CHAPTER IV

INTEGRALS OF MOTION FOR THE TODA SYSTEM

The aim of this chapter is to use two approaches: the ARS method, and the direct calculation method, in order to identify the integrals of motion for the Toda system [14,17].

ARS METHOD

Consider two special cases of boundary conditions : The free end lattice with three non-equal-mass particles and the fixed-end lattice for two particles.

Free-end lattice

Let us consider a free-end-lattice with three, non-equal mass, particles. The Hamiltonian governing the system reads

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{p_3^2}{2} + e^{\epsilon(q_1-q_2)} + e^{q_2-q_3} . \tag{4.1}$$

We introduce the following change of variables:

$$a_1 = \frac{1}{2}e^{\epsilon(q_1-q_2)/2} , \quad a_2 = \frac{1}{2}e^{(q_2-q_3)/2} ,$$
$$b_k = -\frac{p_k}{2m_k}, \tag{4.2}$$

$k = 1,2,3$ with m_k normalized to one. In terms of the a_k, b_k's, the equations of motion become

$$\dot{a}_1 = \epsilon a_1(b_2-b_1),$$
$$\dot{a}_2 = -a_2[(1+m_2)b_2 + m_1b_1],$$
$$\dot{b}_1 = \frac{2\epsilon a_1^2}{m_1}, \qquad \dot{b}_2 = \frac{2}{m_2}(a_2^2-\epsilon a_1^2), \qquad m_1, m_2, \epsilon > 0 \qquad (4.3)$$

where we use the total momentum $m_1b_1 + m_2b_2 + b_3 = 0$ to eliminate b_3.

In order to investigate the dominant behavior around a singularity in the complex-time plane we put:

$$a_1 = \alpha\tau^p, \qquad a_2 = \beta\tau^q,$$

where $\tau = t - t_o$ with t_o the position of the complex pole.

One find the three cases are available

i) $a_1 \sim \alpha\tau^p$, $a_2 \sim \beta\tau^{-1}$, $p > -1$,

(dominant terms: $\dot{a}_1 = \epsilon a_1b_2$, $\dot{a}_2 = -a_2(1+m_2)b_2$,

$\dot{b}_1 = \frac{2\epsilon a_1^2}{m_1}$, $\dot{b}_2 = \frac{2a_2^2}{m_2}$),

where α = arbitrary constant,

$$p = \frac{\epsilon}{1+m_2}, \quad 2p = \text{integer } (>0). \qquad (4.4)$$

so that the dominant behavior of b_1 does not introduce a branch point.

ii) $a_1 \sim \alpha\tau^{-1}$, $a_2 \sim \beta\tau^q$, $q > -1$,

(dominant terms: $\dot{a}_1 = \epsilon a_1(b_2-b_1)$,

$\dot{a}_2 = -a_2[(1+m_2)b_2 + m_1b_1]$,

$\dot{b}_1 = \frac{2\epsilon a_1^2}{m_1}$,

$\dot{b}_2 = -\frac{2\epsilon a_1^2}{m_2}$)

where β = arbitrary constant,

$$q = \frac{m_1}{\epsilon(m_1+m_2)}, \quad 2q = \text{integer } (>0), \qquad (4.5)$$

iii) $a_1 \sim \alpha\tau^{-1}$, $a_2 \sim \beta\tau^{-1}$;

(dominant terms:
$$\dot a_1 = \epsilon a_1(b_2-b_1),$$
$$\dot a_2 = -a_2[(1+m_2)b_2 + m_1 b_1],$$
$$\dot b_1 = \frac{2\epsilon a_1^2}{m_1},$$
$$\dot b_2 = \frac{2}{m_2}(a_2^2-\epsilon a_1^2).\ )$$

where
$$\alpha^2 = \frac{-m_1(1+m_2+\epsilon)}{2\epsilon^2(1+m_1+m_2)}, \quad \beta^2 = \frac{-[m_1+\epsilon(m_1+m_2)]}{2\epsilon(1+m_1+m_2)}. \tag{4.6}$$

Now following the ARS algorithm we investigate the resonances. Looking for higher order terms of the form, we have

case i)
$$a_1 = \alpha\tau^p + \gamma\tau^{p+r},\ a_2 = \beta\tau^{-1} + \delta\tau^{r-1},$$
$$b_1 = \frac{2\epsilon}{m_1(2p+1)}\alpha^2\tau^{2p+1} + \xi\tau^{2p+1+r},$$
$$b_2 = \frac{1}{1+m_2}\tau^{-1} + \theta\tau^{r-1}.$$

We find the resonance equation:
$$r(2p+r+1)(r-2)(r+1) = 0. \tag{4.7}$$

The solutions of Eq.(4.1) have the Painlevé property in this case with 3 arbitrary constants: t_0, α, and one entering at $r = 2$.

case ii)
$$a_1 = \alpha\tau^{-1} + \gamma\tau^{r-1},\ a_2 = \beta\tau^{q} + \delta\tau^{r+q},$$
$$b_1 = -\frac{2\epsilon\alpha^2}{m_1}\tau^{-1} + \xi\tau^{r-1},$$
$$b_2 = \frac{2\epsilon\alpha^2}{m_2}\tau^{-1} + \theta\tau^{r-1}.$$

We find the resonance equation:
$$r(r-1)(r-2)(r+1) = 0. \tag{4.8}$$

In this case we have the Painlevé property with 4 arbitrary constants: t_0, α, and two entering at $r = 1$ and 2.

