การขึ้นต่อกันระหว่างสมการเชิงฟังก์ชันแบบโคชี

นายวัชรพล พิมพ์เสริฐ

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรดุษฎีบัณฑิต

สาขาวิชาคณิตศาสตร์ ภาควิชาคณิตศาสตร์

คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย

ปีการศึกษา 2551

DEPENDENCE AMONG CAUCHY-TYPE FUNCTIONAL EQUATIONS

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A Dissertation Submitted in Partial Fulfillment of the Requirements
for the Degree of Doctor of Philosophy Program in Mathematics
Department of Mathematics
Faculty of Science
Chulalongkorn University
Academic Year 2008

Thesis Title	Dependence among Cauchy-Type Functional Equations
By	Mr. Watcharapon Pimsert
Field of Study	Mathematics
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Accepted by the Faculty of Science, Chulalongkorn University in Partial Fulfillment of the Requirements for the Doctoral Degree.

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วัชรพล พิมพ์เสริฐ : การขึ้นต่อกันระหว่างสมการเชิงฟังก์ชันแบบโคชี. (DEPENDENCE AMONG CAUCHY-TYPE FUNCTIONAL EQUATIONS) อ.ที่ปรึกษาวิทยานิพนธ์หลัก: รศ.ดร. พัฒนี อุดมกะวานิช, อ.ที่ปรึกษาวิทยานิพนธ์ร่วม: ศ. ดร. วิเชียร เลาหโกศล, 74 หน้า.

ในปีค.ศ. 1988 ของ ดอมเบรสได้พิจารณาการขึ้นต่อกันของสมการเชิงฟังก์ชันแบบโคชีทั้งสี่แบบ โดยการแก้สมการเชิงฟังก์ชันที่เรียกว่า สมการเชิงฟังก์ชันแบบยูนิเวอร์ซอลโคชี ซึ่งสมการดังกล่าวครอบคลุมทั้งสี่รูปแบบของสมการเชิงฟังก์ชันแบบโคชี เมื่อพิจารณาโดเมนของฟังก์ชันผลเฉลยที่เป็นริงที่บรรจุเอกลักษณ์และหารลงตัวด้วยสอง และเรนจ์ของฟังก์ชันผลเฉลยเป็นสกิวฟิลด์

และในปี ค.ศ. 1988 และ 2005 คอนราด เจ ฮูเวอรส์และพาลานิบพัน คันนับพันได้พิสูจน์ว่าสมการเชิงฟังก์ชันทั้งสามแบบได้แก่ $f(x+y) - f(x) - f(y) = f(1/x + 1/y)$, $f(x+y) - f(xy)= f(1/x + 1/y)$, และ $f(xy) = f(x) + f(y)$ มีความสมมูลกันซึ่งกันและกันในแง่ที่ว่ามีฟังก์ชันผลเฉลยชุดเดียวกัน โดยฟังก์ชันผลเฉลยดังกล่าวที่สนใจนั้นเป็นฟังก์ชันค่าจริงและมีโดเมนคือเซตของจำนวนจริงบวก

จากงานของดอมเบรส เราจะพบว่า 0 มีส่วนสำคัญเป็นอย่างมากในขั้นตอนการแก้สมการ และเป็นผลทำให้ฟังก์ชันลอกการิทึมไม่ปรากฏในชุดฟังก์ชันผลเฉลย เนื่องมาจากโดเมนที่พิจารณาบรรจุ 0 ดังนั้นในงานวิจัยแรก เราจึงพิจารณาสมการเชิงฟังก์ชันแบบยูนิเวอร์ซอลโคชีอีกครั้งสำหรับกรณีที่โดเมนของฟังก์ชันผลเฉลยเป็นเซตของจำนวนจริงบวกและเรนจ์เป็นเซตของจำนวนเชิงซ้อน และจากงานของฮูเวอรส์ เราได้พิจารณาหาผลเฉลยสำหรับสมการเชิงฟังก์ชันรูปหนึ่งที่ขยายมาจากสมการของฮูเวอรส์

ภาควิชา คณิตศาสตร์	ลายมือชื่อนิสิต......วัชรพล พิมพ์เสริฐ......
สาขาวิชา คณิตศาสตร์	ลายมือชื่ออ.ที่ปรึกษาวิทยานิพนธ์หลัก......พัฒนี อุดมกะวานิช......
ปีการศึกษา 2551	ลายมือชื่ออ.ที่ปรึกษาวิทยานิพนธ์ร่วม..........

4773834923 : MAJOR MATHEMATICS
KEYWORDS : FUNCTIONAL EQUATION / CAUCHY FUNCTIONAL EQUATION

WATCHARAPON PIMSERT : DEPENDENCE AMONG CAUCHY-TYPE FUNCTIONAL EQUATIONS. ADVISOR : ASSOC. PROF. PATANEE UDOMKAVANICH, Ph.D., CO-ADVISOR : PROF.VICHIAN LAOHAKOSOL, Ph.D., 74 pp.

In 1988, Jean Dhombres investigated various kinds of independence among the four forms of the classical Cauchy functional equation as well as solved completely a functional equation, called the universal Cauchy functional equation, which contains all the four forms of the Cauchy functional equation. He took a ring which is divisible by 2 and possesses a unit as the domain of solution functions and a skew-field as their range.

In 1999 and 2005, Konrad J. Heuvers and Palanippan Kannappan showed that the three functional equations: $f(x+y)-f(x)-f(y) = f\left(\frac{1}{x}+\frac{1}{y}\right)$, $f(x+y)-f(xy) = f\left(\frac{1}{x}+\frac{1}{y}\right)$ and $f(xy) = f(x)+f(y)$ with $f : \mathbb{R}^+ \to \mathbb{R}$, are equivalent in the sense that a solution of one equation is also a solution of another.

The work of Dhombres does not include the case of logarithmic function because his domain of solution functions contains 0, the condition which plays a vital role in his work. Our first objective is to complement the work of Dhombres by re-investigating all his results for solution functions whose domain is the set of positive real numbers and whose range is the complex numbers. Our second objective is to solve an extension of one of the functional equations considered by Heuvers.

Department: Mathematics
Field of Study: Mathematics
Academic Year: 2008

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Advisor's Signature......P. Udomk......
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ACKNOWLEDGEMENTS

I would like to express my profound gratitude and deep appreciation to Associate Professor Dr. Patanee Udomkavanich and Professor Dr. Vichian Laohakosol, my thesis advisor and co-advisor, respectively, for their advice and encouragement. Sincere thanks and deep appreciation are also extended to Associate Professor Dr. Wicharn Lewkeeratiyutkul, the chairman, Associate Professor Dr. Paisan Nakmahachalasint, Assistant Professor Dr. Sajee Pianskool and Associate Professor Dr. Virool Boonyasombat, committee members, for their comments and suggestions. Also, I thank all teachers who have taught me all along.

In particular, I would like to express my deep gratitude to my parents, brother, sister and Charinthip Hengkrawit, my wife, for their encouragement throughout my graduate study.

CONTENTS

CHAPTER I

INTRODUCTION

An equation in which unknowns are functions is called a "functional equation". The best known and most basic functional equation is the **Cauchy functional equation**

$$f(x+y) = f(x) + f(y), \tag{C_1}$$

containing two variables x, y and one unknown function f of one variable. The functional equation (C_1) was solved by Cauchy in 1821 for $f : \mathbb{R} \to \mathbb{R}$ under a continuity assumption to have a general solution of the form $f(x) = ax$, where $a = f(1)$. This form of solution holds under a number of other assumptions, such as boundedness, monotonicity and measurability. However, without any additional assumption, the Cauchy functional equation (C_1) may possess arbitrarily wild solutions, the fact established by Hamel in 1905 using the notion of a basis of $\mathbb{R}$, which bears his name.

We give an illustration of this fact by constructing uncountably many non-continuous solutions to (C_1) for $f : \mathbb{R} \to \mathbb{R}$. Recall that a subset $H \subset \mathbb{R}$ is a Hamel basis for $\mathbb{R}$ if every $x \in \mathbb{R}$ can be written uniquely in the form

$$x = \sum_{i=1}^{n} r_i h_i,$$

for some $n \in \mathbb{N}$ and $r_i \in \mathbb{Q}$, $h_i \in H$ for $i = 1, \ldots, n$. Consider the class of functions given by

$$f_g(x) = r_1 g(h_1) + \cdots + r_n g(h_n),$$

where $g : H \to \mathbb{R}$. Clearly, each f_g is a solution of (C_1) over $\mathbb{R}$ and since the choice of g is arbitrary, the desired conclusion follows.

The Cauchy functional equation arises naturally in a good deal of contexts. For example, it is the homomorphism property of additive groups and so in many situations it is sometimes referred to as an *additivity property*; it is one of the two properties of linearity in the subject of functional analysis.

To solve a functional equation, not only an equation like (C_1) needs to be specified, but the domain and range of the solution functions are also critical. This can be seen from the following examples.

Example 1. Consider the functional equation

$$f(xy) = f(x) + f(y).$$

Let us first find all solution functions $f : \mathbb{R} \to \mathbb{R}$. Substituting y by 0, we have $f(0) = f(x) + f(0)$ yielding $f(x) = 0$, i.e. the zero function is the only solution. However, if we look for solutions whose domain is the set $\mathbb{R} \setminus \{0\}$, besides the zero function, the logarithmic function $f(x) = \log|x|$ is another solution showing that this functional equation has at least two distinct solutions for functions with domain $\mathbb{R} \setminus \{0\}$.

In this example, we observe that the set of all solutions in the domain $\mathbb{R} \setminus \{0\}$ has more elements than the set of all solutions in the larger domain $\mathbb{R}$. This brings about another delicate point about solving functional equations. It might seem that, for any functional equation, the following is true : if D_1 and D_2 are subsets of real numbers and $D_1 \subset D_2$, then the set of solutions of that equation in the domain D_2 is not greater than the set of all solutions of the same equation in the domain D_1. Unfortunately, this is not true in general as evident from the solutions of $f(xy) = f(x) + f(y)$ in the domain $D = \{1\}$, in which the zero function is the only solution.

The next example demonstrates the effect from the range of the solution functions.

Example 2. Consider the functional equation

$$1 + f(n)f(n+1) = 2n^2\,(f(n+1) - f(n)) \quad (n \in \mathbb{N}).$$

We wish to find all solutions $f : \mathbb{N} \to M$ with different ranges $M = \mathbb{N}$ and $M = \mathbb{Q}$. This is a problem taken from one of the Romanian Mathematical Olympiads so the solution is a little involved.

