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SOLVING LINEAR PROGRAMMING PROBLEM WITH UNCERTAINTY: PROBABILITY INTERVAL AND RANDOM SET PARAMETERS

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ในวิทยานิพนธ์ฉบับนี้เราศึกษาความสัมพันธ์ระหว่างข้อมูลแบบช่วงความน่าจะเป็นและแบบเซตสุ่ม และการแก้ปัญหากำหนดการเชิงเส้นที่มีความไม่แน่นอนโดยมีพารามิเตอร์แบบช่วงความน่าจะเป็นและแบบเซตสุ่ม เราพบเงื่อนไขในการตรวจสอบว่าข้อมูลแบบช่วงความน่าจะเป็นที่มีอยู่จะสามารถนำมาสร้างเป็นข้อมูลแบบเซตสุ่มที่ให้ข้อมูลเหมือนเดิมได้หรือไม่ ซึ่งถ้าข้อมูลแบบช่วงความน่าจะเป็นสอดคล้องกับเงื่อนไขที่เราพบ เราจะสามารถเปลี่ยนปัญหากำหนดการเชิงเส้นที่ประกอบด้วยพารามิเตอร์ทั้งสองชนิดให้กลายเป็นปัญหาที่มีพารามิเตอร์แบบเซตสุ่มเท่านั้น โดยเราจะใช้ทฤษฎีการตัดสินใจของข้อมูลแบบเซตสุ่มมาหาคำตอบของปัญหานั้น ส่วนในกรณีที่ข้อมูลแบบช่วงความน่าจะเป็นไม่สอดคล้องกับเงื่อนไขเหล่านั้น เราจะหาคำตอบของปัญหาโดยวิธีการหาคำตอบในรูปแบบวิธีการมองในแง่ดีและวิธีการมองในแง่ร้าย ในท้ายสุดนี้เราได้นำเสนอขั้นตอนวิธีสำหรับการตรวจสอบเงื่อนไขของข้อมูลแบบช่วงความน่าจะเป็นดังกล่าวและการหาค่าการแจกแจงที่เหมาะสมเพื่อใช้ในการหาคำตอบในแต่ละวิธี

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In this thesis, we concentrate on the relationship of probability intervals and random sets. Furthermore, we are interested in solving uncertain linear programming problems with probability interval and random set parameters. We discover the conditions to verify when a given probability interval obtains the same information as a random set information. If these conditions are satisfied, we can transform a problem that contains both types of uncertainty into a problem which has only the random set information. In addition, we use an idea from decision making theory with random sets for solving this problem. If a probability interval does not satisfy these conditions, we can solve the problem for finding the optimistic and pessimistic expected recourse values. Finally, we present an algorithm for checking these conditions and constructing appropriate distributions for each of the optimistic and pessimistic approaches.

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CHAPTER I
INTRODUCTION

In real-life situations, we normally make a decision under uncertain information. There are many interpretations of uncertainty, such as random set, possibility distribution, probability interval, p-box and cloud [7, 10, 16, 17]. However, we concentrate on the relationship between a probability interval and a random set. A probability interval is an interval presenting lower and upper bounds of a probability of each element in a considerable set while a random set is a set of probabilities which are bounded below and above by belief and plausibility functions. From these definitions, a probability interval is easier to understand than a random set because a random set information requires the knowledge about belief and plausibility functions. Even though random sets are difficult to present and a user may not clearly understand, using a random set to represent data is more appropriate than using a probability interval in a situation where a user does not want the others to understand his/her probability interval information clearly. The following example shows a situation that using a random set is more suitable than using a probability interval for a user.

Example 1.1. *Suppose we are a banker of a horse-racing who has a probability interval information about famous horse, A, B, C and D, as follows.*

$$p(\{A \textit{ will be the winner}\}) \in \left[\frac{1}{4}, \frac{1}{2}\right], \quad p(\{B \textit{ will be the winner}\}) \in \left[\frac{1}{4}, \frac{7}{16}\right],$$

$$p(\{C \textit{ will be the winner}\}) \in \left[\frac{1}{8}, \frac{3}{8}\right], \quad p(\{D \textit{ will be the winner}\}) \in \left[\frac{1}{8}, \frac{5}{16}\right].$$

If we show this information for an on-looker to bid on a winner, he/she may guess that A will be the winner. If A is the winner, we must pay a jackpot to this on-looker. From this information, we may see that A has the most probability to be the winner. However, it is not the good idea for us to show this information for *the on-looker. In fact, this probability interval information can be conveyed to the unique random set which has the same information; i.e.,*

- *the belief that A will be the winner is* $\frac{1}{4}$ *which equal to the belief that B will be the winner,*
- *the belief that C will be the winner is* $\frac{1}{8}$ *which equal to the belief that D will be the winner,*
- *the belief that one of a horse in the set* $\{A, B, C\}$ *will be the winner for sure is* $\frac{11}{16}$*, and*
- *the belief that one of a horse in the set* $\{A, C, D\}$ *will be the winner for sure is* $\frac{9}{16}$*.*

We can see that people could understand the probability interval presentation of this information more than the random set's one. On the other hand, an on-looker would understand the random set representation of this information only when he/she has the knowledge on this subject.

The details on how to transform this probability interval to the random set are explained on page 49. Therefore, if we present this random set information instead of the probability interval information, it may be harder for the on-looker to interpret which horse will have more chance to win. Thus, the random set information will gain more benefit for us.

Hence the first objective of our study is to find the conditions of a set $\mathcal{P}_{PI}$ of probabilities generated from a probability interval when it has the same information as a set $\mathcal{P}_{RS}$ of probabilities generated from a random set. We discover the conditions of a probability interval that makes $\mathcal{P}_{PI} = \mathcal{P}_{RS}$ which we will discuss later in Chapter III. There was no research on this topic, as we had worked on the literature review. In the study of Lemmer, Kyburg and Deneoeux, they found an algorithm for constructing a random set which is an approximation of a given probability interval. Destercke, Dubois and Chojnacki [5] explained that Lemmer and Kyburg [8] worked on an inner approximated transformation of a set $\mathcal{P}_{PI}$ into a set $\mathcal{P}_{RS}$, which means $\mathcal{P}_{RS} \subset \mathcal{P}_{PI}$. Later, Deneoeux [4] studied the transformation of a set $\mathcal{P}_{PI}$ into a set $\mathcal{P}_{RS}$ using an outer approximation ($\mathcal{P}_{PI} \subset \mathcal{P}_{RS}$). Therefore, we are interested in finding the conditions that a given probability interval and a random set obtain the same information ($\mathcal{P}_{PI} = \mathcal{P}_{RS}$).

In addition, solving uncertain linear programming problems with probability interval and random set parameters is another objective of our work. We have studied the technique for solving linear optimizations under uncertainty problems from Thipwiwatpojana [15], which is used to find a pessimistic and an optimistic solutions for solving this type of problems. In this study, we use pessimistic and optimistic approaches for solving uncertain linear programming problems with probability interval and random set parameters. However, before solving the problem, all probability interval parameters must be verified whether they satisfy the conditions in Chapter III. Then, we can transform the problem to be the problem with only random set parameters, if the conditions are satisfied. We use decision making approaches for random sets presented by Hung T. Nguyen [11] to solve the problem. The solution approaches from [11] are not only a pessimistic or an optimistic solution but also all of the convex combinations between a pessimistic and an optimistic solutions. The details of these approaches can be found in Chapter IV.

In the next chapter, the necessary background knowledge used in this thesis, which are probability intervals, random sets and a recourse model are provided in Chapter II. After that, we discuss the conditions of a given probability interval which can be represented as a random set in Chapter III. Later, a method for solving linear programming problems with probability interval and random set parameters is shown in Chapter IV. Finally, we provide the conclusion of this thesis in Chapter V.

CHAPTER II
PRELIMINARIES

We provide the mathematical definitions of a probability interval and a random set in this chapter. The review of a recourse model is in the last part of this chapter. We begin with the definition of a probability interval in Section 2.1 and follow by the definition of a random set in Section 2.2. Other technical terms and notation are also given where needed. Finally, we finish this chapter with the review of a recourse model in Section 2.3.

2.1 Probability interval

In this thesis, we use the notation set $X = \{x_1, x_2, \ldots, x_n\}$ to be the set of realizations of an uncertainty information, unless stated otherwise.

Definition 2.1. *probability interval (see [2])*

Given $X = \{x_1, x_2, \ldots, x_n\}$ *as the set of realizations of an uncertainty information and a family of intervals* $L = \{[l_i, u_i], i = 1, 2, \ldots, n \mid 0 \leq l_i \leq u_i \leq 1\}$, *we define the set* $\mathcal{P}_L$ *of probability distributions on* X *as*

$$\mathcal{P}_L = \{p \mid l_i \leq p(\{x_i\}) \leq u_i, \sum_{i=1}^{n} p(\{x_i\}) = 1, \forall i = 1, 2, \ldots, n\},$$

where $p(\{x_i\})$ *is a probability density of* $\{x_i\}$.

The set L is called a set of probability intervals, or a ***probability interval****, in short. While, the set* $\mathcal{P}_L$ *is the set of all possible probabilities associated with L.*

Definition 2.2. *proper probability interval*

A probability interval $L = \{[l_i, u_i], i = 1, 2, \ldots, n \mid 0 \leq l_i \leq u_i \leq 1\}$ *such that*

$$\sum_{i=1}^{n} l_i \leq 1 \leq \sum_{i=1}^{n} u_i, \tag{2.1}$$

is called a ***proper probability interval****.*

The probability interval satisfying the condition (2.1) will guarantee that $\mathcal{P}_L$ is nonempty. The empty set is not usable, so we consider only a proper probability interval in this thesis. Moreover, it is well known that for all probability functions, p, in a nonempty set $\mathcal{P}_L$

$$\inf_{p\in\mathcal{P}_L} p(A) \leq p(A) \leq \sup_{p\in\mathcal{P}_L} p(A),\ \forall A \in P(X),$$

where $P(X)$ is the power set of set X. We define

$$l(A) = \inf_{p\in\mathcal{P}_L} p(A) \text{ and } u(A) = \sup_{p\in\mathcal{P}_L} p(A),\ \forall A \in P(X).$$

The next definition is the condition of a proper probability interval which guarantees that for each i, the lower bound, l_i, and/or the upper bound, u_i, can be reached by some probabilities in the set $\mathcal{P}_L$.

Definition 2.3. *reachability*

A proper probability interval $L = \{[l_i, u_i], i = 1, \ldots, n\}$ *is called* ***reachable*** *if*

$$\sum_{j\neq i} l_j + u_i \leq 1 \text{ and } \sum_{j\neq i} u_j + l_i \geq 1,\ \forall i. \tag{2.2}$$

It was proved by L.M. De Campos et al. [2] that if a proper probability interval satisfies the condition (2.2), then $l(\{x_i\}) = l_i$ and $u(\{x_i\}) = u_i$, for all i. In addition, we can compute the lower bound and upper bound of a nonempty set set A of X by using the values l_i and u_i from a reachable probability interval as follows.

$$l(A) = \max(\sum_{x_i\in A} l_i, 1 - \sum_{x_i\in A^c} u_i), \tag{2.3}$$

$$u(A) = \min(\sum_{x_i\in A} u_i, 1 - \sum_{x_i\in A^c} l_i),\ \forall A \in P(X), \tag{2.4}$$

where A^c is the complement set of the set A.

We use $l(A)$ and $u(A)$ to represent the bounds of probabilities of a set A; i.e.,

$\{p \mid l(A) \leq p(A) \leq u(A), \forall A \in P(X)\}$. However, this set is the same as the set $\mathcal{P}_L$. We use the notation $\mathcal{P}_{PI}$ instead of $\mathcal{P}_L$ when our probability interval is reachable.

$$\mathcal{P}_{PI} = \{p \mid l(A) \leq p(A) \leq u(A), \forall A \in P(X)\}.$$

We discuss a random set in the next section. We begin with the definitions of a σ-algebra, a measurable space, a measurable mapping, and a probability space that can be found in many standard probability measure text books [1, 12, 13]. After that, we use them to provide the definition of a random set. However, in this thesis we consider only a finite random set. A random set can be represented by belief and plausibility measures. The definitions of belief and plausibility measures, which are closely related to a random set, are presented afterwards.

2.2 Random set

Definition 2.4. *σ-algebra and measurable space (see [1])*

*Let Ω be a nonempty set. A **σ-algebra** on Ω, denoted by σ_Ω, is a family of subsets of Ω that satisfies the following properties:*

- $\varnothing \in \sigma_\Omega$,
- $B \in \sigma_\Omega \Rightarrow B^c \in \sigma_\Omega$, *and*
- $B_i \in \sigma_\Omega$, *for any countable (or finite) subset B_i of $\sigma_\Omega \Rightarrow \cup_i B_i \in \sigma_\Omega$.*

*A pair (Ω, σ_Ω) is called a **measurable space**.*

Definition 2.5. *measure, measure space, probability space and probability measure (see [1])*

*Let (Ω, σ_Ω) be a measurable space. By a **measure** on this space, we mean a function $\mu : \sigma_\Omega \to [0, \infty]$ with the properties:*

- $\mu(\varnothing) = 0$, *and*
- *if $B_i \in \sigma_\Omega, \forall i = 1, 2, \ldots$, are disjoint, then* $\mu\Big(\bigcup_{i=1}^{\infty} B_i\Big) = \sum_{i=1}^{\infty} \mu(B_i)$.

*We refer to the triple $(\Omega, \sigma_\Omega, \mu)$ as a **measure space**. If $\mu(\Omega) = 1$, we refer to it as a **probability space** and write it as $(\Omega, \sigma_\Omega, Pr_\Omega)$, where Pr_Ω is a **probability measure**.*

Definition 2.6. *measurable mapping (see [1])*

Let (Ω, σ_Ω) *and* (U, σ_U) *be measurable spaces. A function* $f : \Omega \to U$ *is said to be a* $(\sigma_\Omega, \sigma_U)$***-measurable mapping*** *if* $f^{-1}(A) = \{\omega \in \Omega : f(\omega) \in A\} \in \sigma_\Omega$*, for each* $A \in \sigma_U$.

Definition 2.7. *random set (see [11])*

Let $(\Omega, \sigma_\Omega, Pr_\Omega)$ *be a probability space and* (F, σ_F) *be a measurable space, where* $F \subseteq \sigma_U, U \neq \varnothing$*, and* (U, σ_U) *is a measurable space. A* ***random set*** Γ *is a* $(\sigma_\Omega, \sigma_F)$*-measurable mapping*

$$\begin{aligned} \Gamma : \Omega &\to F \\ \omega &\mapsto \Gamma(\omega). \end{aligned}$$

When X is finite, we can use a ***basic probability assignment function*** 'm' over $P(X)$ to represent a random set

$$m(E) = p(\{\omega, \Gamma(\omega) = E\}),\ \forall E \in P(X),$$

such that $\sum_{E \in P(X)} m(E) = 1$ and $m(\varnothing) = 0$. A set $E \in P(X)$, where $m(E) \geq 0$ is called a ***focal element*** of m, and denote F as the ***set of focal elements***. Then, we use the order pair (F, m) to define a random set. We should recognize that probability distribution functions, Pr's, and basic probability assignment functions, m's, are different, i.e.,

- When $A \subseteq B,\ Pr(A) \leq Pr(B)$, but it is not necessary that $m(A) \leq m(B)$.
- $Pr(X) = 1$, while it is not necessary that $m(X) = 1$.
- $Pr(A) + Pr(A^c) = 1$, but there is no relationship between $m(A)$ and $m(A^c)$.

Definition 2.8. *belief and plausibility measures (see [1])*

Let X *be a finite set of realizations of an uncertain information. A* ***belief measure****, denoted by* Bel*, is a function*

$$Bel : P(X) \to [0, 1]$$

such that $Bel(\varnothing) = 0, Bel(X) = 1$, *and it contains a super-additive property for all possible families of subsets of* X, *that is,*

$$Bel(A_1 \cup \ldots \cup A_n) \geq \sum_j Bel(A_j) - \sum_{j<k} Bel(A_j \cap A_k) + \ldots + (-1)^{n+1} Bel(A_1 \cap \ldots \cap A_n)$$

where $A_1, A_2, \ldots, A_n \subseteq X$. *The basic property of belief measures is a weaker version of the additive property of probability measures. Thus, for any* $A, A^c \subseteq X$,

$$Bel(A) + Bel(A^c) \leq 1.$$

A ***plausibility measure***, *denoted by* Pl, *is defined by*

$$Pl(A) = 1 - Bel(A^c), \forall A \in P(X).$$

Similarly, $Bel(A) = 1 - Pl(A^c), \forall A \in P(X)$.

However, belief and plausibility measures can be defined by a basic probability assignment function m. Shafer [14] showed that the basic probability assignment function m can be used to formulate belief and plausibility measures as follows,

$$Bel(A) = \sum_{E | E \subseteq A} m(E), \text{ and}$$
$$Pl(A) = 1 - Bel(A^c) = \sum_{E | E \cap A \neq \varnothing} m(E), \ \forall A \in P(X).$$

Therefore, the following set is the set of all possible probabilities that are induced by a random set, and we use the notation, $\mathcal{P}_{RS}$ to represent it.

$$\mathcal{P}_{RS} = \{p \mid \forall A \in P(X), Bel(A) \leq p(A) \leq Pl(A)\}.$$

The meaning of belief and plausibility depend on the context where they were used. We provide the meaning of these functions in a general context as follows.

$Bel(A)$ means that a user's belief that one of the elements in A will happen for sure with proportion $Bel(A)$.

$Pl(A)$ means that a user's belief that one of the elements in A^c may not happen for sure with proportion $Pl(A)$.

In the next section, we explain the details of a recourse model which is used for solving linear programming problems with uncertainty.

2.3 Recourse model

Stochastic programming is known for solving uncertain problems which some of the objective or constraints have uncertain data with a probability interpretation. A recourse model is one of approaches in stochastic programming which we make our decisions now then minimize the expected costs (or utilities) of the consequences of these decisions. A two-stage recourse problem has a general form as

$$\begin{aligned}
\min_x \quad & cx + E[h(x,w)] \\
\text{s.t.} \quad & Ax \geq b \\
& x \geq 0 \\
\text{where} \quad & h(x,w) = \min g_w y \\
\text{s.t.} \quad & W_w y \geq r_w - T_w x \\
& y \geq 0.
\end{aligned}$$

In this problem, x is a vector of decisions that we must take and y is a vector of decisions that represent new actions or consequences of x.

