot,\leqslant)$ is order isomorphic to exactly one of the following:

2.1) $(\{-1,0,1\},+_1,\cdot,\leqslant)$.

2.2) $(\{-1,0,1\},+_2,\cdot,\leqslant)$.

Proof: If $D_K^- = \emptyset$ then $K = \{1,0\}$ which is additively commutative. Assume that $D_K^- \neq \emptyset$. If $LI_K^-(1) = RI_K^-(1)$ then K is additively commutative by Lemma 4.8(4). Assume that $LI_K^-(1) \neq RI_K^-(1)$. Then there exists an $a_o \in D_K^-$ such that $a_o + 1 \neq 1 + a_o$.

<u>Case 1</u>: $a_o + 1 = 1$. By Lemma 4.8(1), $1 + a_o = a_o$. Then $a_o^{-1} = 1 + a_o^{-1} = (a_o+1) + a_o^{-1} = a_o + (1+a_o^{-1}) = a_o + a_o^{-1} \leqslant a_o + 0 = a_o$ and $a_o = 1 + a_o = (a_o^{-1}+1) + a_o = a_o^{-1} + (1+a_o) = a_o^{-1} + a_o \leqslant a_o^{-1} + 0 = a_o^{-1}$. Hence $a_o = a_o^{-1}$, so $a_o^2 = 1$. By Lemma 4.8(5), $D_K^- = a_o D_K^+$. Since $D_K^+ = \{1\}$, $D_K^- = \{a_o\}$. Therefore $K = \{a_o, 0, 1\}$. Define $f: (K,+,\cdot,\leqslant) \to (\{-1,0,1\},+_1,\cdot,\leqslant)$ by $f(0) = 0$, $f(1) = 1$ and $f(a_o) = -1$. Hence f is an order isomorphism.

<u>Case 2</u>: $a_o + 1 = a_o$. Using a proof similar to the proof of Case 1 we can show that $(K,+,\cdot,\leqslant)$ is order isomorphic to $(\{-1,0,1\},+_2,\cdot,\leqslant)$.#

<u>Definition 4.12</u>. A system $(K,+,\cdot,\leqslant)$ is called an <u>ordered distributive seminear-field of infinity type</u> if $(K,+,\cdot)$ is a distributive seminear-field of infinity type and $\leqslant$ is a total order on K satisfying the following properties:

(i) For any $x,y,z \in K$, $x \leqslant y$ and $z \leqslant \infty$ imply that $x + z \leqslant y + z$, $z + x \leqslant z + y$, $xz \leqslant yz$ and $zx \leqslant zy$.

(ii) $1 < \infty$.

Let K be an order distributive seminear-field of infinity type. The <u>finite part</u> of K, denoted by D_K^f, is $\{x \in K \mid x < \infty\}$ and the <u>infinite part</u> of K, denoted by D_K^i, is $\{x \in K \mid x > \infty\}$. Note that

$D^f_K \neq \emptyset$ and $K = D^f_K \cup \{\infty\} \cup D^i_K$.

Remark 4.13. Let K be an ordered distributive seminear-field of infinity type. It is easily shown that the following statements hold:

(1) For any $x \in D^f_K$, $y \in D^i_K$, $xy, yx, y^{-1} \in D^i_K$.

(2) $D^f_K \subseteq xD^i_K$ for all $x \in D^i_K$.

(3) $(D^f_K, \cdot, \leqslant)$ is a totally ordered group.

(4) If K is complete then D^f_K is complete.

(5) $D^f_K \subseteq \mathrm{LCor}_K(1)$ if and only if $D^f_K \subseteq \mathrm{RCor}_K(1)$.

(6) $D^i_K \subseteq \mathrm{LCor}_K(1)$ if and only if $D^i_K \subseteq \mathrm{RCor}_K(1)$.

(7) If $x \in D^f_K \cap \mathrm{LCor}_K(1)$ $[D^f_K \cap \mathrm{RCor}_K(1)]$ then $\{y \in D^f_K \mid y \geqslant x\} \subseteq \mathrm{LCor}_K(1)$ $[\mathrm{RCor}_K(1)]$.

(8) If $x \in D^i_K \cap \mathrm{LCor}_K(1)$ $[D^i_K \cap \mathrm{RCor}_K(1)]$ then $\{y \in D^i_K \mid y \leqslant x\} \subseteq \mathrm{LCor}_K(1)$ $[\mathrm{RCor}_K(1)]$.

We shall now determine $\mathrm{LCor}_K(1)$ and $\mathrm{RCor}_K(1)$ for complete ordered distributive seminear-fields of infinity type. First, we shall need some lemmas. The first lemma has been proven in [3], page 86, and the assumption that the addition is commutative was not used in the proof.

Lemma 4.14 ([3]). For any complete ordered distributive seminear-field K of infinity type, if $(H, \cdot)$ is a subgroup of $(D^f_K, \cdot)$ such that

$|H| > 1$ then H has no upper bound in D_K^f.

Lemma 4.15. Let K be a complete ordered distributive seminear-field of infinity type such that $LCor_K(1)$ is a proper subset of K. Then $LCor_K(1) \cap D_K^f = \emptyset$.

Proof: First, we shall show that $(K \setminus LCor_K(1), \cdot)$ is a subgroup of $(K \setminus \{\infty\}, \cdot)$. Let $x, y \in K \setminus LCor_K(1)$. Then $x + 1 \neq \infty$ and $y + 1 \neq \infty$, so $1 + y^{-1} \neq \infty$. It follows that $x + (xy^{-1}+1) + y^{-1} = (x+xy^{-1})+(1+y^{-1}) = x(1+y^{-1})+(1+y^{-1}) = (x+1)(1+y^{-1}) \neq \infty$. Hence $xy^{-1}+1 \neq \infty$, so $xy^{-1} \in K \setminus LCor_K(1)$. Therefore $(K \setminus LCor_K(1), \cdot)$ is a subgroup of $(K \setminus \{\infty\}, \cdot)$.

Then $1 \in K \setminus LCor_K(1)$, so $1 + 1 < \infty$. If $D_K^f = \{1\}$ then $LCor_K(1) \cap D_K^f = \emptyset$, so we are done. Assume that $D_K^f \neq \{1\}$. Choose $a \in D_K^f$ such that $a < 1$. Then $a + 1 \leqslant 1 + 1 < \infty$, so $a \in K \setminus LCor_K(1)$. Hence we have that $|D_K^f \cap (K \setminus LCor_K(1))| > 1$. Now, we have that $(D_K^f \cap (K \setminus LCor_K(1)), \cdot)$ is a subgroup of $(D_K^f, \cdot)$. By Lemma 4.14, $D_K^f \cap (K \setminus LCor_K(1))$ has no upper bound in D_K^f. Then for each $x \in D_K^f$ there exists a $y_x \in D_K^f \cap (K \setminus LCor_K(1))$ such that $x < y_x$. Hence $x + 1 \leqslant y_x + 1 < \infty$ for all $x \in D_K^f$, so $x \notin LCor_K(1)$ for all $x \in D_K^f$. Therefore $LCor_K(1) \cap D_K^f = \emptyset$. #

Lemma 4.16. Let K be a complete ordered distributive seminear-field of infinity type. If $LCor_K(1)$ has an upper bound then $\sup(LCor_K(1)) = \sup(RCor_K(1))$.

Proof: Assume that $LCor_K(1)$ has an upper bound. Let $a_o = \sup (LCor_K(1))$. Then $a_o \geqslant \infty$. Claim that a_o is an upper bound of $RCor_K(1)$. Suppose not. Then there exists an $x \in RCor_K(1)$ such that $x > a_o$. Then $x > \infty$ and $x \notin LCor_K(1)$. It follows that $x^{-1} \notin RCor_K(1)$. Since $x \in RCor_K(1)$, so $x^{-1} \in LCor_K(1)$, hence $\infty < x^{-1} \leqslant a_o < x$. By Remark 4.13(8), $x^{-1} \in RCor_K(1)$, a contradiction. Hence we have the claim. Let $b_o = \sup (RCor_K(1))$. Then $b_o \leqslant a_o$. Using a proof similar to the proof of the above claim we get that b_o is an upper bound of $LCor_K(1)$. Hence $a_o \leqslant b_o$. Therefore $a_o = b_o$. #

Theorem 4.17. Let K be a complete order distributive seminear-field of infinity type. Then $LCor_K(1)$ is exactly one of the following sets:

(1) $\{\infty\}$.