case iii) $a_1 = \alpha\tau^{-1} + \gamma\tau^{r-1}$, $a_2 = \beta\tau^{-1} + \delta\tau^{r-1}$,

$$b_1 = -\frac{2\epsilon\alpha^2}{m_1}\tau^{-1} + \xi\tau^{r-1},$$

$$b_2 = \frac{2}{m_2}(\epsilon\alpha^2 - \beta^2)\tau^{-1} + \theta\tau^{r-1}.$$

We find the resonance equation:

$$(r-2)(r+1)(r^2-r-2M) = 0, \tag{4.9}$$

where
$$M = \frac{[m_1 + \epsilon(m_1+m_2)](1+m_2+\epsilon)}{\epsilon m_2(1+m_1+m_2)},$$

$$= \frac{(1+p)(1+q)}{1-pq},$$

from Eq.(4.4) and Eq.(4.5). For the Painlevé property, $r^2 - r - 2M = 0$, which must have integer roots only, and this implies that

$$M = \frac{(1+p)(1+q)}{1-pq} = \frac{n(n+1)}{2} \quad n = 0,1,2,\ldots \tag{4.10}$$

with $2p$ and $2q$ positive integers. Eq.(4.10) has five possible solutions:

a) $p = q = \frac{1}{2}$, with $n = 2$ and resonances at $r = -2, -1, 2, 3$. Eqs.(4.4) and (4.5) give

$$m_1 = \frac{\epsilon(2\epsilon-1)}{2-\epsilon}, \quad m_2 = 2\epsilon-1, \quad 1 < \epsilon < 2. \tag{4.11}$$

b) $p = \frac{1}{2}$, $q = 1$, or $p = 1$, $q = \frac{1}{2}$, both with $n = 3$ and resonances at $r = -3, -1, 2, 4$. For $p = \frac{1}{2}$, $q = 1$, Eqs.(4.4) and (4.5) give

$$m_1 = \frac{\epsilon(\epsilon-1)}{2-\epsilon}, \quad m_2 = \epsilon-1, \quad 1 < \epsilon < 2. \tag{4.12}$$

The case $p = 1$, $q = \frac{1}{2}$ leads to Eq.(4.12) with $\epsilon \to 2\epsilon$.

c) $p = \frac{1}{2}$, $q = \frac{3}{2}$, or $p = \frac{3}{2}$, $q = \frac{1}{2}$, both with $n = 5$

and resonances at $r = -5,-1, 2, 6$. For $p = \frac{1}{2}$, $q = \frac{3}{2}$. Eqs.(4.4) and (4.5) give

$$m_1 = \frac{3\epsilon(2\epsilon-1)}{2-3\epsilon}, \quad m_2 = 2\epsilon-1, \quad 1 < \epsilon < 2. \tag{4.13}$$

The case $p = \frac{3}{2}$, $q = \frac{1}{2}$ lead to Eq.(4.13) with $\epsilon \to \epsilon/3$.

For the free-end lattice with three masses

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{p_3^2}{2} + e^{\epsilon(q_1-q_2)} + e^{(q_2-q_3)}.$$

There are three families of lattices which have the Painlevé property :

a) $m_1 = \frac{\epsilon(2\epsilon-1)}{2-\epsilon}$, $m_2 = 2\epsilon-1$, $1 < \epsilon < 2$.

b) $m_1 = \frac{\epsilon(\epsilon-1)}{2-\epsilon}$, $m_2 = \epsilon-1$, $1 < \epsilon < 2$.

c) $m_1 = \frac{3\epsilon(2\epsilon-1)}{2-3\epsilon}$, $m_2 = 2\epsilon-1$, $1 < \epsilon < 2$.

Fixed-end lattice

Let us consider a fixed-end lattice with two nonequal masses. The form of the Hamiltonian governing the system is

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + e^{-\delta q_1} + e^{\epsilon(q_1-q_2)} + e^{q_2}. \tag{4.14}$$

We introduce the following change of variables:

$$a_1 = \frac{1}{2}e^{-\delta q_1/2}, \quad a_2 = \frac{1}{2}e^{\epsilon(q_1-q_2)/2}, \quad a_3 = \frac{1}{2}e^{q_2/2},$$
$$b_1 = -\frac{p_1}{2m_1}, \quad b_2 = -\frac{p_2}{2m_2}. \tag{4.15}$$

In terms of the a_k, b_k's, the equations of motion become

$$\dot{a}_1 = \delta a_1 b_1, \quad \dot{b}_1 = \frac{2}{m_1}(\epsilon a_3^2 - \delta a_1^2),$$
$$\dot{a}_2 = -a_2 b_2, \quad \dot{b}_2 = \frac{2}{m_2}(a_2^2 - \epsilon a_3^2),$$
$$\dot{a}_3 = \epsilon a_3(b_2 - b_1). \tag{4.16}$$

In order to investigate the dominant behavior around a singularity in the complex-time plane, we put

$$a_1 = \alpha \tau^p, \quad a_2 = \beta \tau^q, \quad a_3 = \gamma \tau^s,$$

where $\tau = t - t_o$ with t_o the position of the complex pole.

One finds that three cases are available

i) $a_1 \sim \alpha \tau^{-1}$, $a_2 \sim \beta \tau^{-1}$, $a_3 \sim \gamma \tau^s$,

(dominant terms: $\dot{a}_1 = \delta a_1 b_1$, $\dot{a}_2 = -a_2 b_2$, $\dot{a}_3 = \epsilon a_3 (b_2 - b_1)$

$\dot{b}_1 = -\frac{2\delta a_1^2}{m_1}$, $\dot{b}_2 = \frac{2a_2^2}{m_2}$)

where γ = arbitrary constant

$$\alpha^2 = -\frac{m_1}{2\delta}, \quad \beta^2 = -\frac{m_2}{2},$$

$$s = \epsilon(1 + \frac{1}{\delta}). \tag{4.17}$$

ii) $a_1 \sim \alpha \tau^p$, $a_2 \sim \beta \tau^q$, $a_3 \sim \gamma \tau^{-1}$,

(dominant terms: $\dot{a}_1 = \delta a_1 b_1$, $\dot{a}_2 = -a_2 b_2$, $\dot{a}_3 = \epsilon a_3 (b_2 - b_1)$,

