CHAPTER III

INTEGRALS OF MOTION FOR THE HÉNON-HEILES SYSTEM

In this chapter, we focus on the Hénon-Heiles Hamiltonian system. The cases of integrable systems with the Painlevé property have been considered by Grammaticos and co-workers in ref[13] and Bountis and co-workers in ref[14]. In the first section of this chapter we are devoted to the study of integrability of the Hénon-Heiles Hamiltonian with the Painlevé property. The integrable cases of the Hénon-Heiles Hamiltonian by direct calculation method will be studied in the last section.

ARS METHOD

By starting with a general expression of the Hénon-Heiles hamiltonian:

$$H = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + ax^2 + by^2) + dx^2 y - \frac{1}{3}ey^3. \tag{3.1}$$

the corresponding equations of motion are

$$\begin{aligned} \ddot{x} &= -ax - 2dxy, \\ \ddot{y} &= -by - dx^2 + ey^2. \end{aligned} \tag{3.2}$$

In order to investigate the leading order behavior around a singularity in the complex-time plane we put:

$$x = A\tau^{\alpha} \quad , \quad y = B\tau^{\beta}, \tag{3.3}$$

where $\tau = t - t_0$ with t_0 the position of the complex pole.

One finds that two cases of the dominant behavior are available

$$\text{i) } A = \pm(3/d)\sqrt{2+e/d}\ , \quad \alpha = -2,$$
$$B = -3d \ , \quad \beta = -2. \tag{3.4}$$

(dominant terms : $\ddot{x} = -2dxy$, $\ddot{y} = -dx^2 + ey^2$)

$$\text{ii) } A = \text{arbitrary} \ , \quad \alpha = \frac{1 \pm \sqrt{1-48d/e}}{2},$$
$$B = 6/e \ , \quad \beta = -2. \tag{3.5}$$

(dominant terms : $\ddot{x} = -2dxy$, $\ddot{y} = ey^2$).

The Painlevé criterion requires that α and β be integers, which restricts the values of d/e in the case 2.

Now, following ARS algorithm, we investigate the resonances. Looking for higher order terms of the form, we have

$$x = A\tau^{\alpha} + C\tau^{\alpha+r},$$
$$y = B\tau^{\beta} + D\tau^{\beta+r}. \tag{3.6}$$

Substituting Eq.(3.6) into the dominant part of the equations of motion, ones finds all possible r

For case i:

$$\det Q(r) = \det\begin{pmatrix} (r-2)(r-3) + 2dB & 2dA \\ 2dA & (r-2)(r-3) - 2eB \end{pmatrix} = 0,$$

$$r = -1,\ 6,\ \frac{5}{2} \pm \frac{1}{2}\sqrt{1-24(1+e/d)}, \tag{3.7}$$

and for case ii:

$$\det Q(r) = \det\begin{pmatrix} (r-\alpha)(r+\alpha-1) + 2dB & 2dA \\ & (r-2)(r-3) - 2eB \end{pmatrix} = 0,$$

$$r = -1,\ 0,\ 6, \pm\sqrt{1-48(1-d/e)}. \qquad (3.8)$$

The root -1 is associated with the arbitrariness of the pole position. The root 0 in case ii is related to the arbitrariness of A. For the system to be integrable according to the Painlevé criterion the values of r must be integers

There are four possible cases for having the Painlevé property:
$e = -d, a = b$; $e = -2d, a \neq b$; $e = -6d, a \neq b$; $e = -16d, b = 16$. (3.9)

DIRECT CALCULATION METHOD

Now we look for the integrable cases by the direct calculation method. Consider the case of a constant of motion quardratic in the velocities.

The compatibility condition Eq.(2.19) reduces to

$$\begin{aligned} &6\alpha(2d+e) = 0, && 2\beta(6d+e) + 4\alpha(a-b) = 0, \\ &(-5e-2d)\zeta = 0, && 4d(\gamma+\xi) + \beta(4a-b) = 0, \\ &k(b-a) = 0, && \delta(2b-a) - 2k(d-e) = 0, \\ &6\alpha d = 0, && 3\zeta d = 0. \end{aligned} \qquad (3.10)$$

Equation (3.10) leads to the cases:

a: $e = -6d$, $a \neq b$, $\delta = 0$, $\alpha = 0$, $k = 0$.

The equations of g_i reduce to

$$g_0 = \beta y + \gamma,$$
$$g_1 = -\beta x,$$
$$g_2 = \xi . \qquad (3.11)$$

The integration for the h is straightforward:

$$h = a\beta x^2 y + 2\beta d x^2 y^2 + \gamma a x^2 + 2\gamma d x^2 y - (b/2)\beta x^2 y - (d/4)\beta x^4 + (e/2)\beta x^2 y^2 + b\xi y^2 - (2e/3)\xi y,$$

and gives the condition

$$-2a\beta + (b/2)\beta + 2d\xi - 2d\gamma = 0. \qquad (3.12)$$

Now the constant of motion is written as

$$C = (\beta y + \gamma)\dot{x}^2 - \beta x\dot{x}\dot{y} + \xi\dot{y}^2 + h. \qquad (3.13)$$

In this case, Green [15] has obtained the corresponding integral of motion ($d = 1$, $\beta = 4$, $\xi = 0$, $\gamma = -(4a-b)$),

$$G = x^4 + 4x^2y^2 - 4\dot{x}(\dot{x}y - \dot{y}x) + 4ax^2y + (4a-b)(\dot{x}^2 + ax^2).$$

b: $e = -2d$, $\beta = 0$, $\gamma = \xi$, $\alpha = 0$, $\delta = 0$, $k = 0$, $a \neq b$.

