CHAPTER I

PRELIMINARIES

In this chapter, we shall give some notations, definitions and theorems used in this thesis. Our notations are:

$\mathbb{Z}$ = the set of all intergers,

$\mathbb{Z}^+$ = the set of all positive intergers,

$\mathbb{Z}^-$ = the set of all negative intergers,

$\mathbb{Z}_0^+ = \mathbb{Z}^+ \cup \{0\}$,

$\mathbb{Q}$ = the set of all rational numbers,

$\mathbb{Q}^+$ = the set of all positive rational numbers,

$\mathbb{Q}_0^+ = \mathbb{Q}^+ \cup \{0\}$,

$\mathbb{R}$ = the set of all real numbers,

$\mathbb{R}^+$ = the set of all positive real numbers,

$\mathbb{R}^-$ = the set of all negative real numbers and

$\mathbb{R}_0^+ = \mathbb{R}^+ \cup \{0\}$.

In this thesis, if we do not give the definition of a binary operation or an order on a subset of $\mathbb{R}$ then we shall mean the usual binary operation and the usual order on it.

Definition 1.1. Let $(P,\leqslant)$ be a partially ordered set. The opposite partial order (or the dual partial order) on P is the relation $\leqslant_{opp}$ on P defined as follows: For any $a,b \in P$, $a \leqslant_{opp} b$ if and only if $b \leqslant a$.

Definition 1.2. Let $(P,\leqslant)$ be a partially ordered set. P is a lattice if sup $\{x,y\}$ and inf $\{x,y\}$ exist for all $x,y \in P$ and P is said to be complete if every subset of P which has a lower bound has an infimum.

In [3], page 5, it was shown that a partially ordered set P is complete if and only if every subset of P which has an upper bound has a supremum.

Definition 1.3. Let $(P,\leqslant)$ be a totally ordered set. A nonempty subset S of P is called strongly dense in P if for any $x,y \in P$, $x < y$ implies that there exists a $z \in S$ such that $x < z < y$

For any nonempty subset A of a partially ordered set P, let

$$U(A) = \{x \in P \mid a \leqslant x \text{ for all } a \in A\} \text{ and}$$

$$L(A) = \{x \in P \mid x \leqslant a \text{ for all } a \in A\}.$$

If $A = \{a_1,a_2,\ldots,a_n\}$, denote $U(A) = U(a_1,a_2,\ldots,a_n)$ and $L(A) = L(a_1,a_2,\ldots,a_n)$.

Definition 1.4. A partially ordered set P is called upper [lower] directed if $U(a,b)$ [$L(a,b)$] is nonempty for all $a,b \in P$, directed if it is both upper and lower directed.

Example 1.5. Let X be a nonempty set such that $|X| \geqslant 2$. Then the set of all proper subsets of X is lower directed but not upper directed and the set of all nonempty subsets of X is upper directed but not lower directed with respect to set inclusion.

<u>Definition 1.6</u>. A subset C of a partially ordered set P is called <u>convex</u> if for any $x,y \in C$, $z \in P$, $x \leqslant z \leqslant y$ implies $z \in C$.

Note that the intersection of convex subsets of a partially ordered set is also convex.

<u>Definition 1.7</u>. Let $(P,\leqslant)$ and $(P',\leqslant')$ be partially ordered sets. A map $f: P \to P'$ is called <u>isotone</u> if for any $x,y \in P$, $x \leqslant y$ implies $f(x) \leqslant' f(y)$, an <u>order isomorphism</u> if f is bijective and f, f^{-1} are isotone.

<u>Remark 1.8</u>. Let $(P,\leqslant)$ and $(P',\leqslant')$ be totally ordered sets and $f: P \to P'$ an isotone bijection. Then f^{-1} is isotone.

<u>Proof</u>: Let $x,y \in P'$ be such that $x <' y$. If $f^{-1}(y) < f^{-1}(x)$ then $y = f(f^{-1}(y)) < f(f^{-1}(x)) = x$, a contradiction. Hence $f^{-1}(x) < f^{-1}(y)$. Therefore f^{-1} isotone. #

<u>Proposition 1.9</u>. Let $(P,\leqslant)$ and $(P',\leqslant')$ be partially ordered sets, $f: P \to P'$ an isotone map and C' a convex subset of P'. Then $f^{-1}(C')$ is a convex subset of P.

<u>Proof</u>: Let $x,y \in f^{-1}(C')$ and $z \in P$ be such that $x \leqslant z \leqslant y$. Then $f(x) \leqslant' f(z) \leqslant' f(y)$. But C' is convex, so $f(z) \in C'$. Hence $z \in f^{-1}(C')$. Therefore $f^{-1}(C')$ is convex. #

<u>Definition 1.10</u>. A partial order $\leqslant$ on a semigroup $(S,\cdot)$ is said to be <u>compatible</u> if for any $x,y,z \in S$, $x \leqslant y$ implies $xz \leqslant yz$ and $zx \leqslant zy$.

Definition 1.11. A system $(S,\cdot,\leqslant)$ is a partially [totally] ordered semigroup if $(S,\cdot)$ is a semigroup and $\leqslant$ is a compatible partial [total] order on S.

Example 1.12. (1) $(\mathbb{Z},+,\leqslant)$, $(\mathbb{Q},+,\leqslant)$ and $(\mathbb{R},+,\leqslant)$ are totally ordered groups.