$\dot{b}_1 = \frac{2\epsilon a_3^2}{m_1}$, $\dot{b}_2 = -\frac{2\epsilon a_3^2}{m_2}$)

where α = arbitrary constant

β = arbitrary constant

$$p = \frac{\delta m_2}{\epsilon(m_1 + m_2)} = -\frac{2\epsilon\delta\gamma^2}{m_1},$$

$$q = \frac{m_1}{\epsilon(m_1 + m_2)} = -\frac{2\epsilon\gamma^2}{m_2},$$

$$\gamma^2 = -\frac{m_1 m_2}{2\epsilon^2(m_1 + m_2)} \tag{4.18}$$

iii) $a_1 \sim \alpha \tau^{-1}$, $a_2 \sim \beta \tau^{-1}$, $a_3 \sim \gamma \tau^{-1}$,

(dominant terms: $\dot{a}_1 = \delta a_1 b_1$, $\dot{a}_2 = -a_2 b_2$, $\dot{a}_3 = \epsilon a_3 (b_2 - b_1)$,

$$\dot{b}_1 = \frac{2}{m_1}(\epsilon a_3^2 - \delta a_1^2),\ \dot{b}_2 = \frac{2}{m_2}(a_2^2 - \epsilon a_3^2)\quad)$$

where $\epsilon\gamma^2 - \delta\alpha^2 = \frac{m_1}{2\delta}$, $\epsilon\gamma^2 - \beta^2 = \frac{m_2}{2}$. (4.19)

Following the ARS algorithm we investigate the resonances. Looking for higher order terms of the form

case i) $a_1 = \alpha\tau^{-1} + A\tau^{r-1},\ a_2 = \beta\tau^{-1} + B\tau^{r-1}$

$a_3 = \gamma\tau^{s} + C\tau^{r+s},$

$b_1 = \frac{2\delta\alpha^2}{m_1}\tau^{-1} + \theta\tau^{r-1},$

$b_2 = -\frac{2\beta^2}{m_2}\tau^{-1} + \xi\tau^{r-1}.$

We find the resonance equation:

$$r(r-2)^2(r+1)^2 = 0 \tag{4.20}$$

case ii) $a_1 = \alpha\tau^{p} + A\tau^{r+p},\ a_2 = \beta\tau^{q} + B\tau^{r+q},$

$a_3 = \gamma\tau^{-1} + C\tau^{r-1},$

$b_1 = -\frac{2\epsilon\gamma^2}{m_1}\tau^{-1} + \theta\tau^{r-1},$

$b_2 = \frac{2\epsilon\gamma^2}{m_2}\tau^{-1} + \xi\tau^{r-1}.$

We find the resonance equation:

$$r^2(r-1)(r-2)(r+1) = 0 \tag{4.21}$$

case iii) $a_1 = \alpha\tau^{-1} + A\tau^{r-1},\ a_2 = \beta\tau^{-1} + B\tau^{r-1},$

$a_3 = \gamma\tau^{-1} + C\tau^{r-1},$

$b_1 = \frac{2}{m_1}(-\epsilon\gamma^2 + \delta\alpha^2)\tau^{-1} + \theta\tau^{r-1},$

$b_2 = \frac{2}{m_2}(\epsilon\gamma^2 - \beta^2)\tau^{-1} + \xi\tau^{r-1}.$

We find the resonance equation:

$$r(r+1)(r-2)(r^2 - r - 2 - 4M) = 0$$

where $$M = \frac{\epsilon^2\gamma^2\delta}{m_1} + \frac{\epsilon^2\gamma^2}{m_2\delta},$$
$$= -\epsilon\left(\frac{p}{2} + \frac{q}{2\delta}\right), \qquad (4.22)$$

from Eqs.(4.18) and (4.19). For the Painlevé property, $r^2 - r - 2 - 4M = 0$, which must have integer roots, only and this implies that

$$M = -\epsilon\left(\frac{p}{2} + \frac{q}{2\delta}\right) = \frac{(n-2)(n+1)}{4}, \quad n = 0,1,2,... \qquad (4.23)$$

with Eqs.(4.18) and (4.19) and M = 1 :

a) s=1 has the three possible solutions:

$$- \quad \delta = 1, \ \epsilon = \frac{1}{2}, \ m_1 = 3m_2, \qquad (4.24)$$
$$- \quad \delta = 1, \ \epsilon = \frac{1}{2}, \ m_1 = m_2, \qquad (4.25)$$

b) $s = \frac{3}{2}$ has the possible solution:

$$\delta = \frac{1}{2}, \ \epsilon = \frac{1}{2}, \ m_1 = m_2, \qquad (4.26)$$

c) s=2 has the two possible solutions:

$$- \quad \delta = 1, \ \epsilon = 1, \ m_1 = m_2, \qquad (4.27)$$
$$- \quad \delta = \frac{1}{3}, \ \epsilon = \frac{1}{2}, \ m_1 = \frac{m_2}{3}, \qquad (4.28)$$

For the fixed-end lattice with two masses:

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + e^{-\delta q_1} + e^{\epsilon(q_2 - q_1)} + e^{q_2}.$$

There are five families of lattices which have the Painlevé property :

a) $m_1/m_2 = 1, \ \delta = \epsilon = 1,$

b) $m_1/m_2 = 1, \ \delta = 1, \epsilon = 1/2,$

c) $m_1/m_2 = 1/3, \ \delta = 1, \ \epsilon = 1/2$

d) $m_1/m_2 = 1/3, \ \delta = 1/3, \ \epsilon = 1/2,$

e) $m_1/m_2 = 1, \ \delta = 1/2, \epsilon = 1/2 .$

DIRECT CALCULATION METHOD

Free-end lattice

The Hamiltonian governing the system reads

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{p_3^2}{2m_3} + e^{\epsilon(q_1-q_2)} + e^{q_2-q_3} \tag{4.29}$$

We introduce the following change of variables:

$$x = \epsilon(q_1-q_2) \quad , \quad y = q_2-q_1 \ , \ z = m_1q_1+m_2q_2+ m_3q_3 . \tag{4.30}$$