If $f(n+1)f(n) = -1$, then the functional equation gives $f(n+1) = f(n)$ and so $f(n)^2 + 1 = 0$, which is impossible since $M \subset \mathbb{R}$. Thus, the functional equation can be rewritten as

$$\frac{f(n+1) - f(n)}{1 + f(n)f(n+1)} = \frac{1}{2n^2}. \tag{1.1}$$

Let $x_n = \arctan f(n)$. The equation (1.1) becomes

$$\tan(x_{n+1} - x_n) = \frac{1}{2n^2},$$

or equivalently,

$$x_{n+1} - x_n = \arctan\left(\frac{1}{2n^2}\right) + p_n\pi \quad (p_n \in \mathbb{Z}).$$

By the identity

$$\sum_{k=1}^{n-1} \arctan\left(\frac{1}{2k^2}\right) = \sum_{k=1}^{n-1} (\arctan(2k+1) - \arctan(2k-1)) = \arctan(2n-1) - \frac{\pi}{4},$$

we have

$$\arctan(2n-1) - \frac{\pi}{4} = \sum_{k=1}^{n-1} ((x_{k+1} - x_k) - p_k\pi) = x_n - x_1 - q\pi,$$

for some $q \in \mathbb{Z}$. Thus, $x_n = \arctan(2n-1) + x_1 - \frac{\pi}{4} + q\pi$, yielding

$$f(n) = \tan(x_n) = \frac{2n-1+\tan\left(x_1 - \frac{\pi}{4}\right)}{1-(2n-1)\tan\left(x_1 - \frac{\pi}{4}\right)} = \frac{(f(1)+1)n-1}{f(1)-n(f(1)-1)},$$

with $a := \tan(x_1) = f(1) \notin \{\frac{n}{n-1} : n \in \mathbb{N}, n \geq 2\}$.

For $M = \mathbb{N}$, if $f(1) \geq 2$, then $f(3) = \frac{3f(1)+2}{3-2f(1)} < 0$, which is not in $\mathbb{N}$, showing that $f(1) = 1$ and the solution is $f(n) = 2n-1$ which is easily checked to be the only solution. However, for $M = \mathbb{Q}$, the above calculation continues to hold yielding

$$f(n) = \frac{(a+1)n-1}{a-n(a-1)},$$

where $a := f(1) \in \mathbb{Q} \setminus \{\frac{n}{n-1} : n \in \mathbb{N}, n \geq 2\}$, which gives infinitely many solutions.

There are certain other functional equations which can be transformed into the Cauchy functional equation (C_1).Three most important such functional equations are

$$f(xy) = f(x)f(y) \qquad (C_2)$$

$$f(x+y) = f(x)f(y) \qquad (C_3)$$

$$f(xy) = f(x) + f(y). \qquad (C_4)$$

These four equations will be referred to as **the four versions of the classical Cauchy functional equation**.

To see that these three equations can be transformed to (C_1), let us first take logarithm in (C_3) which gives

$$\log f(x+y) = \log f(x) + \log f(y).$$

This is the Cauchy functional equation in $\log f$. In another direction, if we put $f(e^u) = g(u)$ in (C_4), we get

$$g(u+v) = f(e^{u+v}) = f(e^u) + f(e^v) = g(u) + g(v),$$

which is the Cauchy functional equation in g. This last substitution also turns (C_2) into

$$g(u+v) = f(e^{u+v}) = f(e^u)\, f(e^v) = g(u)g(v),$$

which is the Cauchy type (C_3).

The three functional equations (C_2), (C_3) and (C_4) also appear naturally and separately in various scientific contexts. Under suitable assumptions, typical solutions of (C_2), (C_3) and (C_4) are, respectively,

$$f(x) = x^b,\ \ f(x) = e^{cx},\ \ f(x) = d\log x,$$

where b, c, d are arbitrary constants. Owing to such typical forms of solutions, they are often referred to as the *Cauchy power, the Cauchy exponential* and *the Cauchy logarithm* equations, respectively.

As observed in the case of the Cauchy functional equation (C_1), in general, solutions of the equations of Cauchy type are abundant, for example, through appropriate change of variables, uncountably many non-continuous solutions of (C_3) are e.g.

$$f_G(x) = \exp\left(R_1 G(h_1) + \cdots + R_n G(h_n)\right),$$

where $\log x = R_1 h_1 + \cdots + R_n h_n$ is the unique representation of $x \in \mathbb{R}^+$ with respect to the Hamel basis, H, and $G : H \to \mathbb{R}$.

In 1988, Jean Dhombres discovered inter-relations among the four forms of the Cauchy functional equation (C_1), (C_2), (C_3) and (C_4). One such relation is the s-independence defined as follows:

Definition. Let $(\alpha), (\beta)$ be two distinct equations taken from (C_1), (C_2), (C_3) and (C_4). The pair $\{(\alpha), (\beta)\}$ is ***s*-independent over** (X, Y) if the only common solution functions $f : X \to Y$ to (α) and (β) are either the zero function or the identity function.

In the case $X = Y$, we simply say $\{(\alpha), (\beta)\}$ are ***s*-independent over** X.

Dhombres, [5], stated the following results without proof:

- The pairs of equations $\{(C_1),(C_3)\}$ and $\{(C_2),(C_3)\}$ are s-independent over a ring.
- The pairs of equations $\{(C_1),(C_4)\}$ and $\{(C_2),(C_4)\}$ are s-independent over $(\mathbb{R}^+, \mathbb{R})$ and they are a fortiori s-independent over $\mathbb{R}$ or $\mathbb{C}$.

Another relation considered by Dhombres in the work is:

Definition. Let i, j be two distinct elements of $\{1, 2, 3, 4\}$.

1. The functional equations C_i and C_j are **alien relative to** (X, Y) if each solution function $f : X \to Y$ of the functional equation $C_i + C_j$ is also a solution of the system C_i and C_j.

2. The functional equations C_i and C_j are **weakly alien relative to** (X, Y) if each function $f : X \to Y$ which is a **non-constant** solution of $C_i + C_j$ is also a solution of the system C_i and C_j.

Dhombres proved the following results directly from solutions of the corresponding functional equations.

Theorem. Let X be ring divisible by 2 and Y a ring satisfying the following two properties: for each $y \in Y$,

1. if $y^3 = y$, then $y \in \{-1, 0, 1\}$ and
2. if $y^2 = 0$, then $y = 0$.

Then the functional equations C_1 and C_2 are alien relative to (X, Y).

Proposition. Let X be ring divisible by 2 and Y a ring. Let $f : X \to Y$ be such that $f(0) = 0$. Then the functional equations C_1 and C_2 are alien relative to (X, Y).

Theorem. Let X be a ring with unit and divisible by 2 and Y a field. The equation $\pm C_1, \pm C_2$ are weakly alien relative to (X, Y).

Theorem. Let X be a ring with unit and divisible by 2 and Y a skew-field. Let $k, l \in Y \setminus \{0\}$.

1. the equation kC_1 and lC_2 are weakly alien relative to (X, Y)
2. the equation kC_1 and lC_2 are alien relative to (X, Y) if and only if $k = l$.

To treat the remaining solutions of other types of the Cauchy functional equation, Dhombres resorted to solving the following functional equation

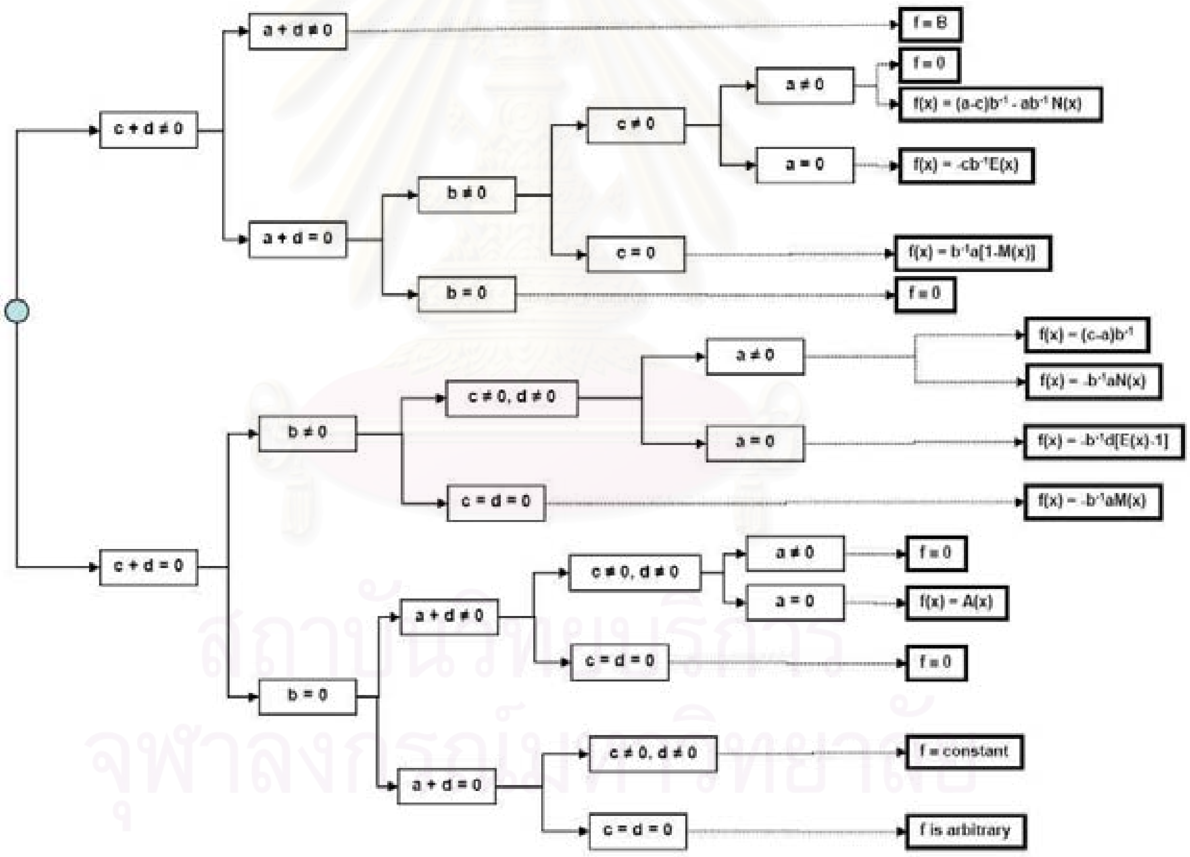
$$af(xy) + bf(x)f(y) + cf(x + y) + d(f(x) + f(y)) = 0,$$

which will be termed as a **universal Cauchy functional equation**. He considered the solution function f defined over a ring which is divisible by 2 and possesses a unit, while the values of f are in a skew-field. The constants a, b, c and d belong to this skew-field and commute with all elements of the skew-field.

Regarding solutions to the universal Cauchy functional eauation, Dhombres proved:

Theorem. Let X be a ring with unit divisible by 2 and Y a skew-field. Let a, b, c, d be elements of the center of Y. The following diagram, starting from the values of these four elements, gives all the solutions $f : X \to Y$ of the functional equation:

$$af(xy) + bf(x)f(y) + cf(x+y) + d\{f(x) + f(y)\} = 0.$$



In the diagram,

A denotes any solutions of $f(x+y) = f(x) + f(y)$,

M denotes any solutions of $f(xy) = f(x)f(y)$,

E denotes any solutions of $f(x+y) = f(x)f(y)$,

N denotes any solutions of $f(x+y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$.