(2) Let $\mathbb{C}$ be the additive group of complex numbers. For any $a + bi$, $c + di \in \mathbb{C}$, define $a + bi \leqslant c + di$ if and only if either $a < c$ or $a = c$ and $b \leqslant d$. Then $(\mathbb{C},+,\leqslant)$ is a partially ordered group.

(3) In $\mathbb{Q}^+$, define $a \leqslant b$ if and only if $\frac{b}{a}$ is an integer. Then $(\mathbb{Q}^+,\cdot,\leqslant)$ is a partially ordered group.

(4) Let $G = \{(x,y) \mid x,y \in \mathbb{R}\}$. For any $u,v,x,y \in \mathbb{R}$, define $(u,v)\cdot(x,y) = (u+x,\ e^x v+y)$ and $(u,v) \leqslant (x,y)$ if and only if either $u < x$ or $u = x$ and $v \leqslant y$. Then $(G,\cdot,\leqslant)$ is a partially ordered group.

(5) Let $M = \left\{ \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} / \ a,b,c \in \mathbb{R} \right\}$.

Define $\begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \leqslant \begin{bmatrix} 1 & d & f \\ 0 & 1 & e \\ 0 & 0 & 1 \end{bmatrix}$ if and only if either 1) $a < d$, 2) $a = d$ and $b < e$ or 3) $a = d$, $b = e$ and $c + bd \leqslant ab + f$. Then $(M,\circ,\leqslant)$ is a partially ordered group.

Definition 1.13. Let $(G,\cdot,\leqslant)$ be a totally ordered group. G is called Archimedian if for any $x,y \in G \setminus \{1\}$, $x < y$ implies that there exists an $n \in \mathbb{Z}$ such that $y < x^n$.

In [3], page 10, it was shown that a totally ordered group G is Archimedean if and only if for any $x,y \in G \setminus \{1\}$, $x < y$ implies that there exists an $n \in \mathbb{Z}$ such that $y^n < x$.

<u>Proposition 1.14</u> ([3]). Let G be a complete totally ordered group. Then G is Archimedean.

<u>Theorem 1.15</u> ([3]). Any complete totally ordered group is order isomorphic to exactly one of the following:

(1) $(\{1\},\cdot,\leqslant)$.

(2) $(\{2^n \mid n \in \mathbb{Z}\},\cdot,\leqslant)$.

(3) $(\mathbb{R}^+,\cdot,\leqslant)$.

<u>Proposition 1.16</u> ([1]). For any partially ordered group G and $a_o \in G$, G is directed if and only if $U(x,a_o) \neq \emptyset$ for all $x \in G$.

<u>Proposition 1.17</u> ([1]). Let G be a partially ordered group. Then the following statements hold:

(1) If G contains an element a_o such that $a_o \geqslant 1$ and $U(a_o)$ generates G then G is directed. Also, if G is directed and $a_o \in G$ then for any $b \in G$ there exist $y,z \in U(a_o)$ such that $b = yz^{-1}$.

(2) G is a lattice if and only if $\sup\{x,1\}$ exists for all $x \in G$.

(3) If G is a lattice then for any $x,y,z \in G$, $\sup\{xz,yz\} = (\sup\{x,y\})\cdot z$ and $\sup\{zx,zy\} = z\cdot\sup\{x,y\}$.

<u>Definition 1.18</u>. A subset A of a group G is called an <u>O-set</u> of G

if it satisfies the following conditions:

(1) $A \cap A^{-1} = \{1\}$.

(2) $A^2 \subseteq A$.

(3) $xAx^{-1} \subseteq A$ for all $x \in G$.

Note that for any group G, $\{1\}$ is an O-set of G and for any partially ordered group G, $\{x \in G \mid x \geqslant 1\}$ which is called the <u>positive cone</u> of G is an O-set of G.

<u>Theorem 1.19</u> ([1]). A subset A of a group G is an O-set of G if and only if there exists a compatible partial order on G such that A is the positive cone induced by $\leqslant$.

<u>Proof</u>: Assume that A is an O-set of G. Define a relation $\leqslant$ on G as follows: For $x,y \in G$, $x \leqslant y$ if and only if $x^{-1}y \in A$. We shall show that $\leqslant$ is a partial order on G. Since $1 \in A$, $\leqslant$ is reflexive. Let $x,y \in G$ be such that $x \leqslant y$ and $y \leqslant x$. Then $x^{-1}y, y^{-1}x \in A$. But $x^{-1}y = (y^{-1}x)^{-1}$, so $x^{-1}y \in A \cap A^{-1}$. Hence $x = y$. Thus $\leqslant$ is anti-symmetric. Let $x,y,z \in G$ be such that $x \leqslant y$ and $y \leqslant z$. Then $x^{-1}y, y^{-1}z \in A$. Since $A^2 \subseteq A$, so $x^{-1}z = (x^{-1}y)(y^{-1}z) \in A$ and thus we get that $x \leqslant z$. Hence $\leqslant$ is transitive. Therefore $\leqslant$ is a partial order on G.