The equations of motion are derived from the Lagrange equations:

$$\frac{\partial}{\partial t}(m_1\dot{q}_1) = -\epsilon X, \quad \frac{\partial}{\partial t}(m_2\dot{q}_2) = \epsilon X-Y,$$

$$\frac{\partial}{\partial t}(m_3\dot{q}_3) = Y, \tag{4.31}$$

$$\text{with } X = e^{\epsilon(q_1-q_2)} = e^{x} \ , \quad Y = e^{q_2-q_3} = e^{y}. \tag{4.32}$$

From Eq.(4.30) the equations of the motion of x and y:

$$\ddot{x} = \epsilon(\ddot{q}_1 - \ddot{q}_2) \qquad = \frac{\epsilon}{m_2}\left[Y - \frac{\epsilon(m_1+m_2)X}{m_1}\right],$$

$$\ddot{y} = \ddot{q} - \ddot{q}_3 \qquad = \frac{\epsilon}{m_2}\left[X - \frac{(m_2+m_3)Y}{\epsilon m_3}\right. \tag{4.33}$$

Eq. (4.33) can be written:

$$\ddot{x} = Y - \alpha X \ , \quad \ddot{y} = X - \beta Y \tag{4.34}$$

$$\text{with} \quad \alpha = \frac{\epsilon(m_1+m_2)}{m_1} \quad , \quad \beta = \frac{m_2+m_3}{\epsilon m_3}$$

For the system Eq.(4.31) to be completely integrable it is sufficient that the system Eq.(4.34) be integrable.

Now we will look for a second integral polynomial in velocities. At order 3, the form of the integral is

$$C = f_o\dot{x}^3 + f_1\dot{x}^2\dot{y} + f_2\dot{x}\dot{y}^2 + f_3\dot{y}^3 + g_o\dot{x} + g_1\dot{y}.$$

In this case, we can restrict ourselves to constant f_i's.

The compatibility condition, necessary for the integration of the equations for the g_i 's, Eq.(2.25), is considerably simplified from Eq.(4.34). We are led to the conditions

$$3f_o = \beta f_1, \quad 3f_3 = \alpha f_2. \tag{4.35}$$

One can easily integrate the equations for the g_i's:

$$\begin{aligned} g_o &= (\alpha\beta-1)f_1 X + 2(\beta f_2 - f_1)Y, \\ g_1 &= 2(\alpha f_1 - f_2)X + (\alpha\beta-1)f_2 Y. \end{aligned} \tag{4.36}$$

The next compatiblility condition Eq.(2.26) is an identity in terms of the independent functions X^2, XY, Y^2. The coefficients f_1 and f_2 must satisfy three linear equations:

$$\begin{gathered} \alpha f_1(3-\alpha\beta) - 2f_2 = 0 \quad , \quad -2f_1 + \beta f_2(3-\alpha\beta) = 0 \ , \\ f_1(2\alpha-\alpha\beta-1) + f_2(2\beta-\alpha\beta-1) = 0. \end{gathered} \tag{4.37}$$

This system has a nontrivial solution (f_1, f_2) whenever α and β satisfy the equation

$$(\beta-\alpha)(\alpha\beta-1) = 0, \tag{4.38}$$

or, equivalently,

$$\begin{gathered} 2(2\alpha-\alpha\beta-1) - (2\beta-\alpha\beta-1)(\alpha^2\beta - 3\alpha) = 0 \\ (2\alpha-1-\alpha\beta)(\alpha\beta-3) - 2(2\beta-\alpha\beta-1) = 0 \end{gathered} \tag{4.39}$$

The condition $\alpha\beta = 1$ is impossible because the change of variables (Eq.(4.34)) is not defined.

Thus $\alpha = \beta$ (=1) , and the second equation (4.39) then reads

$$(\alpha-1)^2(\alpha+1)(\alpha^2-\alpha-2) = 0. \qquad (4.40)$$

When $\alpha = \beta = 2$ ($f_2 = -f_1$), we can define

$$m_1 = \frac{\epsilon(2\epsilon-1)}{2-\epsilon} \quad , \quad m_2 = 2\epsilon-1 \quad , \quad m_3 = 1 \qquad (4.41)$$

The system (4.33) is integrable and admits apart from the energy. a second constant of motion cubic in the velocities

$$C = 2\dot{x}^3 + 3\dot{x}^2\dot{y} - 3\dot{x}\dot{y}^2 - 2\dot{y}^3 + 9(e^x - 2e^y)\dot{x} + 9(2e^x - e^y)\dot{y}. \qquad (4.42)$$

The values of the parameters m_i and ϵ correspond to the first case in Eq.(4.11).

Let us consider an integral of order 4 in the velocities

$$C = f_0\dot{x}^4 + f_1\dot{x}^3\dot{y} + f_2\dot{x}^2\dot{y}^2 + f_3\dot{x}\dot{y}^3 + f_4\dot{y}^4 + g_0\dot{x}^2 + g_1\dot{x}\dot{y} + g_2\dot{y}^2 + h(x,y)$$

We can restrict ourselves to constant f_i's.

From the compatibility condition for the integrability of the g_i's (Eq.(2.30)) reduces to

$$4f_0 = \beta f_1 \quad , \quad \alpha f_3 = 4f_4 . \qquad (4.43)$$

Using Eq.(2.29), the equations for the g_i's can be integrated to give

$$g_o = (\alpha\beta-1)f_1 X + 2f_2\beta Y - 3f_1 Y,$$
$$g_1 = 3f_1\alpha X - 2f_2 X - 2f_2 Y - 3f_3\beta Y,$$
$$g_2 = 2f_2\alpha X - 3f_3 X - f_3 Y + 4f_4\beta Y.$$