(2) $D_K^i \cup \{\infty\}$.

(3) K.

Proof: Assume that $\{\infty\} \subset LCor_K(1) \subset K$. By Lemma 4.15, $LCor_K(1) \cap D_K^f = \emptyset$. Hence $LCor_K(1) \subseteq D_K^i \cup \{\infty\}$. If $LCor_K(1)$ has no upper bound then $D_K^i \subseteq LCor_K(1)$ by Remark 4.13 (8), hence $LCor_K(1) = D_K^i \cup \{\infty\}$, so we are done. Assume that $LCor_K(1)$ has an upper bound. Let $a_o = \sup (LCor_K(1))$. By Lemma 4.16, $a_o = \sup (RCor_K(1))$.

Claim that $D_K^f = \{1\}$. Suppose not. Let $d \in D_K^f \setminus \{1\}$. Then $a_o d \neq a_o$.

Case I: $a_o d < a_o$. Then there exists an $r \in LCor_K(1)$ such that $r > a_o d$. Then $rd^{-1} > a_o$, so $rd^{-1} \notin LCor_K(1)$. Thus $rd^{-1}+1 \neq \infty$, so $1 + r^{-1}d \neq \infty$. Hence $r^{-1}d \notin RCor_K(1)$. Since $a_o = \sup (RCor_K(1))$, by Remark 4.13 (8), we get that $r^{-1}d \geqslant a_o$, hence $r^{-1} \geqslant a_o d^{-1}$. Since $r \in LCor_K(1)$, $r^{-1} \in RCor_K(1)$. Thus $r^{-1} \leqslant a_o$, so $a_o d^{-1} \leqslant a_o$. Hence $a_o \leqslant a_o d$, a contradiction.

Case II: $a_o < a_o d$. Then $a_o d^{-1} < a_o$ and using the same proof as in Case I we get a contradiction.

Hence we have the claim.

By Lemma 4.15, $LCor_K(1) \cap D_K^f = \emptyset$, hence $1 + 1 < \infty$. Thus $1 + 1 \in D_K^f = \{1\}$, so $1 + 1 = 1$. We shall show that $D_K^i \subseteq LCor_K(1)$. Let $x \in D_K^i$. Then $x + 1 \geqslant \infty$ and $x + 1 = x + (1+1) = (x+1) + 1$. If $x + 1 > \infty$ then $1 = 1 + (x+1)^{-1} \geqslant 1 + \infty = \infty$, a contradiction. Hence $x + 1 = \infty$, so $x \in LCor_K(1)$. Therefore $D_K^i \subseteq LCor_K(1)$. But $LCor_K(1) \subseteq D_K^i \cup \{\infty\}$. Hence $LCor_K(1) = D_K^i \cup \{\infty\}$. #

From Theorem 4.17, there are three types of complete ordered distributive seminear-fields of infinity type. A complete ordered distributive seminear-field K of infinity type is called a <u>type I distributive seminear-field of infinity type</u> if $LCor_K(1) = \{\infty\}$, a <u>type II distributive seminear-field of infinity type</u> if $LCor_K(1) = D_K^i \cup \{\infty\}$, a <u>type III distributive seminear-field of infinity type</u> if $LCor_K(1) = K$.

Remark 4.18. Let K be a type I distributive seminear-field of infinity type. Then $LCor_K(1) = \{\infty\}$. This implies that $x + y \neq \infty$ for all $x,y \in K \setminus \{\infty\}$. Then we have that $x + y < \infty$ for all $x,y \in D_K^f$. By Remark 4.13 (3) and (4), $(D_K^f,+,\cdot,\leqslant)$ is a complete ordered distributive ratio seminear-ring.

Theorem 4.19. Let K be a type I distributive seminear-field of infinity type such that $1 + 1 = 1$. Then $D_K^i = \emptyset$.

Proof: Suppose not. Let $x \in D_K^i$. Since $LCor_K(1) = \{\infty\}$, $x + 1 > \infty$. Since $x + 1 = x + (1+1) = (x+1) + 1$, so $1 = 1 + (x+1)^{-1} > \infty$, a contradiction. Hence $D_K^i = \emptyset$. #

Remark 4.20. From Remark 4.18 and Theorem 4.19, we get that K is order isomorphic to one of the complete ordered distributive ratio seminear-rings D in Theorem 2.32 with an ∞ element adjoined and $a < \infty$ for all $a \in D$.

Theorem 4.21. Let K be a type I distributive seminear-field of infinity type such that for any $x,y,z \in K$, $x \leqslant y$ implies $z + x \leqslant z + y$ or for any $x,y,z \in K$, $x \leqslant y$ implies $x + z \leqslant y + z$. Then $D_K^i = \emptyset$.

Proof: Assume that for any $x,y,z \in K$, $x \leqslant y$ implies $z + x \leqslant z + y$. Suppose that $D_K^i \neq \emptyset$. Let $x \in D_K^i$. Then $x + 1 \leqslant x + \infty = \infty$. But $LCor_K(1) = \{\infty\}$, so $x + 1 > \infty$, a contradiction. Hence $D_K^i = \emptyset$. If K has a property that for any $x,y,z \in K$, $x \leqslant y$ implies

$x + z \leqslant y + z$ then using a proof similar to the above proof we get that $D_K^i = \emptyset$. #

Remark 4.22. Let K be a type I distributive seminear-field of infinity type such that $1 + 1 \neq 1$. Assume that for any $x,y,z \in K$, $x \leqslant y$ implies $z + x \leqslant z + y$ or for any $x,y,z \in K$, $x \leqslant y$ implies $x + z \leqslant y + z$. From Remark 4.18 and Theorem 4.21, we get that K is order isomorphic to one of the complete ordered distributive ratio seminear-rings D in Theorem 2.39 with an ∞ element adjoined and $a < \infty$ for all $a \in D$.

Remark 4.23. Let K be a type II distributive seminear-field of infinity type. Then $LCor_K(1) = D_K^i \cup \{\infty\}$, so $xy^{-1}+1 = \infty = x^{-1}y + 1$ for all $x \in D_K^f$, $y \in D_K^i$. Hence we have that $x + y = y + x = \infty$ for all $x \in D_K^f$, $y \in D_K^i$ and $x + y < \infty$ for all $x,y \in D_K^f$. By Remark 4.13 (3) and (4), D_K^f is a complete ordered distributive ratio seminear-ring.

We shall now classify all type II distributive seminear-fields of infinity type subject to the condition that for any $x,y,z \in K$, $x \leqslant y$ implies $x + z \leqslant y + z$ or for any $x,y,z \in K$, $x \leqslant y$ implies $z + x \leqslant z + y$.

Theorem 4.24. Let K be a type II distributive seminear-field of infinity type such that D_K^f is additively commutative. Then K is additively commutative.

Proof: If $D_K^i = \emptyset$ then we are done. Assume that $D_K^i \neq \emptyset$. Let $x,y \in D_K^i$.

Case 1: $xy^{-1} \in D_K^f$. Then $x + y = (xy^{-1}+1)y = (1+xy^{-1})y = y + x$.

Case 2: $xy^{-1} \in D_K^i$. Since $LCor_K(1) = D_K^i \cup \{\infty\}$, $xy^{-1}+ 1 = \infty$ and $yx^{-1}+ 1 = (xy^{-1})^{-1}+ 1 = \infty$. Then $x + y = (xy^{-1}+1)y = \infty = \infty \cdot x = (yx^{-1}+1)x = y + x$.

Hence K is additively commutative. #

Remark 4.25. Let K be a type II distributive seminear-field of infinity type such that K is additively commutative and for any $x,y,z \in K$, $x \leqslant y$ implies $x + z \leqslant y + z$. In [3], all such K were classified. So from now on, we shall study all such K which are not additively commutative. By the above theorem D_K^f must be isomorphic to either (4),(5),(8) or (9) in Theorem 2.32.