To show that $\leqslant$ is compatible, let $x,y,z \in G$ be such that $x \leqslant y$. Then $x^{-1}y \in A$. Since $(zx)^{-1}(zy) = x^{-1}y \in A$, $zx \leqslant zy$. Since $z^{-1}Az \subseteq A$, so $(xz)^{-1}(yz) = z^{-1}(x^{-1}y)z \in A$, hence $xz \leqslant yz$. Therefore $\leqslant$ is compatible.

Clearly, $A = \{x \in G \mid x \geqslant 1\}$.

The converse follows from the above note.

Note that from Theorem 1.19, $\leqslant$ is the unique compatible partial order on G such that A is the positive cone. To prove this, assume that $\overset{*}{\leqslant}$ is a compatible partial order on G such that $A = \{x \in G \mid x \overset{*}{\geqslant} 1\}$. We shall show that $\leqslant = \overset{*}{\leqslant}$. Let $x,y \in G$ be such that $x \leqslant y$. Then $x^{-1}y \in A$, so $x^{-1}y \overset{*}{\geqslant} 1$. Hence $y \overset{*}{\geqslant} x$. Thus $\leqslant \subseteq \overset{*}{\leqslant}$. Similarly, $\overset{*}{\leqslant} \subseteq \leqslant$. Therefore $\leqslant = \overset{*}{\leqslant}$.

Let G be a group. Then the set of all O-sets of G and the set of all compatible partial orders on G are partially ordered sets with respect to set inclusion.

The following corollary is obtained from Theorem 1.19.

Corollary 1.20. Let G be a group, $\mathcal{A}$ the set of all O-sets of G and $\mathcal{B}$ the set of all compatible partial orders on G. Then $\mathcal{A}$ and $\mathcal{B}$ are order isomorphic.

Proof: Define $\phi : \mathcal{A} \to \mathcal{B}$ as follows: Let $A \in \mathcal{A}$. Then Theorem 1.19 determines a unique compatible partial order $\leqslant_A$ on G. Define $\phi(A) = \leqslant_A$. Clearly, ϕ is a bijection.

To show that ϕ is isotone, let $A,B \in \mathcal{A}$ be such that $A \subseteq B$. Then there exist compatible partial orders $\leqslant_A$ and $\leqslant_B$ such that $A = \{x \in G \mid x \geqslant_A 1\}$ and $B = \{x \in G \mid x \geqslant_B 1\}$. We shall show that $\leqslant_A \subseteq \leqslant_B$. Let $x,y \in G$ be such that $x \leqslant_A y$. Then $x^{-1}y \in A$. Since $A \subseteq B$, so $x^{-1}y \in B$, hence $x \leqslant_B y$. Thus $\leqslant_A \subseteq \leqslant_B$, so $\phi(A) \subseteq \phi(B)$. Hence ϕ is isotone.

To show that ϕ^{-1} is isotone, let $\leqslant, \overset{*}{\leqslant} \in \mathcal{B}$ be such that $\leqslant \subseteq \overset{*}{\leqslant}$. Let $x \in \phi^{-1}(\leqslant)$. Since $\phi^{-1}(\leqslant) = \{y \in G \mid y \geqslant 1\}$, $x \geqslant 1$, so

$x \overset{*}{\geqslant} 1$. But $\psi^{-1}(\overset{*}{\leqslant}) = \{ y \in G \mid y \overset{*}{\geqslant} 1 \}$, so $x \in \psi^{-1}(\overset{*}{\leqslant})$. Thus $\psi^{-1}(\leqslant) \subseteq \psi^{-1}(\overset{*}{\leqslant})$. Hence ψ^{-1} is isotone.

Therefore ψ is an order isomorphism. #

Theorem 1.21 ([1]). Let P be a semigroup with identity 1. Then there exists a partially ordered group G having P as its positive cone if and only if

(i) P is cancellative,

(ii) $Pa = aP$ for all $a \in P$ and

(iii) for any $a,b \in P$, $ab = 1$ implies $a = b = 1$.

Proof: Since U(1) of a partially ordered group G has properties (i) - (iii), so if P is isomorphic to U(1) then P also has properties (i) - (iii).

Conversely, assume that P satisfies properties (i) - (iii). By properties (i) and (ii) of P, we get that for any $a,x \in P$ there exists a unique $x_a \in P$ such that $xa = ax_a$. Clearly, $a_a = a$ and $1_a = 1$ for all $a \in P$. For any $a,b,x,y \in P$, $a(xy)_a = (xy)a = x(ya) = x(ay_a) = (xa)y_a = ax_ay_a$ and $(ab)x_{ab} = x(ab) = (xa)b = (ax_a)b = a(x_ab) = ab(x_a)_b$, so by property (i) of P, $(xy)_a = x_ay_a$ and $x_{ab} = (x_a)_b$ for all $a,b,x,y \in P$.

Define a relation $\sim$ on $P \times P$ as follows: For $a,b,c,d \in P$, $(a,b) \sim (c,d)$ if and only if $ad_b = cb$. We shall show that $\sim$ is an equivalence relation. Clearly, $\sim$ is reflexive. Let $a,b,c,d \in P$ be such that $(a,b) \sim (c,d)$. Then $ad_b = cb$, so we get that $add_{bd} = ad(d_b)_d = ad_bd = cbd = c(bd)_{bd} = cb_{bd}d_{bd} = cb_dd_{bd}$. Hence $ad = cb_d$, so $(c,d) \sim (a,b)$. Therefore $\sim$ is symmetric. Let $a,b,c,d,e,f \in P$ be such that $(a,b) \sim (c,d)$ and $(c,d) \sim (e,f)$. Then $ad_b = cb$ and $cf_d = ed$, so we get that $af_b d_b = a(fd)_b = a(df_d)_b = ad_b(f_d)_b = cb(f_d)_b = cf_d b = edb = ebd_b$. Thus $af_b = eb$, so $(a,b) \sim (e,f)$. Hence $\sim$ is transitive. Therefore $\sim$ is an equivalence relation.