The second compatibility condition for the integrability of h, Eq.(2.32), is an identity in terms of the independent functions X^2, XY, Y^2. We obtain a system in terms of f_1, f_2, f_3, α, β :

$$f_1(-2+3\alpha-\alpha\beta) + f_2(2\beta-2\alpha) + f_3(-3\beta+\alpha\beta+2) = 0,$$
$$-12f_1 + 12\beta f_2 - 6\beta^2 f_3 = 0,$$
$$-6\alpha^2 f_1 + 12\alpha f_2 - 12f_3 = 0. \tag{4.44}$$

If we stick to the $f_3 = 0$ then Eq.(4.44) gives the conditions

$$(\alpha\beta + 2) = 0 \quad \text{or} \quad (\beta - 2)(\alpha - 1) = 0,$$
$$(\alpha - 2)(\alpha - 1) = 0. \tag{4.45}$$

Eq.(4.45) can be written

$$\text{a) } \alpha = 1 \;, \quad \beta = 2 \qquad \text{or b) } \alpha = 2, \; \beta = 1, \tag{4.46}$$

The two cases a) and b) are related by changing ϵ in 2ϵ, a scaling on x (x into x/2)

The case b) writes, in terms of ϵ, m_1, m_2, m_3.

$$m_1 = \frac{\epsilon(\epsilon-1)}{2-\epsilon} \;, \quad m_2 = (\epsilon-1) \;, \quad m_3 = 1 \;, \quad 1 < \epsilon < 2$$

In this case, the value of constant C is

$$C = \dot{x}^4 + 4\dot{x}^3\dot{y} + 4\dot{x}^2\dot{y} + 4(e^x - e^y)\dot{x}^2 + 8(2e^x - e^y)\dot{x}\dot{y} + 16e^x\dot{y}^2$$
$$+ 4e^{2y}. \tag{4.47}$$

The values of the parameters m_i and ϵ correspond to the second case provided by the ARS method.

Let us consider the case of order 6 in the velocities. The form of a sixth-order constant is

$$\begin{aligned} C = {} & e_o\dot{x}^6 + e_1\dot{x}^5\dot{y} + e_2\dot{x}^4\dot{y}^2 + e_3\dot{x}^3\dot{y}^3 + e_4\dot{x}^2\dot{y}^4 + e_5\dot{x}\dot{y}^5 \\ & + e_6\dot{y}^6 + f_o\dot{x}^4 + f_1\dot{x}^3\dot{y} + f_2\dot{x}^2\dot{y}^2 + f_3\dot{x}\dot{y}^3 + f_4\dot{y}^4 + g_o\dot{x}^2 \\ & + g_1\dot{x}\dot{y} + g_2\dot{y}^2 + h. \end{aligned}$$

We can restrict ourselves to constant e_i's.

From the compatibility condition for the integrability of the f_i's (2.38) gives

$$6e_o = \beta e_1 \quad , \quad \alpha e_5 = 6e_6. \tag{4.48}$$

Using Eq.(2.37), the equations for f_i's can be integrated to give

$$\begin{aligned} f_o &= e_1(\alpha\beta-1)X + (2e_2\beta - 5e_1)Y, \\ &\equiv A_oX + B_oY, \\ f_1 &= (5e_1\alpha - 2e_2)X + (3e_3\beta - 4e_2)Y, \\ &\equiv A_1X + B_1Y, \\ f_2 &= (4e_2\alpha - 3e_3)X + (4e_4\beta - 3e_3)Y, \\ &\equiv A_2X + B_2Y, \\ f_3 &= (3e_3\alpha - 4e_4)X - 2e_4Y, \\ &\equiv A_3X + B_3Y, \\ f_4 &= 2e_4\alpha X + (6e_6\beta - e_5)Y, \\ &\equiv A_4X + B_4Y, \end{aligned} \tag{4.49}$$

The next compatibility condition Eq.(2.40) reduces to

$$4B_\circ - \beta B_1 = 0,$$

$$\alpha A_3 - 4A_4 = 0,$$

$$(\beta A_1 - B_1) + 2(\alpha B_2 - A_2) + 3(\beta A_3 - B_3) + 4(\alpha B_\circ - A_\circ)$$

$$= (\alpha B_3 - A_3) + 2(\beta A_2 - B_2) + 3(\alpha B_1 - A_1) + 4(\beta A_4 - B_4). \quad (4.50)$$

It is possible to calculate g :

$$g_\circ = \tfrac{1}{2}(4A_\circ\alpha - A_1)X^2 + [4(B_\circ\alpha - A_\circ) + (A_1\beta - B_1)]XY$$

$$+ \tfrac{1}{2}[2\beta B_2 - 3B_1]Y^2,$$

$$\equiv C_\circ X^2 + D_\circ XY + E_\circ Y^2,$$

$$g_1 = \tfrac{1}{2}(3\alpha A_1 - 2A_2)X^2 + [3(\alpha B_1 - A_1) + 2(\beta A_2 - B_2)$$

$$+ 4(A_\circ - \alpha B_\circ) - (A_1\beta - B_1)]XY + \tfrac{1}{2}(3\beta B_3 - 2B_2)Y^2,$$

$$\equiv C_1 X^2 + D_1 XY + E_1 Y^2,$$

$$g_2 = (\alpha A_2 - \tfrac{3}{2}A_3)X^2 + (4\beta A_4 + \alpha B_3 - A_3 - 4B_4)XY$$

$$+ (2\beta B_4 - \tfrac{B_3}{2})Y^2,$$

$$\equiv C_2 X^2 + D_2 XY + E_2 Y^2. \quad (4.51)$$

The last compatibility condition (Eq.(2.42)) will give the conditions:

$$2E_\circ = \beta E_1,$$

$$2C_2 = \alpha C_1,$$

$$4(D_\circ - \alpha E_\circ) - 2(\beta D_1 - E_1) - (D_1 - \alpha E_1) - 2(E_2 - \beta D_2) = 0,$$

$$2(C_\circ - \alpha D_\circ) + (D_1 - \beta C_1) + 2(\alpha D_1 - C_1) + 4(\beta C_2 - D_2) = 0. \quad (4.52)$$