Finally, B is an element of Y subject only to

$$(a + bB + c + 2d)B = 0.$$

Making use of the last theorem, Dhombres established the following dependence relations.

Proposition. Let X be a ring with unit and divisible by 2, Y a field with characteristic $\neq 2$. Then the equations C_1, C_2 and C_3 are pairwise alien.

Motivated by the work of Dhombres, we note that his analysis is based heavily on the fact that 0 belongs to the domain of the solution functions. This rules out the case of logarithmic function, whose domain of existence does not contain 0. Our first objective in this direction is to complement the work of Dhombres by re-investigating all his results mentioned above for solution functions whose domain is the set of positive real numbers, $\mathbb{R}^+$, and whose range is the set of complex numbers,$\mathbb{C}$. This should recover the missing case of the logarithmic function.

Our second objective deals with another aspect of the Cauchy functional equation. In 1999 and 2005, Konrad J. Heuvers and Palaniappan Kannappan considered the three functional equations:

$$f(x+y) - f(x) - f(y) = f\left(\frac{1}{x} + \frac{1}{y}\right), \tag{1.2}$$

$$f(x+y) - f(xy) = f\left(\frac{1}{x} + \frac{1}{y}\right), \tag{1.3}$$

$$f(xy) = f(x) + f(y), \tag{1.4}$$

when $f : \mathbb{R}^+ \to \mathbb{R}$. Heuvers, [7], proved:

Theorem. The functional equation (1.2) is equivalent to the logarithmic functional equation (1.4) for $f : \mathbb{R}^+ \to \mathbb{R}$ in the sense that a solution of one equation is also a solution of the other.

In a related work, they later proved:

Theorem. The functional equation (1.3) for $f : \mathbb{R}^+ \to \mathbb{R}$ is equivalent to the logarithmic equation (1.4).

Through our observation, Heuvers's original proof in [7] is incomplete. We aim to give a correct proof of this result as well as to solve a functional equation extending to the equation (1.2).

Unless stated otherwise, throughout the entire thesis our solution functions have $\mathbb{R}^+$ *as a domain and* $\mathbb{C}$ *as a range.*

We now briefly describe the contents of the thesis. In Chapter II, we investigate the ZI-independence relation, called s-independence by Dhombres, among the four types of the Cauchy functional equation.

In Chapter III, we introduce the definitions of alien and weakly alien relations and solve the universal Cauchy functional equation.

In Chapter IV, we investigate the alien and weakly alien relations among the four types of the Cauchy functional equation, together with multiples of those equations.

In Chapter V, we solve the functional equation $af(x+y) + bf(x) + cf(y) = df\left(1/x + 1/y\right)$.

Let us end this introduction with remarks about basic references consulted either implicitly or explicitly during the course of this research.

The book ON FUNCTIONS AND FUNCTIONAL EQUATIONS by Jaroslav Smital provides an elementary and easy-to-read introduction to the subject of functional equations.

The book FUNCTIONAL EQUATIONS: A Problem Solving Approach by B. J. Venkatachala is also elementary and deals mainly with solving functional equations appearing in various high-school competitions.

The book LECTURES ON FUNCTIONAL EQUATIONBS AND THEIR APPLICATIONS by J. Aczél is well-known as a classic to the subject whose modern version is the book FUNCTIONAL EQUATIONS IN SEVERAL VARIABLES by J. Aczél and J. Dhombres.

The book SOME ASPECTS OF FUNCTIONAL EQUATIONS by J. Dhombres treats different aspects of the Cauchy functional equation and its applications quite thoroughly.

The book A SHORT COURSE ON FUNCTIONAL EQUATIONS by J. Aczél contains interesting applications of the Cauchy functional equation in the social and behavioral sciences.

The book FUNCTIONAL EQUATIONS AND INEQUALITIES IN SEVERAL VARIABLES by S. Czerwik is more modern and deals mainly with functional equations in linear spaces.

CHAPTER II

ZI-INDEPENDENCE

We recall the four versions of the classical Cauchy functional equation as the followings:

$$f(x+y) = f(x) + f(y) \tag{C_1}$$

$$f(xy) = f(x)f(y) \tag{C_2}$$

$$f(x+y) = f(x)f(y) \tag{C_3}$$

$$f(xy) = f(x) + f(y) \tag{C_4}$$

The following independence notion was first introduced by Dhombres in [5]. There he used the word s-independence instead of ZI-independence.

Definition 2.0.1. Let $(\alpha), (\beta)$ be two distinct equations taken from (C_1), (C_2), (C_3) and (C_4). The pair $\{(\alpha), (\beta)\}$ are **ZI-independent over** (X, Y) if the only common solution functions $f : X \to Y$ to (α) and (β) are either the zero function or the identity function.

We complement Dhombres's work here by investigating ZI-independence among all four versions of the Cauchy functional equation for solution functions sending the positive real numbers into the complex field.

2.1 The Results

We first prove some auxiliary lemmas with less restriction on the domain and codomain.

Lemma 2.1.1. *Let X be a set, Y an integral domain and $f : X \to Y$. Then $f \equiv 0$ and $f \equiv 2$ are the only solutions of the functional equation*

$$f(x) + f(y) = f(x)f(y). \tag{2.1}$$

Proof. Putting $y = x$, we obtain

$$2f(x) = f(x)^2.$$

Thus, for each $x \in X$, either $f(x) = 0$ or $f(x) = 2$.

We proceed to show that either $f \equiv 0$ or $f \equiv 2$. Suppose that there exists x_0 such that $f(x_0) = 2$. Then,

$$f(x) + 2 = 2f(x) \quad (x \in X),$$

implying that $f \equiv 2$. □

Lemma 2.1.2. *Let Y be a set and $f : \mathbb{R}^+ \to Y$. If f satisfies*

$$f(xy) = f(x + y), \tag{2.2}$$

then f is a constant function.

Proof. Putting $y = 1$ in (2.2), we get

$$f(x) = f(x + 1).$$

Substituting $y+1$ for y in (2.2), we obtain,

$$f(xy+x) = f(x(y+1)) = f(x+(y+1)) = f(x+y+1) = f(x+y) = f(xy).$$

Let z, w be distinct elements. Then, by above equation,

$$\begin{aligned} f(z) &= f(w+(z-w)) \\ &= f\left((z-w)(z-w)^{-1}w + (z-w)\right) \\ &= f\left((z-w)(z-w)^{-1}w\right) \\ &= f(w). \end{aligned}$$

□

Our first two main results read:

Theorem 2.1.3. *The pairs* $\{(C_1),(C_3)\}, \{(C_2),(C_4)\}$ *and* $\{(C_1),(C_4)\}$ *are ZI-independent over* $(\mathbb{R}^+, \mathbb{C})$ *while* $\{(C_2),(C_3)\}$ *is not.*

Proof. Observe that both pairs of equation $\{(C_1),(C_3)\}$ and $\{(C_2),(C_4)\}$ lead to the equation (2.1). It thus follows from Lemma 2.1.1 that

$$f \equiv 0 \text{ or } f \equiv 2.$$

However, by direct checking, $f \equiv 2$ is not a solution of any of (C_1), (C_2), (C_3) or (C_4). Hence the pairs (C_1,C_3), and (C_2,C_4) are ZI-independent over $(\mathbb{R}^+, \mathbb{C})$.

Both pairs of equation $\{(C_2),(C_3)\}$ and $\{(C_1),(C_4)\}$ yield the equation (2.2). Hence, by Lemma 2.1.2, their solutions must be constant functions. Direct checking shows that

$$f \equiv 0 \text{ or } f \equiv 1$$

are the only solutions of the pair $\{(C_2),(C_3)\}$ and $f \equiv 0$ is the only solution of the pair $\{(C_1),(C_4)\}$. Thus, the pair $\{(C_1),(C_4)\}$ is ZI-independent over $(\mathbb{R}^+, \mathbb{C})$ while $\{(C_2),(C_3)\}$ are not. $\square$

Theorem 2.1.4. *The pair* $\{(C_3),(C_4)\}$ *is ZI-independent over* $(\mathbb{R}^+, \mathbb{C})$.

Proof. As is well-known, see e.g. [1] or [10], the equation (C_3) yields

$$f(q) = f(1)^q \text{ for all } q \in \mathbb{Q}^+.$$

Replacing x and y by 1 in (C_4), we obtain $f(1) = f(1) + f(1)$, which implies $f(1) = 0$ and so

$$f(q) = 0 \text{ for all } q \in \mathbb{Q}^+.$$

Let ζ be a positive irrational number. If $\zeta > 1$, then, by (C_3),

$$f(\zeta) = f(\zeta - 1 + 1) = f(\zeta - 1)f(1) = 0.$$

If $\zeta < 1$, then, by (C_4),

$$0 = f(1) = f\left(\zeta\frac{1}{\zeta}\right) = f(\zeta) + f\left(\frac{1}{\zeta}\right).$$

Using the above facts, we deduce $f(\zeta) = -f(\frac{1}{\zeta}) = 0$. Therefore, f is the zero function which implies the ZI-independence of the pair (C_3,C_4). $\square$

It is well-known that for a solution of (C_1) over $\mathbb{Q}$, there is a constant $c \in \mathbb{R}$ such that

$$f(x) = cx \quad (x \in \mathbb{Q}). \tag{2.3}$$

However, a general form of the solution to (C_1) over $\mathbb{R}$ is much more complex; see e.g.

Chapter 2 of [10]. Indeed, assuming the axiom of choice there are uncountably many non-continuous solution functions to the Cauchy functional equation (C_1), a fact proved in 1905 by Georg Hamel using Hamel bases. Following the work in Chapter 2 of [10], an example of such a class of functions satisfying (C_1) is given by

$$f_g(x) = r_1 g(h_1) + \cdots + r_n g(h_n),$$

where $H := \{h_i\}$ is a Hamel basis of $\mathbb{R}$, $x = r_1 h_1 + \cdots + r_n h_n$ ($r_i \in \mathbb{Q}$) is the unique representation of $x \in \mathbb{R}$ with respect to H, and g is any function defined over H. In the same manner, a particular class of uncountably many non-continuous functions satisfying (C_2) is given by

$$f_G(x) = \exp\left(R_1 G(h_1) + \cdots + R_n G(h_n)\right),$$

where $\log x = R_1 h_1 + \cdots + R_n h_n$ is the unique representation of $x \in \mathbb{R}^+$ with respect to the Hamel basis, H, and G is any function defined over H. It seems likely that there may be a number of common solutions to (C_1) and (C_2) and to get a meaningful result about their ZI-independence, some condition(s) may be necessary. To do so, we first note a simple lemma based on the following fact, Corollary 4 in page 15 of [3].

Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a solution of (C_1). If the image of f is not dense in $\mathbb{R}$, then $f(x) = cx$ for some constant c.

Lemma 2.1.5. *Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a solution of (C_1) and assume the image of f is not dense in $\mathbb{R}$.*

1. *If $f(1) = 1$, then f is the identity function.*

2. *If $f(1) = 0$, then f is the zero function.*

Proof. Using the fact just mentioned, we deduce $f(x) = cx$ for some constant c. The values $f(1) = 1$, respectively, $f(1) = 0$ yield $c = 1$, respectively, $c = 0$ $\square$

Our final result reads:

Theorem 2.1.6. *The pair of functional equations* $\{(C_1), (C_2)\}$ *is ZI-independent over* $(\mathbb{R}^+, K)$ *where* $\mathbb{C} \supset K = K_x + iK_y$ *and either* K_x *or* K_y *are non-dense subsets in* $\mathbb{R}$.

Proof. Let $f : \mathbb{R}^+ \to K$ be a function satisfies (C_1) and (C_2). Substituting x and y in (C_2) by 1, we obtain

$$f(1) = 0 \quad \text{or} \quad f(1) = 1.$$

If $f(1) = 0$, using (C_2), f is the zero function. Assume that $f(1) = 1$. In this case, we express

$$f(x) = u(x) + iv(x),$$

where u and v are real-valued functions on $\mathbb{R}^+$. Thus $1 = f(1) = u(1) + iv(1)$, which implies

$$u(1) = 1 \quad \text{and} \quad v(1) = 0. \tag{2.4}$$

Consequently,

$$u(q) = q \text{ and } v(q) = 0 \quad \text{for all } q \in \mathbb{Q}^+.$$

Since K_x or K_y are not dense in $\mathbb{R}$, the image of u or the image of v cannot be dense in $\mathbb{R}$. We consider each case seperately.

Case 1: the image of u is not dense in $\mathbb{R}$.

By Lemma 2.1.5 and (2.4), u is the identity function so $f(x) = x + iv(x)$. By (C_2),

$$\begin{aligned} xy + iv(xy) &= (x + iv(x))(y + iv(y)) \\ &= xy - v(x)v(y) + i\{xv(y) + yv(x)\}. \end{aligned}$$

Hence $v(x)v(y) = 0$ for all $x, y \in \mathbb{R}^+$. Consequently, $v \equiv 0$, which implies that f is the identity function.

Case 2: the image of v is not dense in $\mathbb{R}$.

By Lemma 2.1.5 and (2.4), v is the zero function. Hence f is a real-valued function. By (C_2), for each $x \in \mathbb{R}^+$,

$$f(x) = f((\sqrt{x})^2) = f(\sqrt{x})^2 \geq 0,$$

and hence the image of f is not dense in $\mathbb{R}$. Using Lemma 2.1.5, we obtain that f is the identity function. Therefore $\{(C_1),(C_2)\}$ are ZI-independent over $(\mathbb{R}^+, K)$. $\square$

CHAPTER III

THE UNIVERSAL CAUCHY FUNCTIONAL EQUATION

In this Chapter, the notions of alien and weakly alien relations among all solutions of the four versions of the Cauchy functional equation are extended. To do so, we first solve the functional equation

$$af(xy) + bf(x)f(y) + cf(x+y) + d(f(x) + f(y)) = 0,$$

which contains all the four forms of the classical Cauchy functional equation. This complements an earlier work of Dhombres in 1988 where the same functional equation was solved for solutions whose domains contain zero, which leaves out the logarithmic function. Here not only the logarithmic function is recovered but the analysis is entirely different and is based on solving appropriate difference equations.

3.1 Introduction

Recall that the four versions of the Cauchy functional equation are

$$f(x+y) = f(x) + f(y) \qquad (C_1)$$

$$f(xy) = f(x)f(y) \qquad (C_2)$$

$$f(x+y) = f(x)f(y) \qquad (C_3)$$

$$f(xy) = f(x) + f(y) \qquad (C_4)$$

The following notions of dependence relations among the above four equations were first introduced in [5].

Definition 3.1.1. Let i, j be two distinct elements of $\{1, 2, 3, 4\}$.

1. The functional equations C_i and C_j are **alien relative to** (X, Y) if each solution function $f : X \to Y$ of the functional equation $C_i + C_j$ is also a solution of the system C_i and C_j.

2. The functional equations C_i and C_j are **weakly alien relative to** (X, Y) if each function $f : X \to Y$ which is a **non-constant** solution of $C_i + C_j$ is also a solution of the system C_i and C_j.

Such relations will be investigated among all solution functions, sending the set of positive reals into the complex field, of the four versions of the Cauchy functional equation. Instead of directly checking all possible solutions from each type of the above four Cauchy functional equations, we encompass them into one single functional equation, referred to as a **universal Cauchy functional equation**,

$$af(xy) + bf(x)f(y) + cf(x + y) + d\{f(x) + f(y)\} = 0, \tag{3.1}$$

where a, b, c, d are four parameters belonging to the range of the solution functions. The four forms of the classical Cauchy functional equation correspond, respectively, to:

1. $a = b = 0$ and $c = 1 = -d$.

   Then (3.1) is the classical Cauchy functional equation in additive form

$$f(x + y) = f(x) + f(y), \tag{3.2}$$

and we denote a general solution of (3.2) by A.

2. $a = 1 = -b$ and $c = d = 0$.

   Then (3.1) is the classical Cauchy functional equation in multiplicative form

$$f(xy) = f(x)f(y), \tag{3.3}$$

   and we denote a general solution of (3.3) by M.

3. $a = 1 = -d$ and $b = c = 0$.

   Then (3.1) is the classical Cauchy functional equation in the cross multiplicative-additive form

$$f(xy) = f(x) + f(y), \tag{3.4}$$

   and we denote a general solution of (3.4) by L which includes the logarithmic function.

4. $c = 1 = -b$ and $a = d = 0$.

   Then (3.1) is the classical Cauchy functional equation in the cross additive-multiplicative form

$$f(x + y) = f(x)f(y), \tag{3.5}$$

   and we denote a general solution of (3.5) by E.

The universal Cauchy functional equation (3.1) was first treated in 1988 by Dhombres, [5], in his investigation of the inter-relations among the four forms of the classical Cauchy functional equation. Dhombres solved (3.1) for a solution function defined over a ring, which is divisible by 2 possessing a unit, and its range is contained in a skew-field containing the parameters a, b, c, d which also commute with all elements of the skew-field. The method of Dhombres is elementary in nature but several crucial uses are made of the presence of 0 in its domain which inevitably leaves out the important logarithmic function from a possible solution. In order to recover the logarithmic function, here we take instead the positive reals, $\mathbb{R}^+$, as our domain of solution functions and by so doing Dhombres's technique is no longer valid.

Our approach is first to substitute $y = 1$ into the functional equation (3.1) which turns it into a first-order difference equation, in the variable x, with constant coefficients. The case where $c = 0$ is particularly easy to solve without having to resort to the method of difference equations and is dealt with in the next section. If $c \neq 0$, we split the proof into two parts, called the first sub-case of $c \neq 0$ and the second sub-case of $c \neq 0$. Both parts are solved by strategically treating appropriate segregating conditions. The first part, corresponding to the condition $a + bf(1) + d = 0$, is solved by elementary means. The second part, which is hardest among the three modes of attack, is solved through the technique of difference equations. Since solutions of such difference equations generally involve periodic functions of period 1, we need to impose a natural restriction that the solution function is finite-valued over the unit interval. In the course of the proof, whenever the functional equation is reduced to any one of the four forms of Cauchy functional equation mentioned above, the solution will be denoted by the respective symbols A, M, L or E. This is in conformity with the customary practice because without appropriate restrictions, solutions of these four forms of Cauchy functional equation are quite numerous, see e.g. pp. 35-38 of [1] or Section 2.2 of [3]. Our main result is:

Theorem 3.1.2. *Let $a, b, c, d \in \mathbb{C}$; A denote a solution of* (3.2)*, M denote a solution* (3.3)*, L denote a solution of* (3.4) *and E denote a solution of* (3.5)*. If $f : \mathbb{R}^+ \to \mathbb{C}$ satisfies the universal Cauchy functional equation*

$$af(xy) + bf(x)f(y) + cf(x + y) + d\{f(x) + f(y)\} = 0, \tag{3.6}$$

then all possible solution functions are displayed in Figure 3.1, Figure 3.2 and Figure 3.3, respectively.

3.2 Case $c = 0$

If $c = 0$, the equation (3.6) is particularly easy to solve without having to resort to the method of difference equations and we deal with it first.

Theorem 3.2.1. *Let* $a, b, d \in \mathbb{C}$ *;* M *denote a solution* (3.3)*, and let* L *denote a solution of* (3.4)*. If* $f : \mathbb{R}^+ \to \mathbb{C}$ *satisfies the functional equation*

$$af(xy) + bf(x)f(y) + d\{f(x) + f(y)\} = 0, \tag{3.7}$$

then all possible solution functions are displayed in Figure 3.1.

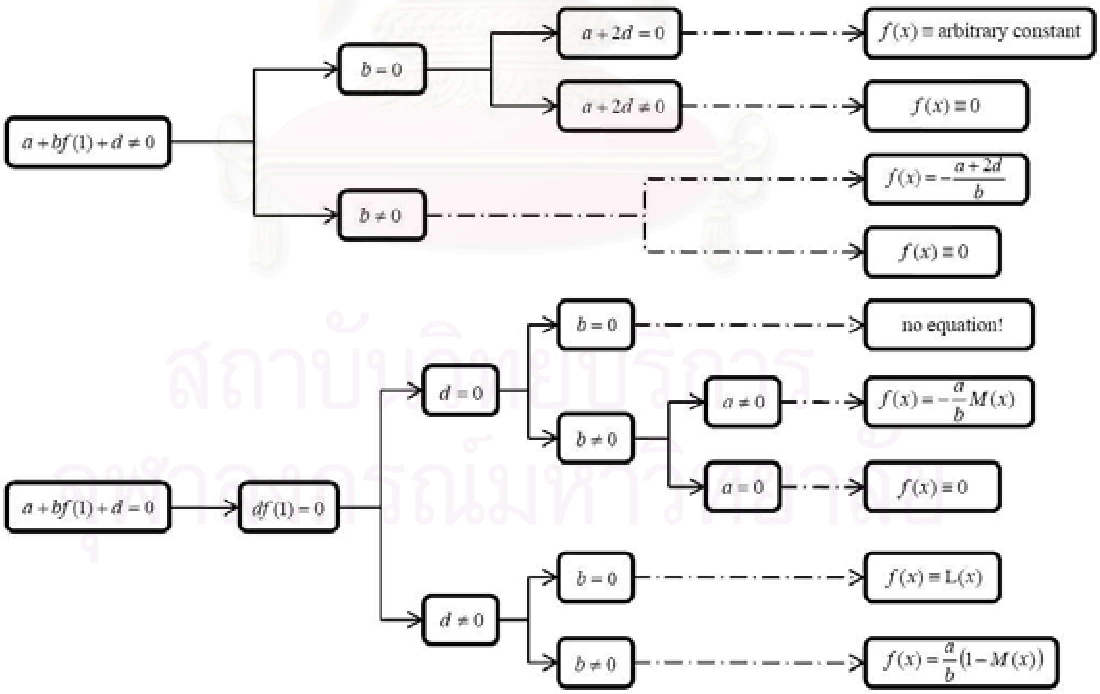


Figure 3.1: Case $c = 0$

Proof. Putting $y = 1$ into (3.7), we get

$$\begin{aligned} 0 &= af(x) + bf(x)f(1) + d\{f(x) + f(1)\} \\ &= \{a + bf(1) + d\}\, f(x) + df(1). \end{aligned} \tag{3.8}$$

We distinguish two separate cases corresponding to $a + bf(1) + d \neq 0$ or otherwise.