Let $G = \frac{P \times P}{\sim}$. Define an operation $\cdot$ on G by $[(a,b)]\cdot[(c,d)] = [(ac_b, db)]$ for all $a,b,c,d \in P$. We shall show that $\cdot$ is well-defined. Let $v,w,x,y \in P$ be such that $(v,w) \in [(a,b)]$ and $(x,y) \in [(c,d)]$. Then $(a,b) \sim (v,w)$ and $(c,d) \sim (x,y)$, so $aw_b = vb$ and $cy_d = xd$. Hence we have that $(ac_b)(yw)_{db} = a(c(yw)_d)_b = a(cy_d w_d)_b = a(xdw_d)_b = a(xwd)_b = a(wx_w d)_b = aw_b(x_w d)_b = vb(x_w d)_b = v(x_w d)b = (vx_w)db$. Thus $(ac_b, db) \sim (vx_w, yw)$ which implies that $[(a,b)]\cdot[(c,d)] = [(ac_b, db)] = [(vx_w, yw)] = [(v,w)]\cdot[(x,y)]$. Therefore $\cdot$ is well-defined.

To show that $\cdot$ is associative, let $a,b,c,d,e,f \in P$. Then

$$\begin{aligned}([(a,b)]\cdot[(c,d)])[(e,f)] &= [(ac_b e_{db}, fdb)] \\ &= [(a(ce_d)_b, (fd)b)] \\ &= [(a,b)]\cdot[(ce_d, fd)] \\ &= [(a,b)]([(c,d)]\cdot[(e,f)]).\end{aligned}$$

Hence $\cdot$ is associative.

Clearly, $[(1,1)]$ is the identity of G and $[(b,a)]$ is the inverse of $[(a,b)]$ for all $a,b \in P$. Therefore G is a group. Define $i: P \to G$ by $i(a) = [(a,1)]$ for all $a \in P$. Let $a,b \in P$ be such that $i(a) = i(b)$. Then $(a,1) \sim (b,1)$, so $a = b$. Hence i is injective. For any $a,b \in P$, $i(ab) = [(ab,1)] = [(a,1)][(b,1)] = i(a)i(b)$, so i is a homomorphism. Therefore i is a monomorphism.

We shall show that $i(P)$ is an O-set of G. Let $\alpha \in i(P) \cap i(P)^{-1}$. Then $\alpha = i(a) = i(b)^{-1}$ for some $a,b \in P$. Thus $i(ab) = i(a)i(b) = [(1,1)]$, so $ab = 1$. By property (iii) of P, $a = b = 1$. Thus $\alpha = [(1,1)]$. Hence $i(P) \cap i(P)^{-1} = \{[(1,1)]\}$. For any $a,b \in P$, $i(a)i(b) = [(a,1)][(b,1)] = [(ab,1)] = i(ab) \in i(P)$. Hence $i(P)^2 \subseteq i(P)$. Let $a,b,x \in P$. Then $[(a,b)]i(x)[(a,b)]^{-1} = [(a,b)][(x,1)][(b,a)] = [(ax_b b,ab)]$. Since $aP = Pa$, $ax_b = ya$ for some $y \in P$. Then we have that $ax_b b = yab$, so $(ax_b b,ab) \sim (y,1)$. It follows that $[(a,b)]i(x)[(a,b)]^{-1} = [(y,1)] = i(y) \in i(P)$. Hence $\alpha i(P)\alpha^{-1} \subseteq i(P)$ for all $\alpha \in G$. Therefore $i(P)$ is an O-set of G. By Theorem 1.19, there exists a compatible partial order on G such that $i(P)$ is the positive cone. #

Remark 1.22. It follows from Theorem 1.21 that if a semigroup P with 1 satisfies properties (i) - (iii) then there exists a partially ordered group G having P as its positive cone up to isomorphism and G which is defined as above is the smallest partially ordered group having P as its positive cone and it is also directed.

To prove this, let $(G', \leqslant')$ be a partially ordered group and $j: P \to G'$ a monomorphism such that $j(P)$ is the positive cone of G'. Since $xa = ax_a$ for all $a,x \in P$, $j(x_a) = j(a)^{-1}j(x)j(a)$ for all $a,x \in P$. Define $f: G \to G'$ by $f([(a,b)]) = j(a)j(b)^{-1}$ for all $a,b \in P$. We shall show that f is well-defined. Let $c,d \in P$ be such that $(c,d) \in [(a,b)]$. Then $(a,b) \sim (c,d)$, so $ad_b = cb$. Hence $(j(a)j(b)^{-1}j(d))j(b) = j(a)j(d_b) = j(ad_b) = j(cb) = j(c)j(b)$. Thus $j(a)j(b)^{-1}j(d) = j(c)$, so $j(a)j(b)^{-1} = j(c)j(d)^{-1}$. Therefore f is well-defined. For each $a,b,c,d \in P$, $f([(a,b)]\cdot[(c,d)]) =$

$f([(ac_b,db)]) = j(ac_b)j(db)^{-1} = j(a)j(c_b)j(b)^{-1}j(d)^{-1} =$

$j(a)(j(b)^{-1}j(c)j(b))j(b)^{-1}j(d)^{-1} = j(a)j(b)^{-1}j(c)j(d)^{-1} =$

$f([(a,b)])f([(c,d)])$. Hence f is a homomorphism.