However, in this thesis we study about solving linear programming problem with uncertainty as the following form

$$\left.\begin{aligned}
\min_x \quad & cx \\
\text{s.t.} \quad & \widehat{A}x \geq \widehat{b} \\
& Bx \geq d \\
& x \geq 0.
\end{aligned}\right\} \tag{2.5}$$

where $\widehat{A}$ and $\widehat{b}$ can be random sets or probability intervals. We cannot use the two-stage recourse model to solve it because we do not know the exact probability density mass value of each realization. We present how to solve the problem (2.5) in Chapter IV.

In the next chapter, we provide how to obtain the conditions of a given reachable probability interval that can be represented as a random set, $\mathcal{P}_{PI} = \mathcal{P}_{RS}$, and the proof of all conditions.

CHAPTER III
WHEN A REACHABLE PROBABILITY INTERVAL IS A RANDOM SET

Our objective of this thesis is to find the conditions of a given reachable probability interval L that would obtain $\mathcal{P}_{PI} = \mathcal{P}_{RS}$. Therefore, we must consider the conditions of a given reachable probability interval when it receives the following equations:

$$Bel(A) = \max(\sum_{x_i \in A} l_i, 1 - \sum_{x_i \in A^c} u_i), \text{ and}$$
$$Pl(A) = \min(\sum_{x_i \in A} u_i, 1 - \sum_{x_i \in A^c} l_i).$$

However, we can consider only when $Bel(A) = l(A)$ which we will obtain $\mathcal{P}_{PI} = \mathcal{P}_{RS}$ as the following details. Beginning with computing $l(A^c)$, we can calculate the value of $l(A^c)$ by using Equation (2.3), as follows

$$l(A^c) = \max(\sum_{x_i \in A^c} l_i, 1 - \sum_{x_i \in A} u_i).$$

If $l(A^c) = \sum_{x_i \in A^c} l_i$, then we gain $u(A) = 1 - \sum_{x_i \in A} u_i$ from Equation (2.4). Otherwise, if $l(A^c) = 1 - \sum_{x_i \in A} u_i$, then we also have $u(A) = \sum_{x_i \in A^c} l_i$. Therefore, we gain $l(A^c) + u(A) = 1$ or $u(A) = 1 - l(A^c)$ that can be computed in the similar way as $Pl(A)$, $Pl(A) = 1 - Bel(A^c)$. Consequently, we need to verify only the conditions when $l(A)$ has the same value as $Bel(A)$.

3.1 A specific assignment function to obtain $l(A) = Bel(A)$

If we want $l(A) = Bel(A)$; $\forall A \subseteq X$, the basic assignment function 'm' requires to have a specific pattern. We categorize this specific pattern by considering all sizes of the nonempty subset A of X when setting $l(A) = Bel(A)$.

First, consider when $|A| = 1$, that is, $A = \{x_i\}$ for each $i = 1, \dots, n$. The equation $Bel(A) = \sum_{E, E \subseteq A} m(E)$ can be written as $Bel(\{x_i\}) = m(\{x_i\})$. Moreover, we get $l(\{x_i\}) = l_i$ and $u(\{x_i\}) = u_i$ from the reachable property. Therefore, we must set $Bel(\{x_i\}) = m(\{x_i\}) = l(\{x_i\})$, for each i.

Second, consider when $|A| = 2$, that is, $A = \{x_i, x_j\}$. From $Bel(\{x_i, x_j\}) = m(\{x_i\}) + m(\{x_j\}) + m(\{x_i, x_j\})$ and $l(\{x_i, x_j\}) = Bel(\{x_i, x_j\})$, we get

$$m(\{x_i, x_j\}) = l(\{x_i, x_j\}) - m(\{x_i\}) - m(\{x_j\}) = l(\{x_i, x_j\}) - l_i - l_j.$$

Then, when $|A| = 3$, that is, $A = \{x_i, x_j, x_k\}$. From $Bel(\{x_i, x_j, x_k\}) = m(\{x_i\}) + m(\{x_j\}) + m(\{x_k\}) + m(\{x_i, x_j\}) + m(\{x_i, x_k\}) + m(\{x_j, x_k\}) + m(\{x_i, x_j, x_k\})$ and $l(\{x_i, x_j, x_k\}) = Bel(\{x_i, x_j, x_k\})$, we get

$$\begin{aligned} m(\{x_i, x_j, x_k\}) &= l(\{x_i, x_j, x_k\}) - \sum_{\hat{i}, \hat{j} \in \{i,j,k\}} m(\{x_{\hat{i}}, x_{\hat{j}}\}) - \sum_{\hat{i} \in \{i,j,k\}} m(\{x_{\hat{i}}\}) \\ &= l(\{x_i, x_j, x_k\}) - \sum_{\hat{i}, \hat{j} \in \{i,j,k\}} m(\{x_{\hat{i}}, x_{\hat{j}}\}) - \sum_{\hat{i} \in \{i,j,k\}} l_{\hat{i}}. \end{aligned}$$

Therefore, we can compute $m(A)$ when $1 \leq |A| \leq n - 1$ by the mathematical induction, as follows

$$\begin{aligned} m(\{\underbrace{x_i, x_j, \dots, x_s}_{|A|}\}) = l(\{\underbrace{x_i, x_j, \dots, x_s}_{|A|}\}) - \sum m(\{\underbrace{x_i, x_j, \dots, x_r}_{|A|-1}\}) - \dots \\ - \sum_{\hat{i}, \hat{j} \in \{i,j,\dots,s\}} m(\{x_{\hat{i}}, x_{\hat{j}}\}) - \sum_{\hat{i} \in \{i,j,\dots,s\}} l_{\hat{i}}. \end{aligned}$$

Finally, consider when $A = X$, since there is the property of random set that $\sum_{E \in P(X)} m(E) = 1$, we obtain

$$m(X) = 1 - \sum l_i - \sum m(\{x_i, x_j\}) - \dots - \sum m(\{\underbrace{x_i, x_j, \dots, x_s}_{n-1}\}).$$

Therefore, if we want to gain $l(A) = Bel(A)$ for all nonempty set $A \in P(X)$, we must compute the basic assignment function m as the following pattern:

$$\left.\begin{array}{rcl} m(\{x_i\}) &=& l_i \\ m(\{x_i, x_j\}) &=& l(\{x_i, x_j\}) - l_i - l_j \\ m(\{x_i, x_j, x_k\}) &=& l(\{x_i, x_j, x_k\}) - \sum\limits_{\widehat{i},\widehat{j} \in \{i,j,k\}} m(\{x_{\widehat{i}}, x_{\widehat{j}}\}) - \sum\limits_{\widehat{i} \in \{i,j,k\}} l_{\widehat{i}} \\ && \vdots \\ m(\underbrace{\{x_i, x_j, \ldots, x_s\}}_{n-1}) &=& l(\{x_i, x_j, \ldots, x_s\}) - \sum m(\{\underbrace{x_i, x_j, \ldots, x_r}_{n-2}\}) - \ldots \\ && - \sum\limits_{\widehat{i} \in \{i,j,\ldots,s\}} l_{\widehat{i}} \\ m(X) &=& 1 - \sum l_i - \sum m(\{x_i, x_j\}) - \ldots - \sum m(\{\underbrace{x_i, x_j, \ldots, x_s}_{n-1}\}). \end{array}\right\} \tag{3.1}$$

We provide the conditions to earn $m(A) \geq 0, \forall A \subseteq X$ in the next section.

3.2 Conditions of a reachable probability interval to be a random set

From the previous section, the nonnegativity of the basic assignment function m in the system (3.1) have been not yet verified. However, it is obvious that $m(A)$ always has the nonnegative value for all $A \in P(X)$ because of the reachable property when the size of X is one or two as follows.

When $X = \{x_1\}$, we get $m(\{x_1\}) = 1$ from the reachable probability intervals that $p(\{x_1\}) \in [1, 1]$.

When $X = \{x_1, x_2\}$, let $L = \{[l_1, u_1], [l_2, u_2]\}$ be a reachable probability interval. By the definition of the reachable probability interval, we obtain $l_2 + u_1 \leq 1$ and $u_1 + l_2 \geq 1$, therefore $l_2 + u_1 = 1$. From $l_1 \leq u_1$, we then get $l_2 + l_1 \leq l_2 + u_1 = 1$, so $l_2 + l_1 \leq 1$. Let $m(\{x_1\}) = l_1$ and $m(\{x_2\}) = l_2$. So $m(X) = m(\{x_1, x_2\}) = 1 - l_1 - l_2 \geq 0$.

When $|X| \geq 3$, using the system (3.1) for a given probability interval, it may turn out that $m(A) < 0$. Thus, we need to find the conditions of a probability interval to make sure that $m(A) \geq 0$. We separate the conditions that we found into two groups. The first group is the conditions that the reachable probability interval is enough for constructing the random set. The second group is the extending conditions of the first group. These conditions are stated in the Theorems 3.2, 3.4 and 3.5. We need Lemma 3.1 in order to prove these theorems.

Lemma 3.1. *Let* $X = \{x_1, x_2, \ldots, x_n\}$ *with* $n \geq 3$ *and let* $L = \{[l_i, u_i], i = 1, 2, \ldots, n \mid 0 \leq l_i \leq u_i \leq 1\}$ *be a reachable probability interval. If* $m(\{x_i\}) = l_i$ *and if there exists an index* i *such that* $\sum_{j \neq i} l_j + u_i = 1$ *then* $m(A) = 0, \forall A \in P(X \setminus \{x_i\})$ *with* $|A| \geq 2$.

Proof. WLOG, we use the index $i = 1$. Therefore, $l_2 + l_3 + \ldots + l_n + u_1 = 1$. We will prove by the mathematical induction on $|A|$.

The basic step. When $|A| = 2$. Let set $A = \{x_{j_1}, x_{j_2}\}$, where $j_1 < j_2$. By the assumption, we get

$$\begin{aligned} l_{j_1} + l_{j_2} &= 1 - l_2 - l_3 - \ldots - l_{j_1 - 1} - l_{j_1 + 1} - \ldots - l_{j_2 - 1} - l_{j_2 + 1} - l_n - u_1 \\ &> 1 - u_2 - u_3 - \ldots - u_{j_1 - 1} - u_{j_1 + 1} - \ldots - u_{j_2 - 1} - u_{j_2 + 1} - u_n - u_1. \end{aligned}$$

Hence, $l(\{x_{j_1}, x_{j_2}\}) = l_{j_1} + l_{j_2}$ by Equation (2.3). Moreover, we get $m(\{x_{j_1}, x_{j_2}\}) = 0$ since we want $Bel(\{x_{j_1}, x_{j_2}\}) = l(\{x_{j_1}, x_{j_2}\})$ and we know that $Bel(\{x_{j_1}, x_{j_2}\}) = m(\{x_{j_1}\}) + m(\{x_{j_2}\}) + m(\{x_{j_1}, x_{j_2}\})$.

The inductive step. Let $m(A) = 0, \forall A \in P(X \setminus \{x_1\})$, where $|A| = k, k \leq n-2$. Consider $l_2 + l_3 + \ldots + l_n = 1 - u_1$, we get $l(\{x_2, x_3, \ldots, x_n\}) = l_2 + l_3 + \ldots + l_n$. From

$$m(\{\underbrace{x_i, x_j, \ldots, x_s}_{n-1}\}) = l(\{\underbrace{x_i, x_j, \ldots, x_s}_{n-1}\}) - \sum m(\{\underbrace{x_i, x_j, \ldots, x_r}_{n-2}\}) - \ldots - \sum_{\widehat{i} \in \{i, j, \ldots, s\}} l_{\widehat{i}}$$

and $m(A) = 0$ for all $A \in P(X \setminus \{x_1\})$, where $|A| = k, k \leq n-2$, we have $m(\{x_2, x_3, \ldots, x_n\}) = l(\{x_2, x_3, \ldots, x_n\}) - \sum_{i \in \{2, 3, \ldots, n\}} l_i = 0$.

By the mathematical induction, we can conclude that if there exists an index i such that $\sum_{j \neq i} l_j + u_i = 1$, then $m(A) = 0$, $\forall A \in P(X \setminus \{x_i\})$ with $|A| \geq 2$. □

The next theorem shows that if we have a reachable probability interval which has at most two indices, say i_1, i_2, such that $\sum_{j \neq i_1} l_j + u_{i_1} < 1$ and $\sum_{j \neq i_2} l_j + u_{i_2} < 1$, then we can construct the unique random set that has the same information as this probability interval.

Theorem 3.2. *Let* $X = \{x_1, x_2, \ldots, x_n\}$*,* $n \geq 3$*, and let* $L = \{[l_i, u_i] \mid 0 \leq l_i \leq u_i \leq 1, i = 1, 2, \ldots, n\}$ *be a reachable probability interval. If there are at most two indices, say* i_1, i_2*, such that* $\sum_{j \neq i_1} l_j + u_{i_1} < 1$ *and* $\sum_{j \neq i_2} l_j + u_{i_2} < 1$*, then we can construct the unique random set that has the same information as the probability interval* L*, i.e.,* $\mathcal{P}_{RS} = \mathcal{P}_{PI}$*, which means* $Bel(A) = l(A)$ *and* $Pl(A) = u(A)$*,* $\forall A \in P(X)$.

Proof. We organize our proof into the following three cases.

Case 1: There are two indices, say i_1, i_2, such that

$$\sum_{j \neq i_1} l_j + u_{i_1} < 1 \text{ and } \sum_{j \neq i_2} l_j + u_{i_2} < 1.$$

WLOG, let $i_1 = 1$ and $i_2 = 2$.

1. When $n = 3$, since $\sum_{j \neq 1} l_j + u_1 < 1$ and $\sum_{j \neq 2} l_j + u_2 < 1$, we get $\sum_{j \neq 3} l_j + u_3 = 1$ by the property of reachable probability intervals.

   Let $m(\{x_i\}) = l_i$. By Lemma 3.1, we get $m(\{x_1, x_2\}) = 0$. Next, we will show that $m(\{x_1, x_3\}), m(\{x_2, x_3\})$ and $m(X)$ have nonnegative values, to complete this part. Since $l_1 + l_3 < 1 - u_2$ and $l_2 + l_3 < 1 - u_1$, we then get $l(\{x_1, x_3\}) = 1 - u_2$ and $l(\{x_2, x_3\}) = 1 - u_1$, using Equation (2.3). Hence,

$$m(\{x_1, x_3\}) = l(\{x_1, x_3\}) - l_1 - l_3 = 1 - u_2 - l_1 - l_3 = 1 - (\underbrace{u_2 + l_1 + l_3}_{<1}) \geq 0.$$

$$m(\{x_2, x_3\}) = l(\{x_2, x_3\}) - l_2 - l_3 = 1 - u_1 - l_2 - l_3 \geq 0.$$

$$\begin{aligned} m(X) &= 1 - \sum m(\{x_i\}) - \sum m(\{x_i, x_j\}) \\ &= 1 - l_1 - l_2 - l_3 - (1 - u_2 - l_1 - l_3 + 1 - u_1 - l_2 - l_3) \\ &= \underbrace{u_2 + u_1 + l_3}_{\geq 1} - 1 \geq 0. \end{aligned}$$

2. When $n \geq 4$, we have $i = 1, 2$ such that $\sum_{j \neq i} l_j + u_i < 1$. Therefore,

$$\begin{aligned} l_2 + l_3 + \ldots + l_n &< 1 - u_1 \\ l_1 + l_3 + \ldots + l_n &< 1 - u_2. \end{aligned} \tag{3.2}$$

So, $l(\{x_2,x_3,\ldots,x_n\}) = 1-u_1$ and $l(\{x_1,x_3,\ldots,x_n\}) = 1-u_2$ by Equation (2.3). For $i \neq 1,2$, we get $\sum_{j\neq i} l_j + u_i = 1$. Set $m(\{x_i\}) = l_i$, we obtain $m(A) = 0$, $\forall A \in P(X \setminus \{x_i\})$ $i \neq 1,2$, by Lemma 3.1. Therefore, we have

$$m(A) = 0,\ \forall A \in P(X) \setminus \{\{x_3,\ldots,x_n\},\{x_1,x_3,\ldots,x_n\},\{x_2,x_3,\ldots,x_n\},X\}$$

Hence, we must find the value of

$$m(\{x_3,\ldots,x_n\}), m(\{x_1,x_3,\ldots,x_n\}), m(\{x_2,x_3,\ldots,x_n\}) \text{ and } m(X).$$

Due to the system of Equations (3.1), we obtain that

$$\begin{aligned}
&l_3+l_4+\ldots+l_n+m(\{x_3,\ldots,x_n\}) = l(\{x_3,\ldots,x_n\})\\
&l_1+l_3+l_4+\ldots+l_n+m(\{x_3,\ldots,x_n\})+m(\{x_1,x_3,\ldots,x_n\}) = 1-u_2\\
&l_2+l_3+l_4+\ldots+l_n+m(\{x_3,\ldots,x_n\})+m(\{x_2,x_3,\ldots,x_n\}) = 1-u_1\\
&\sum l_i+m(\{x_3,\ldots,x_n\})+m(\{x_1,x_3,\ldots,x_n\})+m(\{x_2,x_3,\ldots,x_n\})+m(X) = 1.
\end{aligned}$$

Thus, we can write this system of equations in the form of matrix notation by using $m(\{x_3,\ldots,x_n\}), m(\{x_1,x_3,\ldots,x_n\}), m(\{x_2,x_3,\ldots,x_n\})$ and $m(X)$ as unknown variables.

$$\begin{bmatrix} 1&0&0&0\\ 1&1&0&0\\ 1&0&1&0\\ 1&1&1&1 \end{bmatrix}\begin{bmatrix} m(\{x_3,\ldots,x_n\})\\ m(\{x_1,x_3,\ldots,x_n\})\\ m(\{x_2,x_3,\ldots,x_n\})\\ m(X) \end{bmatrix} = \begin{bmatrix} l(\{x_3,\ldots,x_n\})-l_3-l_4-\ldots-l_n\\ 1-u_2-l_1-l_3-l_4-\ldots-l_n\\ 1-u_1-l_2-l_3-l_4-\ldots-l_n\\ 1-l_1-l_2-l_3-\ldots-l_n \end{bmatrix}$$

$$\begin{aligned}
\begin{bmatrix} m(\{x_3,\ldots,x_n\})\\ m(\{x_1,x_3,\ldots,x_n\})\\ m(\{x_2,x_3,\ldots,x_n\})\\ m(X) \end{bmatrix} &= \begin{bmatrix} 1&0&0&0\\ -1&1&0&0\\ -1&0&1&0\\ 1&-1&-1&1 \end{bmatrix}\begin{bmatrix} l(\{x_3,\ldots,x_n\})-l_3-l_4-\ldots-l_n\\ 1-u_2-l_1-l_3-l_4-\ldots-l_n\\ 1-u_1-l_2-l_3-l_4-\ldots-l_n\\ 1-l_1-l_2-l_3-\ldots-l_n \end{bmatrix}\\
&= \begin{bmatrix} l(\{x_3,\ldots,x_n\})-l_3-l_4-\ldots-l_n\\ -l(\{x_3,\ldots,x_n\})+1-u_2-l_1\\ -l(\{x_3,\ldots,x_n\})+1-u_1-l_2\\ l(\{x_3,\ldots,x_n\})-1+u_1+u_2 \end{bmatrix}.
\end{aligned}$$

Since $l(A) = \max(\sum_{x_i\in A} l_i, 1-\sum_{x_i\in A^c} u_i)$, we have
$m(\{x_3,\ldots,x_n\}) = l(\{x_3,\ldots,x_n\}) - l_3 - l_4 - \ldots - l_n \geq 0$ and
$m(X) = l(\{x_3,\ldots,x_n\}) - 1 + u_1 + u_2 \geq 1 - u_1 - u_2 - 1 + u_1 + u_2 = 0$.
We must show $m(\{x_1,x_3,\ldots,x_n\}) \geq 0$ and $m(\{x_2,x_3,\ldots,x_n\}) \geq 0$ to complete the proof of this case.