Case 1. $a + bf(1) + d \neq 0$.

Here, (3.8) immediately yields a constant solution, namely,

$$f(x) = \frac{-df(1)}{a + bf(1) + d} = \text{constant}.$$

To distinguish what a constant solution looks like, putting $f \equiv$ constant k into (3.7), we get

$$k(a + bk + 2d) = 0.$$

If $b = 0$ and $a + 2d = 0$, then f is arbitrary. If $b = 0$ but $a + 2d \neq 0$, then f is the zero function. If $b \neq 0$, then f is the zero function or $f(x) = \frac{-(a+2d)}{b}$.

Case 2. $a + bf(1) + d = 0$.

Here, (3.8) yields

$$df(1) = 0,$$

and there are two sub-cases.

- Sub-case 2A : $d = 0$.

If $b = 0$, then $a = 0$, and so we have no functional equation.

If $b \neq 0$ and $a \neq 0$, then the starting equation (3.7) becomes

$$af(xy) + bf(x)f(y) = 0,$$

equivalently,

$$-\frac{b}{a}f(xy) = \left\{-\frac{b}{a}f(x)\right\}\left\{-\frac{b}{a}f(y)\right\}.$$

Referring to (3.3), we infer that f is of the form

$$f(x) = -\frac{a}{b}M(x).$$

If $b \neq 0$ but $a = 0$, then the starting equation (3.7) is

$$bf(x)f(y) = 0$$

yielding f as the zero function.

- Sub-case 2B: $d \neq 0$. This gives $f(1) = 0$ and so $a + d = 0$.

If $b = 0$, then the original equation (3.7) becomes

$$af(xy) - a\{f(x) + f(y)\} = 0.$$

Since $a = -d \neq 0$, we get

$$f(xy) = f(x) + f(y)$$

yielding as solution

$$f \equiv L.$$

If $b \neq 0$, the starting equation (3.7) now reads

$$af(xy) + bf(x)f(y) - a\{f(x) + f(y)\} = 0,$$

which is equivalent to

$$1-\frac{b}{a}f(xy)=\left\{1-\frac{b}{a}f(x)\right\}\left\{1-\frac{b}{a}f(y)\right\}.$$

Referring to (3.3), we infer that

$$f(x)=\frac{a}{b}\left\{1-M(x)\right\}.$$

The solution functions displayed in Figure 3.1 are confirmed by direct checking. $\square$

3.3 The first sub-case of $c \neq 0$

In this section we solve the first sub-case of $c \neq 0$.

Theorem 3.3.1. *Let a, b, c ($\neq 0$), $d \in \mathbb{C}$. Assume that $f : \mathbb{R}^+ \to \mathbb{C}$ satisfies the universal Cauchy functional equation* (3.6). *If $a + bf(1) + d = 0$, then all possible solution functions are displayed in Figure 3.2, where $K = -\frac{df(1)}{c}$.*

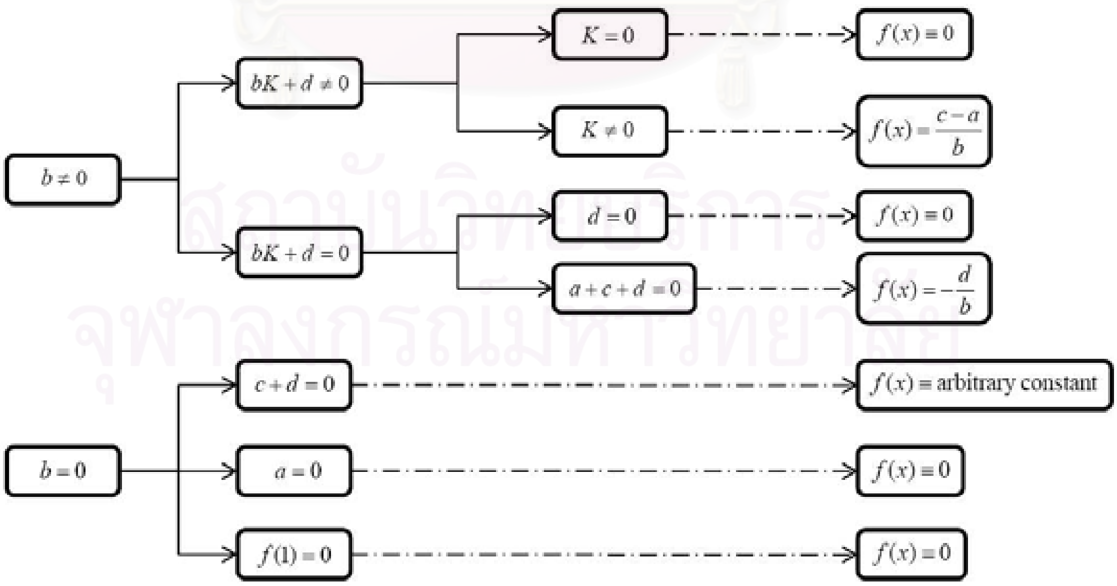


Figure 3.2: Case $c \neq 0$ and $a + bf(1) + d = 0$

Proof. Putting $y = 1$ into (3.6), we get

$$\begin{aligned} 0 &= \{a + bf(1) + d\} f(x) + cf(x+1) + df(1) \\ &= cf(x+1) + df(1), \end{aligned} \tag{3.9}$$

yielding, for all $x \in \mathbb{R}^+$,

$$f(x+1) = -\frac{df(1)}{c} =: K, \text{ say,} \tag{3.10}$$

with

$$bf(1) = -(a+d).$$

To determine the values of f over the unit interval, we divide into two cases corresponding to $b = 0$ or $b \neq 0$.

Case 1. $b \neq 0$. Then

$$f(1) = -\frac{a+d}{b}. \tag{3.11}$$

Putting $x \in (0,1)$ and $y = \frac{1}{x}$ in the original equation (3.6), we get

$$\begin{aligned} 0 &= af(1) + bf(x)f\left(\frac{1}{x}\right) + cf\left(x + \frac{1}{x}\right) + d\left\{f(x) + f(\frac{1}{x})\right\} \\ &= af(1) + bf(x)K + cK + d\{f(x) + K\}, \end{aligned}$$

and so

$$(bK + d)\, f(x) = -(af(1) + K(c+d)) \quad (x \in (0,1))\,. \tag{3.12}$$

We subdivide further into two sub-cases.

- **Sub-case 1.1.** $bK + d \neq 0$.

There are two more possibilities, $K = 0$ and $K \neq 0$.

$\odot$ Possibility 1: $K = 0$.

This gives $d \neq 0$ and the definition (3.10) of K shows that

$$f(x) = 0 \ \ (x > 1)\,.$$

Using again the definition (3.10) together with $d \neq 0$, we infer that

$$f(1) = 0 \text{ and so } f(x) = 0 \text{ for all } x \geq 1.$$

As for $x \in (0, 1)$, referring to (3.12) we have

$$df(x) = 0.$$

The solution function is then

$$f \equiv 0.$$

$\odot$ Possibility 2: $K \neq 0$.

Taking $x, y > 1$ in the original equation (3.6) and using the definition (3.10) of K, we get

$$K(a + bK + c + 2d) = 0.$$

Since $K \neq 0$ and $b \neq 0$, we deduce that

$$K = -\frac{a + c + 2d}{b}.$$

Equating $bK + d$ and using the value of $f(1)$ in (3.11), we get

$$-(a+c+d) = bK + d \quad (\neq 0)$$
$$= b(\frac{-df(1)}{c}) + d = \frac{d}{c}(a+d+c),$$

i.e.,

$$c + d = 0. \tag{3.13}$$

Using this last relation (3.13) and (3.10), we have

$$f(x) = K = \frac{-df(1)}{c} = f(1) \quad (x > 1)\,. \tag{3.14}$$

Observe also that (3.13) together with $a + c + d = -(bK + d) \neq 0$ show that $a \neq 0$. Taking $x \in (0, 1)$ and $y = \frac{1}{x}$ in the original equation (3.6), we have

$$af(1) + bf(x)f\left(\frac{1}{x}\right) + cf\left(x + \frac{1}{x}\right) + df(x) + df\left(\frac{1}{x}\right) = 0.$$

Taking (3.14) into account, we get

$$af(1) + bf(x)f(1) + cf(1) + df(x) + df(1) = 0 \qquad (x \in (0,1))\,. \tag{3.15}$$

Putting $x \in (0, 1)$ and $y = 1$ into the original equation (3.6), and also using (3.14), we have

$$af(x) + bf(x)f(1) + cf(1) + df(x) + df(1) = 0 \qquad (x \in (0,1))\,. \tag{3.16}$$

Solving (3.15) and (3.16) yields

$$a(f(1) - f(x)) = 0 \quad (x \in (0,1))\,.$$

Since $a \neq 0$, this implies

$$f(x) = f(1) \quad \text{for all } x \in (0,1).$$

The solution function is thus

$$f(x) = f(1) = -\frac{a+d}{b} = \frac{c-a}{b} \qquad \left(x \in \mathbb{R}^+\right).$$

• **Sub-case 1.2.** $bK + d = 0$.

Using the definition (3.10) of K and the value of $f(1)$ in (3.11), we get

$$0 = bK + d = b\left(\frac{-df(1)}{c}\right) + d = b\left(\frac{d}{c}\right)\left(\frac{a+d}{b}\right) + d = \frac{d}{c}\,(a+d+c)\,.$$

This last relation entails two possibilities

$$d = 0 \text{ or } a + d + c = 0.$$

⊙ Possibility 1: $d = 0$.