To show that f is injective, let $a,b,c,d \in P$ be such that $f([(a,b)]) = f([(c,d)])$. Then $j(a)j(b)^{-1} = j(c)j(d)^{-1}$, so we get that $j(ad_b) = j(a)j(d_b) = (j(a)j(b)^{-1}j(d))j(b) = j(c)j(b) = j(cb)$. Since j is injective, so $ad_b = cb$ and we have that $[(a,b)] = [(c,d)]$. Hence f is injective.

To show that f is isotone, let $\alpha,\beta \in G$ be such that $\alpha \leqslant \beta$. Since $i(P)$ is the positive cone of G, $\alpha^{-1}\beta \in i(P)$. Then $\alpha^{-1}\beta = i(a) = [(a,1)]$ for some $a \in P$. Thus $f(\alpha)^{-1}f(\beta) = f(\alpha^{-1}\beta) = f([(a,1)]) = j(a) \in j(P)$. Since $j(P)$ is the positive cone of G', $f(\alpha) \leqslant' f(\beta)$. Hence f is isotone.

To show that for any $\alpha,\beta \in G$, $f(\alpha) \leqslant' f(\beta)$ implies $\alpha \leqslant \beta$, let $\alpha,\beta \in G$ be such that $f(\alpha) \leqslant' f(\beta)$. Then $f(\alpha^{-1}\beta) = f(\alpha)^{-1}f(\beta) \in j(P)$, so $f(\alpha^{-1}\beta) = j(a) = f([(a,1)])$ for some $a \in P$. Since f is injective, $\alpha^{-1}\beta = [(a,1)] = i(a) \in i(P)$. Hence $\alpha \leqslant \beta$.

Therefore f is an order monomorphism. For each $a \in P$, $(f \circ i)(a) = f(i(a)) = f([(a,1)]) = j(a)$, so we get that $f \circ i = j$. To show that f is the unique order monomorphism such that $f \circ i = j$, let $g: G \to G'$ be an order monomorphism such that $g \circ i = j$. Then for each $a,b \in P$, $f([(a,b)]) = j(a)j(b)^{-1} = (g \circ i)(a)(g \circ i)(b)^{-1} = g([(a,1)])g([(b,1)])^{-1} = g([(a,1)][1,b)]) = g([(a,b)])$. Hence $f = g$.

<u>Definition 1.23</u>. A system $(S,+,\cdot)$ is called a <u>distributive seminear-ring</u> if

(i) $(S,+)$ and $(S,\cdot)$ are semigroups and

(ii) $(x+y)z = xz + yz$ and $z(x+y) = zx + zy$ for all $x,y,z \in S$.

The operations $+$ and $\cdot$ are called the addition and multiplication of the distributive seminear-ring, respectively.

A subset H of S is called a subseminear-ring of S if $(H,+,\cdot)$ is a distributive seminear-ring.

Definition 1.24. Let $(S,+,\cdot)$ be a distributive seminear-ring. An element e of S is called a multiplicative [additive] identity of S if e is the identity of the semigroup $(S,\cdot)$ $[(S,+)]$. An element a of S is called a multiplicative [additive] zero of S if a is the zero of the semigroup $(S,\cdot)[(S,+)]$. S is called additively commutative if $(S,+)$ is a commutative semigroup, multiplicatively [additively] cancellative if $(S,\cdot)[(S,+)]$ is a cancellative semigroup and 0-multiplicatively cancellative if for any $x,y,z \in S$, $xy = xz$ and $x \neq 0$ imply that $y = z$ and for any $x,y,z \in S, yx = zx$ and $x = \emptyset$ imply that $y=z$

Definition 1.25. An equivalence relation ρ on a distributive seminear-ring S is called a congruence on S if for any $x,y,z \in S$, $x \rho y$ implies $(x+z)\rho(y+z)$, $(z+x)\rho(z+y)$, $(xz)\rho(yz)$ and $(zx)\rho(zy)$.

Given a distributive seminear-ring S and a congruence ρ on S, let $S_{/\rho}$ denote the set of all congruence classes of ρ.

Definition 1.26. Let S and S' be distributive seminear-rings. A map $f: S \to S'$ is called a homomorphism of S into S' if for any $x,y \in S$, $f(x+y) = f(x)+f(y)$ and $f(xy) = f(x)f(y)$.

The following notation will be used in the thesis : For a distributive seminear-ring S and $x \in S$, let

$$LI_S(x) = \{y \in S \mid y + x = x\} \text{ and}$$

$$RI_S(x) = \{y \in S \mid x + y = x\}.$$

Definition 1.27. Let $(S,+,\cdot)$ be a distributive seminear-ring. S is called a distributive ratio seminear-ring if $(S,\cdot)$ is a group, a distributive near-ring if $(S,+)$ is a group, a distributive seminear-field if $(S,\cdot)$ is a group with zero. A subset H of S is called a ratio subseminear-ring if $(H,+,\cdot)$ is a distributive ratio seminear-ring. A subnear-ring and a subseminear-field are defined similarly.