Consider when $l(\{x_3,\ldots,x_n\}) = l_3 + l_4 + \ldots + l_n$, so

$$\begin{aligned} m(\{x_1,x_3,\ldots,x_n\}) &= -l(\{x_3,\ldots,x_n\}) + 1 - u_1 - l_2 \\ &= -(\underbrace{l_2 + l_3 + l_4 + \ldots + l_n + u_1}_{<1}) - 1 \geq 0. \end{aligned}$$

When $l(\{x_3,\ldots,x_n\}) = 1 - u_1 - u_2$, then

$$\begin{aligned} m(\{x_1,x_3,\ldots,x_n\}) &= -l(\{x_3,\ldots,x_n\}) + 1 - u_1 - l_2 \\ &= -1 + u_1 + u_2 + 1 - u_1 - l_2 = u_2 - l_2 \geq 0. \end{aligned}$$

Hence $-l(\{x_3,\ldots,x_n\}) + 1 - u_2 - l_1 \geq 0$, and $m(\{x_1,x_3,\ldots,x_n\}) \geq 0$. We can obtain $m(\{x_2,x_3,\ldots,x_n\}) \geq 0$ in a similar fashion.

Case 2: There is only one index, say i_1, such that $\sum_{j \neq i_1} l_j + u_{i_1} < 1$. WLOG, let $i_1 = 1$. Then we have $l_2 + l_3 + \ldots + l_n < 1 - u_1$. So, $l(\{x_2,x_3,\ldots,x_n\}) = 1 - u_1$. For $i \neq 1$, we get $\sum_{j \neq i} l_j + u_i = 1$. Let $m(\{x_i\}) = l_i$, we obtain $m(A) = 0$, $\forall A \in P(X \setminus \{x_i\})$ $\forall i \neq 1$, by Lemma 3.1. Therefore, we have

$$m(A) = 0,\ \forall A \in P(X) \setminus \{\{x_2,x_3,\ldots,x_n\}, X\}.$$

So, we must show $m(\{x_2,x_3,\ldots,x_n\}) \geq 0$ and $m(X) \geq 0$ to complete the proof. The second last equation of the system (3.1) provides

$$m(\{\underbrace{x_i,x_j,\ldots,x_s}_{n-1}\}) = l(\{x_i,x_j,\ldots,x_s\}) - \sum m(\{\underbrace{x_i,x_j,\ldots,x_r}_{n-2}\}) - \ldots - \sum_{\widehat{i} \in \{i,j,\ldots,s\}} l_{\widehat{i}},$$

which means

$$\begin{aligned} m(\{x_2,x_3,\ldots,x_n\}) &= l(\{x_2,x_3,\ldots,x_n\}) - \sum_{i \in \{2,3,\ldots,n\}} l_i \\ &= 1 - u_1 - l_2 - l_3 - \ldots - l_n \\ &= 1 - (\underbrace{u_1 + l_2 + l_3 + \ldots + l_n}_{<1}) > 0. \end{aligned}$$

From the last equation of (3.1), we also have

$$m(X) = 1 - \sum l_i - \sum m(\{x_i,x_j\}) - \ldots - \sum m(\{\underbrace{x_i,x_j,\ldots,x_s}_{n-1}\}).$$

Therefore,

$$\begin{aligned} m(X) &= 1 - \sum l_i - m(\{x_2, x_3, \ldots, x_n\}) \\ &= 1 - \sum l_i - (1 - u_1 - l_2 - l_3 - \ldots - l_n) = u_1 - l_1 \geq 0. \end{aligned}$$

Case 3: There is no index i such that $\sum_{j \neq i} l_j + u_i < 1$. So, we get

$$\sum_{j \neq i} l_j + u_i = 1, \ \forall i.$$

Let $m(\{x_i\}) = l_i$, we obtain $m(A) = 0, \ \forall A \in P(X \setminus \{x_i\}) \ \forall i$, by Lemma 3.1. Therefore, we have $m(A) = 0, \ \forall A \in P(X) \setminus X$, and $m(X) = 1 - \sum l_i \geq 0$.

By all of these cases, we can conclude that when there are at most two indices, say i_1, i_2, such that $\sum_{j \neq i_1} l_j + u_{i_1} < 1$ and $\sum_{j \neq i_2} l_j + u_{i_2} < 1$, we can construct the unique random set that has the same information such that $\mathcal{P}_{RS} = \mathcal{P}_{PI}$, i.e.,$Bel(A) = l(A)$ and $Pl(A) = u(A), \ \forall A \in P(X)$. □

Theorem 3.2 cannot guarantee that if the reachable probability interval with more than two indices i satisfies $\sum_{j \neq i} l_j + u_i < 1$, then we can construct the random set. If there are three indices, say i_1, i_2 and i_3, such that

$$\sum_{j \neq i_1} l_j + u_{i_1} < 1, \sum_{j \neq i_2} l_j + u_{i_2} < 1 \text{ and} \sum_{j \neq i_3} l_j + u_{i_3} < 1 \tag{3.3}$$

and the rest holds with equalities $\sum_{j \neq i} l_j + u_i = 1, \ i \neq i_1, i_2, i_3$, then the following example explains that the reachability is not enough to guarantee that there is the random set which has the same information as the given probability interval.

Example 3.3. *Let the set of realizations $X = \{x_1, x_2, x_3\}$ with a reachable probability interval $L = \{[l_1, u_1] = \left[\frac{1}{12}, \frac{3}{12}\right], \ [l_2, u_2] = \left[\frac{5}{12}, \frac{7}{12}\right], \ [l_3, u_3] = \left[\frac{2}{12}, \frac{5}{12}\right]\}$. It is easily seen that $\sum_{j \neq i} l_j + u_i < 1, \ \forall i = 1, 2, 3$. Set $m(\{x_i\}) = l_i$. Since $l_i + l_j < 1 - u_k$, we then get $l(\{x_i, x_j\}) = 1 - u_k$, by Equation (2.3). Thus,*

$$m(\{x_i, x_j\}) = l(\{x_i, x_j\}) - l_i - l_j = 1 - u_k - l_i - l_j = 1 - (\underbrace{u_k + l_i + l_j}_{<1}) \geq 0.$$

Finally, consider the value of $m(X)$

$$
\begin{aligned}
m(X) &= 1 - \sum m(\{x_i\}) - \sum m(\{x_i, x_j\}) \\
&= 1 - l_1 - l_2 - l_3 - (1 - u_1 - l_2 - l_3 + 1 - u_2 - l_1 - l_3 + 1 - u_3 - l_1 - l_2) \\
&= l_1 + l_2 + l_3 + u_1 + u_2 + u_3 - 2 = \sum_{i=1}^{3} l_i + \sum_{i=1}^{3} u_i - 2 = \frac{8}{12} + \frac{15}{12} - 2 = \frac{-1}{12} < 0.
\end{aligned}
$$

We can see from Example 3.3 that we do not know that $\sum_{i=1}^{3} l_i + \sum_{i=1}^{3} u_i \geq 2$ or $\sum_{i=1}^{3} l_i + \sum_{i=1}^{3} u_i < 2$, in general. Thus, we could not conclude that $m(X)$ is nonnegative. Therefore, the reachability and conditions in Theorem 3.2 are not enough to guarantee that we can construct a random set.

However, we explain the sufficient conditions of a reachable probability interval to be a random set when it satisfies Equation (3.3) in Appendix B. In the next section, we extend the condition (3.3) to be more general and provide the proof of the probability interval satisfying an extended condition (3.4) and the sufficient conditions (I) or (II) can be represented as a random set.

3.3 Extended conditions of a reachable probability interval to be a random set

The condition (3.3) can be extended by letting t be the number of index i that satisfy the condition $\sum_{j \neq i} l_j + u_i < 1$, so the extended condition is

$$\sum_{j \neq i_k} l_j + u_{i_k} < 1,\ \forall k = 1, \ldots, t \text{ and } 3 \leq t \leq n. \tag{3.4}$$

We define the following index sets for understandability

$$
\begin{aligned}
I &= \{1, \ldots, n\},\\
I_t &= \{i_1, \ldots, i_t\},\\
J &= I \setminus I_t,\\
X_J &= \{x_i,\ \forall i \in J\},\\
Y &= \{K \cup X_J \mid K \subseteq I_t\},\\
Y' &= \{K \cup X_J \mid K \subseteq I_t \text{ and } |K| = t-1\} = \{X \setminus \{x_{i_k}\} \mid k = 1, \ldots, t\}.
\end{aligned}
$$

Moreover, we extend the sufficient conditions of a reachable probability interval receiving from Appendix B corresponding the condition (3.4) as follows

(I) (I.1) $l(A) = \sum_{x_i \in A} l_i,\ \forall A \in Y - Y',$ and

(I.2) $l(X \setminus \{x_{i_k}\}) = 1 - u_{i_k},\ \forall i_k \in I_t,$ and

(I.3) $(t-2)\sum_{i\in I} l_i + \sum_{i\in I} u_i - \sum_{j\in J} u_j + \sum_{j\in J} l_j \geq t-1.$

(II)(II.1) $l(A) = 1 - \sum_{x_i \in A^c} u_i,\ \forall A \in Y,$ and

(II.2) only one of the following

(II.2.1) $t \leq n-2$, or

(II.2.2) $t = n-1$ and $\sum_{j \neq i_k} u_j + l_{i_k} = 1,\ i_k \in J.$

If a given reachable probability interval satisfies either condition (I) or (II), this probability interval can be represented as a random set. In addition, Theorems 3.4 and 3.5 prove the statement that there is a unique random set which has the same information as the given reachable probability interval.

Theorem 3.4. *Let $X = \{x_1, x_2, \ldots, x_n\}, n \geq 3,$ and let $L = \{[l_i, u_i] \mid 0 \leq l_i \leq u_i \leq 1, i = 1, 2, \ldots, n\}$ be a reachable probability interval. Suppose the conditions (3.4) and (I) hold. Then we can construct a unique random set that has the same information as the probability interval L, i.e., $\mathcal{P}_{RS} = \mathcal{P}_{PI}$, which means $Bel(A) = l(A)$ and $Pl(A) = u(A)$, $\forall A \in P(X)$.*

Proof. See Appendix for the case when $t = 3$.
Let $3 < t \leq n$, such that $\sum_{j \neq i_k} l_j + u_{i_k} < 1$, $\forall k = 1, \ldots, t$.
WLOG, let $i_1 = 1, i_2 = 2, \ldots, i_t = t$. For $j \in J = \{t+1, \ldots, n\}$, we get $\sum_{i \neq j} l_i + u_j = 1$.
Set $m(\{x_i\}) = l_i$, $\forall i = 1, \ldots, n$, we obtain $m(A) = 0, \forall A \in P(X \setminus \{x_j\}), j \in J$, by Lemma 3.1. Therefore, we have $m(A) = 0, \forall A \in P(X) \setminus Y$. Hence, we must find the value of $m(Y)$. Due to the system of Equations (3.1), we obtain

$$\left.\begin{array}{rcl}
m(\{x_{t+1}, \ldots, x_n\}) & = & l(\{x_{t+1}, \ldots, x_n\}) - l_{t+1} - \ldots - l_n \\
m(\{x_i, x_{t+1}, \ldots, x_n\}) & = & l(\{x_i, x_{t+1}, \ldots, x_n\}) - m(\{x_{t+1}, \ldots, x_n\}) \\
 & & -l_i - l_{t+1} - \ldots - l_n \\
m(\{x_i, x_j, x_{t+1}, \ldots, x_n\}) & = & l(\{x_i, x_j, x_{t+1}, \ldots, x_n\}) - m(\{x_{t+1}, \ldots, x_n\}) \\
 & & -m(\{x_i, x_{t+1}, \ldots, x_n\}) - m(\{x_j, x_{t+1}, \ldots, x_n\}) \\
 & & -l_i - l_j - l_{t+1} - \ldots - l_n \\
m(\{x_i, x_j, x_k, x_{t+1}, \ldots, x_n\}) & = & l(\{x_i, x_j, x_k, x_{t+1}, \ldots, x_n\}) \\
 & & -m(\{x_{t+1}, \ldots, x_n\}) - \sum_{\widehat{i} \in \{i,j,k\}} m(\{x_{\widehat{i}}, x_{t+1}, \ldots, x_n\}) \\
 & & - \sum_{\widehat{i},\widehat{j} \in \{i,j,k\}} m(\{x_{\widehat{i}}, x_{\widehat{j}}, x_{t+1}, \ldots, x_n\}) \\
 & & -l_i - l_j - l_k - l_{t+1} - \ldots - l_n \\
 & & \vdots \\
m(X \setminus \{x_i\}) & = & l(X \setminus \{x_i\}) - m(\{x_{t+1}, \ldots, x_n\}) - \sum_j m(\{x_j, x_{t+1}, \ldots, x_n\}) \\
 & & -\sum_{j,k} m(\{x_j, x_k, x_{t+1}, \ldots, x_n\}) - l_1 - \ldots - l_{i-1} \\
 & & -l_{i+1} - \ldots - l_n \\
m(X) & = & 1 - m(\{x_{t+1}, \ldots, x_n\}) - \sum_{i=1}^{t} m(\{x_i, x_{t+1}, \ldots, x_n\}) \\
 & & -\sum_{i,j} m(\{x_i, x_j, x_{t+1}, \ldots, x_n\}) \\
 & & -\ldots - \sum_{i=1}^{t} m(X \setminus \{x_i\}) - \sum_{i=1}^{n} l_i \\
 & & \text{where } i, j, k = 1, \ldots, t \text{ and } i \neq j \neq k.
\end{array}\right\} \quad (3.5)$$

We obtain from the condition (I.1) that $l(\{x_{t+1}, \ldots, x_n\}) = l_{t+1} + \ldots + l_n$. Therefore, by considering the value of $m(\{x_{t+1}, \ldots, x_n\})$, we have

$$\begin{aligned}
m(\{x_{t+1}, \ldots, x_n\}) &= l(\{x_{t+1}, \ldots, x_n\}) - l_{t+1} - \ldots - l_n \\
&= l_{t+1} + \ldots + l_n - l_{t+1} - \ldots - l_n = 0
\end{aligned}$$

Consequently, we receive $m(A) = 0$, $\forall A \in Y - Y'$.
Therefore, we must show $m(X \setminus \{x_i\}) \geq 0$, $\forall i = 1, \ldots, t$, and $m(X) \geq 0$ to complete the proof.

Consider the value of $m(X \setminus \{x_i\})$.

$$\begin{aligned} m(X \setminus \{x_i\}) &= l(X \setminus \{x_i\}) - l_1 - \ldots - l_{i-1} - l_{i+1} - \ldots - l_n \\ &= 1 - u_i - l_1 - \ldots - l_{i-1} - l_{i+1} - \ldots - l_n, \text{ (by the condition (I.2))} \\ &= 1 - (\underbrace{\sum_{j \neq i} l_j + u_i}_{<1}) > 0 \end{aligned}$$

We know that $\sum_{j \neq i} l_j + u_i < 1, \ \forall i = 1, \ldots, t$, by the condition (3.4). So we obtain $m(X \setminus \{x_i\}) > 0$, $\forall i = 1, \ldots, t$.

Next, let consider $m(X)$.

$$\begin{aligned} m(X) &= 1 - \sum_{i=1}^{t} m(X \setminus \{x_i\}) - \sum_{i=1}^{n} l_i \\ &= 1 - \sum_{i=1}^{t} (1 - u_i - l_1 - \ldots - l_{i-1} - l_{i+1} - \ldots - l_n) - \sum_{i=1}^{n} l_i \\ &= 1 - \sum_{i=1}^{t} (1 - u_i - l_1 - \ldots - l_{i-1} - l_{i+1} - \ldots - l_n - l_i + l_i) - \sum_{i=1}^{n} l_i \\ &= 1 - t + \sum_{i=1}^{t} u_i + t \sum_{i=1}^{n} l_i - \sum_{i=1}^{t} l_i - \sum_{i=1}^{n} l_i \\ &= 1 - t + \sum_{i \in I_t} u_i + (t-1) \sum_{i \in I} l_i - \sum_{i \in I_t} l_i \\ &= 1 - t + (\sum_{i \in I} u_i - \sum_{j \in J} u_j) + (t-1) \sum_{i \in I} l_i - \sum_{i \in I} l_i + \sum_{j \in J} l_j \\ &= (t-2) \sum_{i \in I} l_i + \sum_{i \in I} u_i - \sum_{j \in J} u_j + \sum_{j \in J} l_j - t + 1 \geq 0 \end{aligned}$$

We obtain $(t-2)\sum_{i \in I} l_i + \sum_{i \in I} u_i - \sum_{j \in J} u_j + \sum_{j \in J} l_j \geq t - 1$, by the condition (I.3).