The equation (3.10) thus yields

$$f(x) = 0 \qquad (x > 1)\,. \tag{3.17}$$

Taking $x > 1, y = \frac{1}{x}$ in the original equation (3.6), using (3.17), $d = 0$ and the value

$f(1)$ in (3.11) give

$$0 = af(1) = a\left(-\frac{a}{b}\right) = -\frac{a^2}{b}, \text{ i.e., } a = 0.$$

The original equation (3.6) reduces to

$$bf(x)f(y) + cf(x+y) = 0,$$

or equivalently,

$$\frac{-b}{c}f(x+y) = \left(\frac{-b}{c}f(x)\right)\left(\frac{-b}{c}f(y)\right).$$

Referring to (3.5), we infer that the solution function is

$$f(x) = \frac{-c}{b}E(x).$$

Since $a = 0 = d$, the relation (3.11) tells us that

$$0 = f(1) = \frac{-c}{b}E(1),$$

i.e.,

$$E(1) = 0.$$

Together with (3.17), we deduce that

$$E(x) = 0 \qquad (x \geq 1).$$

We turn our attention now to the open unit interval. For each $x \in (0,1)$, since there exists $n \in \mathbb{N}$ such that $nx > 1$, using the additive-multiplicative equation of E we have

$$0 = E(nx) = E(x)^n, \tag{3.18}$$

i.e.,

$$E(x) = 0 \quad (x \in (0,1)).$$

The solution function is thus

$$f \equiv 0.$$

⊙ Possibility 2: $a + c + d = 0$.

We may assume without loss of generality that $d \neq 0$, for otherwise the analysis in Possibility 1 applies. The function value in (3.11) is

$$f(1) = -\frac{a+d}{b} = \frac{c}{b} \neq 0, \tag{3.19}$$

and so (3.10) becomes

$$f(x) = \frac{-df(1)}{c} = \frac{-d}{b} \qquad (x > 1). \tag{3.20}$$

Substituting $x \in (0,1), y = \frac{1}{x}$ into the original equation (3.6), we have

$$af(1) + bf(x)f\left(\frac{1}{x}\right) + cf\left(x + \frac{1}{x}\right) + d\left\{f(x) + f\left(\frac{1}{x}\right)\right\} = 0,$$

i.e.,

$$f(x)\left\{bf\left(\frac{1}{x}\right) + d\right\} = -\left\{af(1) + cf\left(x + \frac{1}{x}\right) + df\left(\frac{1}{x}\right)\right\}.$$

Taking (3.19) and (3.20) into account and simplifying, we get

$$0 = f(x)\left\{b\left(\frac{-d}{b}\right) + d\right\} = \frac{-ac + cd + d^2}{b} = \frac{a^2}{b}$$

yielding

$$a = 0 \text{ and so } c = -d.$$

The original equation (3.6) reduces to

$$bf(x)f(y) - df(x+y) + d\{f(x) + f(y)\} = 0, \tag{3.21}$$

which is equivalent to

$$\frac{b}{d}f(x+y) + 1 = \left(\frac{b}{d}f(x) + 1\right)\left(\frac{b}{d}f(y) + 1\right).$$

Referring to the additive-multiplicative form of the Cauchy function equation (3.5), we deduce that

$$f(x) = \frac{d}{b}\left(E(x) - 1\right).$$

Using (3.20), we get

$$-\frac{d}{b} = f(x) = \frac{d}{b}\left(E(x) - 1\right), \qquad (x > 1)$$

which implies that

$$E(x) = 0 \;\; (x > 1)\,.$$

Similarly, using (3.19) leads to

$$E(1) = 0.$$

By the same arguments as in (3.18), we must have $E \equiv 0$ and we conclude that the solution function is

$$f(x) \equiv -\frac{d}{b} \quad (x \in \mathbb{R}^+).$$

Case 2. $b = 0$. Then

$$0 = a + bf(1) + d = a + d \tag{3.22}$$

and the original equation (3.6) becomes

$$af(xy) + cf(x+y) - a\{f(x) + f(y)\} = 0. \tag{3.23}$$

Substituting $y = 1$ gives $f(x+1) = \frac{af(1)}{c}$, i.e.,

$$f(x) = \frac{af(1)}{c} \qquad (x > 1). \tag{3.24}$$

Putting $x \in (0,1), y = \frac{1}{x} > 1$, into (3.23) and using (3.24) yield

$$af(x) = af(1)\left(2 - \frac{a}{c}\right). \tag{3.25}$$

Putting $y = x > 1$ into (3.23) and using (3.24) give

$$0 = \frac{af(1)}{c}(a + c - 2a) = \frac{af(1)}{c}(c + d).$$

This leaves us three possibilities corresponding to

$$c + d = 0, a = 0 \text{ and } f(1) = 0.$$

⊙ Possibility 1: $c + d = 0$.

Then $0 \neq c = -d = a$ and (3.24) yields

$$f(x) = f(1) \qquad (x > 1).$$

However, since $a \neq 0$ (for otherwise we are in Possibility 2), (3.25) shows that $f(x) = f(1) \quad (x < 1)$. The solution function is thus

$$f \equiv \text{arbitrary constant.}$$

⊙ Possibility 2: $a = 0$.

Thus (3.22) yields $d = 0$ and the original equation (3.6) reduces considerably to $cf(x+y) = 0$, showing that the solution function is $f \equiv 0$.

⊙ Possibility 3: $f(1) = 0$.

The relations (3.24) and (3.25) lead immediately to the solution function $f \equiv 0$. The results in Figure 3.2 is confirmed by directly checking all the solution functions found. □

3.4 The second sub-case of $c \neq 0$

We finally come to the second sub-case of $c \neq 0$, which is the hardest of the three.

Theorem 3.4.1. *Let a, b, c $(\neq 0)$, $d \in \mathbb{C}$; let A denote a solution of* (3.2)*, and let E denote a solution of* (3.5)*. Assume that $f : \mathbb{R}^+ \to \mathbb{C}$ is a finite-valued function over* $(0, 1]$ *and satisfies the universal Cauchy functional equation* (3.6)*. If $a + bf(1) + d \neq 0$, then all possible solution functions are displayed in Figure 3.3, where*

$$\omega(1) = -\frac{cf(1)(bf(1) + c + 2d)}{(bf(1) + d)(bf(1) + c + d)}.$$

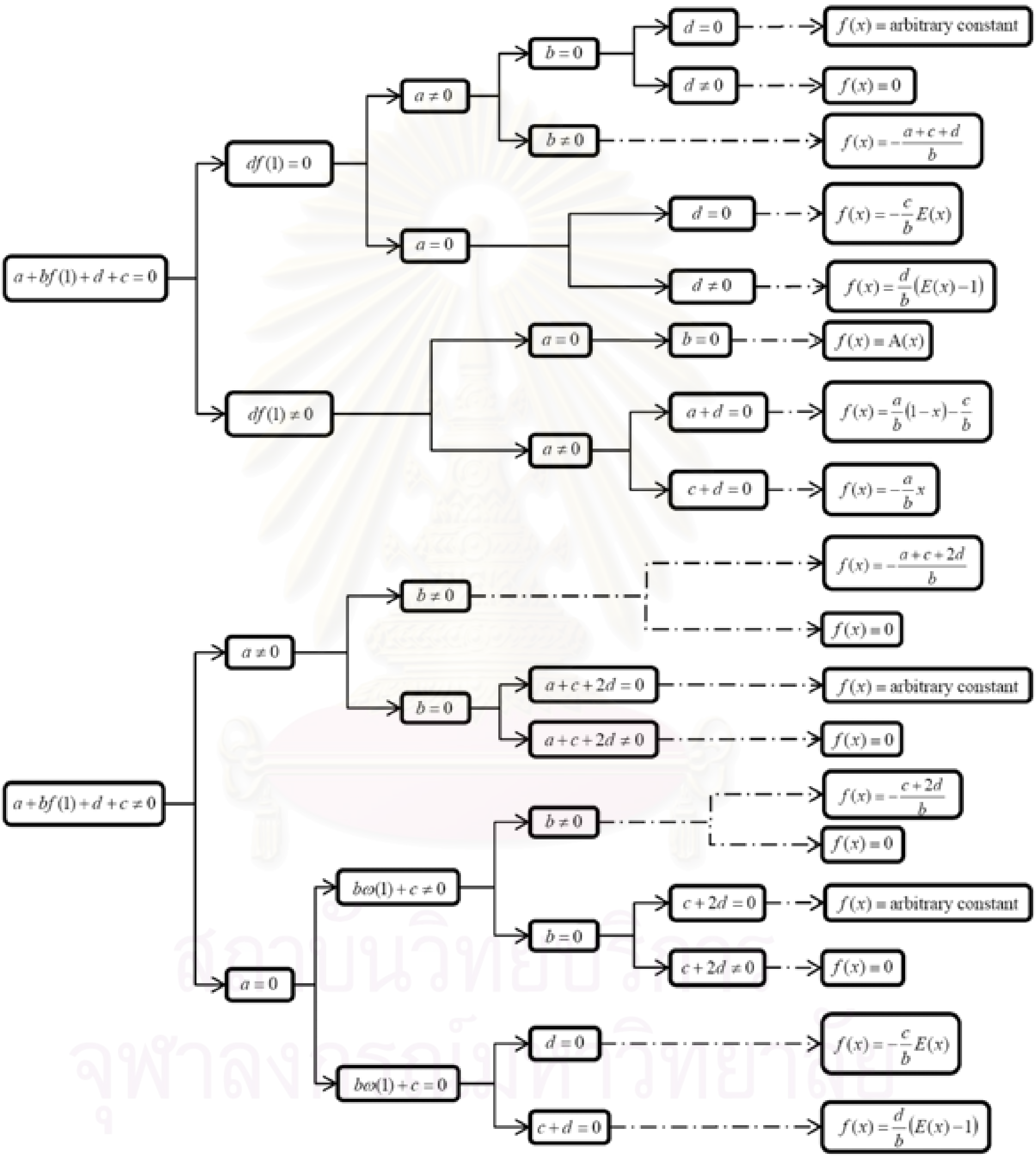


Figure 3.3: Case $c \neq 0$ and $a + bf(1) + d \neq 0$

Proof. Putting $y = 1$ into (3.6), we get

$$\{a + bf(1) + d\} f(x) + cf(x+1) = -df(1), \tag{3.26}$$

We now treat two distinct cases corresponding to $a+bf(1)+d+c = 0$ and otherwise.

Case 1: $a + bf(1) + d + c = 0$.

Thus $a+bf(1)+d = -c$, and (3.26) leads to a non-homogeneous first order difference equation with constant coefficients

$$cf(x+1) - cf(x) = -df(1). \tag{3.27}$$

The general solution of (3.27) is of the form, see e.g. [9],

$$f(x) = \omega(x) - \frac{df(1)}{c}x, \tag{3.28}$$

where $\omega(x)$ denotes a periodic function of period 1. We sub-divide into two sub-cases corresponding to $df(1) = 0$ or otherwise.

- **Sub-case 1.** $df(1) = 0$.