Remark 4.26. Let $F = \{(2^n,0) \mid n \in \mathbb{Z}\} \cup \{\infty\} \cup \{(2^n,1) \mid n \in \mathbb{Z}\}$ and $H = \{(2^n,0) \mid n \in \mathbb{Z}\} \cup \{\infty\} \cup \{(\sqrt{2^n},1) \mid n$ is an odd interger$\}$. The following are examples of type II distributive seminear-fields of infinity type:

(1) $(F,+_\ell,\cdot,\leqslant_1)$ where $+_\ell$, $\cdot$ and $\leqslant_1$ are defined as follows: $w +_\ell \infty = \infty +_\ell w = w\cdot\infty = \infty\cdot w = \infty$ for all $w \in F$ and for each $x,y \in \{2^n \mid n \in \mathbb{Z}\}$,

$$(x,0) +_\ell (y,0) = (x,0),$$

$$(x,1) +_\ell (y,1) = (x,1),$$

$$(x,0) +_\ell (y,1) = (y,1) +_\ell (x,0) = \infty,$$

$$(x,0)\cdot(y,0) = (xy,0),$$

$$(x,1)\cdot(y,1) = (xy,0),$$

$$(x,0)\cdot(y,1) = (xy,1),$$

$$(y,1)\cdot(x,0) = (yx,1),$$

$$(x,0) <_1 \infty <_1 (y,1),$$

$(x,0) \leqslant_1 (y,0)$ if and only if $x \leqslant y$ and

$(x,1) \leqslant_1 (y,1)$ if and only if $x \leqslant y$.

(2) $(F,+_\ell,\cdot,\leqslant_2)$ where $+_\ell$ and $\cdot$ are given in (1) and $\leqslant_2$ is defined as follows: For each $x,y \in \{2^n | n \in \mathbb{Z}\}$,

$$(x,0) <_2 \infty <_2 (y,1),$$

$(x,0) \leqslant_2 (y,0)$ if and only if $x \leqslant y$ and

$(x,1) \leqslant_2 (y,1)$ if and only if $y \leqslant x$.

(3) $(F,+_\ell,*,\leqslant_1)$ where $+_\ell$ and $\leqslant_1$ are given in (1) and $*$ is defined as follows: $w*\infty = \infty*w = \infty$ for all $w \in F$ and for each $x,y \in \{2^n | n \in \mathbb{Z}\}$,

$$(x,0)*(y,0) = (xy,0),$$

$$(x,1)*(y,1) = (x^{-1}y,0),$$

$(x,0)*(y,1) = (x^{-1}y,1)$ and

$$(y,1)*(x,0) = (yx,1).$$

(4) $(F,+_\ell,*,\leqslant_2)$ where $+_\ell$ and $*$ are given in (3) and $\leqslant_2$ is given in (2).

(5) $(F,+_r,\cdot,\leqslant_1)$ where $\cdot$ and $\leqslant_1$ are given in (1) and $+_r$ is defined as follows: $w +_r \infty = \infty +_r w = \infty$ for all $w \in F$ and for each $x,y \in \{2^n | n \in \mathbb{Z}\}$,

$$(x,0) +_r (y,0) = (y,0),$$

$$(x,1) +_r (y,1) = (y,1) \text{ and}$$

$$(x,0) +_r (y,1) = (y,1) +_r (x,0) = \infty.$$

(6) $(F,+_r,\cdot,\leqslant_2)$ where $+_r$ and $\cdot$ are given in (5) and $\leqslant_2$ is given in (2).

(7) $(F,+_r,*,\leqslant_1)$ where $+_r$ and $\leqslant_1$ are given in (5) and $*$ is given in (3).

(8) $(F,+_r,*,\leqslant_2)$ where $+_r$ and $*$ are given in (7) and $\leqslant_2$ is given in (2).

(9) $(H,+_\ell,\cdot,\leqslant_1)$ where $+_\ell$, $\cdot$ and $\leqslant_1$ are similar to those defined in (1).

(10) $(H,+_\ell,\cdot,\leqslant_2)$ where $+$, $\cdot$ and $\leqslant_2$ are similar to those defined in (2).

(11) $(H,+_r,\cdot,\leqslant_1)$ where $+_r$, $\cdot$ and $\leqslant_1$ are similar to those defined in (5).

(12) $(H,+_r,\cdot,\leqslant_2)$ where $+_r$, $\cdot$ and $\leqslant_2$ are similar to those defined in (6).

Note that $(H,+_\ell,*,\leqslant_1)$ where $+_\ell$, $*$ and $\leqslant_1$ are similar to those defined in (3) is order isomorphic to $(F,+_\ell,*,\leqslant_1)$ by the map $\infty \mapsto \infty$, $(2^n,0) \mapsto (2^n,0)$ and $(2^n,1) \mapsto (\sqrt{2^{2n-1}},1)$.

<u>Lemma 4.27.</u> Let K be a type II distributive seminear-field of infinity type such that for any $x,y,z \in K$, $x \leqslant y$ implies $x+z \leqslant y+z$

or for any $x,y,z \in K$, $x \leqslant y$ implies $z + x \leqslant z + y$. If $D_K^i \neq \emptyset$ then $D_K^i = xD_K^f = D_K^f x$ for all $x \in D_K^i$.

<u>Proof</u>: First, we shall assume that for any $x,y,z \in K$, $x \leqslant y$ implies $x + z \leqslant y + z$. Assume that $D_K^i \neq \emptyset$. Let $x \in D_K^i$. Then $xD_K^f \subseteq D_K^i$. Suppose that $xD_K^f \subset D_K^i$. Let $y \in D_K^i \setminus xD_K^f$. Since $y = x(x^{-1}y)$, so $x^{-1}y \notin D_K^f$, hence $x^{-1}y \in D_K^i$. Since $\mathrm{LCor}_K(1) = D_K^i \cup \{\infty\}$, $y + x = x(x^{-1}y+1) = \infty = y\cdot\infty = y((x^{-1}y)^{-1}+1) = y(y^{-1}x+1) = x + y$. If $x \geqslant y$ then $x + y \geqslant y + y = y(1+1) > \infty$, a contradiction. If $y > x$ then $y + x \geqslant x + x = x(1+1) > \infty$, a contradiction. Hence $xD_K^f = D_K^i$. Similarly, $D_K^f x = D_K^i$.

If K has a property that for any $x,y,z \in K$, $x \leqslant y$ implies $z + x \leqslant z + y$ then using a proof similar to the above proof we get that $D_K^i = xD_K^f = D_K^f x$ for all $x \in D_K^i$. #

<u>Theorem 4.28</u>. Let $(K,+,\cdot,\leqslant)$ be a type II distributive seminear-field of infinity type such that for any $x,y,z \in K$, $x \leqslant y$ implies $x + z \leqslant y + z$ or for any $x,y,z \in K$, $x \leqslant y$ implies $z+x \leqslant z+y$. Assume that $(D_K^f,+,\cdot,\leqslant)$ is order isomorphic to $(\{2^n \mid n \in \mathbb{Z}\},+_\ell,\cdot,\leqslant)$. Then $(K,+,\cdot,\leqslant)$ is order isomorphic to exactly one of the following:

(1) $(\{2^n \mid n \in \mathbb{Z}\} \cup \{\infty\},+_\ell,\cdot,\leqslant)$.

(2) $(F,+_\ell,\cdot,\leqslant_1)$.

(3) $(F,+_\ell,\cdot,\leqslant_2)$.

(4) $(F,+_\ell,*,\leqslant_1)$.

(5) $(F,+_\ell,*,\leqslant_2)$.

(6) $(H,+_\ell,\cdot,\leqslant_1)$.

(7) $(H,+_\ell,\cdot,\leqslant_2)$.