□

When $l(A) = 1 - \sum_{x_i \in A^c} u_i, \forall A \in Y$ (the condition (II.1)), the following theorem shows that we can construct the random set.

Theorem 3.5. *Let* $X = \{x_1, x_2, \ldots, x_n\}, n \geq 5$. *Let* $L = \{[l_i, u_i] \mid 0 \leq l_i \leq u_i \leq 1, i = 1, 2, \ldots, n\}$ *be a reachable probability interval. Suppose the conditions*

(3.4) and (II) hold. Then we can construct a unique random set that has the same information as the probability interval L, i.e., $\mathcal{P}_{RS} = \mathcal{P}_{PI}$*, which means* $Bel(A) = l(A)$ *and* $Pl(A) = u(A)$*,* $\forall A \in P(X)$*.*

Proof. We present the proof into 2 cases.

Case 1: Suppose the conditions (II.2.1) and (3.4) hold. Let $t \leq n-2$ such that $\sum_{j \neq i_k} l_j + u_{i_k} < 1$, $\forall k = 1, \ldots, t$.

WLOG, let $i_1 = 1, i_2 = 2, \ldots, i_t = t$. For $j \in J$, we get $\sum_{i \neq j} l_i + u_j = 1$.

Set $m(\{x_i\}) = l_i$, $\forall i = 1, \ldots, n$. We obtain $m(A) = 0, \forall A \in P(X \setminus \{x_j\}), j \in J$, by Lemma 3.1. Therefore, we have $m(A) = 0, \forall A \in P(X) \setminus Y$. The values of $m(A); A \in Y$ must be nonnegative to achieve our purpose. We can find the values of $m(A); A \in Y$ by using the system of Equations (3.5).

First, consider $m(\{x_{t+1}, \ldots, x_n\})$.

$$\begin{aligned} m(\{x_{t+1}, \ldots, x_n\}) &= l(\{x_{t+1}, \ldots, x_n\}) - l_{t+1} - \ldots - l_n \\ &= 1 - u_1 - u_2 - \ldots - u_t - l_{t+1} - \ldots - l_n \geq 0 \end{aligned}$$

Since $l(\{x_{t+1}, \ldots, x_n\}) = 1 - u_1 - u_2 - \ldots - u_t$ from the condition (II.1), we obtain $1 - u_1 - u_2 - \ldots - u_t \geq l_{t+1} + \ldots + l_n$, by Equation (2.3).

Second, consider $m(\{x_i, x_{t+1}, \ldots, x_n\})$.

$$\begin{aligned} m(\{x_i, x_{t+1}, \ldots, x_n\}) &= l(\{x_i, x_{t+1}, \ldots, x_n\} - m(\{x_{t+1}, \ldots, x_n\}) - l_i - l_{t+1} - \ldots - l_n \\ &= 1 - u_1 - \ldots - u_{i-1} - u_{i+1} - \ldots - u_t - 1 + u_1 + u_2 + \ldots + u_t \\ &+ l_{t+1} + \ldots + l_n - l_i - l_{t+1} - \ldots - l_n \\ &= u_i - l_i > 0 \end{aligned}$$

Hence, $m(\{x_i, x_{t+1}, \ldots, x_n\}) > 0$, $\forall i = 1, \ldots, t$.

Next, we will prove by the mathematical induction on the size of A that $m(A) = 0$, $\forall A \in Y$ with $|A| \geq n - t + 2$ to complete the proof of this case.

Basic step. When $|A| = n - t + 2$, let a set $A = \{x_i, x_j, x_{t+1}, \ldots, x_n\}$, where

$i, j = 1, \ldots, t$ and $i < j$. Using the system of Equations (3.5), we obtain

$$\begin{aligned} m(\{x_i, x_j, x_{t+1}, \ldots, x_n\}) &= l(\{x_i, x_j, x_{t+1}, \ldots, x_n\}) - l_i - l_j - l_{t+1} - \ldots - l_n \\ &\quad - m(\{x_{t+1}, \ldots, x_n\}) - m(\{x_i, x_{t+1}, \ldots, x_n\}) - m(\{x_j, x_{t+1}, \ldots, x_n\}) \\ &= 1 - u_1 - \ldots - u_{i-1} - u_{i+1} - \ldots - u_{j-1} - u_{j+1} - \ldots - u_t \\ &\quad - l_i - l_j - l_{t+1} - \ldots - l_n - 1 + u_1 + u_2 + \ldots + u_t + l_{t+1} \\ &\quad + \ldots + l_n - u_i + l_i - u_j + l_j = 0. \end{aligned}$$

Inductive step. Let $m(A) = 0$, $\forall A \in Y$ with $n - t + 2 \leq |A| \leq n - 1$
Using the system of Equations (3.5), we obtain

$$\begin{aligned} m(X) &= 1 - \sum_{i=1}^{n} l_i - m(\{x_{t+1}, \ldots, x_n\}) - \sum_{i=1}^{t} m(\{x_i, x_{t+1}, \ldots, x_n\}) \\ &= 1 - l_1 - \ldots - l_n - 1 + u_1 + u_2 + \ldots + u_t + l_{t+1} + \ldots + l_n - u_1 + l_1 \\ &\quad - \ldots - u_t + l_t = 0 \end{aligned}$$

By the mathematical induction, we can conclude that $m(A) = 0$, $\forall A \in Y$ with $|A| \geq n - t + 2$.

Case 2: Suppose the conditions (II.2.2) and (3.4) hold. Let $t = n - 1$ such that $\sum_{j \neq i_k} l_j + u_{i_k} < 1$, $\forall k = 1, \ldots, t$ and $\sum_{j \neq i_j} u_j + l_{i_j} = 1, i_j \in J$. Since there are $n - 1$ indices of i such that $\sum_{j \neq i} l_j + u_i < 1$, we obtain that there exists only one index i_j that $\sum_{j \neq i_j} l_j + u_{i_j} = 1$, by the reachable probability interval.

WLOG, let $i_j = n$. Set $m(\{x_i\}) = l_i \ \forall i = 1, \ldots, n$, we obtain $m(A) = 0, \forall A \in P(X \setminus \{x_n\})$, by Lemma 3.1.

Therefore, we must find the value of $m(A); A \in Y$ where $Y = \{K \cup \{x_n\} \mid K \subseteq I_t\}$.

Due to the system of Equations (3.1), we obtain

$$\begin{aligned}
m(\{x_i, x_n\}) &= l(\{x_i, x_n\}) - l_i - l_n \\
m(\{x_i, x_j, x_n\}) &= l(\{x_i, x_j, x_n\}) - \sum_{\widehat{i},\widehat{j} \in \{i,j,n\}} m(\{x_{\widehat{i}}, x_{\widehat{j}}\}) - \sum_{\widehat{i} \in \{i,j,n\}} l_{\widehat{i}} \qquad (3.6) \\
&\vdots \\
m(\{\underbrace{x_i, x_j, \ldots, x_n}_{n-1}\}) &= l(\{x_i, x_j, \ldots, x_n\}) - \sum m(\{\underbrace{x_i, x_j, \ldots, x_r}_{n-2}\}) - \ldots - \sum_{\widehat{i} \in \{i,j,\ldots,n\}} l_{\widehat{i}} \\
m(X) &= 1 - \sum l_i - \sum m(\{x_i, x_n\}) - \ldots - \sum m(\{\underbrace{x_i, x_j, \ldots, x_n}_{n-1}\}).
\end{aligned}$$

where $i, j = 1, \ldots, t$ and $i \neq j$.

First, consider the values of $m(\{x_i, x_n\})$.
Since $l(A) = 1 - \sum_{x_i \in A^c} u_i$, $m(\{x_i, x_n\}) = 1 - \sum_{j \neq i,n} u_j - l_i - l_n > 0$.
Next, we will prove by the mathematical induction on the size of A that $m(A) = 0$, $\forall A \in Y$ with $|A| \geq 3$ to complete the proof of this case.

Basic step. When $|A| = 3$, let set $A = \{x_i, x_j, x_n\}$, where $i, j = 1, \ldots, n-1$ and $i < j$. By Equation (3.6), we get

$$\begin{aligned}
m(\{x_i, x_j, x_n\}) &= 1 - \sum_{k \in I \setminus \{i,j,n\}} u_k - (1 - \sum_{k \in I \setminus \{i,n\}} u_k - l_i - l_n + 1 - \sum_{k \in I \setminus \{j,n\}} u_k - l_j - l_n) - \sum_{k \in \{i,j,n\}} l_k \\
&= 1 - \sum_{k \in I \setminus \{i,j,n\}} u_k - 1 + \sum_{k \in I \setminus \{i,n\}} u_k + l_i + l_n - 1 + \sum_{k \in I \setminus \{j,n\}} u_k + l_j + l_n - \sum_{k \in \{i,j,n\}} l_k \\
&= u_j + \sum_{k \in I \setminus \{j,n\}} u_k + l_n - 1 \\
&= \underbrace{(\sum_{i \neq n} u_i + l_n)}_{1} - 1 = 0
\end{aligned}$$

We get $\sum_{i \neq n} u_i + l_n = 1$, by the condition (II.2.2).

Inductive step. Let $m(A) = 0$, $\forall A \in Y$ with $3 \leq |A| \leq n-1$.

Consider $m(X)$,

$$
\begin{aligned}
m(X) &= 1 - \sum_{i=1}^{n} l_i - \sum_{i=1}^{n-1} m(\{x_i, x_n\}) \\
&= 1 - \sum_{i=1}^{n} l_i - \sum_{i=1}^{n-1} (1 - \sum_{i \in I \setminus \{i,n\}} u_i - l_i - l_n) \\
&= 1 - \sum_{i=1}^{n} l_i - \sum_{i=1}^{n-1} (1 - \sum_{i=1}^{n} u_i + u_i + u_n - l_i - l_n) \\
&= 1 - \sum_{i=1}^{n} l_i - n + 1 + (n-1) \sum_{i=1}^{n} u_i - \sum_{i=1}^{n-1} u_i - (n-1)u_n + \sum_{i=1}^{n-1} l_i + (n-1) l_n \\
&= 2 - n + (n-2) \sum_{i=1}^{n} u_i - (n-2)u_n + (n-2) l_n \\
&= 2 - n + (n-2)(\sum_{i=1}^{n} u_i - u_n + l_n) \\
&= 2 - n + (n-2)\underbrace{(\sum_{i \neq n} u_i + l_n)}_{1} = 2 - n + n - 2 = 0.
\end{aligned}
$$

By the mathematical induction, we can conclude that $m(A) = 0$, $\forall A \in Y$ with $|A| \geq 3$.

By these two cases, we can conclude that if there are exact t indices, say $i_1, \ldots, i_t$, such that $\sum_{j \neq i_k} l_j + u_{i_k} < 1$, $\forall k = 1, \ldots, t$ as well as $t \leq n-1$ and $l(A) = 1 - \sum_{x_i \in A^c} u_i$, $\forall A \in Y$, we can construct the unique random set such that $\mathcal{P}_{RS} = \mathcal{P}_{PI}$, i.e., $Bel(A) = l(A)$ and $Pl(A) = u(A)$, $\forall A \in P(X)$. When $t = n-1$, the probability interval must satisfy the condition $\sum_{j \neq i_j} u_j + l_{i_j} = 1$, $i_j \in J$ to guarantee that we can construct the random set. □

We provide an algorithm for checking the conditions of a given probability interval and constructing the random set which has the same information.

3.4 Probability interval to random set algorithm

Algorithm 3.6. *Algorithm for transforming a given probability interval to a random set*

$X := \{x_1, x_2, ..., x_n\}; n >= 3;$

$p(\{x_i\}) \in [l_i, u_i];$

$m(A) := 0; \forall A \in P(X)$

$sumL := 0; sumU := 0; sumUJ := 0; sumLJ := 0;$

$sumUT := 0; t := 0; max := 0;$

for $i = 1, \ldots, n$

  $sumL := sumL + l_i;$

  $sumU := sumU + u_i;$

  $m(\{x_i\}) := l_i;$

  $i := i + 1;$

end

[Step 1 Check for proper probability interval.]

if $(sumL > 1)$ or $(sumU < 1)$

  return "Your data are not a proper probability interval."(Stop)

else

  for $i = 1, \ldots, n$

[Step 2 Check for reachable probability interval.]

    $a_i := sumL - l_i + u_i$

    if $(a_i > 1)$ or $(sumU - u_i + l_i < 1)$

      return "Your data are not a reachable probability interval."(Stop)

    end

[Step 3 Check the conditions of probability interval.]

    if $a_i < 1$

      $t := t + 1;$

      keep index i in I_t;

      $sumUT := sumUT + u_i;$

    else

keep index i in X_j;

$sumUJ := sumUJ + u_i$;

$sumLJ := sumLJ + l_i$;

end

$i := i + 1$;

end

[Step 4 Construct the random set.]

if $t = 2$

$max := sumL - l_{I_{t_1}} - l_{I_{t_2}}$;

if $max < 1 - u_{I_{t_1}} - u_{I_{t_2}}$

$max := 1 - u_{I_{t_1}} - u_{I_{t_2}}$;

end

$m(X \setminus \{x_{I_{t_1}}, x_{I_{t_2}}\}) := max - sumL + l_{I_{t_1}} + l_{I_{t_2}}$;

$m(X \setminus \{x_{I_{t_1}}\}) := -max + 1 - u_{I_{t_1}} - l_{I_{t_2}}$;

$m(X \setminus \{x_{I_{t_2}}\}) := -max + 1 - u_{I_{t_2}} - l_{I_{t_1}}$;

$m(X) := max - 1 + u_{I_{t_1}} + u_{I_{t_2}}$;

elseif $t = 1$

$m(X \setminus \{x_{I_{t_1}}\}) := 1 - u_{I_{t_1}} - sumL + l_{I_{t_1}}$;

$m(X) := u_{I_{t_1}} - l_{I_{t_1}}$;

elseif $t = 0$

$m(X) := 1 - sumL$;

else

if $l(A) = \sum_{x_i \in A} l_i, \forall A \in Y - Y'$

$b := (t-2)\sum_{i \in I} l_i + \sum_{i \in I} u_i - \sum_{j \in J} u_j + \sum_{j \in J} l_j - t + 1$;

if $b \geq 0$

for $i = 1, \ldots, j$

$m(X \setminus \{x_{I_{t_i}}\}) := 1 - a_{I_{t_i}}$;

end

$m(X) := b$;

else

return "Your data do not satisfy the condition (I.3)."(Stop)

end

elseif $l(A) = 1 - \sum_{x_i \in A^c} u_i, \ \forall A \in Y$

if $t \leq n-2$

$m(X_J) := 1 - sumUT - sumLJ;$

for $i = 1, \ldots, j$

$m(X_J \cup \{x_{I_{t_i}}\}) := u_{I_{t_i}} - l_{I_{t_i}};$

end

elseif $t = n-1$

if $sumU - u_k + l_k = 1, k \in X_J$

for $i = 1, \ldots, j$

$m(\{x_k, x_{I_{t_i}}\}) := 1 - sumU + u_{I_{t_i}} + u_k - l_{I_{t_i}} - l_k;$

end

else

return "Your data do not satisfy the condition (II.2.2)." (Stop)

end

else

return "Your data do not satisfy the condition (II)." (Stop)

end

else

return "Your data do not satisfy the condition (I) and (II)." (Stop)

end

end

return $m(A); \forall A \in P(X)$

end

CHAPTER IV
LINEAR PROGRAMMING PROBLEM WITH PROBABILITY INTERVAL AND RANDOM SET PARAMETERS

This chapter, we offer a method for solving a linear programming problem with probability interval and random set parameters which is in the form

$$\left.\begin{array}{rc} \min_x & c_1x_1 + c_2x_2 + \ldots + c_nx_n \\ \text{s.t.} & \widehat{a}_{11}x_1 + \widehat{a}_{12}x_2 + \ldots + \widehat{a}_{1n}x_n \geq \widehat{b}_1, \\ & \widehat{a}_{21}x_1 + \widehat{a}_{22}x_2 + \ldots + \widehat{a}_{2n}x_n \geq \widehat{b}_2, \\ & \vdots \\ & \widehat{a}_{k1}x_1 + \widehat{a}_{k2}x_2 + \ldots + \widehat{a}_{kn}x_n \geq \widehat{b}_k, \\ & Bx \geq d, \\ & x \geq 0, \end{array}\right\} \tag{4.1}$$

where $\widehat{a}_{ij}$ and $\widehat{b}_i$, $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, n$, can be random sets or probability intervals. Moreover, we present a code from SAGE which is a free open-source mathematics software for each method of solving a linear programming problem with probability interval and random set parameters. We will use Problem A to explain the method for solving a linear programming with probability interval and random set parameters.