The nine equations (4.48), (4.50) and (4.52) summarize in terms of α, β and the e_i's. They form a system of nine equations for the six unknowns e_i. It is possible to show that in order to get a solution to this system, α and β have to take the following values.

a) $\alpha = 2$, $\beta = 2$: This integral is the product of the constant of degree 3 in the velocities.

b) $\alpha = 2$, $\beta = 1$: This integral is the product of the constant of degree 4 in the velocities.

c) $\alpha = \frac{2}{3}$, $\beta = 2$: This corresponds to

$$m_1 = \frac{3\epsilon(2\epsilon-1)}{2-3\epsilon}, \quad m_2 = 2\epsilon - 1, \quad m_3 = 1, \quad \frac{1}{2} < \epsilon < \frac{2}{3}$$

We finally find the values of e_i's, the functions f_i 's and the functions g_i's

$$e_0 = 4, \; e_1 = 12, \; e_2 = 13, \; e_3 = 6, \; e_4 = 1,$$

$$f_0 = 4X - 8Y, \quad f_1 = 14X - 16Y, \quad f_2 = \frac{50}{3}X - 10Y,$$

$$f_3 = 8X - 2Y, \quad f_4 = \frac{4}{3}X,$$

$$g_0 = -\frac{5}{3}X^2 + \frac{20}{3}XY + 4Y^2,$$

$$g_1 = -\frac{8}{3}X^2 + 6XY + 4Y^2,$$

$$g_2 = -\frac{8}{9}X + \frac{4}{3}XY + Y^2.$$

The values of the constants are

$$\begin{aligned} C = {} & 4\dot{x}^6 + 12\dot{x}^5\dot{y} + 13\dot{x}^4\dot{y}^2 + 6\dot{x}^3\dot{y}^3 + \dot{x}^2\dot{y}^4 + 4(e^x - 2e^y)\dot{x}^4 \\ & + (14e^x - 16e^y)\dot{x}^3\dot{y} + 10(\tfrac{5}{3}e^x - e^y)\dot{x}^2\dot{y}^2 + 2(4e^x - e^y)\dot{x}\dot{y}^3 \\ & + \tfrac{4}{3}e^x\dot{y}^4 + (-\tfrac{5}{3}e^{2x} + \tfrac{20}{3}e^{x+y} + 4e^{2y})y + \tfrac{4}{27}e^{3x} + \tfrac{4}{9}e^{2x+y} \\ & + (-\tfrac{8}{3}e^{2x} + 6e^{x-y} + 4e^{2y})\dot{x}\dot{y} + (-\tfrac{8}{9}e^{2x} + \tfrac{4}{3}e^{x+y} + e^{2y})\dot{y}^2. \end{aligned}$$

For the cases of the free end lattice, there are three integrable cases which can be calculated by the direct calculation method.

a) $m_1 = \frac{\epsilon(2\epsilon-1)}{2-\epsilon}$, $m_2 = 2\epsilon-1$, $\frac{1}{2} < \epsilon < 2$,

b) $m_1 = \frac{\epsilon(\epsilon-1)}{2-\epsilon}$, $m_2 = \epsilon-1$, $1 < \epsilon < 2$,

c) $m_1 = \frac{3\epsilon(2\epsilon-1)}{2-3\epsilon}$, $m_2 = 2\epsilon-1$, $1 < \epsilon < 2$,

(case c see appendix)

Every case of integrability predicted by the ARS method was recovered by the direct calculation of the integrals of motion.

Fixed-end lattice

The form of the Hamiltonian governing the system is

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + e^{-\delta x} + e^{\epsilon(x-y)} + e^{y}. \tag{4.53}$$

We put $X = e^{-\delta x}$, $D = e^{\epsilon(x-y)}$, $Y = e^{y}$. (4.54)

The equations of motion read

$$\ddot{x} = \frac{1}{m_1}(\delta e^{-\delta x} - \epsilon e^{\epsilon(x-y)}),$$

$$\ddot{y} = \frac{1}{m_2}(\epsilon e^{\epsilon(x-y)} - e^{y}),$$

after a scaling in time

$$\ddot{x} = (\delta X - \epsilon D), \qquad \ddot{y} = a(\epsilon D - Y), \quad a = m_1/m_2. \tag{4.55}$$

Let us begin with a quartic constant (Eq.(2.27)). We can restrict ourselves to constant f_i.

The first compatibility condition (Eq. (2.30)) reduces to

$$f_1 = 0, \quad f_3 = 0, \quad 2f_\circ + f_2(1-a) = 0. \tag{4.56}$$

We integrate and find the functions g_i :

$$\begin{aligned} g_\circ &= 4f_\circ X + 4f_\circ D + 2f_2 aY, \\ &= F_\circ(X + D) + F_2 Y, \\ g_1 &= 4f_\circ \;D \; - 2f_2 \;aD, \\ &= (F_\circ \; - F_2 \;)D, \\ g_2 &= -2f_2 X, \\ &= (F_2/a)X. \end{aligned} \tag{4.57}$$

The last compatibility condition (Eq.(2.32)) will give

$$\begin{aligned} &(F_\circ - F_2)(\delta^2 + 2\epsilon^2 - 3\epsilon\delta) = 0, \\ &F_\circ(3 - a) + F_2(a - 1) = 0, \\ &F_2(-2\epsilon + a - \epsilon a + 2\epsilon^2) + F_\circ(\epsilon a - a) = 0. \end{aligned} \tag{4.58}$$

This system of Eqs.(4.56) and (4.58) is satisfied for

$F_\circ \neq F_2$:

and for $F_\circ = 0$; $a = 1$, $\epsilon = 1$, or $\epsilon = \frac{1}{2}$

and $\delta/\epsilon = 2$, or $\delta/\epsilon = 1$.