Proof: If $D_K^i = \emptyset$ then K is order isomorphic to (1). Assume that $D_K^i \neq \emptyset$. For simplicity, we shall assume that $D_K^f = \{2^n \mid n \in \mathbb{Z}\}$. Let $a \in D_K^i$. By Lemma 4.27, $D_K^i = aD_K^f = a^{-1}D_K^f$. Then $a^2 \in D_K^f$. Also, we get that $a^2 = 2^N$ and $2a = a2^M$ for some $N,M \in \mathbb{Z}$. It follows by induction that $2^na = a2^{nM}$ for all $n \in \mathbb{Z}_0^+$. Since $2^{-1}a = a2^{-M}$, it follows by induction that $2^na = a2^{nM}$ for all $n \in \mathbb{Z}^-$. Hence we have that $2^na = a2^{nM}$ for all $n \in \mathbb{Z}$. From $2^N = a^2 = a^{-1}a^2a = a^{-1}2^Na = a^{-1}a2^{NM} = 2^{NM}$, we get that $N = 0$ or $M = 1$.

Case 1: $M = 1$: Then $2^na = a2^n$ for all $n \in \mathbb{Z}$.

Subcase 1.1: $N = 2k$ for some $k \in \mathbb{Z}$. Let $b = 2^{-k}a$. Then $b \in D_K^i$ and $b^2 = (2^{-k}a)(2^{-k}a) = 2^{-2k}a^2 = 2^{-2k}2^N = 1$. By Lemma 4.27, $D_K^i = bD_K^f$. Since $2b = 2(2^{-k}a) = 2^{-k+1}a = a2^{-k+1} = a2^{-k}2 = 2^{-k}a2 = b2$, it follows by induction that $2^nb = b2^n$ for all $n \in \mathbb{Z}$.

Subcase 1.1.1: $b < b2$. It follows by induction that $b2^n < b2^{n+1}$ for all $n \in \mathbb{Z}_0^+$. Since $b2^{-1} < b$, by induction, we get that $b2^n < b2^{n+1}$ for all $n \in \mathbb{Z}^-$. Hence $b2^n < b2^{n+1}$ for all $n \in \mathbb{Z}$. This implies that for any $m,n \in \mathbb{Z}$, $m < n$ implies $b2^m < b2^n$. Define $f: (K,+,\cdot,\leqslant) \to (F,+_\ell,\cdot,\leqslant_1)$ by

$$f(x) = \begin{cases} (x,0) & \text{if } x \in D_K^f, \\ \infty & \text{if } x = \infty, \\ (d,1) & \text{if } x = bd \text{ for some } d \in D_K^f. \end{cases}$$

Clearly, f is a bijection.

To show that f is a homomorphism, let $x,y \in K \setminus \{\infty\}$.

Case I: $x,y \in D_K^f$. Then $f(x+y) = f(x) = (x,0) = (x,0) +_\ell (y,0) = f(x) +_\ell f(y)$ and $f(xy) = (xy,0) = (x,0)(y,0) = f(x)f(y)$.

Case II: $x,y \in D_K^i$. Then $x = bc$ and $y = bd$ for some $c,d \in D_K^f$. Also, $x + y = b(c+d) = bc$ and $xy = bcbd = bbcd = cd$. Hence $f(x+y) = (c,1) = (c,1) +_\ell (d,1) = f(x) +_\ell f(y)$ and $f(xy) = (cd,0) = (c,1)(d,1) = f(x)f(y)$.

Case III: $x \in D_K^i$ and $y \in D_K^f$. Then $x = bd$ for some $d \in D_K^f$. Also, $f(x+y) = f(\infty) = \infty = (d,1) +_\ell (y,0) = f(x) +_\ell f(y)$ and $f(xy) = (dy,1) = (d,1)(y,0) = f(x)f(y)$.

Case IV: $x \in D_K^f$ and $y \in D_K^i$. This proof is similar to the proof of Case III.

Hence f is a homomorphism.

To show that f is isotone, let $x,y \in K \setminus \{\infty\}$ be such that $x < y$. It is easy to see that if $x < y < \infty$ or $x < \infty < y$ then $f(x) <_1 f(y)$. Assume that $y > x > \infty$. Then $x = b2^m$ and $y = b2^n$ for some $m,n \in \mathbb{Z}$. Thus $b2^m < b2^n$, so $m < n$. It follows that $f(x) = (2^m,1) <_1 (2^n,1) = f(y)$. Hence f is isotone.

Therefore f is an order isomorphism.

Subcase 1.1.2: $b2 < b$. It follows by induction that $b2^{n+1} < b2^n$ for all $n \in \mathbb{Z}$. From this we get that for any $m,n \in \mathbb{Z}$, $m < n$ implies $b2^n < b2^m$. Define $g: (K,+,\cdot,\leqslant) \to (F,+_\ell,\cdot,\leqslant_2)$ by

$$g(x) = \begin{cases} (x,0) & \text{if } x \in D_K^f, \\ \infty & \text{if } x = \infty, \\ (d,1) & \text{if } x = bd \text{ for some } d \in D_K^f. \end{cases}$$

Clearly, g is a bijection. Using the same proof as in Subcase 1.1.1 we get that g is a homomorphism.

To show that g is isotone, let $x,y \in K \setminus \{\infty\}$ be such that $x < y$. It is easy to see that if $x < y < \infty$ or $x < \infty < y$ then $g(x) <_2 g(y)$. Assume that $y > x > \infty$. Then $x = b2^m$ and $y = b2^n$ for some $m,n \in \mathbb{Z}$. Thus $b2^m < b2^n$, so $n < m$. It follows that $g(x) = (2^m,1) <_2 (2^n,1) = g(y)$. Hence g is isotone.

Therefore g is an order isomorphism.

<u>Subcase 1.2</u>: $N = 2k - 1$ for some $k \in \mathbb{Z}$. Let $b = 2^{-k}a$. Then $b \in D_K^i$ and $b^2 = (2^{-k}a)(2^{-k}a) = 2^{-2k}a^2 = 2^{-2k}2^N = 2^{-1}$. By Lemma 4.27, $D_K^i = bD_K^f$. Using the same proof as in Subcase 1.1 we get that $2^nb = b2^n$ for all $n \in \mathbb{Z}$.

<u>Subcase 1.2.1</u>: $b < b2$. Using the same proof as in Subcase 1.1.1 we get that for any $m,n \in \mathbb{Z}$, $m < n$ implies $b2^m < b2^n$. Define $h: (K,+,\cdot,\leqslant) \to (H,+_\ell,\cdot,\leqslant_1)$ by

$$h(x) = \begin{cases} (x,0) & \text{if } x \in D_K^f, \\ \infty & \text{if } x = \infty, \\ (\sqrt{2^{2n-1}},1) & \text{if } x = b2^n \text{ for some } n \in \mathbb{Z}. \end{cases}$$

Clearly, h is a bijection. Using a proof similar to the proof of Subcase 1.1.1 we get that h is isotone and for any $x,y \in K \setminus \{\infty\}$ if $x,y \in D_K^f$ or $x \in D_K^f$ and $y \in D_K^i$ or $x \in D_K^i$ and $y \in D_K^f$ then

$h(x+y) = h(x) +_\ell h(y)$ and $h(xy) = h(x)h(y)$. Let $x,y \in D_K^i$. Then $x = b2^m$ and $y = b2^n$ for some $m,n \in \mathbb{Z}$. Also, $x + y = b(2^m+2^n) = b2^m$ and $xy = b2^m b2^n = bb2^{m+n} = 2^{m+n-1}$. Thus $h(x+y) = (\sqrt{2^{2m-1}},1) = (\sqrt{2^{2m-1}},1) +_\ell (\sqrt{2^{2n-1}},1) = h(x) +_\ell h(y)$ and $h(xy) = (2^{m+n-1},0) = (\sqrt{2^{2m-1}},1)(\sqrt{2^{2n-1}},1) = h(x)h(y)$. Hence h is a homomorphism. Therefore h is an order isomorphism.

<u>Subcase 1.2.2</u>: $b2 < b$. Using a proof similar to the proof of Subcase 1.2.1 we get that $(K,+,\cdot,\leqslant)$ is order isomorphic to $(H,+_\ell,\cdot,\leqslant_2)$.

<u>Case 2</u>: $N = 0$. Then $a^2 = 2^N = 1$. Since $2^n a = a2^{nM}$ for all $n \in \mathbb{Z}$, $2 = (2a)a = (a2^M)a = a(2^M a) = a(a2^{M^2}) = a^2 2^{M^2} = 2^{M^2}$. Hence $M = \pm 1$. By Subcase 1.1, we get that if $M = 1$ then $(K,+,\cdot,\leqslant)$ is order isomorphic to either (2),(3),(6) or (7). Assume that $M = -1$. Then $2^n a = a2^{-n}$ for all $n \in \mathbb{Z}$.