Problem A:

$$
\begin{aligned}
\min_{x} \quad & 2x_1 + 4x_2 + 2.5x_3 \\
s.t. \quad & \widehat{a}_{11}x_1 + \widehat{a}_{12}x_2 + 4x_3 \geq \widehat{b}_1 && (4.2)\\
& \widehat{a}_{21}x_1 + \widehat{a}_{22}x_2 + \widehat{a}_{23}x_3 \geq \widehat{b}_2 && (4.3)\\
& x_1 + 2x_3 \geq 16 \\
& 6x_1 + 8x_2 + 4x_3 \geq 128 \\
& x_3 \leq 80 \\
& x_1, x_2, x_3 \geq 0,
\end{aligned}
$$

where

$$
\widehat{a}_{11} = \begin{cases} 3 & , \quad p(\{3\}) \in [1/6, 1/2] \\ 4 & , \quad p(\{4\}) \in [1/3, 2/3] \\ 5 & , \quad p(\{5\}) \in [1/6, 1/2] \end{cases}
$$

$$
\begin{aligned}
\widehat{a}_{12} = \{1,2,3\} \text{ where } & m(\{1,2\}) = 1/8,\ m(\{2,3\}) = 1/4, \\
& m(\{1,3\}) = 1/4,\ m(X) = 3/8
\end{aligned}
$$

$$
\widehat{b}_1 = \begin{cases} 64 & , \quad p(\{64\}) \in [1/4, 1/2] \\ 68 & , \quad p(\{68\}) \in [1/4, 7/16] \\ 72 & , \quad p(\{72\}) \in [1/8, 3/8] \\ 76 & , \quad p(\{76\}) \in [1/8, 5/16] \end{cases}
$$

$$
\begin{aligned}
\widehat{a}_{21} = \{1,2,3\} \text{ where } & m(\{1\}) = 2/5,\ m(\{2\}) = 1/10,\ m(\{3\}) = 1/5, \\
& m(\{1,2\}) = 1/5,\ m(\{2,3\}) = 1/10
\end{aligned}
$$

$$
\widehat{a}_{22} = \begin{cases} 3 & , \quad p(\{3\}) \in [1/16, 7/16] \\ 4 & , \quad p(\{4\}) \in [1/8, 1/2] \\ 5 & , \quad p(\{5\}) \in [3/16, 9/16] \\ 6 & , \quad p(\{6\}) \in [1/4, 5/8] \end{cases}
$$

$$\widehat{a}_{23} = \{1,2,3\} \text{ where } m(\{1\}) = 1/4,\ m(\{1,3\}) = 5/16,$$
$$m(\{2,3\}) = 3/16,\ m(X) = 1/4$$

$$\widehat{b}_2 = \begin{cases} 70 & , \quad p(\{70\}) \in [1/8, 7/16] \\ 80 & , \quad p(\{80\}) \in [3/16, 1/2] \\ 90 & , \quad p(\{90\}) \in [5/16, 9/16]. \end{cases}$$

In general, the pessimistic and optimistic expected recourse approaches are used for solving a linear programming problem with uncertainty. First of all, we must consider the probability interval parameters. If they satisfy the conditions stated in the previous chapter, we can transform this problem to the problem which has only random set parameters and use the idea of decision making of random set to solve it. Otherwise, we must solve this problem with two types of parameters after we find appropriate distributions for each approach for solving the problem.

We used the program in Appendix A for checking the conditions of probability interval parameters and found that all of them could be constructed the random set as shown in Table 4.1.

Table 4.1. *The corresponding random set from probability interval parameters in Problem A*

probability interval	$\widehat{a}_{11}$	$\widehat{b}_1$	$\widehat{a}_{22}$	$\widehat{b}_2$
set of realizations	$X = \{3,4,5\}$	$X = \{64,68,72,76\}$	$X = \{3,4,5,6\}$	$X = \{70,80,90\}$
mass function	$m(\{3\}) = 1/6$	$m(\{64\}) = 1/4$	$m(\{3\}) = 1/16$	$m(\{70\}) = 2/16$
	$m(\{4\}) = 2/6$	$m(\{68\}) = 1/4$	$m(\{4\}) = 2/16$	$m(\{80\}) = 3/16$
	$m(\{5\}) = 1/6$	$m(\{72\}) = 1/8$	$m(\{5\}) = 3/16$	$m(\{90\}) = 5/16$
	$m(X) = 1/3$	$m(\{76\}) = 1/8$	$m(\{6\}) = 4/16$	$m(\{80,90\}) = 1/16$
		$m(\{64,72,76\}) = 1/16$	$m(X) = 3/8$	$m(\{70,90\}) = 1/16$
		$m(\{64,68,72\}) = 1/16$		$m(\{70,80\}) = 1/8$
		$m(X) = 1/8$		$m(X) = 1/8$

Therefore, Problem A can be solved as a linear programming problem that has only random set parameters. However, we will explain how to solve the problem with two types of parameters before solving the problem with only random set parameters.

4.1 Solving linear programming problems with probability interval and random set parameters

The pessimistic and optimistic expected recourse approaches commonly used for solving a linear programming problem with probability interval and random set parameters. It is necessary to find the lowest and largest density functions of probability intervals and random sets before applying these approaches to our problem. The lowest density function will provide the smallest expected value among all probabilities in the set M_{PI} (or M_{RS}) of all probability density functions satisfying the probability interval (or random set) information. Similarly, the largest density function will provide the largest expected value among all probabilities in M_{PI} (or M_{RS}). More details on solving linear programming problems with generalized uncertainty can be found in Thipwiwatpojana [15].

4.1.1 The lowest and largest density functions generated by a random set

Given a random set for a set of realizations $X = \{x_1, x_2, \ldots, x_n\}$, and an evaluation function θ on X, where $\theta(x_1) \leq \theta(x_2) \leq \ldots \leq \theta(x_n)$, the lowest and the largest expected values of θ can be evaluated by using the following density functions $\underline{f}$ and $\overline{f}$, respectively, where

$$\left.\begin{array}{rcl} \underline{f}(x_1) & = & Bel(\{x_1, x_2, \ldots, x_n\}) - Bel(\{x_2, x_3, \ldots, x_n\}) \\ & \vdots & \\ \underline{f}(x_i) & = & Bel(\{x_i, x_{i+1}, \ldots, x_n\}) - Bel(\{x_{i+1}, x_{i+2}, \ldots, x_n\}) \\ & \vdots & \\ \underline{f}(x_n) & = & Bel(\{x_n\}), \end{array}\right\} \tag{4.4}$$

and

$$\left.\begin{array}{rcl}\overline{f}(x_1) & = & Bel(\{x_1\}) \\ & \vdots & \\ \overline{f}(x_i) & = & Bel(\{x_1,x_2,\ldots,x_i\}) - Bel(\{x_1,x_2,\ldots,x_{i-1}\}) \\ & \vdots & \\ \overline{f}(x_n) & = & Bel(\{x_1,x_2,\ldots,x_n\}) - Bel(\{x_1,x_2,\ldots,x_{n-1}\}). \end{array}\right\} \quad (4.5)$$

Nguyen [11] proved that $\underline{f}$ in (4.4) obtains the lowest density function of θ. Moreover, it was showed that $\overline{f}$ in (4.5) obtains the largest density function of θ by Thipwiwatpojana [15]. Before we construct an algorithm for finding the lowest and largest density functions generated by a random set, we provide the algorithm for computing the belief measure as follows.

Algorithm 4.2. *Algorithm for finding the belief value from a given mass function*

```
#m is the set of domains of mass functions
#f is the set of values of mass functions
def Bel(k):
    b=0
    for j in range(len(m)):
        if m[j].issubset(k):
            b=b+f[j]
    return b
```

We use Algorithm 4.2 for computing the lowest and largest density functions generated by a random set as follows.

Algorithm 4.3. *Algorithm for finding* $\underline{f}$ *generated by a random set*

```
#n is the set of the domain of mass function used to compute
the lowest density function
A=[] #keep the values of belief function of set n
B=[] #keep the values of the lowest density function of a random set
for i in range(len(n)):
```

```
    A.append(Bel(n[i]))
for i in range(len(n)-1):
    B.append(A[i]-A[i+1])
B.append(A[len(n)-1])
print 'The lowest density function=',B
```

Algorithm 4.4. *Algorithm for finding $\overline{f}$ generated by a random set*

```
#p is the set of the domain of mass function used to compute
the largest density function
C=[] #keep the values of belief function of set p
D=[0] #keep the values of the largest density function of a random set
for i in range(len(p)):
    C.append(Bel(p[i]))
D[0]=C[0]
for i in range(0,len(p)-1):
    D.append(C[i+1]-C[i])
print 'The largest density function=',D
```

In this program, it is necessary to give the information of the random set in a special form. For example, if we have the random set information of the set $X = \{0, 1, 2, 3, 4\}$ as $m(\{0\}) = 1/25$, $m(\{1\}) = 2/25$, $m(\{2\}) = 4/25$, $m(\{3\}) = 6/25$, $m(\{4\}) = 1/25$, $m(\{0, 1\}) = 3/25$, $m(\{1, 2\}) = 2/25$, $m(\{0, 1, 2\}) = 1/25$, $m(\{1, 2, 3\}) = 3/25$, $m(\{0, 2, 3, 4\}) = 1/25$ and $m(X) = 1/25$, then we must type the information as the following in order to be able to use this program.

```
m=[{0},{1},{2},{3},{4},{0,1},{1,2},{0,1,2},{1,2,3},{0,2,3,4},{0,1,2,3,4}]
f=[1/25,2/25,4/25,6/25,1/25,3/25,2/25,1/25,3/25,1/25,1/25]
n=[{0,1,2,3,4},{1,2,3,4},{2,3,4},{3,4},{4}]
p=[{0},{0,1},{0,1,2},{0,1,2,3},{0,1,2,3,4}]
```

the result by using Algorithms 4.3 and 4.4 are

```
The lowest density function = [7/25, 7/25, 4/25, 6/25, 1/25]
The largest density function = [1/25, 1/5, 7/25, 9/25, 3/25].
```

The result for finding the lowest and largest density functions generated by random sets in Problem A is

```
a11 -> The lowest density function = [1/2, 1/3, 1/6],
       The largest density function = [1/6, 1/3, 1/2]
a12 -> The lowest density function = [3/4, 1/4, 0],
       The largest density function = [0, 1/8, 7/8]
b1 -> The lowest density function = [1/2, 1/4, 1/8, 1/8],
      The largest density function = [1/4, 1/4, 3/16, 5/16]
a21 -> The lowest density function = [3/5, 1/5, 1/5],
       The largest density function = [2/5, 3/10, 3/10]
a22 -> The lowest density function = [7/16, 1/8, 3/16, 1/4],
       The largest density function = [1/16, 1/8, 3/16, 5/8]
a23 -> The lowest density function = [13/16, 3/16, 0],
       The largest density function = [1/4, 0, 3/4]
b2 -> The lowest density function = [7/16, 1/4, 5/16],
      The largest density function = [1/8, 5/16, 9/16].
```

Next, we will show how to obtain the lowest and largest density functions generated by a probability interval.

4.1.2 The lowest and largest density functions generated by a probability interval

Consider $X = \{x_1, x_2, \ldots, x_n\}$ which has the set $\mathcal{P}_L$ of probability distributions on X as $\mathcal{P}_L = \{p \mid l_i \leq p(\{x_i\}) \leq u_i, \sum_{i=1}^{n} p(\{x_i\}) = 1, \forall i = 1, 2, \ldots, n\}$ where $x_1 \leq x_2 \leq \ldots \leq x_n$. It was shown in [15] that there is an optimal solution of the problem (4.6)

$$\min_{p \in \mathcal{P}_L} / \max_{p \in \mathcal{P}_L} \; x_1 p(\{x_1\}) + x_2 p(\{x_2\}) + \ldots + x_n p(\{x_n\}) \tag{4.6}$$

by using a greedy algorithm.

The lowest density function, $\underline{f}$, is an optimal solution for the problem

$$\min_{p \in \mathcal{P}_L} \; x_1 p(\{x_1\}) + x_2 p(\{x_2\}) + \ldots + x_n p(\{x_n\}),$$

and the largest density function, $\overline{f}$, is an optimal solution for the problem

$$\max_{p \in \mathcal{P}_L} x_1 p(\{x_1\}) + x_2 p(\{x_2\}) + \ldots + x_n p(\{x_n\}).$$

The following is the greedy algorithm for finding the lowest and largest density functions generated by a probability interval.

Algorithm 4.5. *Algorithm for finding $\underline{f}$ generated by a probability interval*

```
# l is a set of lower bound
# u is a set of upper bound
L=[0,0] #initial list for keeping the value of the lowest density function
L[0]=u[0]
sum=1-L[0]
for i in range(2,len(l)):
    L.append(l[i])
    sum=sum-l[i]
L[1]=sum
print 'The lowest density function =',L
```

Algorithm 4.6. *Algorithm for finding $\overline{f}$ generated by a probability interval*

```
U=[] #initial list for keeping the value of the largest density function
sum=1-u[len(u)-1]
for i in range(len(u)-2):
     U.append(l[i])
     sum=sum-l[i]
U.append(sum)
U.append(u[len(u)-1])
print 'The largest density function =',U
```

For example, if we have the probability interval information of the set $X = \{x_1, x_2, x_3, x_4\}$ as $L = \{[l_1, u_1] = [1/4, 1/2], [l_2, u_2] = [1/4, 7/16], [l_3, u_3] = [1/8, 3/8], [l_4, u_4] = [1/8, 5/16]\}$, we must type the information as the following in order to be able to use this program.

```
l=[1/4,1/4,1/8,1/8]
u=[1/2,7/16,3/8,5/16]
```

the result by using Algorithms 4.5 and 4.6 are

```
The lowest density function = [1/2, 1/4, 1/8, 1/8]
The largest density function = [1/4, 1/4, 3/16, 5/16].
```

The result for finding the lowest and largest density functions generated by probability intervals in Problem A is

```
a11 -> The lowest density function = [1/2, 1/3, 1/6],
       The largest density function = [1/6, 1/3, 1/2]
b1 -> The lowest density function = [1/2, 1/4, 1/8, 1/8],
      The largest density function = [1/4, 1/4, 3/16, 5/16]
a22 -> The lowest density function = [7/16, 1/8, 3/16, 1/4],
       The largest density function = [1/16, 1/8, 3/16, 5/8]
b2 -> The lowest density function = [7/16, 1/4, 5/16],
      The largest density function = [1/8, 5/16, 9/16].
```

After we can find the lowest and largest density functions generated by a probability interval and a random set, we can find the pessimistic and optimistic expected recourse values as follows.

4.1.3 Pessimistic and optimistic expected recourse models

Consider Problem (4.1), we assume there are α_{ij} realizations for each $\widehat{a}_{ij}$; i.e., $a_{ij}^1, a_{ij}^2, \ldots, a_{ij}^{\alpha_{ij}}$ with probability density mass values $g_{ij}(a_{ij}^1), g_{ij}(a_{ij}^2), \ldots, g_{ij}(a_{ij}^{\alpha_{ij}})$, respectively and there are β_i realizations for each $\widehat{b}_i$; i.e.,$b_i^1, b_i^2, \ldots, b_i^{\beta_i}$, with probability density mass values $h_i(b_i^1), h_i(b_i^2), \ldots, h_i(b_i^{\beta_i})$, respectively, where $\sum_k^{\alpha_{ij}} g_{ij}(a_{ij}^k) = 1$ and $\sum_k^{\beta_i} h_i(b_i^k) = 1$. Let $K_i = \prod_{j=1}^n \alpha_{ij}\beta_i$ be the number of scenarios with respect to the i^{th} constraint of Problem (4.1). We can transform Problem (4.1) to the expected recourse model as follows

$$\left.\begin{array}{lc}
\min_x & c_1x_1+c_2x_2+\ldots+c_nx_n+ \\
& s_1\sum_{L=1}^{K_1} f_1^L w_1^L + s_2\sum_{L=1}^{K_2} f_2^L w_2^L+\ldots+s_m\sum_{L=1}^{K_m} f_m^L w_m^L \\
\text{s.t.} & w_1^1 \geq b_1^1 - a_{11}^1x_1 - a_{12}^1x_2 - \ldots - a_{1n}^1x_n, \\
& \vdots \\
& w_1^{K_1} \geq b_1^{\beta_1} - a_{11}^{\alpha 11}x_1 - a_{12}^{\alpha 12}x_2 - \ldots - a_{1n}^{\alpha 1n}x_n, \\
& \vdots \\
& w_k^1 \geq b_k^1 - a_{k1}^1x_1 - a_{k2}^1x_2 - \ldots - a_{kn}^1x_n, \\
& \vdots \\
& w_k^{K_k} \geq b_k^{\beta_k} - a_{k1}^{\alpha k1}x_1 - a_{k2}^{\alpha k2}x_2 - \ldots - a_{kn}^{\alpha kn}x_n \\
& Bx \geq d, \\
& x, w \geq 0,
\end{array}\right\} \quad (4.7)$$

where $w_i = \max\{0, b_i - a_{i1}x_1 - a_{i2}x_2 - \ldots - a_{in}x_n\}$ is a recourse variable, s_i is the positive penalty price for each i^{th} constraint of (4.1), f_i^L is the joint probability for w_i^L that $f_i^L = h_i(b_i^l)g_{i1}(-a_{i1}^{l1})g_{i2}(-a_{i2}^{l2})\ldots g_{in}(-a_{1n}^{ln})$ with the assumption that all variables are independent and L is the scenario $(l, l_1, l_2, \ldots, l_n)$.

Let M_i be the set of all joint probabilities satisfying the information on the i^{th} constraint of (4.1), therefore $M_i = \{f_i \mid \sum_{L=1}^{K_i} f_i^L = 1, f_i^L = h_i^L g_{i1}^L g_{i2}^L \ldots g_{in}^L\}$. Moreover, we define Ξ as the the feasible set of the constraints in (4.7) and $M := \{f = (f_1, f_2, \ldots, f_m) \mid f_i \in M_i, i = 1, 2, \ldots, k\}$.

The optimistic expected recourse value is the objective value of the problem (4.8).

$$\left.\begin{array}{c}
\min_{f\in M}\min_{(x,w)\in\Xi} \quad c_1x_1+c_2x_2+\ldots+c_nx_n+ \\
s_1\sum_{L=1}^{K_1} f_1^L w_1^L + s_2\sum_{L=1}^{K_2} f_2^L w_2^L+\ldots+s_k\sum_{L=1}^{K_k} f_k^L w_k^L.
\end{array}\right\} \quad (4.8)$$

On the other hands, the pessimistic expected recourse value is the objective value of the problem (4.9).

$$\left.\begin{array}{c}
\max_{f\in M}\min_{(x,w)\in\Xi} \quad c_1x_1+c_2x_2+\ldots+c_nx_n+ \\
s_1\sum_{L=1}^{K_1} f_1^L w_1^L + s_2\sum_{L=1}^{K_2} f_2^L w_2^L+\ldots+s_k\sum_{L=1}^{K_k} f_k^L w_k^L.
\end{array}\right\} \quad (4.9)$$

It was also shown in Thipwiwatpojana [15] that finding the solutions of the problems (4.8) and (4.9) are equivalent to finding the solutions of the problems (4.10) and (4.11), respectively.

$$\min_{(x,w)\in\Xi} c_1x_1 + c_2x_2 + \ldots + c_nx_n + s_1\underline{f}_1w_1 + s_2\underline{f}_2w_2 + \ldots + s_k\underline{f}_kw_k. \tag{4.10}$$

$$\min_{(x,w)\in\Xi} c_1x_1 + c_2x_2 + \ldots + c_nx_n + s_1\overline{f}_1w_1 + s_2\overline{f}_2w_2 + \ldots + s_k\overline{f}_kw_k. \tag{4.11}$$

The following algorithm is the program of finding the optimistic and pessimistic expected values of Problem A.