The equations of g_i reduces to

$g_0 = \gamma$, $g_1 = 0$, $g_2 = \gamma$,

The equations for the h can be integrated to give

$$h = \gamma a x^2 + 2d\gamma x^2 y + \gamma b y^2 - (2/3)\gamma e y^3.$$

Now the constant of motion is written as

$$C = \gamma\{\dot{x}^2 + \dot{y}^2 + x^2(a+2dy) + y^2(b-(2/3)ey)\}. \quad (3.14)$$

If $\gamma = \frac{1}{2}$, $C = \frac{1}{2}\{\dot{x}^2 + \dot{y}^2 + ax^2 + by^2\} + dx^2y - (1/3)ey^3$.

This is the Hénon-Heiles hamiltonian. C is not a second constant of motion, independent of the hamiltonian. This means that e = -2d does not lead to integrability.

c: $e = -d$, $a = b$, $\beta = 0$, $\gamma = \xi$.

The equations of g_i reduces to

$$g_o = \gamma, \; g_1 = k, \; g_2 = \gamma. \quad (3.15)$$

The equation for the h can be integrated to give

$$h = a\gamma x^2 + 2\gamma dx^2y + kbxy + (d/3)kx^3 - kexy^2 + \gamma by^2 + (2d/3)\gamma y^3.$$

The constant of motion for this case is written as

$$C = \gamma\dot{x}^2 + k\dot{x}\dot{y} + \gamma\dot{y}^2 + a\gamma x^2 + 2\gamma dx^2y + akxy + (d/3)kx^3 - kexy^2 + a\gamma y^2 - (2e/3)\gamma y^3.$$

Now we look for an integral of motion which contains velocities up to the fourth power.

The first compatibility condition (2.30) gives

$A = B = E = F = 0.$

Then the equations of f_i are

$$f_\circ = Cy + D,$$
$$f_1 = -Cx + G,$$
$$f_2 = Hy + I,$$
$$f_3 = -Hx + J,$$
$$f_4 = K. \qquad (3.16)$$

Now we consider the case $e = -16d$ and for the relation Eq.(2.30). The conditions read

$$H = J = K = C = 0.$$

One can integrate the equations for the g_i's:

$$g_\circ = 2aDx^2 + 4dDx^2 y + q,$$
$$g_1 = -\frac{4}{3}dDx^3 + l. \qquad (3.17)$$

where q , l are two integration constants.

We proceed now to first order Eq.(2.31)

$$2g_\circ\ddot{x} + g_1\ddot{y} + \frac{\partial h}{\partial x} = 0,$$
$$g_1\ddot{x} + \frac{\partial h}{\partial y} = 0. \qquad (3.18)$$

The compatibility condition Eq.(2.32) is

$$-\frac{\partial^2 h}{\partial x \partial y} = \frac{\partial}{\partial y}(2g_\circ\ddot{x} + g_1\ddot{y}) = \frac{\partial}{\partial x}(g_1\ddot{x}) , \qquad (3.19)$$

which amounts to

$$e = -16d \quad \text{or} \quad D = 0,$$
$$b = 16a \quad \text{or} \quad D = 0,$$
$$q = 0,$$
$$l = 0.$$

Integration of Eq.(3.18) gives for h:

$$h = -\frac{4}{3}dDx^4(ay+dy^2) + a^2Dx^4 - \frac{2}{3}d^2Dx^6.$$

and finally the integral of motion becomes (with D = 3):

$$C = 3\dot{x}^4 + 6(a+2dy)\dot{x}^2x^2 - 4dx^3\dot{x}\dot{y} - 4dx^4(ay+dy^2) + 3a^2x^4 - \frac{2}{3}d^2x^6.$$

There are three possible cases with the direct calculation method, e = -d, e = -6d, e = -16d.

For the cases e = -d, e = -6d, integrals of motion can be found by the case of a constant of motion quardratic in the velocities.

For the case e = -16d, integrals of motion can be found by the case of a constant of motion order 4 in the velocities.

In this chapter we focus on the Hénon-Heiles hamiltonian system.

$$H = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + ax^2 + by^2) + dx^2y - \frac{1}{3}ey^3.$$

Following the works of Bountis, Segur, and Vivaldi in ref.[14] and Chang, Tabor, and Wriss in ref.[16] we have shown, using the ARS method, that this system possesses the Painlevé property for the parameter values:

a) e = -d, a = b,

b) e = -6d, $a \neq b$,

c) e = -16d, b = 16a,

d) e = -2d, $a \neq b$.

The ARS method gives only the sufficient condition for integrability. One still needs some method to check that the system is integrable. Direct calculation of integral of motion has been suggested in ref.[4]. Using this method, we have calculated the integrals of motion in the cases a) and c). The integrals of motion in case b) was calculated by Green [15]. But for case d), direct calculation method does not give the new integral of motion. This means that the condition e = -2d does not lead to the integrability. The ARS method, although gives only the sufficient condition, provides a useful and convenient way for studying the question of integrability. Supplemented by direct calculation of the constants of motion using various methods we find an effective tool for the investigation of integrability of a dynamical system.