<u>Subcase 2.1</u>: $a < a2$. Using the same proof as in Subcase 1.1.1 we get that for any $m,n \in \mathbb{Z}$, $m < n$ implies $a2^m < a2^n$. Define $i: (K,+,\cdot,\leqslant) \to (F,+_\ell,*,\leqslant_1)$ by

$$i(x) = \begin{cases} (x,0) & \text{if } x \in D_K^f, \\ \infty & \text{if } x = \infty, \\ (d,1) & \text{if } x = ad \text{ for some } d \in D_K^f. \end{cases}$$

Clearly, i is a bijection. Using the same proof as Subcase 1.1.1 we get that i is an isotone, $i(x+y) = i(x) +_\ell i(y)$ for all $x,y \in K$ and if $x,y \in D_K^f$ or $x \in D_K^i$ and $y \in D_K^f$ then $i(xy) = i(x)*i(y)$. Let $x,y \in K \setminus \{\infty\}$. If $x,y \in D_K^i$ then $x = ac$ and $y = ad$ for some $c,d \in D_K^f$, so $xy = (ac)(ad) = a(ca)d = a(ac^{-1})d = c^{-1}d$ and hence $i(xy) = (c^{-1}d,0) = (c,1)*(d,1) = i(x)*i(y)$. If $x \in D_K^f$ and $y \in D_K^i$ then $y = ad$ for some $d \in D_K^f$, so $xy = xad = ax^{-1}d$, and hence $i(xy) = (x^{-1}d,1) = (x,0)*(d,1) = i(x)*i(y)$. Hence i is a homomorphism. Therefore i is an order isomorphism.

Subcase 2.2: $a2 < a$. Using a proof similar to the proof of Subcase 2.1 we get that $(K,+,\cdot,\leqslant)$ is order isomorphic to $(F,+_\ell,*,\leqslant_2)$.

Finally, we shall show that (1) to (7) are not order isomorphic to each others. Clearly, (1) is not order isomorphic to any of the others. Since (4) and (5) are not multiplicatively commutative, (4) and (5) are not order isomorphic to any of the others.

To show that (2) is not order isomorphic to (3), suppose not. Let $f: (F,+_\ell,\cdot,\leqslant_1) \to (F,+_\ell,\cdot,\leqslant_2)$ be an order isomorphism. Let $f(2,0) = (2^N,0)$ and $f(2,1) = (2^P,1)$ for some $N,P \in \mathbb{Z}$. Since $(2,1) <_1 (2^2,1)$, $(2^P,1) = f(2,1) <_2 f(2^2,1) = f((2,1)(2,0)) = f(2,1)f(2,0) = (2^P,1)(2^N,0) = (2^{P+N},1)$. Hence $2^{P+N} < 2^P$, so $N < 0$. Since $(2,0) <_1 (2^2,0)$, so $(2^N,0) = f(2,0) <_2 f(2^2,0) = f((2,0)(2,0)) = f(2,0)f(2,0) = (2^N,0)(2^N,0) = (2^{2N},0)$. Hence $2^N < 2^{2N}$, so $N > 0$, a contradiction. Therefore (2) is not order isomorphic to (3).

To show that (2) is not order isomorphic to (6), suppose not. Let $f: (F,+_\ell,\cdot,\leqslant_1) \to (H,+_\ell,\cdot,\leqslant_1)$ be an order isomorphicm. Let $f(2,0) = (2^N,0)$ and $f(2,1) = (\sqrt{2}^P,1)$ where $N,P \in \mathbb{Z}$ and P is odd. Then $(2^{2N},0) = (2^N,0)(2^N,0) = f(2,0)f(2,0) = f((2,0)(2,0)) = f(2^2,0) =$

$f((2,1)(2,1)) = f(2,1)f(2,1) = (\sqrt{2}^P,1)(\sqrt{2}^P,1) = (2^P,0)$, so we have that $P = 2N$ which is even, a contradiction. Hence (2) is not order isomorphic to (6).

The proofs that (2) and (3) are not order isomorphic to (7) and (3) is not order isomorphic to (6) are similar to the proof that (2) is not order isomorphic to (6). The proofs that (4) is not order isomorphic to (5) and (6) is not order isomorphic to (7) are similar to the proof that (2) is not order isomorphic to (3). #

Theorem 4.29. Let $(K,+,\cdot,\leqslant)$ by a type II distributive seminear-field of infinity type such that for any $x,y,z \in K$, $x \leqslant y$ implies $x+z \leqslant y+z$ or for any $x,y,z \in K$, $x \leqslant y$ implies $z+x \leqslant z+y$. Assume that $(D_K^f,+,\cdot,\leqslant)$ is order isomorphic to $(\{2^n \mid n \in \mathbb{Z}\},+_r,\cdot,\leqslant)$. Then $(K,+,\cdot,\leqslant)$ is order isomorphic to exactly one of the following:

(1) $(\{2^n \mid n \in \mathbb{Z}\} \cup \{\infty\},+_r,\cdot,\leqslant)$.

(2) $(F,+_r,\cdot,\leqslant_1)$.

(3) $(F,+_r,\cdot,\leqslant_2)$.

(4) $(F,+_r,*,\leqslant_1)$.

(5) $(F,+_r,*,\leqslant_2)$.

(6) $(H,+_r,\cdot,\leqslant_1)$.

(7) $(H,+_r,\cdot,\leqslant_2)$.

Proof: The proof is similar to the proof of Theorem 4.28. #

Remark 4.30. Let $f: (\mathbb{R}^+, \cdot) \to (\mathbb{R}^+, \cdot)$ be an isomorphism such that $f \circ f = I_{\mathbb{R}^+}$ and let A be an O-set of $\mathbb{R}^+$ such that $(\mathbb{R}^+, \cdot, \leqslant_A)$ is order isomorphic to $(\mathbb{R}^+, \cdot, \leqslant)$ where $\leqslant_A$ is the compatible partial order having A as its positive cone.

Let $K(f,A) = (\mathbb{R}^+ \times \{0\}) \cup \{\infty\} \cup (\mathbb{R}^+ \times \{1\})$ where f and A are defined as above. Then the following are examples of type II distributive seminear-fields of infinity type:

(1) $(K(f,A), +_\ell, \cdot, \leqslant_3)$ where $+_\ell$ is defined as in Remark 4.26 (1) and $\cdot$ and $\leqslant_3$ are defined as follows: $w \cdot \infty = \infty \cdot w = \infty$ for all $w \in K(f,A)$ and for any $x, y \in \mathbb{R}^+$,

$$(x,0)\cdot(y,0) = (xy,0),$$

$$(x,1)\cdot(y,1) = (f(x)y,0),$$

$$(x,0)\cdot(y,1) = (f(x)y,1),$$

$$(y,1)\cdot(x,0) = (yx,1),$$

$$(x,0) <_3 \infty <_3 (y,1),$$

$(x,0) \leqslant_3 (y,0)$ if and only if $x \leqslant y$ and

$(x,1) \leqslant_3 (y,1)$ if and only if $x \leqslant_A y$.

(2) $(K(f,A), +_r, \cdot, \leqslant_3)$ where $+_r$ is defined as in Remark 4.26 (5) and $\cdot$ and $\leqslant_3$ are given in (1).

Proof: (1) It is clear that $+_\ell$ is associative and $\cdot$ is distributive over $+_\ell$ in $K(f,A)$. Let $(x,i),(y,j),(z,k) \in K(f,A)$ where $x,y,z \in \mathbb{R}^+$ and $i,j,k \in \{0,1\}$. Then

$$[(x,i)(y,j)](z,k)=(x,i)[(y,j)(z,k)]=\begin{cases}(xyz,0) & \text{if } i=j=k=0,\\ (f(xy)z,1) & \text{if } i=j=0 \text{ and } k=1,\\ (f(x)yz,1) & \text{if } i=k=0 \text{ and } j=1,\\ (xyz,0) & \text{if } j=k=0 \text{ and } i=1,\\ (f(x)yz,1) & \text{if } i=j=1 \text{ and } k=0,\\ (f(xy)z,0) & \text{if } i=k=1 \text{ and } j=0,\\ (xf(y)z,0) & \text{if } j=k=1 \text{ and } i=0,\\ (xf(y)z,1) & \text{if } i=j=k=1.\end{cases}$$

Hence $\cdot$ is associative.