Begin with computing the joint probability of each constraint. Since the joint probability of each constraint depends on the number of uncertain parameters in the constraint, there is no unique algorithm for computing the joint probability. However, we can compute the maximum and minimum joint probabilities of each constraint at the same time. We use the largest density function, $\overline{f}$, from Algorithms 4.4 and 4.6 and the lowest density function, $\underline{f}$, from Algorithms 4.3 and 4.5 for computing the maximum and minimum joint probability.

Consequently, we use the maximum and minimum joint probability for finding the pessimistic and optimistic recourse values, respectively. We use Problem A to explain how to use the algorithm for finding the pessimistic and optimistic expected recourse values.

Consider Problem A, there are two constraints containing uncertain parameters, constraints (4.2) and (4.3). However, the number of uncertain parameters in these constraints are not equal. Then, we can not use the same algorithm to find the joint probability. There are three uncertain parameters in constraint (4.2), so an algorithm finding the joint probability has three loops. Thus, there are four loops in the algorithm finding the joint probability of constraint (4.3) with the same reason, as follows.

Algorithm 4.7. *Algorithm for finding the maximum and minimum joint probabilities of constraint (4.2) in Problem A*

```
minf1=[] #keep the minimum joint probability of each scenario
```

```
maxf1=[] #keep the maximum joint probability of each scenario
A11=[] #keep index of parameter a11 of each scenario
A12=[] #keep index of parameter a12 of each scenario
B1=[] #keep index of parameter b1 of each scenario
for i in range(len(mpa11)):
    for j in range(len(mpa12)):
        for k in range(len(mpb1)):
            A11.append(i)
            A12.append(j)
            B1.append(k)
            minf1.append(Mpa11[i]*Mpa12[j]*mpb1[k])
            maxf1.append(mpa11[i]*mpa12[j]*Mpb1[k])
print 'The minimum joint probability =',minf1
print 'The maximum joint probability =',maxf1
```

Algorithm 4.8. *Algorithm for finding the maximum and minimum joint probabilities of constraint (4.3) in Problem A*

```
minf2=[] #keep the minimum joint probability of each scenario
maxf2=[] #keep the maximum joint probability of each scenario
A21=[] #keep index of parameter a21 of each scenario
A22=[] #keep index of parameter a22 of each scenario
A23=[] #keep index of parameter a23 of each scenario
B2=[] #keep index of parameter b2 of each scenario
for i in range(len(mpa21)):
    for j in range(len(mpa22)):
        for k in range(len(mpa23)):
            for l in range(len(mpb2)):
                A21.append(i)
                A22.append(j)
                A23.append(k)
                B2.append(l)
```

```
                minf2.append(Mpa21[i]*Mpa22[j]*Mpa23[k]*mpb2[l])
                maxf2.append(mpa21[i]*mpa22[j]*mpa23[k]*Mpb2[l])
print 'The minimum joint probability =',minf2
print 'The maximum joint probability =',maxf2
```

Before we use Algorithms 4.7 and 4.8, we must type the information of the lowest and largest density functions from Algorithms 4.3, 4.4, 4.5 and 4.6 as follows. In the algorithm, we use variables Mp and mp to represent $\overline{f}$ and $\underline{f}$, respectively.

```
#The largest density function
#constrain (4.2) in Problem A
Mpa11=[1/6, 1/3, 1/2] #The largest density function of parameter a11
Mpa12=[0, 1/8, 7/8]
Mpb1= [1/4, 1/4, 3/16, 5/16]
#constrain (4.3) in Problem A
Mpa21=[2/5, 3/10, 3/10]
Mpa22=[1/16, 1/8, 3/16, 5/8]
Mpa23=[1/4, 0, 3/4]
Mpb2=[1/8, 5/16, 9/16]
#The lowest density function
#constrain (4.2) in Problem A
mpa11=[1/2, 1/3, 1/6]
mpa12=[3/4, 1/4, 0]
mpb1= [1/2, 1/4, 1/8, 1/8]
#constrain (4.3) in Problem A
mpa21=[3/5, 1/5, 1/5]
mpa22=[7/16, 1/8, 3/16, 1/4]
mpa23=[13/16, 3/16, 0]
mpb2= [7/16, 1/4, 5/16]
```

The next algorithm is an algorithm for solving Problem A. If we want to find the optimistic expected recourse value, we set the value of f1 and f2 in the algorithm as f1=minf1 and f2=minf2. Otherwise, if we want to find the pessimistic expected recourse value, we set f1=maxf1 and f2=maxf2.

Algorithm 4.9. *Algorithm for solving Problem A*

```
# Declare the value of each parameter and variable
a11=[3,4,5] # value of a11
a12=[1,2,3] # value of a12
b1=[64,68,72,76] # value of b1
a21=[1,2,3] # value of a21
a22=[3,4,5,6] # value of a22
a23=[1,2,3] # value of a23
b2=[70,80,90] # value of b2
s1=1 # value of penalty price for constraint (4.2)
s2=1 # value of penalty price for constraint (4.3)
# Construct the object for LP problem
p = MixedIntegerLinearProgram(maximization=False)
# Declare variable
x = p.new_variable(dim=1)
w1 = p.new_variable(dim=1) # recourse variable for constraint (4.2)
w2 = p.new_variable(dim=1) # recourse variable for constraint (4.3)
# Add objective function
a=2*x[0]+4*x[1]+2.5*x[2]
b=0
c=0
for i in range(len(f1)):
    b=b+(f1[i]*w1[i])
for i in range(len(f2)):
    c=c+(f2[i]*w2[i])
p.set_objective(a+s1*b+s2*c)
# Add constraints
p.add_constraint(x[0]+2*x[2]>=16)
```

```
p.add_constraint(6*x[0]+8*x[1]+4*x[2]>=128)
p.add_constraint(x[2]<=80)
for i in range(len(f1)):
    p.add_constraint(w1[i]>= b1[B1[i]]-a11[A11[i]]*x[0]-
    a12[A12[i]]*x[1]-4*x[2])
for i in range(len(f2)):
    p.add_constraint(w2[i]>= b2[B2[i]]-a21[A21[i]]*x[0]-
    a22[A22[i]]*x[1]-a23[A23[i]]*x[2])
p.show()
```

The result of this algorithm is shown in Table 4.11 in the last part of this chapter.

4.2 Solving problem with only random set parameters

We use the decision making of random set for solving this type of the problem. There are three approaches for solving the decision making based on distribution functions of random sets, which are

(a) expectation with respect to a distribution function

(b) expectation of a function of a random set

(c) maximum entropy distribution.

The detail of this can be found in Hung T. Nguyen [11]. However, there are two of these approaches that relate with our objective, i.e., the approaches (a) and (b). The result of using the approach (a) is the same as finding the lowest density function generated by a random set. Therefore, we obtain the optimistic expected recourse value when we use the distribution function from this approach. The distribution function of a random set used the approach (b) is in the form of

$$f^*(x_i) = \rho\overline{f}(x_i) + (1-\rho)\underline{f}(x_i), \tag{4.12}$$

where $\rho \in [0, 1]$. The value of this distribution function depends on the parameter ρ. If we use $\rho = 1$, then we obtain the largest density function. If $\rho = 0$, then we obtain the lowest density function. However, we can use any value of $\rho \in [0, 1]$. This leads to the results not only pessimistic or optimistic ones but also all of the results that a supply may want to know. Therefore, solving a problem with only random set parameters is to solve the following problem

$$\min_{(x,w)\in\Xi} c_1x_1 + c_2x_2 + \ldots + c_nx_n + s_1f_1^*w_1 + s_2f_2^*w_2 + \ldots + s_kf_k^*w_k. \tag{4.13}$$

We must find the values of f^* before using the Algorithm 4.9 for solving a problem with only random set parameters. However, we must define the value of ρ before finding f^*. For example, if we set $\rho = 0.5$, then we can evaluate f^* using the following program.

Algorithm 4.10. *Algorithm for finding f^**

```
p=0.5
f1=[]
f2=[]
for i in range(len(minf1)):
    f1.append((p*maxf1[i])+((1-p)*minf1[i]))
for i in range(len(minf2)):
    f2.append((p*maxf2[i])+((1-p)*minf2[i]))
```

Table 4.11. *The result of Problem A using different values ρ*

ρ	*0 (optimistic)*	*0.25*	*0.5*	*0.75*	*1 (pessimistic)*
objective value	*70.93052*	*79.08351*	*85.07783*	*89.62387*	*93.625*
x_1	*6.85714*	*10.0*	*17.42857*	*20.85714*	*20.83333*
x_2	*8.7619*	*8.0*	*6.28571*	*4.71429*	*5.5*
x_3	*4.57143*	*4.0*	*0.0*	*0.0*	*0.0*

CHAPTER V
CONCLUSION

In this thesis, we study the relationship of probability intervals and random sets. We discovered the conditions of a given probability interval which can be represented as a unique random set with the same information, i.e., a reachable probability interval containing at most two indices, say i_1, i_2, such that $\sum_{j \neq i_1} l_j + u_{i_1} < 1$ and $\sum_{j \neq i_2} l_j + u_{i_2} < 1$ as we state in Theorem 3.2. However, a reachable property is not enough for t indices of i satisfying the condition $\sum_{j \neq i} l_j + u_i < 1$ when $2 < t \leq n$. We extend the conditions of a reachable probability interval, and we find that the reachable probability interval must satisfy either one of the following conditions for guarantee that there is a random set which has the same information.

(I) (I.1) $l(A) = \sum_{x_i \in A} l_i, \ \forall A \in Y - Y', \text{ and}$

(I.2) $l(X \setminus \{x_{i_k}\}) = 1 - u_{i_k}, \ \forall i_k \in I_t, \text{ and}$

(I.3) $(t-2) \sum_{i \in I} l_i + \sum_{i \in I} u_i - \sum_{j \in J} u_j + \sum_{j \in J} l_j \geq t - 1.$

(II)(II.1) $l(A) = 1 - \sum_{x_i \in A^c} u_i, \ \forall A \in Y$, and

(II.2) only one of the following

(II.2.1) $t \leq n - 2$, or

(II.2.2) $t = n - 1$ and $\sum_{j \neq i_k} u_j + l_{i_k} = 1, \ i_k \in J.$

We construct the algorithm for checking and constructing a random set from a given probability interval.

In addition, we solved the uncertain linear programming problems with probability interval and random set parameters. In general, we use the optimistic and pessimistic approaches for solving these type of problems. If all of the probability interval parameters satisfy our conditions, we can transform the problem to the problem which has only random set parameters. So, we can use an idea from decision making theory with random sets for solving this problem. We explain all of these solving in Chapter IV.

APPENDIX

Appendix A: Algorithm for constructing a random set from a given probability interval

```
# I is a set of all indices
# l is a set of lower bound
# u is a set of upper bound
sumL=sum(l) #sum all of lower bounds
sumU=sum(u) #sum all of upper bounds
sumUT=0 #sum of upper bounds that index i satisfy '<'
sumUJ=0 #sum of upper bounds that index i satisfy '='
sumLJ=0 #sum of lower bounds that index i satisfy '='
t=0 #number of index i satisfy '<'
r=0 #number of l(A)=sum l(A)
s=0 #number of l(A)=1-sum u(A^c)
J=[] # index i that satisfy '='
max=0
e=0 #check that P.I. is reachable probability interval
a=[]
It=[] # index i that satisfy '<'
# Step 1 : Check for proper probability interval
if sumL>1 or sumU<1: #check proper probability interval
    print "Your data are not proper probability interval."
    e=1
else:
    for i in range(len(I)):
        # Step 2 : Check for reachable probability interval
        a.append(sumL-l[i]+u[i]) #first property of reachability
        if a[i]>1 or sumU-u[i]+l[i]<1:
             print "Your data are not reachable probability interval."
             e=1
```

```
          break
     # Step 3 : Check the conditions of probability interval
     elif a[i]<1:
         t=t+1
         It.append(i) #keep index i that satisfy '<' in list It
         sumUT=sumUT+u[i]
     else:
         J.append(i) #keep index i that satisfy '=' in list J
         sumUJ=sumUJ+u[i]
         sumLJ=sumLJ+l[i]
print 't=',t
# Step 4 : Construct the random set.
if t==2 and e==0: #return mass functions that only two indices
 satisfy '<'
   max=sumL-l[It[0]]-l[It[1]]
   if max<1-u[It[0]]-u[It[1]]:
       max=1-u[It[0]]-u[It[1]]
   print 'm(X\{x_',It[0],',x_',It[1],'}) =',max-sumL+l[It[0]]+l[It[1]]
   print 'm(X\{x_',It[0],'}) =',-max+1-u[It[0]]-l[It[1]]
   print 'm(X\{x_',It[1],'}) =',-max+1-u[It[1]]-l[It[0]]
   print 'm(X) =',max-1+u[It[0]]+u[It[1]]
elif t==1 and e==0 : #return mass functions that only one indix
 satisfy '<'
   print 'm(X\{x_',It[0],'}) =',1-u[It[0]]-sumL+l[It[0]]
   print 'm(X) =' ,u[It[0]]-l[It[0]]
elif t==0 and e==0: #return mass functions that all indices satisfy '='
   print 'm(X) =' ,1-sumL
elif t>2 and e==0:
   p=list(I.subsets()) #list of power set of I
   IT=Set(It) #set of all index i that satisfy '<'
   XJ=Set(J) #set of all index i that satisfy '='
```

```
Y=[] #set of set that J is subset
L=[] #list of value of [sum of l(i),x(i) in A] of set Y
U=[] #list of value of [1- sum of u(i),x(i) in A^c] of set Y
for j in range(len(p)):
    if XJ.issubset(p[j]) and len(p[j])>=2:
        Y.append(p[j])
for i in range(len(Y)):
    L.append(0) #initial value of sum of l(i),x(i) in A
    U.append(1-sumU) #initial value of 1- sum of u(i),x(i) in A^c
for i in range(len(Y)):
    for j in range(len(Y[i])):
        L[i]=L[i]+l[Y[i][j]] #compute sum of l(i),x(i) in A
        U[i]=U[i]+u[Y[i][j]] #compute 1- sum of u(i),x(i) in A^c
for i in range(len(Y)):
#check that it satisfies the condition (I) or (II)
    if L[i]>=U[i]:
        r=r+1
    if U[i]>=L[i]:
        s=s+1
if r==(len(Y)-t-1) or t==len(I):
#all of l(A) satisfy sum l(A);[condition (I.1)]
    b=(t-2)*sumL+sumU-sumUJ+sumLJ-t+1
    if b>=0:
        for i in range(t):
             print 'm(X\{x_',It[i],'}) =' ,1-a[It[i]]
        print 'm(X) =' ,b
    else:
        print "You data do not satisfy the condition (I.3)"
elif s==len(Y):  #all of l(A) satisfy 1-sum u(A^c);
[condition (II.1)]
    if t<= (len(I)-2):
```

```
                print 'm(',XJ,'})=',1-sumUT-sumLJ
                for i in range(t):
                     print 'm(',XJ,'+x_',It[i],'})=',u[It[i]]-l[It[i]]
            elif t== (len(I)-1): #satisfy condition (II.2.2)
                if sumU-u[J[0]]+l[J[0]]==1:
                      for i in range(t):
                          print 'm(x_',J[0],',x_',It[i],'})=',
                          1-sumU+u[It[i]]+u[J[0]]-l[It[i]]-l[J[0]]
                else:
                      print "Your data do not satisfy the condition
                       (II.2.2)."
            else:
                print "Your data do not satisfy the condition (II)."
        else:
            print "Your data do not satisfy the condition (I) and (II)."
    else:
        print "Your probability interval can't convey to random set."
```

In this program, it is necessary to give the information of the probability interval that we want to convert to a random set with the same information. For example, if we have the probability interval information as $L = \{[l_1, u_1] = [1/4, 1/2], [l_2, u_2] = [1/4, 7/16], [l_3, u_3] = [1/8, 3/8], [l_4, u_4] = [1/8, 5/16]\}$, then we must type the information as the following for using this program

```
I=Set([0,1,2,3])
l=[1/4,1/4,1/8,1/8]
u=[1/2,7/16,3/8,5/16].
```

The result by using this algorithm is

```
t= 2
m( X \ {x_ 1 ,x_ 3 }) = 0
m( X \ {x_ 1 }) = 1/16
```

```
m( X \ {x_ 3 }) = 1/16
m( X ) = 1/8.
```

Appendix B: Sufficient conditions of a reachable probability interval

Consider $X = \{x_1, x_2, \ldots, x_n\}, n \geq 3$, and let $L = \{[l_i, u_i] \mid 0 \leq l_i \leq u_i \leq 1, i = 1, 2, \ldots, n\}$ be reachable probability interval. Suppose the condition (3.3) holds. We consider all possible cases to verify the sufficient conditions of this probability interval, which $\mathcal{P}_{IP} = \mathcal{P}_{RS}$.

Proof. WLOG, let $i_1 = 1, i_2 = 2, i_3 = 3$ and set $m(\{x_i\}) = l_i$, $\forall i = 1, \ldots, n$.

In case $n = 3$, consider $X = \{x_1, x_2, x_3\}$. Since $l_i + l_j < 1 - u_k$, we then get $l(\{x_i, x_j\}) = 1 - u_k$ from Equation (2.3). So,

$$m(\{x_i, x_j\}) = l(\{x_i, x_j\}) - l_i - l_j = 1 - u_k - l_i - l_j = 1 - (\underbrace{u_k + l_i + l_j}_{<1}) \geq 0.$$

Now, let's consider $m(X)$.

$$\begin{aligned} m(X) &= 1 - \sum m(\{x_i\}) - \sum m(\{x_i, x_j\}) \\ &= 1 - l_1 - l_2 - l_3 - (1 - u_1 - l_2 - l_3 + 1 - u_2 - l_1 - l_3 + 1 - u_3 - l_1 - l_2) \\ &= l_1 + l_2 + l_3 + u_1 + u_2 + u_3 - 2 = \sum_{i=1}^{3} l_i + \sum_{i=1}^{3} u_i - 2. \end{aligned} \tag{5.1}$$

As we want $m(X) \geq 0$, the probability interval must have the condition that $\sum l_i + \sum u_i \geq 2$.

In case $n = 4$, we get $\sum_{j \neq 4} l_j + u_4 = 1$ by the property of reachable probability intervals.

By Lemma 3.1, we get $m(\{x_1, x_2\}) = m(\{x_1, x_3\}) = m(\{x_2, x_3\}) = m(\{x_1, x_2, x_3\}) = 0$.