Finally, we find three distinct cases:

$$\text{a) } m_1/m_2 = 1, \; \delta = \epsilon = 1, \tag{4.59}$$

$$\text{b) } m_1/m_2 = 1, \; \delta = 1, \; \epsilon = 1/2, \tag{4.60}$$

$$\text{c) } m_1/m_2 = 1, \; \delta = 1/2, \; \epsilon = 1/2. \tag{4.61}$$

(equivalent to $\delta = 2, \epsilon = 1$)

The values of the constants are

$$\text{a) } C = \frac{\dot{x}^2\dot{y}^2}{2} + e^y\dot{x}^2 - e^{x-y}\dot{x}\dot{y} + e^{-x}\dot{y}^2 + \frac{e^{2(x-y)}}{2} + e^x + 2e^{y-x} + e^{-y},$$

b) $C = \frac{\dot{x}^2\dot{y}^2}{2} + e^{y}\dot{x}^2 - e^{(x-y)/2}\dot{x}\dot{y} + e^{-x}\dot{y}^2 + \frac{e^{x-y}}{2} + 2e^{y-x}$,

c) $C = \frac{\dot{x}^2\dot{y}^2}{2} + e^{y}\dot{x}^2 - e^{(x-y)/2}\dot{x}\dot{y} + e^{-x/2}\dot{y}^2 + 2e^{y-x/2} + \frac{e^{x-y}}{2} + e^{-y/2}$.

The values of the parameters m_i , δ and ϵ correspond to the case [Eqs.(4.27), (4.24) and (4.26)] provided by the ARS method.

Now Let us consider the case of a constant of order 6 in the velocities. The first compatibility condition gives a system of e_i's:

$$e_1 = e_5 = 0,$$

$$6e_0 + (2-a)2e_2 + (1-a)3e_3 + (1-2a)2e_4 = 0. \qquad (4.62)$$

The integration of the equations for the f_i's is straightforward:

$$\begin{aligned}
f_0 &= 6e_0X + 6e_0D + 2ae_2Y,\\
&\equiv E_0X + E_0D + aE_2Y,\\
f_1 &= 6e_0D + 3ae_3Y - 2ae_2D,\\
&\equiv E_0D + aE_3Y - aE_2D,\\
f_2 &= 4e_2X + 4e_2D - 3ae_3D - 2ae_2D + 6e_0D + 4e_4aY,\\
&\equiv 2E_2X + [(2-a)E_2 - aE_3 + E_0]D + 2E_4aY,\\
f_3 &= 3e_3X - 2e_4D,\\
&\equiv E_3X - E_4D,\\
f_4 &= 2e_4D,\\
&\equiv E_4D.
\end{aligned}$$

The second compatibility gives relations on E_i:

$$[E_0\epsilon(-5\epsilon\delta + 2\delta^2 + 4\epsilon^2) + E_2\epsilon(4(\epsilon-\delta)^2 + 5a\delta\epsilon - 2a\delta^2 - 4a\epsilon^2) + E_4(\epsilon-\delta)^2(-4a\epsilon+\delta)]$$
$$(\epsilon-\delta) = 0, \qquad (4.63)$$

$$3E_0(a-3) + E_2(9a-4-3a^2) + E_4(1-3a) = 0,$$

$$(1-\epsilon)[E_0(1-\epsilon)(-1+3\epsilon) + E_2(1-\epsilon)(-3a\epsilon+a+4\epsilon^2) + E_4(3-4a-4\epsilon+4a\epsilon)\epsilon^2] = 0. \tag{4.64}$$

The Eqs.(4.62),(4.63) and (4.64) are satisfied for

$$a = 1,\ \epsilon = 1,\ \delta = 1;$$

$$a = 1,\ \epsilon = 1/2,\ \delta = 1/2;$$

$$a = 1,\ \epsilon = 1,\ \delta = 2;$$

which corresponds to the cases Eqs.(4.59) - (4.61).

For $a = 1/3$, $\epsilon = 1/2$, the conditions Eqs.(4.62) and (4.64) are satisfied and

$$E_2 = -6E_0\ ,\quad E_4 = 27E_0.$$

In order to satisfy Eq.(4.63) for these values of a and ϵ, δ must take one of the following values:

$$\delta = 1,\ 1/3,\ 2/3,\ 1/2.$$

The system (4.62), (4.63) and (4.64) are satisfied. The calculation of the g_i's reads

$$g_0 = 2E_0X^2 + 4E_0XD + \frac{D^2}{2}(4E_0-aE_0+a^2E_2) + 2a^2E_4Y^2 + 4aE_2XY + \frac{DY}{\epsilon}(4a\epsilon E_2+aE_0-a^2E_2),$$

$$g_1 = \frac{(3\delta-4\epsilon)}{\epsilon-\delta}(aE_2-E_0)DX + \frac{D^2}{2}(7E_0-7aE_2+3a^2E_2-3aE_0) + YD\left(\frac{a^2E_2}{\epsilon}-\frac{aE_0}{\epsilon}-3a^2E_2+3aE_0+4a\epsilon E_2-4a^2\epsilon E_4\right),$$

$$g_2 = \frac{E_4D^2}{2} - \frac{XD}{\epsilon}(E_4\delta-4a\epsilon E_4) + 4aE_4XY + 2E_2X^2. \tag{4.65}$$

Now we can make explicit the values of g_i in Eq.(4.65) for cases of values of a, ϵ and δ solutions of Eqs.(4.62),(4.63) and (4.64) and check whether relation (2.42) holds or not