Example 1.28. (1) Let $(G,\cdot)$ be a group. Define $x + y = x$ $[x+y = y]$ for all $x,y \in G$. Then $(G,+,\cdot)$ is a distributive ratio seminear-ring.

(2) Let $(G,+)$ be a group (not necessarily abelian) with identity 0. Define $x \cdot y = 0$ for all $x,y \in G$. Then $(G,+,\cdot)$ is a distributive near-ring.

(3) Let R be a noncommutative ring and G an additively noncommutative distributive near-ring in (2). Then $(G \times R,+,\cdot)$ is a multiplicatively and additively noncommutative distributive near-ring where + and $\cdot$ are defined by $(a,b)+(c,d) = (a+c,b+d)$ and $(a,b)\cdot(c,d) = (ac,bd)$ for all $a,c \in G, b,d \in R$.

(4) $(\{0,1\},+,\cdot)$ where + and $\cdot$ are defined by the operation tables:

+	0	1
0	0	1
1	1	1

·	0	1
0	0	0
1	0	1

is a distributive seminear-field and is called the Boolean distributive seminear-field.

(5) Let $(G,\cdot)$ be a group with zero a. Define

5.1) $x + y = a$ for all $x,y \in G$ or

5.2) $$x + y = \begin{cases} a & \text{if } x \neq y, \\ x & \text{if } x = y \end{cases}$$

for all $x,y \in G$.

Then $(G,+,\cdot)$ is a distributive seminear-field.

(6) Let D be a distributive ratio seminear-ring, 0 and ∞ symbols not representing any element of D. We can adjoin 0 to D and ∞ to D to get distributive seminear-fields $D \cup \{0\}$ and $D \cup \{\infty\}$ by defining $x + 0 = 0 + x = x$, $x \cdot 0 = 0 \cdot x = 0 \cdot 0 = 0 + 0 = 0$ and $x + \infty = \infty + x = \infty + \infty = x \cdot \infty = \infty \cdot x = \infty \cdot \infty = \infty$ for all $x \in D$

Remark 1.29. Let D be a distributive ratio seminear-ring such that $1 + 1 = 1$. For each $x \in D$, $LI_D(x)$ and $RI_D(x)$ are both nonempty since $x + x = x$. The following results will be used in the thesis:

(1) For any $x \in D$, $LI_D(x) = xLI_D(1)$.

(2) If $x \in LI_D(1)$ then $x^n \in LI_D(1)$ for all $n \in \mathbb{Z}^+$.

The statements (1) and (2) are also true for $RI_D(x)$ for all $x \in D$.

Remark 1.30. For any distributive near-ring R, $ab + cd = cd + ab$ for all $a,b,c,d \in R$ and for any distributive seminear-field K, if $1 + 1 = 1$ then K cannot be a skew field.

Definition 1.31. Let D and D' be distributive ratio seminear-rings and $f: D \to D'$ a homomorphism. The kernel of f, denoted by ker f, is $\{x \in D \mid f(x) = 1'\}$.

<u>Definition 1.32</u>. Let D be a distributive ratio seminear-ring. A subset C of D is called a <u>C-set</u> of D if C is a multiplicative normal subgroup of D such that $(x+1)^{-1}(x+y)$, $(1+x)^{-1}(y+x) \in C$ for all $x \in D$, $y \in C$. A C-set C of D is a <u>prime C-set</u> in D if $(x+1)^{-1}(x+y)$, $(1+x)^{-1}(y+x) \notin C$ for all $x \in D$, $y \notin C$.

<u>Proposition 1.33</u>. Let D and D′ be distributive ratio seminear-rings and $f: D \to D'$ a homomorphism. Then the following statements hold:

(1) ker f is a C-set of D.

(2) If C′ is a C-set of D′ then $f^{-1}(C')$ is a C-set of D.

(3) If f is onto and C is a C-set of D then f(C) is a C-set of of D′.

<u>Proof</u>: (1) Since f is a homomorphism, ker f is a multiplicative normal subgroup of D. Let $x \in D$ and $y \in \ker f$. Then $f((x+1)^{-1}(x+y)) = (f(x)+1')^{-1}(f(x)+1') = 1'$ so $(x+1)^{-1}(x+y) \in \ker f$. Similarly, $(1+x)^{-1}(y+x) \in \ker f$. Hence ker f is a C-set of D.

(2) Let C′ be a C-set of D. Since C′ is a multiplicative normal subgroup of D′, $f^{-1}(C')$ is a multiplicative normal subgroup of D. Let $x \in D$ and $y \in f^{-1}(C')$. Since C′ is a C-set of D′, so $f((x+1)^{-1}(x+y)) = (f(x)+1')^{-1}(f(x)+f(y)) \in C'$, hence $(x+1)^{-1}(x+y) \in f^{-1}(C')$. Similarly, $(1+x)^{-1}(y+x) \in f^{-1}(C')$. Therefore $f^{-1}(C')$ is a C-set of D.