Hence, we must consider the remaining basic probability assignment functions which are

$m(\{x_1, x_4\}), m(\{x_2, x_4\}), m(\{x_3, x_4\}), m(\{x_1, x_2, x_4\}), m(\{x_1, x_3, x_4\}), m(\{x_2, x_3, x_4\})$ and $m(X)$. If all of them are nonnegative, we can conclude that it is possible to construct a random set which has the same information as this probability interval.

We obtain $l_i+l_j+l_k < 1-u_l$, $\forall l \in 1,2,3$, by the condition (3.3), so $l(\{x_i,x_j,x_k\}) = 1-u_l$ from Equation (2.3).
Due to the condition $Bel(A) = l(A)$, then we obtain the system of equations as follows:

$$\begin{aligned}
&l_1 + l_4 + m(\{x_1,x_4\}) = l(\{x_1,x_4\})\\
&l_2 + l_4 + m(\{x_2,x_4\}) = l(\{x_2,x_4\})\\
&l_3 + l_4 + m(\{x_3,x_4\}) = l(\{x_3,x_4\})\\
&l_1 + l_2 + l_4 + m(\{x_1,x_4\}) + m(\{x_2,x_4\}) + m(\{x_1,x_2,x_4\}) = 1 - u_3\\
&l_1 + l_3 + l_4 + m(\{x_1,x_4\}) + m(\{x_3,x_4\}) + m(\{x_1,x_3,x_4\}) = 1 - u_2\\
&l_2 + l_3 + l_4 + m(\{x_2,x_4\}) + m(\{x_3,x_4\}) + m(\{x_2,x_3,x_4\}) = 1 - u_1\\
&\sum l_i + \sum m(\{x_i,x_j\}) + \sum m(\{x_i,x_j,x_k\}) + m(X) = 1.
\end{aligned}$$

Thus, we can write this system of equations in the form of matrix notation by using unknown variables $m(\{x_1,x_4\}), m(\{x_2,x_4\}), m(\{x_3,x_4\}), m(\{x_1,x_2,x_4\}), m(\{x_1,x_3,x_4\})$, $m(\{x_2,x_3,x_4\})$ and $m(X)$.

$$\begin{bmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 1&1&0&1&0&0&0\\ 1&0&1&0&1&0&0\\ 0&1&1&0&0&1&0\\ 1&1&1&1&1&1&1 \end{bmatrix}
\begin{bmatrix} m(\{x_1,x_4\})\\ m(\{x_2,x_4\})\\ m(\{x_3,x_4\})\\ m(\{x_1,x_2,x_4\})\\ m(\{x_1,x_3,x_4\})\\ m(\{x_2,x_3,x_4\})\\ m(X) \end{bmatrix}
=
\begin{bmatrix} l(\{x_1,x_4\}) - l_1 - l_4\\ l(\{x_2,x_4\}) - l_2 - l_4\\ l(\{x_3,x_4\}) - l_3 - l_4\\ 1 - u_3 - l_1 - l_2 - l_4\\ 1 - u_2 - l_1 - l_3 - l_4\\ 1 - u_1 - l_2 - l_3 - l_4\\ 1 - l_1 - l_2 - l_3 - l_4 \end{bmatrix}$$

$$\begin{bmatrix} m(\{x_1,x_4\}) \\ m(\{x_2,x_4\}) \\ m(\{x_3,x_4\}) \\ m(\{x_1,x_2,x_4\}) \\ m(\{x_1,x_3,x_4\}) \\ m(\{x_2,x_3,x_4\}) \\ m(X) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & -1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} l(\{x_1,x_4\}) - l_1 - l_4 \\ l(\{x_2,x_4\}) - l_2 - l_4 \\ l(\{x_3,x_4\}) - l_3 - l_4 \\ 1 - u_3 - l_1 - l_2 - l_4 \\ 1 - u_2 - l_1 - l_3 - l_4 \\ 1 - u_1 - l_2 - l_3 - l_4 \\ 1 - l_1 - l_2 - l_3 - l_4 \end{bmatrix}$$

$$= \begin{bmatrix} l(\{x_1,x_4\}) - l_1 - l_4 \\ l(\{x_2,x_4\}) - l_2 - l_4 \\ l(\{x_3,x_4\}) - l_3 - l_4 \\ 1 + l_4 - (l(\{x_1,x_4\}) + l(\{x_2,x_4\}) + u_3) \\ 1 + l_4 - (l(\{x_1,x_4\}) + l(\{x_3,x_4\}) + u_2) \\ 1 + l_4 - (l(\{x_2,x_4\}) + l(\{x_3,x_4\}) + u_1) \\ l(\{x_1,x_4\}) + l(\{x_2,x_4\}) + l(\{x_3,x_4\}) + u_3 + u_2 + u_1 - l_4 - 2 \end{bmatrix}.$$

So,

$$m(\{x_1,x_4\}) = l(\{x_1,x_4\}) - l_1 - l_4 \tag{5.2}$$

$$m(\{x_2,x_4\}) = l(\{x_2,x_4\}) - l_2 - l_4 \tag{5.3}$$

$$m(\{x_3,x_4\}) = l(\{x_3,x_4\}) - l_3 - l_4 \tag{5.4}$$

$$m(\{x_1,x_2,x_4\}) = 1 + l_4 - (l(\{x_1,x_4\}) + l(\{x_2,x_4\}) + u_3) \tag{5.5}$$

$$m(\{x_1,x_3,x_4\}) = 1 + l_4 - (l(\{x_1,x_4\}) + l(\{x_3,x_4\}) + u_2) \tag{5.6}$$

$$m(\{x_2,x_3,x_4\}) = 1 + l_4 - (l(\{x_2,x_4\}) + l(\{x_3,x_4\}) + u_1) \tag{5.7}$$

$$m(X) = l(\{x_1,x_4\}) + l(\{x_2,x_4\}) + l(\{x_3,x_4\}) + u_3 + u_2 + u_1 - l_4 - 2 \tag{5.8}$$

Since $l(A) = \max(\sum_{x_i \in A} l_i, 1 - \sum_{x_i \in A^c} u_i)$, Equations (5.2), (5.3) and (5.4) are nonnegative.

Consider Equation (5.5), we do not know the exact values of $l(\{x_1, x_4\})$ and $l(\{x_2, x_4\})$. Thus, we can consider Equation (5.5) in 4 cases.

Case A1: Suppose $l(\{x_1, x_4\}) = l_1 + l_4$ and $l(\{x_2, x_4\}) = l_2 + l_4$.

$$\begin{aligned} m(\{x_1, x_2, x_4\}) &= 1 + l_4 - (l(\{x_1, x_4\}) + l(\{x_2, x_4\}) + u_3) \\ &= 1 + l_4 - l_1 - l_4 - l_2 - l_4 - u_3 = 1 - (\underbrace{u_3 + l_1 + l_2 + l_4}_{<1}) \geq 0. \end{aligned}$$

We have $u_3 + l_1 + l_2 + l_4 < 1$, by the condition (3.3) that $\sum_{j \neq 3} l_j + u_3 < 1$.

Case A2: Suppose $l(\{x_1, x_4\}) = l_1 + l_4$ and $l(\{x_2, x_4\}) = 1 - u_1 - u_3$.

$$\begin{aligned} m(\{x_1, x_2, x_4\}) &= 1 + l_4 - (l(\{x_1, x_4\}) + l(\{x_2, x_4\}) + u_3) \\ &= 1 + l_4 - l_1 - l_4 - 1 + u_1 + u_3 - u_3 = u_1 - l_1 \geq 0. \end{aligned}$$

Case A3: Suppose $l(\{x_1, x_4\}) = 1 - u_2 - u_3$ and $l(\{x_2, x_4\}) = l_2 + l_4$.

$$\begin{aligned} m(\{x_1, x_2, x_4\}) &= 1 + l_4 - (l(\{x_1, x_4\}) + l(\{x_2, x_4\}) + u_3) \\ &= 1 + l_4 - 1 + u_2 + u_3 - l_2 - l_4 - u_3 = u_2 - l_2 \geq 0. \end{aligned}$$

Case A4: Suppose $l(\{x_1, x_4\}) = 1 - u_2 - u_3$ and $l(\{x_2, x_4\}) = 1 - u_1 - u_3$.

$$\begin{aligned} m(\{x_1, x_2, x_4\}) &= 1 + l_4 - (l(\{x_1, x_4\}) + l(\{x_2, x_4\}) + u_3) \\ &= 1 + l_4 - 1 + u_2 + u_3 - 1 + u_1 + u_3 - u_3 = \underbrace{u_1 + u_2 + u_3 + l_4}_{\geq 1} - 1 \geq 0. \end{aligned}$$

We have $u_1 + u_2 + u_3 + l_4 \geq 1$ from the property of a reachable probability interval.

All of these 4 cases, we conclude that $m(\{x_1, x_2, x_4\}) \geq 0$. We can consider Equations (5.6) and (5.7) in the similar way. Moreover, we obtain $m(\{x_1, x_3, x_4\}) \geq 0$ and $m(\{x_2, x_3, x_4\}) \geq 0$.

To complete the case $n = 4$, we must consider Equation (5.8) when we obtain $m(X) \geq 0$. Since we do not know the values of $l(\{x_1, x_4\})$, $l(\{x_2, x_4\})$ and $l(\{x_3, x_4\})$, we consider Equation (5.8) in 8 cases.

Case B1: Suppose $l(\{x_1, x_4\}) = l_1 + l_4, l(\{x_2, x_4\}) = l_2 + l_4$ and $l(\{x_3, x_4\}) = l_3 + l_4$.

$$\begin{aligned} m(X) &= l(\{x_1, x_4\}) + l(\{x_2, x_4\}) + l(\{x_3, x_4\}) + u_3 + u_2 + u_1 - l_4 - 2 \\ &= l_1 + l_4 + l_2 + l_4 + l_3 + l_4 + u_3 + u_2 + u_1 - l_4 - 2 \\ &= \underbrace{l_1 + l_2 + l_3 + l_4}_{<1} + \underbrace{u_1 + u_2 + u_3 + l_4}_{\geq 1} - 2 \end{aligned} \tag{5.9}$$

As we want $m(X) \geq 0$, the probability interval must have the condition that $\sum l_i + \sum u_i - u_4 + l_4 \geq 2$.

Case B2: Suppose $l(\{x_1, x_4\}) = 1 - u_2 - u_3, l(\{x_2, x_4\}) = l_2 + l_4$ and $l(\{x_3, x_4\}) = l_3 + l_4$.

$$\begin{aligned} m(X) &= l(\{x_1, x_4\}) + l(\{x_2, x_4\}) + l(\{x_3, x_4\}) + u_3 + u_2 + u_1 - l_4 - 2 \\ &= 1 - u_2 - u_3 + l_2 + l_4 + l_3 + l_4 + u_3 + u_2 + u_1 - l_4 - 2 \\ &= \underbrace{u_1 + l_2 + l_3 + l_4}_{<1} - 1 < 0 \end{aligned} \tag{5.10}$$

We have $u_1 + l_2 + l_3 + l_4 < 1$, by the condition (3.3) that $\sum_{j \neq 1} l_j + u_1 < 1$. Since $m(X) < 0$, we cannot construct the random set which has the same information as this probability interval in this case.

Case B3: Suppose $l(\{x_1, x_4\}) = l_1 + l_4, l(\{x_2, x_4\}) = 1 - u_1 - u_3$ and $l(\{x_3, x_4\}) = l_3 + l_4$.

$$\begin{aligned} m(X) &= l(\{x_1, x_4\}) + l(\{x_2, x_4\}) + l(\{x_3, x_4\}) + u_3 + u_2 + u_1 - l_4 - 2 \\ &= l_1 + l_4 + 1 - u_1 - u_3 + l_3 + l_4 + u_3 + u_2 + u_1 - l_4 - 2 \\ &= \underbrace{u_2 + l_1 + l_3 + l_4}_{<1} - 1 < 0 \end{aligned} \tag{5.11}$$

Similar to Case B2, we cannot construct the random set which has the same information as this probability interval in this case.

Case B4: Suppose $l(\{x_1, x_4\}) = l_1 + l_4, l(\{x_2, x_4\}) = l_2 + l_4$ and $l(\{x_3, x_4\}) =$

$1 - u_1 - u_2$.

$$m(X) = l(\{x_1, x_4\}) + l(\{x_2, x_4\}) + l(\{x_3, x_4\}) + u_3 + u_2 + u_1 - l_4 - 2$$
$$= l_1 + l_4 + l_2 + l_4 + 1 - u_1 - u_2 + u_3 + u_2 + u_1 - l_4 - 2$$
$$= \underbrace{u_3 + l_1 + l_2 + l_4}_{<1} - 1 < 0 \tag{5.12}$$

We cannot construct the random set which has the same information as this probability interval in this case with the same reason as Case B2 and Case B3.

Case B5: Suppose $l(\{x_1, x_4\}) = l_1 + l_4, l(\{x_2, x_4\}) = 1 - u_1 - u_3$ and $l(\{x_3, x_4\}) = 1 - u_1 - u_2$.

$$m(X) = l(\{x_1, x_4\}) + l(\{x_2, x_4\}) + l(\{x_3, x_4\}) + u_3 + u_2 + u_1 - l_4 - 2$$
$$= l_1 + l_4 + 1 - u_1 - u_3 + 1 - u_1 - u_2 + u_3 + u_2 + u_1 - l_4 - 2 = l_1 - u_1 \leq 0$$

If $l_1 = u_1$, then we can construct the random set. Otherwise, we cannot.

Case B6: Suppose $l(\{x_1, x_4\}) = 1 - u_2 - u_3, l(\{x_2, x_4\}) = l_2 + l_4$ and $l(\{x_3, x_4\}) = 1 - u_1 - u_2$.

$$m(X) = l(\{x_1, x_4\}) + l(\{x_2, x_4\}) + l(\{x_3, x_4\}) + u_3 + u_2 + u_1 - l_4 - 2$$
$$= 1 - u_2 - u_3 + l_2 + l_4 + 1 - u_1 - u_2 + u_3 + u_2 + u_1 - l_4 - 2 = l_2 - u_2 \leq 0$$

If $l_2 = u_2$, then we can construct the random set. Otherwise, we cannot.

Case B7: Suppose $l(\{x_1, x_4\}) = 1 - u_2 - u_3, l(\{x_2, x_4\}) = 1 - u_1 - u_3$ and $l(\{x_3, x_4\}) = l_3 + l_4$.

$$m(X) = l(\{x_1, x_4\}) + l(\{x_2, x_4\}) + l(\{x_3, x_4\}) + u_3 + u_2 + u_1 - l_4 - 2$$
$$= 1 - u_2 - u_3 + 1 - u_1 - u_3 + l_3 + l_4 + u_3 + u_2 + u_1 - l_4 - 2 = l_3 - u_3 \leq 0$$

If $l_3 = u_3$, then we can construct the random set. Otherwise, we cannot.

Case B8: Suppose $l(\{x_1, x_4\}) = 1 - u_2 - u_3, l(\{x_2, x_4\}) = 1 - u_1 - u_3$ and $l(\{x_3, x_4\}) = 1 - u_1 - u_2$.

$$m(X) = l(\{x_1, x_4\}) + l(\{x_2, x_4\}) + l(\{x_3, x_4\}) + u_3 + u_2 + u_1 - l_4 - 2$$
$$= 1 - u_2 - u_3 + 1 - u_1 - u_3 + 1 - u_1 - u_2 + u_3 + u_2 + u_1 - l_4 - 2$$
$$= 1 - (\underbrace{u_1 + u_2 + u_3 + l_4}_{\geq 1}) \leq 0. \tag{5.13}$$

If $u_1 + u_2 + u_3 + l_4 = 1$, then we can construct the random set. Otherwise, we cannot.