Here, $f(x) = \omega(x)$ and substituting this shape of $f(x)$ into the original equation (3.6) yields

$$a\omega(xy) + b\omega(x)\omega(y) + c\omega(x+y) + d\{\omega(x) + \omega(y)\} = 0. \tag{3.29}$$

Replacing x by $x+1$ in (3.29) and using the periodicity of ω lead to

$$0 = a\omega(xy+y) + b\omega(x)\omega(y) + c\omega(x+y) + d\{\omega(x) + \omega(y)\}. \tag{3.30}$$

Comparing (3.29) and (3.30) yields

$$a\{\omega(xy+y)-\omega(xy)\}=0.$$

If $a \neq 0$, then

$$\omega(xy+y)=\omega(xy).$$

Since x and y are arbitrary, we deduce that

$$\omega(z+y)=\omega(z) \text{ for all } z,y\in\mathbb{R}^+$$

and this forces $\omega(x)$ to be constant and so is $f(x)$. To determine this constant, putting $f \equiv k$ into (3.29), we get

$$k(a+bk+c+2d)=0.$$

For $b=0$, from the defining relation of Case 1, i.e., $a+bf(1)+d+c=0$, we get

$$a+d+c=0$$

This together with the relation just found, we deduce

$$kd=0.$$

If $d=0$, then f is an arbitrary constant. If $d\neq 0$, then f is the zero function.

For $b\neq 0$, from $a+bf(1)+d+c=0$, we deduce that

$$f(1)=\frac{-(a+c+d)}{b}.$$

If $a = 0$, from $df(1) = 0$, there are two possibilities

$$d = 0 \text{ or } d \neq 0.$$

⊙ Possibility 1: $d = 0$.

The equation (3.29) becomes

$$b\omega(x)\omega(y) + c\omega(x+y) = 0.$$

Note that $b \neq 0$ by the relation defining Case 1. Thus,

$$-\frac{b}{c}\,\omega(x+y) = \left(-\frac{b}{c}\,\omega(x)\right)\left(-\frac{b}{c}\,\omega(y)\right).$$

Appealing to (3.5), the solution function is

$$f(x) = \omega(x) = -\frac{c}{b}\,E(x), \tag{3.31}$$

where here $E(x)$ must also be periodic of period 1.

⊙ Possibility 2: $d \neq 0$. Thus

$$f(1) = 0.$$

From the identification of Case 1, we have

$$0 = a + bf(1) + c + d = c + d.$$

Here $b \neq 0$ by the same reasoning as in Possibility 1. The equation (3.29) becomes

$$b\omega(x)\omega(y) - d\omega(x+y) + d\{\omega(x) + \omega(y)\} = 0,$$

which is of the same form as (3.21) in the proof of Theorem 3.3.1 and the same analysis leads to

$$f(x) = \omega(x) = \frac{d}{b}(E(x) - 1). \tag{3.32}$$

Checking the solution functions (3.31) and (3.32) shows that the restriction of being periodic with period 1 for $E(x)$ can be discarded.

- **Sub-case 2.** $df(1) \neq 0$.

Substituting $f(x)$ from (3.28) into the original equation (3.6) and simplifying give

$$0 = a\omega(xy) - \frac{adf(1)}{c}xy + b\omega(x)\omega(y) - \frac{bdf(1)}{c}x\omega(y) - \frac{bdf(1)}{c}y\omega(x) + b\left(\frac{df(1)}{c}\right)^2 xy$$
$$+ c\omega(x+y) - df(1)(x+y) + d\omega(x) + d\omega(y) - \frac{d^2f(1)}{c}(x+y).$$

Keeping y fixed for the time being, dividing this last relation by x, letting $x \to \infty$ and using the finite-value assumption of the solution function, we get

$$-\frac{adf(1)}{c}y - \frac{bdf(1)}{c}\omega(y) + b\left(\frac{df(1)}{c}\right)^2 y - df(1) - \frac{d^2f(1)}{c} = 0. \tag{3.33}$$

Now dividing by y and letting $y \to \infty$, we arrive at

$$-\frac{adf(1)}{c} + b\left(\frac{df(1)}{c}\right)^2 = 0.$$

Since $df(1)/c \neq 0$, we get

$$a = \frac{bdf(1)}{c}. \tag{3.34}$$

If $a = 0$, then $b = 0$. The defining condition of Case 1 then implies $c = -d$ which turns the original equation (3.6) into (3.2). The solution function is thus

$$f(x) = A(x).$$

If $a \neq 0$, substituting (3.34) into (3.33) and simplifying, we get

$$0 = -\frac{df(1)}{c}\left\{b\omega(y) + c + d\right\},$$

and so

$$b\omega(y) + c + d = 0.$$

Since $a \neq 0$, the relation (3.34) shows that

$$b \neq 0.$$

Thus

$$\omega(y) = -\frac{c+d}{b},$$

and (3.28) implies that

$$f(x) = -\frac{c+d}{b} - \frac{df(1)}{c}\,x = -\frac{1}{b}(ax + c + d). \tag{3.35}$$

Substituting this shape of f into the original equation (3.6) leads to

$$\begin{aligned} 0 &= a\left\{-\frac{1}{b}(axy + c + d)\right\} + b\left\{-\frac{1}{b}(ax + c + d)\right\}\left\{-\frac{1}{b}(ay + c + d)\right\} \\ &\quad + c\left\{-\frac{1}{b}(a(x+y) + c + d)\right\} + d\left\{-\frac{1}{b}(ax + c + d)\right\} + d\left\{-\frac{1}{b}(ay + c + d)\right\} \\ &= -\frac{1}{b}(a+d)(c+d), \end{aligned}$$

which necessitates that the solution function is of form (3.35) if and only if $a + d = 0$ or $c + d = 0$.

If $a+d=0$, then the solution function is

$$f(x) = -\frac{1}{b}(ax+c+d) = \frac{a}{b}(1-x) - \frac{c}{b}.$$

If $c+d=0$, then the solution function is

$$f(x) = -\frac{1}{b}(ax+c+d) = -\frac{a}{b}x.$$

Case 2: $a+bf(1)+d+c \neq 0$.

The general solution of (3.26) is of the form

$$f(x) = \omega(x)\left(-\frac{a+bf(1)+d}{c}\right)^x - \frac{df(1)}{a+bf(1)+d+c} := \omega(x)P^x - \frac{df(1)}{c(1-P)}, \quad (3.36)$$

where, by the hypothesis of the theorem,

$$P = -\frac{a+bf(1)+d}{c} \neq 0. \quad (3.37)$$

Since $a+bf(1)+d+c \neq 0$, we see that $P \neq 1$. Substituting this shape of f into the original equation (3.6) and simplifying, we arrive at

$$\begin{aligned}0 = {} & a\omega(xy)P^{xy} - a\frac{df(1)}{c(1-P)} + b\omega(x)\omega(y)P^{x+y} - b\frac{df(1)}{c(1-P)}\{\omega(x)P^x + \omega(y)P^y\} \\ & + b\left(\frac{df(1)}{c(1-P)}\right)^2 + c\omega(x+y)P^{x+y} - c\frac{df(1)}{c(1-P)} + d\omega(x)P^x \\ & - d\frac{df(1)}{c(1-P)} + d\omega(y)P^y - d\frac{df(1)}{c(1-P)}.\end{aligned}$$

Separating the constant, called α for short, and the variable parts, we get

$$\begin{aligned}\alpha &:= \frac{df(1)}{c(1-P)}\left\{a - \frac{bdf(1)}{c(1-P)} + c + 2d\right\} \qquad (3.38)\\ &= a\omega(xy)P^{xy} + b\omega(x)\omega(y)P^{x+y} - b\frac{df(1)}{c(1-P)}(\omega(x)P^x + \omega(y)P^y)\\ &+ c\omega(x+y)P^{x+y} + d\omega(x)P^x + d\omega(y)P^y.\end{aligned}$$

Now we distinguish two separate sub-cases corresponding to $a = 0$ or otherwise.

- **Sub-case 1.** $a \neq 0$.

Putting $y = 2$ into (3.38) and using the periodicity of ω, we have

$$\begin{aligned}\alpha &= a\omega(2x)P^{2x} + b\omega(x)\omega(2)P^{x+2} - b\frac{df(1)}{c(1-P)}\left\{\omega(x)P^x + \omega(2)P^2\right\}\\ &+ c\omega(x+2)P^{x+2} + d\omega(x)P^x + d\omega(2)P^2.\\ &= a\omega(2x)P^{2x} + b\omega(x)\omega(1)P^{x+2} - b\frac{df(1)}{c(1-P)}\left\{\omega(x)P^x + \omega(1)P^2\right\}\\ &+ c\omega(x)P^{x+2} + d\omega(x)P^x + d\omega(1)P^2.\end{aligned}$$

Again separating the constant, called β for short, and the variable parts of this last relation, we get

$$\begin{aligned}\beta &:= \alpha + b\frac{df(1)}{c(1-P)}\omega(1)P^2 - d\omega(1)P^2 \qquad (3.39)\\ &= P^x\left(a\omega(2x)P^x + b\omega(x)\omega(1)P^2 - b\frac{df(1)}{c(1-P)}\omega(x) + c\omega(x)P^2 + d\omega(x)\right)\\ &:= P^xF(x),\end{aligned}$$

where

$$F(x) := a\omega(2x)P^x + b\omega(x)\omega(1)P^2 - b\frac{df(1)}{c(1-P)}\omega(x) + c\omega(x)P^2 + d\omega(x). \qquad (3.40)$$

Observe that (3.39) holds for any $x \in \mathbb{R}^+$ and so $P^{x+1}F(x+1) = \beta$ implying that

$$PF(x+1) = F(x). \tag{3.41}$$

From the definition (3.40) and the periodicity of ω, we get

$$\begin{aligned} F(x+1) &= a\omega(2(x+1))P^{x+1} + b\omega(x+1)\omega(1)P^2 - b\frac{df(1)}{c(1-P)}\omega(x+1) \\ &\quad + c\omega(x+1)P^2 + d\omega(x+1) \\ &= F(x) + a\omega(2x)P^x(P-1). \end{aligned}$$

This together with (3.41) show that

$$F(x) = PF(x+1) = P\left\{F(x) + a\omega(2x)P^x(P-1)\right\},$$

i.e.,

$$(1-P)F(x) = a\omega(2x)P^{x+1}(P-1).$$

Since $1 - P \neq 0$, we deduce that

$$F(x) = -a\omega(2x)P^{x+1}.$$

Going back to (3.39), we see that

$$\beta = P^x F(x) = -a\omega(2x)P^{2x+1},$$

i.e.,

$$\omega(2x)P^{2x} = -\frac{\beta}{aP}$$

implying that

$$\omega(z)P^z \equiv \text{constant} \qquad \left(z \in \mathbb{R}^+\right).$$

The solution function, (3.36), is thus

$$f(x) = \omega(x)P^x - \frac{df(1)}{(c(1-P)} \equiv \text{constant}.$$

To determine this constant, putting $f \equiv k$ into (3.26), we have

$$k(a + bk + c + 2d) = 0.$$

If $b = 0$ and $a + c + 2d = 0$, then f is an arbitrary constant. If $b = 0$ but $a + c + 2d \neq 0$, then f is the zero function.