For each $x \in \mathbb{R}^+$, $(x,0)(1,0)=(x,0)$ and $(x,1)(1,0)=(1,0)(x,1)=(x,1)$, hence $(1,0)$ is the multiplicative identity of $K(f,A)$. For each $x \in \mathbb{R}^+$, $(x,0)(x^{-1},0) = (1,0)$ and $(x,1)(f(x)^{-1},1) = (1,0)$, hence every element of $K(f,A)$ has an inverse.

Therefore $(K(f,A),+_\ell,\cdot)$ is a distributive seminear-field of infinity type.

Let $u,v,w \in K(f,A)$ be such that $u \leqslant_3 v$ and $w \leqslant_3 \infty$. Then $u +_\ell w = u \leqslant_3 v = v +_\ell w$ and $w +_\ell u = w = w +_\ell v$. It is clear that if $u \leqslant_3 v \leqslant_3 \infty$ or $u \leqslant_3 \infty \leqslant_3 v$ then $uw \leqslant_3 vw$ and $wu \leqslant_3 wv$. Assume that $\infty <_3 u \leqslant_3 v$. Then $u = (x,1)$, $v = (y,1)$ and $w = (z,0)$ for some $x,y,z \in \mathbb{R}^+$. Hence $x \leqslant_A y$. Thus $xz \leqslant_A yz$ and $f(z)x \leqslant_A f(z)y$. Also, we get that $uw = (x,1)(z,0) = (xz,1) \leqslant_3 (yz,1) = (y,1)(z,0) = vw$ and $wu = (z,0)(x,1) = (f(z)x,1) \leqslant_3 (f(z)y,1) = (z,0)(y,1) = wv$.

Therefore $(K(f,A),+_\ell,\cdot,\leqslant_3)$ is an ordered distributive seminear-field of infinity type.

To show that $K(f,A)$ is complete, let B be a nonempty subset of $K(f,A)$ which has a lower bound. It is clear that if $B \cap (\mathbb{R}^+ \times \{0\}) \neq \emptyset$

then inf (B) exists, so we are done. Assume that $B \cap (\mathbb{R}^+ \times \{0\}) = \emptyset$. If there does not exist a lower bound of B in $\mathbb{R}^+ \times \{1\}$ then $\inf(B) = \infty$, so we are done. Assume that there exists a lower bound of B in $\mathbb{R}^+ \times \{1\}$, say $(y,1)$. Then $B = D \times \{1\}$ where $D \subseteq \mathbb{R}^+$. Then $(y,1) \leqslant_3 (x,1)$ for all $x \in D$, so $y \leqslant_A x$ for all $x \in D$. Since $(\mathbb{R}^+, \cdot, \leqslant_A)$ is order isomorphic to $(\mathbb{R}^+, \cdot, \leqslant)$, $(\mathbb{R}^+, \cdot, \leqslant_A)$ is complete. Then inf (D) exists, say z. Thus $z \leqslant_A x$ for all $x \in D$, so we have that $(z,1) \leqslant_3 (x,1)$ for all $x \in D$. Hence $(z,1)$ is a lower bound of B. Let $(w,1)$ be a lower bound of B. Then $(w,1) \leqslant_3 (x,1)$ for all $x \in D$, so $w \leqslant_A x$ for all $x \in D$. Hence $w \leqslant_A z$, so $(w,1) \leqslant_3 (z,1)$. This implies that $(z,1) = \inf(B)$. Therefore K(f,A) is complete.

Hence K(f,A) is a type II distributive seminear-field of infinity type.

(2) It is similar to the proof of (1). #

<u>Proposition 4.31</u>. Let K(f,A) and K(g,B) be define as in Remark 4.30. Then K(f,A) is order isomorphic to K(g,B) if and only if there exists an $a \in \mathbb{R}^+$ such that $B = A^a$ and $f(x)^a = g(x^a)$ for all $x \in \mathbb{R}^+$.

<u>Proof</u>: Assume that K(f,A) is order isomorphic to K(g,B). Let $\psi : (K(f,A), \cdot, \leqslant_3) \to (K(g,B), \cdot, \overset{*}{\leqslant_3})$ be an order isomorphism. Since ψ is isotone, $\psi(\mathbb{R}^+ \times \{0\}) = \mathbb{R}^+ \times \{0\}$ and $\psi(\mathbb{R}^+ \times \{1\}) = \mathbb{R}^+ \times \{1\}$. Then for each $x \in \mathbb{R}^+$ there exists a unique pair of elements y_x and z_x in $\mathbb{R}^+$ such that $\psi(x,0) = (y_x,0)$ and $\psi(x,1) = (z_x,1)$. Define $\psi_1 : (\mathbb{R}^+, \cdot, \leqslant) \to (\mathbb{R}^+, \cdot, \leqslant)$ and $\psi_2 : \mathbb{R}^+ \to \mathbb{R}^+$ by $\psi_1(x) = y_x$ and

$\phi_2(x) = z_x$ for all $x \in \mathbb{R}^+$. Then ϕ_1 and ϕ_2 are well-defined and bijective.

To show that ϕ_1 is a homomorphism, let $x_1, x_2 \in \mathbb{R}^+$. Then $(y_{x_1x_2}, 0) = \phi(x_1x_2, 0) = \phi((x_1,0)(x_2,0)) = \phi(x_1,0)\phi(x_2,0) = (y_{x_1},0)(y_{x_2},0) = (y_{x_1}y_{x_2},0)$, so $y_{x_1x_2} = y_{x_1}y_{x_2}$. Hence $\phi_1(x_1x_2) = y_{x_1x_2} = y_{x_1}y_{x_2} = \phi_1(x_1)\phi_1(x_2)$.

To show that ϕ_1 is isotone, let $x_1, x_2 \in \mathbb{R}^+$ be such that $x_1 \leqslant x_2$. Then $(x_1,0) \leqslant_3 (x_2,0)$, so $(y_{x_1},0) = \phi(x_1,0) \overset{*}{\leqslant}_3 \phi(x_2,0) = (y_{x_2},0)$. Hence $\phi_1(x_1) = y_{x_1} \leqslant y_{x_2} = \phi_1(x_2)$.

Therefore ϕ_1 is an order isomorphism. By Proposition 1.45, there exists an $a \in \mathbb{R}^+$ such that $\phi_1(x) = x^a$ for all $x \in \mathbb{R}^+$. For each $x \in \mathbb{R}^+$, $(\phi_2(x),1) = (z_x,1) = \phi(x,1) = \phi((1,1)(x,0)) = \phi(1,1)\phi(x,0) = (z_1,1)(y_x,0) = (z_1y_x,1) = (\phi_2(1)\phi_1(x),1)$. Hence $\phi_2(x) = \phi_2(1)\phi_1(x)$ for all $x \in \mathbb{R}^+$.

To prove that $f(x)^a = g(x^a)$ for all $x \in \mathbb{R}^+$, let $x \in \mathbb{R}^+$, Then $(\phi_2(1)\phi_1(f(x)),1) = (\phi_2(f(x)),1) = (z_{f(x)},1) = \phi(f(x),1) = \phi((x,0)(1,1)) = \phi(x,0)\phi(1,1) = (y_x,0)(z_1,1) = (g(y_x)z_1,1) = (g(\phi_1(x))\phi_2(1),1)$. Hence $\phi_2(1)\phi_1(f(x)) = g(\phi_1(x))\phi_2(1)$. Therefore $f(x)^a = \phi_1(f(x)) = g(\phi_1(x)) = g(x^a)$.