When $n \geq 5$, we have $i = 1, 2, 3$ such that $\sum_{j \neq i} l_j + u_i < 1$. Therefore,

$$l_2 + l_3 + l_4 + \ldots + l_n < 1 - u_1,$$
$$l_1 + l_3 + l_4 + \ldots + l_n < 1 - u_2, \text{ and}$$
$$l_1 + l_2 + l_4 + \ldots + l_n < 1 - u_3.$$

So, $l(\{x_2, x_3, x_4, \ldots, x_n\}) = 1 - u_1, l(\{x_1, x_3, x_4, \ldots, x_n\}) = 1 - u_2$ and $l(\{x_1, x_2, x_4, \ldots, x_n\}) = 1 - u_3$, by Equation (2.3).
For $i \neq 1, 2, 3$, we get $\sum_{j \neq i} l_j + u_i = 1$. We obtain $m(A) = 0$, $\forall A \in P(X \setminus \{x_i\})$, by Lemma 3.1. Thus, we have $m(A) = 0$, $\forall A \in P(X) \setminus Y$
where $Y = \{\{x_4, \ldots, x_n\}, \{x_1, x_4, \ldots, x_n\}, \{x_2, x_4, \ldots, x_n\}, \{x_3, x_4, \ldots, x_n\}, \{x_1, x_2, x_4, \ldots, x_n\}, \{x_1, x_3, x_4, \ldots, x_n\}, \{x_2, x_3, x_4, \ldots, x_n\}, X\}$.
So, we must consider the values of $m(A)$, $\forall A \in Y$ when they have positive values. Due to the condition $Bel(A) = l(A)$, we obtain the system of equations as follows:

$$l_4 + \ldots + l_n + m(\{x_4, \ldots, x_n\}) = l(\{x_4, \ldots, x_n\}),$$
$$l_1 + l_4 + \ldots + l_n + m(\{x_4, \ldots, x_n\}) + m(\{x_1, x_4, \ldots, x_n\}) = l(\{x_1, x_4, \ldots, x_n\}),$$
$$l_2 + l_4 + \ldots + l_n + m(\{x_4, \ldots, x_n\}) + m(\{x_2, x_4, \ldots, x_n\}) = l(\{x_2, x_4, \ldots, x_n\}),$$
$$l_3 + l_4 + \ldots + l_n + m(\{x_4, \ldots, x_n\}) + m(\{x_3, x_4, \ldots, x_n\}) = l(\{x_3, x_4, \ldots, x_n\}),$$
$$l_1 + l_2 + l_4 + \ldots + l_n + m(\{x_4, \ldots, x_n\}) + \sum_{i \in \{1,2\}} m(\{x_i, x_4, \ldots, x_n\}) + m(\{x_1, x_2, x_4, \ldots, x_n\}) = 1 - u_3,$$
$$l_1 + l_3 + l_4 + \ldots + l_n + m(\{x_4, \ldots, x_n\}) + \sum_{i \in \{1,3\}} m(\{x_i, x_4, \ldots, x_n\}) + m(\{x_1, x_3, x_4, \ldots, x_n\}) = 1 - u_2,$$
$$l_2 + l_3 + l_4 + \ldots + l_n + m(\{x_4, \ldots, x_n\}) + \sum_{i \in \{2,3\}} m(\{x_i, x_4, \ldots, x_n\}) + m(\{x_2, x_3, x_4, \ldots, x_n\}) = 1 - u_1,$$
$$\sum l_i + m(\{x_4, \ldots, x_n\}) + \sum_{i=1}^{3} m(\{x_i, x_4, \ldots, x_n\}) + \sum_{i,j \in \{1,2,3\}} m(\{x_i, x_j, x_4, \ldots, x_n\}) + m(X) = 1.$$

Thus, we can write this system of equations in the form of matrix notation by using $m(\{x_4, \ldots, x_n\}), m(\{x_1, x_4, \ldots, x_n\}), m(\{x_2, x_4, \ldots, x_n\}), m(\{x_3, x_4, \ldots, x_n\}), m(\{x_1, x_2, x_4, \ldots, x_n\}), m(\{x_1, x_3, x_4, \ldots, x_n\}), m(\{x_2, x_3, x_4, \ldots, x_n\})$ and $m(\{X\})$ as the unknown variables.

$$AU = b$$

$$\text{where } A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, U = \begin{bmatrix} m(\{x_4, \ldots, x_n\}) \\ m(\{x_1, x_4, \ldots, x_n\}) \\ m(\{x_2, x_4, \ldots, x_n\}) \\ m(\{x_3, x_4, \ldots, x_n\}) \\ m(\{x_1, x_2, x_4, \ldots, x_n\}) \\ m(\{x_1, x_3, x_4, \ldots, x_n\}) \\ m(\{x_2, x_3, x_4, \ldots, x_n\}) \\ m(X) \end{bmatrix} \text{ and}$$

$$b = \begin{bmatrix} l(\{x_4, \ldots, x_n\}) - l_4 - \ldots - l_n \\ l(\{x_1, x_4, \ldots, x_n\}) - l_1 - l_4 - \ldots - l_n \\ l(\{x_2, x_4, \ldots, x_n\}) - l_2 - l_4 - \ldots - l_n \\ l(\{x_3, x_4, \ldots, x_n\}) - l_3 - l_4 - \ldots - l_n \\ 1 - u_3 - l_1 - l_2 - l_4 - \ldots - l_n \\ 1 - u_2 - l_1 - l_3 - l_4 - \ldots - l_n \\ 1 - u_1 - l_2 - l_3 - l_4 - \ldots - l_n \\ 1 - l_1 - l_2 - l_3 - \ldots - l_n \end{bmatrix}.$$

Consequently,

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 \end{bmatrix} b$$

$$= \begin{bmatrix} l(\{x_4,\ldots,x_n\}) - l_4 - \ldots - l_n \\ l(\{x_1,x_4,\ldots,x_n\}) - l(\{x_4,\ldots,x_n\}) - l_1 \\ l(\{x_2,x_4,\ldots,x_n\}) - l(\{x_4,\ldots,x_n\}) - l_2 \\ l(\{x_3,x_4,\ldots,x_n\}) - l(\{x_4,\ldots,x_n\}) - l_3 \\ 1 - u_3 + l(\{x_4,\ldots,x_n\}) - l(\{x_1,x_4,\ldots,x_n\}) - l(\{x_2,x_4,\ldots,x_n\}) \\ 1 - u_2 + l(\{x_4,\ldots,x_n\}) - l(\{x_1,x_4,\ldots,x_n\}) - l(\{x_3,x_4,\ldots,x_n\}) \\ 1 - u_1 + l(\{x_4,\ldots,x_n\}) - l(\{x_2,x_4,\ldots,x_n\}) - l(\{x_3,x_4,\ldots,x_n\}) \\ \lambda \end{bmatrix},$$

where $\lambda = -l(\{x_4,\ldots,x_n\}) + l(\{x_1,x_4,\ldots,x_n\}) + l(\{x_2,x_4,\ldots,x_n\}) + l(\{x_3,x_4,\ldots,x_n\}) + u_3 + u_2 + u_1 - 2$.

So,

$$m(\{x_4,\ldots,x_n\}) = l(\{x_4,\ldots,x_n\}) - l_4 - \ldots - l_n, \tag{5.14}$$

$$m(\{x_1,x_4,\ldots,x_n\}) = l(\{x_1,x_4,\ldots,x_n\}) - l(\{x_4,\ldots,x_n\}) - l_1, \tag{5.15}$$

$$m(\{x_2,x_4,\ldots,x_n\}) = l(\{x_2,x_4,\ldots,x_n\}) - l(\{x_4,\ldots,x_n\}) - l_2, \tag{5.16}$$

$$m(\{x_3,x_4,\ldots,x_n\}) = l(\{x_3,x_4,\ldots,x_n\}) - l(\{x_4,\ldots,x_n\}) - l_3, \tag{5.17}$$

$$\begin{aligned} m(\{x_1,x_2,x_4,\ldots,x_n\}) = 1 - u_3 + l(\{x_4,\ldots,x_n\}) - l(\{x_1,x_4,\ldots,x_n\}) \\ - l(\{x_2,x_4,\ldots,x_n\}), \end{aligned} \tag{5.18}$$

$$m(\{x_1, x_3, x_4, \ldots, x_n\}) = 1 - u_2 + l(\{x_4, \ldots, x_n\}) - l(\{x_1, x_4, \ldots, x_n\}) \\ - l(\{x_3, x_4, \ldots, x_n\}), \tag{5.19}$$

$$m(\{x_2, x_3, x_4, \ldots, x_n\}) = 1 - u_1 + l(\{x_4, \ldots, x_n\}) - l(\{x_2, x_4, \ldots, x_n\}) \\ - l(\{x_3, x_4, \ldots, x_n\}), \tag{5.20}$$

$$m(X) = -l(\{x_4, \ldots, x_n\}) + l(\{x_1, x_4, \ldots, x_n\}) + l(\{x_2, x_4, \ldots, x_n\}) \\ + l(\{x_3, x_4, \ldots, x_n\}) + u_3 + u_2 + u_1 - 2. \tag{5.21}$$

Since $l(A) = \max(\sum\limits_{x_i \in A} l_i, 1 - \sum\limits_{x_i \in A^c} u_i)$, Equation (5.14) is nonnegative. Therefore, $m(\{x_4, \ldots, x_n\}) = l(\{x_4, \ldots, x_n\}) - l_4 - \ldots - l_n \geq 0$.

Next, consider Equation (5.15), we separate our consideration into 2 cases.

Case C1: Suppose $l(\{x_1, x_4, \ldots, x_n\}) = l_1 + l_4 + \ldots + l_n$, that is $l_1 + l_4 + \ldots + l_n > 1 - u_2 - u_3$, then we also get $l(\{x_4, \ldots, x_n\}) = l_4 + \ldots + l_n$. (From $l_4 + \ldots + l_n > 1 - u_2 - u_3 - l_1 \geq 1 - u_2 - u_3 - u_1$.)

$m(\{x_1, x_4, \ldots, x_n\}) = l_1 + l_4 + \ldots + l_n - l_4 - \ldots - l_n - l_1 = 0$.

Case C2: Suppose $l(\{x_1, x_4, \ldots, x_n\}) = 1 - u_2 - u_3$, that is $1 - u_2 - u_3 > l_1 + l_4 + \ldots + l_n$,

$$l_1 + l_4 + \ldots + l_n + u_2 + u_3 < 1. \tag{5.22}$$

Case C2.1: Suppose $l(\{x_4, \ldots, x_n\}) = l_4 + \ldots + l_n$.

$$\begin{aligned} m(\{x_1, x_4, \ldots, x_n\}) &= 1 - u_2 - u_3 - l_4 - \ldots - l_n - l_1 \\ &= 1 - (\underbrace{u_2 + u_3 + l_4 + \ldots + l_n + l_1}_{<1, \text{ from } (5.22)}) \geq 0. \end{aligned}$$

Case C2.2: Suppose $l(\{x_4, \ldots, x_n\}) = 1 - u_1 - u_2 - u_3$.

$m(\{x_1, x_4, \ldots, x_n\}) = 1 - u_2 - u_3 - 1 + u_1 + u_2 + u_3 - l_1 = u_1 - l_1 \geq 0$.

Both of these cases, we conclude that $m(\{x_1, x_4, \ldots, x_n\}) \geq 0$. We can consider Equations (5.16) and (5.17) in the similar way. Moreover, we obtain

$m(\{x_2, x_4, \ldots, x_n\}) \geq 0$ and $m(\{x_3, x_4, \ldots, x_n\}) \geq 0$.

Next, consider Equation (5.18). We can consider this equation into 4 cases.

Case D1: Suppose $l(\{x_1, x_4, \ldots, x_n\}) = l_1 + l_4 + \ldots + l_n$ and $l(\{x_2, x_4, \ldots, x_n\}) = l_2 + l_4 + \ldots + l_n$. Then we get $l(\{x_4, \ldots, x_n\}) = l_4 + \ldots + l_n$.

$$\begin{aligned} m(\{x_1, x_2, x_4, \ldots, x_n\}) &= 1 - u_3 + l_4 + \ldots + l_n - l_1 - l_4 - \ldots - l_n - l_2 - l_4 - \ldots - l_n \\ &= 1 - (\underbrace{u_3 + l_1 + l_2 + l_4 + \ldots + l_n}_{<1}) \geq 0. \end{aligned}$$

We have $u_3 + l_1 + l_2 + l_4 + \ldots + l_n < 1$, by the condition (3.3) that $\sum_{j \neq 3} l_j + u_3 < 1$.

Case D2: Suppose $l(\{x_1, x_4, \ldots, x_n\}) = l_1 + l_4 + \ldots + l_n$, then we get $l(\{x_4, \ldots, x_n\}) = l_4 + \ldots + l_n$ and $l(\{x_2, x_4, \ldots, x_n\}) = 1 - u_1 - u_3$.
$m(\{x_1, x_2, x_4, \ldots, x_n\}) = 1 - u_3 + l_4 + \ldots + l_n - l_1 - l_4 - \ldots - l_n - 1 + u_1 + u_3 = u_1 - l_1 \geq 0$.

Case D3: Suppose $l(\{x_1, x_4, \ldots, x_n\}) = 1 - u_2 - u_3$ and $l(\{x_2, x_4, \ldots, x_n\}) = l_2 + l_4 + \ldots + l_n$, then we get $l(\{x_4, \ldots, x_n\}) = l_4 + \ldots + l_n$.
$m(\{x_1, x_2, x_4, \ldots, x_n\}) = 1 - u_3 + l_4 + \ldots + l_n - 1 + u_2 + u_3 - l_2 - l_4 - \ldots - l_n = u_2 - l_2 \geq 0$.

Case D4: Suppose $l(\{x_1, x_4, \ldots, x_n\}) = 1 - u_2 - u_3$ and $l(\{x_2, x_4, \ldots, x_n\}) = 1 - u_1 - u_3$.

Case D4.1: Suppose $l(\{x_4, \ldots, x_n\}) = l_4 + \ldots + l_n$, that is $l_4 + \ldots + l_n > 1 - u_1 - u_2 - u_3$.

$$l_4 + \ldots + l_n + u_1 + u_2 + u_3 > 1. \tag{5.23}$$

Then,

$$\begin{aligned} m(\{x_1, x_2, x_4, \ldots, x_n\}) &= 1 - u_3 + l_4 + \ldots + l_n - 1 + u_2 + u_3 - 1 + u_1 + u_3 \\ &= \underbrace{l_4 + \ldots + l_n + u_1 + u_2 + u_3}_{>1, \text{ from (5.23)}} - 1 \geq 0 \end{aligned}$$

Case D4.2: Suppose $l(\{x_4, \ldots, x_n\}) = 1 - u_1 - u_2 - u_3$.
$m(\{x_1, x_2, x_4, \ldots, x_n\}) = 1 - u_3 + 1 - u_1 - u_2 - u_3 - 1 + u_2 + u_3 - 1 + u_1 + u_3 = 0$.

All of these cases, we conclude that $m(\{x_1, x_2, x_4, \ldots, x_n\}) \geq 0$. We can consider Equations (5.19) and (5.20) in the similar way. Moreover, we obtain $m(\{x_1, x_3, x_4, \ldots, x_n\}) \geq 0$ and $m(\{x_2, x_3, x_4, \ldots, x_n\}) \geq 0$.

Finally, by considering Equation (5.21), we do not know the values of $l(\{x_4,\dots,x_n\})$, $l(\{x_1,x_4,\dots,x_n\})$, $l(\{x_2,x_4,\dots,x_n\})$ and $l(\{x_3,x_4,\dots,x_n\})$.
Let $Z = \{\{x_1,x_4,\dots,x_n\},\{x_2,x_4,\dots,x_n\},\{x_3,x_4,\dots,x_n\}\}$. If there exists $S \in Z$ such that $l(S) = \sum_{x_i\in S} l_i$, then $l(\{x_4,\dots,x_n\}) = l_4 + \dots + l_n$.
Therefore, we can consider Equation (5.21) in to 4 cases.

Case E1: Let for all $S \in Z$, $l(S) = \sum_{x_i\in S} l_i$.

$$\begin{aligned} m(X) &= -l_4 - \dots - l_n + l_1 + l_4 + \dots + l_n + l_2 + l_4 + \dots + l_n + l_3 + l_4 + \dots + l_n \\ &\quad + u_3 + u_2 + u_1 - 2 \\ &= \underbrace{\sum_i l_i}_{\leq 1} + (\underbrace{l_4 + \dots + l_n + u_1 + u_2 + u_3}_{>1,\text{ from (5.23)}}) - 2. \end{aligned} \tag{5.24}$$

Since we want $m(X) \geq 0$, the probability interval must have the condition $\sum l_i + \sum u_i - (u_4 + \dots + u_n) + l_4 + \dots + l_n \geq 2$.

Case E2: There exists only one of $S \in Z$, $l(S) = \sum_{x_i\in S} l_i$. WLOG, let $l(\{x_1,x_4,\dots,x_n\}) = l_1 + l_4 + \dots + l_n$.

$$\begin{aligned} m(X) &= -l_4 - \dots - l_n + l_1 + l_4 + \dots + l_n + 1 - u_1 - u_3 + 1 - u_1 - u_2 + u_3 + u_2 + u_1 - 2 \\ &= l_1 - u_1 \leq 0. \end{aligned}$$

If $l_1 = u_1$, then we can construct the random set. Otherwise, we cannot.

Case E3: There exists only one of $S \in Z$, $l(S) = 1 - \sum\limits_{x_i\in S^c} u_i$. WLOG, let $l(\{x_1,x_4,\dots,x_n\}) = 1 - u_2 - u_3$.

$$\begin{aligned} m(X) &= -l_4 - \dots - l_n + 1 - u_2 - u_3 + l_2 + l_4 + \dots + l_n + l_3 + l_4 + \dots + l_n \\ &\quad + u_3 + u_2 + u_1 - 2 \\ &= (\underbrace{l_2 + \dots + l_n + u_1}_{<1}) - 1 < 0. \end{aligned} \tag{5.25}$$

We have $u_1 + l_2 + \dots + l_n < 1$, by the condition (3.3) that $\sum\limits_{j\neq 1} l_j + u_1 < 1$.
Since $m(X) < 0$, we cannot construct the random set which has the same information as this probability interval in this case.

Case E4: Let for all $S \in Z$, $l(S) = 1 - \sum\limits_{x_i\in S^c} u_i$.

Case E4.1: Suppose $l(\{x_4, \ldots, x_n\}) = l_4 + \ldots + l_n$.

$$\begin{aligned} m(X) &= -l_4 - \ldots - l_n + 1 - u_2 - u_3 + 1 - u_1 - u_3 + 1 - u_1 - u_2 + u_3 + u_2 + u_1 - 2 \\ &= 1 - (\underbrace{l_4 + \ldots + l_n + u_3 + u_2 + u_1}_{>1, \text{ from } (5.23)}) < 0. \end{aligned} \tag{5.26}$$

Case E4.2: Suppose $l(\{x_4, \ldots, x_n\}) = 1 - u_1 - u_2 - u_3$.

$$\begin{aligned} m(X) &= -1 + u_1 + u_2 + u_3 + 1 - u_2 - u_3 + 1 - u_1 - u_3 + 1 - u_1 - u_2 + u_3 + u_2 + u_1 - 2 \\ &= 0. \end{aligned} \tag{5.27}$$

This case, we can construct the random set when all the values of $l(\{x_4, \ldots, x_n\})$, $l(\{x_1, x_4, \ldots, x_n\})$, $l(\{x_2, x_4, \ldots, x_n\})$ and $l(\{x_3, x_4, \ldots, x_n\})$ equal to $1 - \sum_{x_i \in A^c} u_i$.

As the result, the probability interval must have one of the following conditions for guarantee that there is the random set which has the same information.

(i) (i.1) $l(A) = \sum_{x_i \in A} l_i$, $\forall A \in Y - Y'$, and

(i.2) $l(X \setminus \{x_{i_k}\}) = 1 - u_{i_k}$, $\forall i_k \in I_t$, and

(i.3) $\sum_{i \in I} l_i + \sum_{i \in I} u_i - \sum_{j \in J} u_j + \sum_{j \in J} l_j \geq 2$.
(see Equations (5.1), (5.9) and (5.24))

(ii) (ii.1) $l(A) = 1 - \sum_{x_i \in A^c} u_i$, $\forall A \in Y$, and

(ii.2) only one of the following

(ii.2.1) $3 \leq n - 2$ (see Equation (5.27)) or

(ii.2.2) $3 = n - 1$ and $\sum_{j \neq i_k} u_j + l_{i_k} = 1$, $i_k \in J$. (see Equation (5.13))

In the other cases when there exist $A, B \in Y - Y'$ that $l(A) = \sum_{x_i \in A} l_i$ and $l(B) = 1 - \sum_{x_i \in B^c} u_i$, we cannot construct the random set as we can see from Equations (5.10), (5.11), (5.12), (5.25) and (5.26). $\square$

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