(3) Assume that f is onto and C is a C-set of D. Since C is a multiplicative normal subgroup of D and f is onto, f(C) is a multiplicative normal subgroup of D′. Let $y \in D'$ and $a \in C$. Then $y = f(x)$ for some $x \in D$. Since C is a C-set of D, $(y+1')^{-1}(y+f(a)) = (f(x)+f(1))^{-1}(f(x)+f(a)) = f((x+1)^{-1}(x+a)) \in f(C)$. Similarly,

$(1'+y)^{-1}(f(a)+y) \in f(c)$. Hence $f(C)$ is a C-set of D'. #

Definition 1.34. Let R and R′ be distributive near-rings and $f: R \to R'$ a homomorphism. The kernel of f, denoted by ker f, is $\{x \in R \mid f(x) = 0'\}$.

Definition 1.35. Let R be a distributive near-ring. A subset J of R is called an ideal of R if J is an additive normal subgroup of R such that xy, yx $\in$ J for all x $\in$ R, y $\in$ J. An ideal J of R is a prime ideal in R if for any x,y $\in$ R, xy $\in$ J implies that x $\in$ J or y $\in$ J.

Proposition 1.36. Let R and R′ be distributive near-rings and $f: R \to R'$ a homomorphism. Then the following statements hold:

(1) ker f is an ideal of R.

(2) If J′ is an ideal of R′ then $f^{-1}(J')$ is an ideal of R.

(4) If f is onto and J is an ideal of R then f(J) is an ideal of R′.

Proof: (1) Since f is a homomorphism, ker f is an additive normal subgroup of R. Let x $\in$ R and y $\in$ ker f. Then $f(xy) = f(x)f(y) = 0'$, so xy $\in$ ker f. Similarly, yx $\in$ ker f. Hence ker f is an ideal in R.

(2) Let J′ be an ideal of R′. Then $f^{-1}(J')$ is an additive normal subgroup of R. Let x $\in$ R and y $\in f^{-1}(J')$. Then f(y) $\in$ J′, so $f(xy) = f(x)f(y) \in J'$. Hence xy $\in f^{-1}(J')$. Similarly, yx $\in f^{-1}(J')$. Therefore $f^{-1}(J')$ is an ideal in R.

(3) Assume that f is onto and J is an ideal of R. Then $f(J)$ is an additive normal subgroup of R'. Let $y \in R'$ and $a \in J$. Then $y = f(x)$ for some $x \in R$. Then $yf(a) = f(xa) \in f(J)$. Similarly, $f(a)y \in f(J)$. Hence $f(J)$ is an ideal in R'.

Definition 1.37. A quotient ratio seminear-ring of a distributive ratio seminear-ring D is a pair (D', ψ) where D' is a distributive ratio seminear-ring and ψ is a homomorphism of D onto D'. A quotient near-ring of a distributive near-ring is defined similarly.

We shall now give the construction of a quotient ratio seminear-ring of a distributive ratio seminear-ring D and the construction of a quotient near-ring of a distributive near-ring R which appears in [4].

Let C be a C-set of D and $\rho_1 = \{(a,b) \in D \times D \mid a^{-1}b \in C\}$. Then ρ_1 is a congruence on D and $(D/\rho_1, +, \cdot)$ is a distributive ratio seminear-ring where + and $\cdot$ are defined by $[a]+[b] = [a+b]$ and $[a]\cdot[b] = [ab]$. For convenience, we denote D/ρ_1 by D/C. Note that $(D/C, \pi)$ is a quotient ratio seminear-ring of D where π is the natural projection map of D onto D/C.

Let J be an ideal of R and $\rho_2 = \{(a,b) \in R \times R \mid -a + b \in J\}$. Then ρ_2 is a congruence on R and $(R/\rho_2, +, \cdot)$ is a distributive near-ring where + and $\cdot$ are defined as above. For convenience, we denote R/ρ_2 by R/J. Note that $(R/J, \pi)$ is a quotient near-ring of R where π is the natural projection map to R onto R/J.

Remark 1.38. The following statements holds:

(1) A C-set C of a distributive ratio seminear-ring D is prime if and only if $D/_C$ is additively cancellative.

(2) An ideal J of a distributive near-ring R is prime if and only if $R/_J$ is 0-multiplicatively cancellative.

Proof: (1) Assume that C is a prime C-set of D. Let $x,y,z \in D$ be such that $[x]+[z] = [y]+[z]$. Then $(1+x^{-1}z)^{-1}(x^{-1}y+x^{-1}z) = (x+z)^{-1}(y+z) \in C$. By assumption, $x^{-1}y \in C$, so $[x] = [y]$. Similarly, if $[z]+[x] = [z]+[y]$ then $[x] = [y]$. Hence $D/_C$ is additively cancellative.

Conversely, assume that $D/_C$ is additively cancellative. Let $x \in D$ and $y \notin C$. Suppose that $(x+1)^{-1}(x+y) \in C$. Then $[x]+[1] = [x+1] = [x+y] = [x]+[y]$. Hence $[y] = [1]$, so $y \in C$, a contradiction. Thus $(x+1)^{-1}(x+y) \notin C$. Similarly, $(1+x)^{-1}(y+x) \notin C$. Therefore C is a prime C-set of D.

(2) Assume that J is a prime ideal in R. Let $x,y,z \in R$ be such that $[x][z] = [y][z]$ and $z \notin J$. Then $(-x+y)z = -xz + yz \in J$. By assumption, $-x + y \in J$, so $[x] = [y]$. Similarly if $[z][x] = [z][y]$ and $z \notin J$ then $[x] = [y]$. Hence $R/_J$ is 0-multiplicatively cancellative.