Next, we shall show that $B = A^a$. Let $x \in A$. Since A is the positive cone induced by $\leqslant_A$, $1 \leqslant_A x$. Then $(1,1) \leqslant_3 (x,1)$, so $(z_1,1) = \phi(1,1) \overset{*}{\leqslant}_3 \phi(x,1) = (z_x,1)$. Hence $z_1 \leqslant_B z_x$, so we get that

$x^a = \psi_1(x) = \psi_2(1)^{-1}(\psi_2(1)\psi_1(x)) = \psi_2(1)^{-1}\psi_2(x) = z_1^{-1}z_x \in B$. Thus $A^a \subseteq B$. Let $x \in B$. Since B is the positive cone induced by $\leqslant_B$, $1 \leqslant_B x$. Then $(1,1) \overset{*}{\leqslant}_3 (x,1)$. Hence $(\psi_2^{-1}(1),1) = \psi^{-1}(1,1) \leqslant_3 \psi^{-1}(x,1) = (\psi_2^{-1}(x),1)$, so $\psi_2^{-1}(1) \leqslant_A \psi_2^{-1}(x)$. Thus $[\psi_2^{-1}(1)]^{-1}\psi_2^{-1}(x) \in A$. From $(\psi_2^{-1}(x),1) = \psi^{-1}(x,1) = \psi^{-1}((1,1)(x,0)) = \psi^{-1}(1,1)\psi^{-1}(x,0) = (\psi_2^{-1}(1),1)(\psi_1^{-1}(x),0) = (\psi_2^{-1}(1)\psi_1^{-1}(x),1)$, we have that $\psi_2^{-1}(x) = \psi_2^{-1}(1)\psi_1^{-1}(x)$. Also, $x^{1/a} = \psi_1^{-1}(x) = [\psi_2^{-1}(1)]^{-1}\psi_2^{-1}(1)\psi_1^{-1}(x) = [\psi_2^{-1}(1)]^{-1}\psi_2^{-1}(x) \in A$. Then $x \in A^a$. Hence $B \subseteq A^a$. Therefore $B = A^a$.

Conversely, assume that there exists an $a \in \mathbb{R}^+$ such that $B = A^a$ and $f(x)^a = g(x^a)$ for all $x \in \mathbb{R}^+$. Define $\psi\colon (K(f,A),+_\ell,\cdot,\leqslant_3) \to (K(g,B),+_\ell,\cdot,\overset{*}{\leqslant}_3)$ as follows: $\psi(\infty) = \infty$ and for each $x \in \mathbb{R}^+$, $\psi(x,0) = (x^a,0)$ and $\psi(x,1) = (x^a,1)$. It is clear that ψ is a bijection and $\psi(w_1 +_\ell w_2) = \psi(w_1) +_\ell \psi(w_2)$ for all $w_1, w_2 \in K(f,A)$. Let $(x,i),(y,j) \in K(f,A)$ where $x,y \in \mathbb{R}^+$ and $i,j \in \{0,1\}$. Then

$$\psi((x,i)(y,j)) = \psi(x,i)\psi(y,j) = \begin{cases} (x^a y^a, 0) & \text{if } i = j = 0, \\ (f(x)^a y^a, 1) & \text{if } i = 0 \text{ and } j = 1, \\ (x^a y^a, 1) & \text{if } i = 1 \text{ and } j = 0, \\ (f(x)^a y^a, 0) & \text{if } i = j = 1. \end{cases}$$

Hence ψ is a homomorphism

To show that ψ is isotone, let $u,v \in K(f,A)$ be such that $u \leqslant_3 v$. It is clear that if $u \leqslant_3 v \leqslant_3 \infty$ or $u \leqslant_3 \infty \leqslant_3 v$ then $\psi(u) \overset{*}{\leqslant}_3 \psi(v)$. Assume that $\infty <_3 u \leqslant_3 v$. Then $u = (x,1)$ and $v = (y,1)$ for some $x,y \in \mathbb{R}^+$. Thus $x \leqslant_A y$, so $x^{-1}y \in A$. But $B = A^a$, so

$(x^a)^{-1}y^a = (x^{-1}y)^a \in B$. Also, $x^a \leqslant_B y^a$. It follows that $\phi(x,1) = (x^a,1) \overset{*}{\leqslant}_3 (y^a,1) = \phi(y,1)$. Hence ϕ is isotone.

Therefore ϕ is an order isomorphism. #

Theorem 4.32. Let $(K,+,\cdot,\leqslant)$ be a type II distributive seminear-field of infinity type such that for any $x,y,z \in K$, $x \leqslant y$ implies $x+z \leqslant y+z$ or for any $x,y,z \in K$, $x \leqslant y$ implies $z+x \leqslant z+y$. Assume that $(D_K^f,+,\cdot,\leqslant)$ is order isomorphic to $(\mathbb{R}^+,+_\ell,\cdot,\leqslant)$. Then $(K,+,\cdot,\leqslant)$ is order isomorphic to exactly one of the following:

(1) $(\mathbb{R}^+_\infty,+_\ell,\cdot,\leqslant)$.

(2) $(K(f,A),+_\ell,\cdot,\leqslant_3)$ where $f: (\mathbb{R}^+,\cdot) \to (\mathbb{R}^+,\cdot)$ is an isomorphism such that $f \circ f = I_{\mathbb{R}^+}$ and A is an O-set of $\mathbb{R}^+$ such that $(\mathbb{R}^+,\cdot,\leqslant_A)$ is order isomorphic to $(\mathbb{R}^+,\cdot,\leqslant)$ where $\leqslant_A$ is the compatible partial order having A as its positive cone and $+_\ell$, $\cdot$ and $\leqslant_3$ are defined as in Remark 4.30 (1).

Proof: If $D_K^i = \emptyset$ then K is order isomorphic to (1). Assume that $D_K^i \neq \emptyset$. For simplicity, we shall assume that $D_K^f = \mathbb{R}^+$. Let $w \in D_K^i$. By Lemma 4.27, $D_K^i = wD_K^f = w^{-1}D_K^f$. Then $w^2 \in D_K^f$. Let $x,y \in D_K^i$. Then $x = wd_1$ and $y = wd_2$ for some $d_1,d_2 \in D_K^f$. Hence $x+y = w(d_1+d_2) = wd_1 = x$. Thus for each $x,y \in D_K^i$, $x + y = x$. Since $(D_K^f,+,\cdot,\leqslant)$ is order isomorphic to $(\mathbb{R}^+,+_\ell,\cdot,\leqslant)$ and $w^2 \in D_K^f$, $w^2 = d_o^2$ for some $d_o \in D_K^f$. Since $d_o^{-1}w \in D_K^i$, $d_o^{-1}w = wr$ for some $r \in D_K^f$. Therefore $w = d_owr = d_o(d_owr)r = d_o^2wr^2 = w^3r^2$. Hence $1 = w^2r^2 = d_o^2r^2$, so $d_or = 1$. Let

$a = wd_o^{-1}$. Then $a^2 = (wd_o^{-1})(wd_o^{-1}) = w(d_o^{-1}w)d_o^{-1} = w(wr)d_o^{-1} = w^2rd_o^{-1} = d_o^2rd^{-1} = d_or = 1$. Since $a \in D_K^i$, by Lemma 4.27, we get that $D_K^i = aD_K^f = D_K^fa$. Hence for each $d \in D_K^f$ there exists a unique $x_d \in D_K^f$ such that $da = ax_d$.

Define $f: (\mathbb{R}^+, \cdot) \to (\mathbb{R}^+, \cdot)$ by $f(d) = x_d$ for all $d \in \mathbb{R}^+$. So we get that f is well-defined. Let $d_1, d_2 \in \mathbb{R}^+$ be such that $f(d_1) = f(d_2)$. Then $x_{d_1} = x_{d_2}$, so $ax_{d_1} = ax_{d_2}$. Hence $d_1a = d_2a$, so $d_1 = d_2$. Therefore f is injective. Let $r \in \mathbb{R}^+$. Since $aD_K^f = D_K^fa$, $ar = da$ for some $d \in D_K^f$. But $da = ax_d$, so $r = x_d$. Hence $f(d) = x_d = r$. Thus f is onto.