1) $a = 1/3$, $\epsilon = 1/2$, $\delta = 1$. The g_i are

$$g_0 = \frac{2X^2}{3} + \frac{4XD}{3} - \frac{8XY}{3} - 2YD + \frac{D^2}{2} + 2Y^2,$$

$$g_1 = 3D^2 + 2XD - 6YD,$$

$$g_2 = \frac{9D^2}{2} - 6XD + 12XY - 4X^2 \quad (\text{with } 3E_0 = 1)$$

Relation (2.42) is verified, and we then compute the constant

$$C = \frac{\dot{x}^2}{18} - 2\dot{x}^4\dot{y}^2 + 9\dot{x}^2\dot{y}^4 + (e^{-x} - 2e^{y} + e^{(x-y)/2})\frac{\dot{x}^4}{3} + e^{(x-y)/2}\dot{x}^3\dot{y}$$

$$+ (6e^{y} - 3e^{(x-y)/2} - 4e^{-x})\dot{x}^2\dot{y}^2 - 9e^{(x-y)/2}\dot{x}\dot{y}^3 + 9e^{-x}\dot{y}^4$$

$$+ (2e^{-2x} + 4e^{-(x+y)/2} - 8e^{y-x} - 6e^{(x+y)/2} + \frac{3}{2}e^{x-y} + 6e^{2y})\frac{\dot{x}^2}{3}$$

$$+ (3e^{x-y} + 2e^{-(x+y)/2} - 6e^{(x+y)/2})\dot{x}\dot{y} - 12e^{(y-x)/2}$$

$$+ (\frac{9}{2}e^{x-y} - 6e^{-(x+y)/2} + 12e^{y-x} - 4e^{-2x})\dot{y}^2.$$

2) $a = 1/3$, $\epsilon = 1/2$, $\delta = 1/3$. The g_i are

$$g_0 = \frac{2X^2}{3} + \frac{4XD}{3} - \frac{8XY}{3} - 2YD + \frac{D^2}{2} + 2Y^2,$$

$$g_1 = 3D^2 + 6XD - 6YD,$$

$$g_2 = \frac{9D^2}{2} + 6XD + 12XY - 4X^2. \quad (\text{with } 3E_0 = 1)$$

In this case, one can check that relation Eq.(2.42) is verified and then compute the constant

$$
\begin{aligned}
C = {} & \frac{\dot{x}^6}{18} - 2\dot{x}^4\dot{y}^2 + 9\dot{x}^2\dot{y}^4 + (e^{-x/3} - 2e^{y} + e^{(x-y)/2})\frac{\dot{x}^4}{3} + e^{(x-y)/2}\dot{x}^3\dot{y} \\
& + (6e^{y} - 3e^{(x-y)/2} - 4e^{-x/3})\dot{x}^2\dot{y}^2 - 9e^{(x-y)/2}\dot{x}\dot{y}^3 + 9e^{-x/3}\dot{y}^4 \\
& + (2e^{-2x/3} + 4e^{\frac{x}{6}-y/2} - 8e^{y-x/3} - 6e^{(x+y)/2} + \frac{3}{2}e^{x-y} + 6e^{2y})\frac{\dot{x}^2}{3} \\
& + (3e^{x-y} + 6e^{x/6-y/2} - 6e^{(x+y)/2})\dot{x}\dot{y} + e^{x} - \frac{4}{9}e^{-x} - 4e^{2y-x/3} \\
& + (\frac{9}{2}e^{x-y} + 6e^{x/6-y/2} + 12e^{y-x/3} - 4e^{2x/3})\dot{y}^2 - 4e^{x/6+y/2} \\
& + \frac{8}{3}e^{-\frac{2x}{3}+y} - 4e^{-x/6-y/2}.
\end{aligned}
$$

3) $a = 1/3$, $\epsilon = 1/2$, $\delta = 2/3$ and $a = 1/3$, $\epsilon = 1/2$, $\delta = 1/2$.

The relation (2.42) does not hold. This means that, there does not exist any integrable case.

There are five cases with the direct calculation method for the fixed-end lattice.

a) $m_1/m_2 = 1$, $\delta = \epsilon = 1$,

b) $m_1/m_2 = 1$, $\delta = 1$, $\epsilon = 1/2$,

c) $m_1/m_2 = 1/3$, $\delta = 1$, $\epsilon = 1/2$,

d) $m_1/m_2 = 1/3$, $\delta = 1/3$, $\epsilon = 1/2$,

e) $m_1/m_2 = 1$, $\delta = 1/2$, $\epsilon = 1/2$.

In this chapter we review the usefulness of the ARS method and direct calculation of the constant of motion as a tool for identifying integrable cases of Toda lattice:

1) The free-end lattice with three masses:

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{p_3^2}{2m_3} + e^{\epsilon(q_1-q_2)} + e^{q_2-q_3}.$$

Bountis, Segur, and Vivaldi [14] showed that there are three cases for which the system satisfy sufficient condition for possessing the Painleve property.

a) $m_1 = \frac{\epsilon(2\epsilon-1)}{2-\epsilon}$, $m_2 = 2\epsilon-1$, $\frac{1}{2} < \epsilon < 2$,

b) $m_1 = \frac{\epsilon(\epsilon-1)}{2-\epsilon}$, $m_2 = \epsilon-1$, $1 < \epsilon < 2$,

c) $m_1 = \frac{3\epsilon(2\epsilon-1)}{2-3\epsilon}$, $m_2 = 2\epsilon-1$, $1 < \epsilon < 2$.

It can be checked that the system is integrable for these conditions by direct calculation method [4]. The integrals of motion for all cases have been calculated by Dorizzi, Grammaticos, et al.[17] by this method.

2) The fixed-end lattice with two masses:

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + e^{-\delta x} + e^{\epsilon(x-y)} + e^{y}.$$

The ARS method gives only the sufficient condition for integrability. Ramani [18], using this method, has found five cases.

a) $m_1/m_2 = 1$, $\delta = \epsilon = 1$,

b) $m_1/m_2 = 1$, $\delta = 1$, $\epsilon = 1/2$,

c) $m_1/m_2 = 1/3$, $\delta = 1$, $\epsilon = 1/2$.

d) $m_1/m_2 = 1/3$, $\delta = 1/3$, $\epsilon = 1/2$.

e) $m_1/m_2 = 1$, $\delta = 1/2$, $\epsilon = 1/2$.

Dorizzi, Grammaticos et al. [17] checked that the system is integrable by direct calculation method and they have found the integrals of motion for all cases.

From the results we review in this chapter, the ARS method and the direct calculation method can be a most powerful tool for the investigation of integrability of dynamical system.