Conversely, assume that $R/_J$ is 0-multiplicatively cancellative. It is easily shown that for any $\alpha,\beta \in R/_J$, $\alpha\beta = [0]$ implies $\alpha = [0]$ or $\beta = [0]$. Let $x,y \in R$ be such that $xy \in J$. Then $[x][y] = [xy] = [0]$, so $[x] = [0]$ or $[y] = [0]$. Hence $x \in J$ or $y \in J$. Therefore J is a prime ideal in R. #

Theorem 1.39 ([5)]. The smallest ratio subseminear-ring (called the prime distributive ratio seminear-ring) of a distributive ratio seminear-ring is either isomorphic to $(\mathbb{Q}^+,+,\cdot)$ or $\{1\}$.

Theorem 1.40 ([5]). A multiplicative zero of a distributive seminear-field is either an additive identity or an additive zero.

Theorem 1.40 indicates that there are two types of distributive seminear-fields. Let K be a distributive seminear-field with a as its multiplicative zero. If a is the additive identity of K, we call K a distributive seminear-field of zero type and denote a by 0 and if a is the additive zero of K, we call K a distributive seminear-field of infinity type and denote a by ∞.

Theorem 1.41 ([5]). The smallest subseminear-field (called the prime distributive seminear-field) of a distributive seminear-field of zero type is isomorphic to exactly one of the following:

(1) $(\mathbb{Q}_0^+,+,\cdot)$.

(2) $\mathbb{Z}_p$, the set of all congruence classes of $\mathbb{Z}$ modulo a prime p in $\mathbb{Z}^+$.

(3) The Boolean distributive seminear-field.

Let K be a distributive seminear-field of infinity type and $x \in K$. Let

$$\mathrm{LCor}_K(x) = \{y \in K \mid y + x = \infty\} \quad \text{and}$$

$$\mathrm{RCor}_K(x) = \{y \in K \mid x + y = \infty\}.$$

Note that $\mathrm{LCor}_K(x)$ and $\mathrm{RCor}_K(x)$ are nonempty since $\infty + x = x + \infty = \infty$.

Remark 1.42. Let K be a distributive seminear-field of infinity type. Then the following statements hold:

(1) $LCor_K(x) = xLCor_K(1)$ for all $x \in K \setminus \{\infty\}$.

(2) $LCor_K(1) = \{\infty\}$ if and only if $RCor_K(1) = \{\infty\}$.

(3) $LCor_K(1) = K$ if and only if $RCor_K(1) = K$.

(4) $LCor_K(1) \setminus \{\infty\} = (RCor_K(1) \setminus \{\infty\})^{-1}$.

Proposition 1.43 ([3]). Let K be a distributive seminear-field of zero type. If K is not a skew field then $a + b \neq 0$ for all $a,b \in K \setminus \{0\}$.

Proposition 1.44 ([3]). If $f: (\mathbb{R},+,\leqslant) \to (\mathbb{R},+,\leqslant)$ is an order homomorphism then there exists an $a \in \mathbb{R}_0^+$ such that $f(x) = ax$ for all $x \in \mathbb{R}$.

Proposition 1.45. If $f: (\mathbb{R}^+,\cdot,\leqslant) \to (\mathbb{R}^+,\cdot,\leqslant)$ is an order monomorphism then there exists an $a \in \mathbb{R}^+$ such that $f(x) = x^a$ for all $x \in \mathbb{R}^+$ (hence f must be a bijection).

Proof: Assume that $f: (\mathbb{R}^+,\cdot,\leqslant) \to (\mathbb{R}^+,\cdot,\leqslant)$ is an order isomorphism. Define $g: (\mathbb{R},+,\leqslant) \to (\mathbb{R},+,\leqslant)$ by $g(x) = \ln(f(e^x))$ for all $x \in \mathbb{R}$. For each $x,y \in \mathbb{R}$, $g(x+y) = \ln(f(e^{x+y})) = \ln(f(e^x)f(e^y)) = \ln(f(e^x)) + \ln(f(e^y)) = g(x) + g(y)$, so we get that g is a homomorphism. For any $x,y \in \mathbb{R}$, if $x \leqslant y$ then $e^x \leqslant e^y$, so $f(e^x) \leqslant f(e^y)$, hence we have that $g(x) = \ln(f(e^x)) \leqslant \ln(f(e^y)) = g(y)$. Thus g is isotone. By Proposition 1.44, there exists an $a \in \mathbb{R}_0^+$ such that $g(x) = ax$ for all $x \in \mathbb{R}$. If $a = 0$ then $g = 0$, so $f(e^x) = e^{\ln(f(e^x))} = e^{g(x)} = e^0 = 1$

for all $x \in \mathbb{R}$ which contradicts the fact that f is injective. Hence $a \in \mathbb{R}^+$. Let $x \in \mathbb{R}^+$. Then $x = e^y$ for some $y \in \mathbb{R}$. Hence $f(x) = f(e^y) = e^{\ln(f(e^y))} = e^{g(y)} = e^{ay} = x^a$. Therefore $f(x) = x^a$ for all $x \in \mathbb{R}^+$. #