To show that f is a homomorphism, let $d_1, d_2 \in \mathbb{R}^+$. Since $ax_{d_1d_2} = (d_1d_2)a = d_1(d_2a) = d_1(ax_{d_2}) = (d_1a)x_{d_2} = ax_{d_1}x_{d_2}$, $x_{d_1d_2} = x_{d_1}x_{d_2}$. Thus $f(d_1d_2) = x_{d_1d_2} = x_{d_1}x_{d_2} = f(d_1)f(d_2)$. Hence f is a homomorphism.

Therefore f is an isomorphism. For each $d \in \mathbb{R}^+$, $d = da^2 = (da)a = (ax_d)a = a(f(d)a) = a(ax_{f(d)}) = x_{f(d)} = f(f(d)) = f \circ f(d)$, so we have that $f \circ f = I_{\mathbb{R}^+}$.

Let $A = \{d \in D_K^f \mid a \leqslant ad\}$. Then $A \neq \emptyset$ since $1 \in A$. We shall show that A is an O-set of a group D_K^f. Since $A^{-1} = \{d \in D_K^f \mid ad \leqslant a\}$, so $A \cap A^{-1} = \{1\}$. Let $d_1, d_2 \in A$. Thus $a \leqslant ad_1$ and $a \leqslant ad_2$, so we get that $a \leqslant ad_2 \leqslant ad_1d_2$. Hence $d_1d_2 \in A$. Therefore $A^2 \subseteq A$. Since $(D_K^f, \cdot) \simeq (\mathbb{R}^+, \cdot)$, $xAx^{-1} = A$ for all $x \in D_K^f$. Hence A is an O-set of a group D_K^f. By Theorem 1.19, there exists a compatible partial order

$\leqslant_A$ on D_K^f such that A is the positive cone induced by $\leqslant_A$. Since $D_K^f = A \cup A^{-1}$, $\leqslant_A$ is a total order on D_K^f.

To show that $(D_K^f, \leqslant_A)$ is complete, Let H be a nonempty subset of D_K^f which has a lower bound, say $d \in D_K^f$. Let $h \in H$. Then $d \leqslant_A h$, so $hd^{-1} \in A$. Hence $a \leqslant ahd^{-1}$, so $ad \leqslant ah$. Therefore, we have that ad is a lower bound of aH. Since K is complete, so inf (aH) exists, say x. Then $x \in D_K^i$, so $x = ar$ for some $r \in D_K^f$. Claim that $r = \inf(H)$. Since $ar = x \leqslant ah$ for all $h \in H$, so $hr^{-1} \in A$ for all $h \in H$, hence $r \leqslant_A h$ for all $h \in H$. Then r is a lower bound of H. Let $t \in D_K^f$ be a lower bound of H with respect to $\leqslant_A$. Then $t \leqslant_A h$ for all $h \in H$, so $ht^{-1} \in A$ for all $h \in H$. Thus $a \leqslant aht^{-1}$ for all $h \in H$, so $at \leqslant ah$ for all $h \in H$. Since $x = \inf(aH)$, $at \leqslant x = ar$. Hence $rt^{-1} \in A$, so $t \leqslant_A r$. Therefore $r = \inf(H)$ with respect to $\leqslant_A$. Hence $(D_K^f, \leqslant_A)$ is complete.

Therefore $(D_K^f, \cdot, \leqslant_A)$ is a complete totally ordered group. Since D_K^f is uncountable, by Theorem 1.15, $(D_K^f, \cdot, \leqslant_A)$ is order isomorphic to $(\mathbb{R}^+, \cdot, \leqslant)$.

Let K(f,A) be a type II distributive seminear-field of infinity type given in Remark 4.30 (1). Define $\eta: (K,+,\cdot,\leqslant) \to (K(f,A),+_\ell,\cdot,\leqslant_3)$ by

$$\eta(x) = \begin{cases} (x,0) & \text{if } x \in D_K^f, \\ \infty & \text{if } x = \infty, \\ (d,1) & \text{if } x = ad \text{ for some } d \in D_K^f. \end{cases}$$

Clearly, η is well-defined, bijective and $\eta(x+y) = \eta(x) +_\ell \eta(y)$ for all $x,y \in K$.

To show that $\eta(xy) = \eta(x)\eta(y)$ for all $x,y \in K$, let $x,y \in K \setminus \{\infty\}$.

<u>Case 1</u>: $x,y \in D_K^f$. Then $\eta(xy) = (xy,0) = (x,0)(y,0) = \eta(x)\eta(y)$.

<u>Case 2</u>: $x \in D_K^f$ and $y \in D_K^i$. Then $y = ad$ for some $d \in D_K^f$. Hence $\eta(xy) = \eta(xad) = \eta(af(x)d) = (f(x)d,1) = (x,0)(d,1) = \eta(x)\eta(y)$.

<u>Case 3</u>: $x \in D_K^i$ and $y \in D_K^f$. Then $x = ad$ for some $d \in D_K^f$. Hence $\eta(xy) = \eta(ady) = (dy,1) = (d,1)(y,0) = \eta(x)\eta(y)$.

<u>Case 4</u>: $x,y \in D_K^i$. Then $x = ad$ and $y = ar$ for some $d,r \in D_K^f$. Hence $xy = adar = aaf(d)r = f(d)r$, so $\eta(xy) = (f(d)r,0) = (d,1)(r,1) = \eta(x)\eta(y)$.

Therefore η is a homomorphism.

To show that η is isotone, let $x,y \in K$ be such that $x \leqslant y$. It is clear that if $x \leqslant y \leqslant \infty$ or $x \leqslant \infty \leqslant y$ then $\eta(x) \leqslant_3 \eta(y)$. Assume that $\infty < x \leqslant y$. Then $x = ad$ and $y = ar$ for some $d,r \in D_K^f$. So we have that $ad \leqslant ar$ which implies that $rd^{-1} \in A$. Hence $d \leqslant_A r$, so $\eta(x) = (d,1) \leqslant_3 (r,1) = \eta(y)$. Thus η is isotone.

Therefore η is an order isomorphism. #

<u>Theorem 4.33</u>. Let $(K,+,\cdot,\leqslant)$ be a type II distributive seminear-field of infinity type such that for any $x,y,z \in K$, $x \leqslant y$ implies $x + z \leqslant y + z$ or for any $x,y,z \in K$, $x \leqslant y$ implies $z + x \leqslant z + y$. Assume that $(D_K^f,+,\cdot,\leqslant)$ is order isomorphic to $(\mathbb{R}^+,+_r,\cdot,\leqslant)$.

Then $(K,+,\cdot,\leqslant)$ is order isomorphic to exactly one of the following:

(1) $(\mathbb{R}^+_\infty,+_r,\cdot,\leqslant)$.

(2) $(K(f,A),+_r,\cdot,\leqslant_3)$ where $f: (\mathbb{R}^+,\cdot) \to (\mathbb{R}^+,\cdot)$ is an isomorphism such that $f \circ f = I_{\mathbb{R}^+}$ and A is an O-set of $\mathbb{R}^+$ such that $(\mathbb{R}^+,\cdot,\leqslant_A)$ is order isomorphic to $(\mathbb{R}^+,\cdot,\leqslant)$ where $\leqslant_A$ is the compatible partial order having A as its positive cone and $+_r$, $\cdot$ and $\leqslant_3$ are defined as in Remark 4.30 (2).

<u>Proof</u>: The proof is similar to the proof of Theorem 4.32. #

Regarding type III distributive seminear-field of infinity type we cannot classify them and we close this chapter by giving some examples of type III distributive seminear-fields of infinity type. Note that if K is type a III distributive seminear-field of infinity type then $x + y = \infty$ for all $x,y \in K$.

(1) $(R^+_\infty,\cdot,\leqslant)$.

(2) $(\{2^n | n \in \mathbb{Z}\} \cup \{\infty\},\cdot,\leqslant)$.

(3) $((R^+ \times \{0\}) \cup \{\infty\} \cup (\mathbb{R}^+ \times \{1\}),+,\cdot,\leqslant)$

where $\cdot$ and $\leqslant$ are the ones given in Remark 4.26 (1).

(4) $((\{2^n | n \in \mathbb{Z}\} \times \{0\}) \cup \{\infty\} \cup (\{2^n | n \in \mathbb{Z}\} \times \{1\}),+,\cdot,\leqslant)$

where $\cdot$ and $\leqslant$ are the ones given in (3).