If $b \neq 0$, then $f(x)$ is the zero function or

$$f(x) = \frac{-(a + c + 2d)}{b}.$$

- **Sub-case 2.** $a = 0$.

Then (3.38) becomes

$$\begin{aligned}\alpha &= \frac{df(1)}{c(1-P)}\left\{-\frac{bdf(1)}{c(1-P)} + c + 2d\right\} \\ &= b\omega(x)\omega(y)P^{x+y} - b\frac{df(1)}{c(1-P)}\left\{\omega(x)P^x + \omega(y)P^y\right\} + c\omega(x+y)P^{x+y} + d\omega(x)P^x + d\omega(y)P^y.\end{aligned}$$

Rearranging and using the definition of P in (3.37) we get

$$P^xP^y\left\{b\omega(x)\omega(y) + c\omega(x+y)\right\} + \frac{d(c+d)}{c(1-P)}\left\{P^x\omega(x) + P^y\omega(y)\right\} = \alpha. \tag{3.42}$$

Now we consider the possibility whether $|P| = 1$.

●● Possibility 1: $|P| = 1$.

Replacing $y = 1$ in (3.42) and using the periodicity of ω, we have

$$\alpha = P^x P \left\{ b\omega(x)\omega(1) + c\omega(x) \right\} + \frac{d(c+d)}{c(1-P)} \left\{ P^x \omega(x) + P\omega(1) \right\}. \tag{3.43}$$

Replacing $y = M \in \mathbb{N} \setminus \{1\}$ in (3.42) and using the periodicity of ω, we get

$$\alpha = P^x P^M \left\{ b\omega(x)\omega(1) + c\omega(x) \right\} + \frac{d(c+d)}{c(1-P)} \left\{ P^x \omega(x) + P^M \omega(1) \right\}. \tag{3.44}$$

Equating (3.43) and (3.44), we have

$$P^x \omega(x)(b\omega(1) + c)(P - P^M) = \frac{d(c+d)}{c(1-P)} \omega(1)(P^M - P). \tag{3.45}$$

Since $P \neq 1$, there must be an $M \in \mathbb{N}$ such that $P - P^M \neq 0$. Thus,

$$P^x \omega(x)(b\omega(1) + c) = -\frac{d(c+d)}{c(1-P)} \omega(1). \tag{3.46}$$

⊙ For $b\omega(1) + c \neq 0$, (3.46) shows that $P^x \omega(x)$ is a constant function. The solution function, (3.36), is thus

$$f \equiv \text{constant}.$$

To consider this constant, putting $f \equiv k$ into (3.26), we again have

$$k(bk + c + 2d) = 0.$$

If $b = 0$ and $c + 2d = 0$, then f is an arbitrary constant. If $b = 0$ but $c + 2d \neq 0$, then f is the zero function.

If $b \neq 0$, then $f(x)$ is the zero function or

$$f(x) = -\frac{c+2d}{b}.$$

$\odot$ For $b\omega(1) + c = 0$, (3.46) shows that

$$d(c+d)\,\omega(1) = 0.$$

Now, b and $\omega(1)$ are both nonzero, for otherwise the condition of this case entails $c = 0$, contradicting its definition. Thus,

$$d(c+d) = 0$$

implying that either

$$d = 0 \quad \text{or} \quad c + d = 0.$$

In any case, (3.43) implies that $\alpha = 0$ and (3.42) becomes

$$b\omega(x)\omega(y) + c\omega(x+y) = 0,$$

or equivalently,

$$-\frac{b}{c}\,\omega(x+y) = \left(-\frac{b}{c}\,\omega(x)\right)\left(-\frac{b}{c}\,\omega(y)\right).$$

Referring to (3.5), we have

$$\omega(x) = -\frac{c}{b}E(x).$$

If $d = 0$, then the solution function in (3.36) is

$$f(x) = \omega(x)P^x - \frac{df(1)}{c(1-P)} = -\frac{c}{b}E(x)P^x.$$

Since in this case, $P \neq 0$ is arbitrary, the solution function can be put under the form

$$f(x) = -\frac{c}{b}E(x),$$

where this last E is a generic symbol representating a general solution of (3.5).

If $c + d = 0$, the definition of P in (3.37) gives

$$\frac{f(1)}{c(1-P)} = \frac{1}{b}$$

and the solution function, (3.36), is thus

$$f(x) = \omega(x)P^x - \frac{df(1)}{c(1-P)} = -\frac{c}{b}E(x)P^x - \frac{d}{b}$$

and as in the last case, the solution function can be put under the form

$$f(x) = -\frac{c}{b}E(x) - \frac{d}{b} = \frac{d}{b}\left(E(x) - 1\right).$$

•• Possibility 2: $|P| \neq 1$.

Let $P = re^{i\theta}$. Substituting $y = 1$, respectively, $y = M \in \mathbb{N} \setminus \{1\}$ into (3.42) and using the periodicity of ω, we get (3.43), respectively,

$$\alpha = P^x r^M e^{i\theta M}\left\{b\omega(x)\omega(1) + c\omega(x)\right\} + \frac{d(c+d)}{c(1-P)}\left\{P^x\omega(x) + r^M e^{i\theta M}\omega(1)\right\}. \quad (3.47)$$

Equating these last two equations gives

$$P^x\omega(x)(b\omega(1) + c)(re^{i\theta} - r^M e^{i\theta M}) = \frac{d(c+d)}{c(1-P)}\omega(1)(r^M e^{i\theta M} - re^{i\theta}).$$

Since $M \neq 1$, we must have $re^{i\theta} - r^M e^{i\theta M} \neq 0$. This leads us to (3.46) of the last possibility and the same analysis is applicable leading to the same three forms of solution. The results in Figure 3.3 is confirmed by checking the solution functions found above. □

CHAPTER IV

DEPENDENCE RELATIONS

The notions of alien and weakly alien relations among all solutions of the four classical versions of the Cauchy functional equation are re-considered in this chapter, but are investigated here among all solution functions, sending the set of positive reals into the complex field, of the four forms of the Cauchy functional equation.

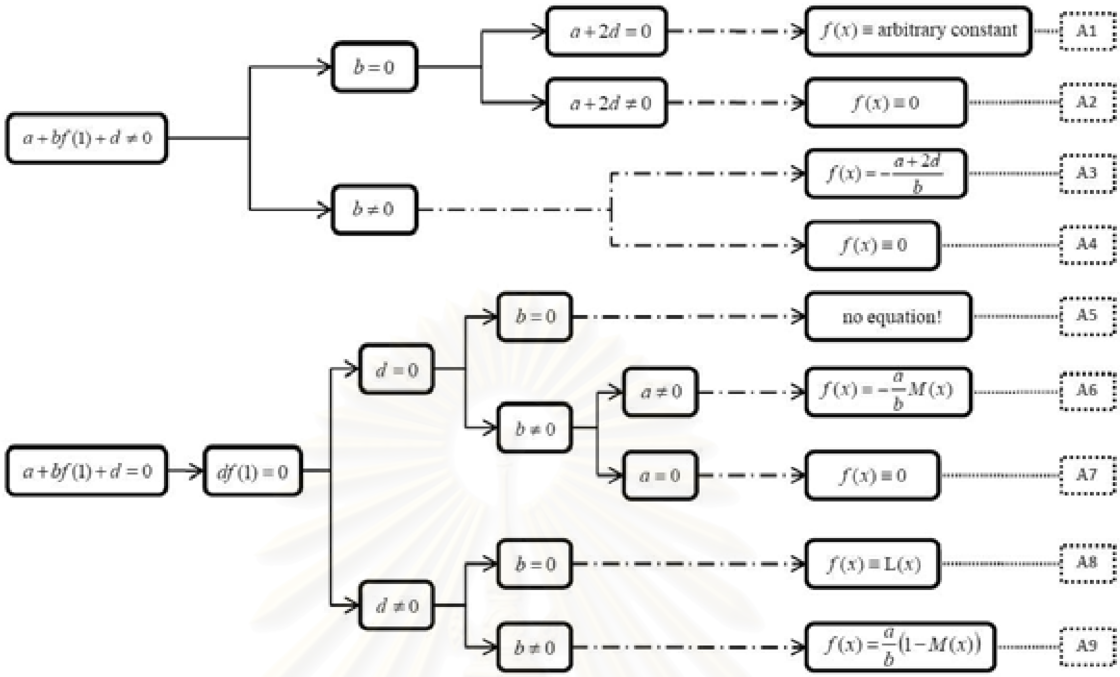
4.1 Known results

We recall the results stated in Chapter III:

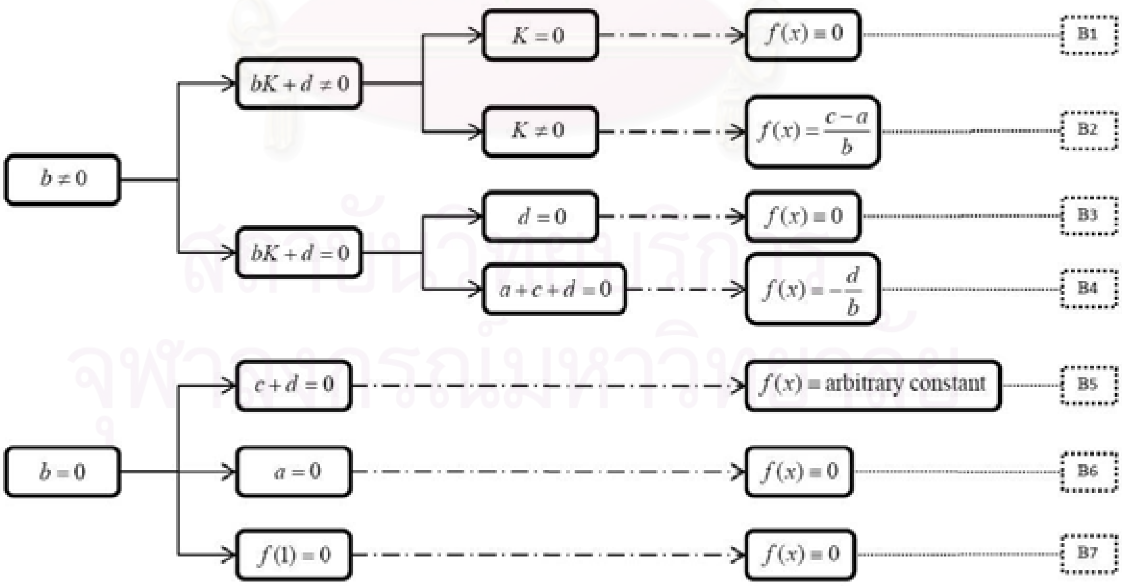
Theorem 4.1.1. *Let $a, b, d \in \mathbb{C}$. If $f : \mathbb{R}^+ \to \mathbb{C}$ satisfies the functional equation*

$$af(xy) + bf(x)f(y) + d\{f(x) + f(y)\} = 0,$$

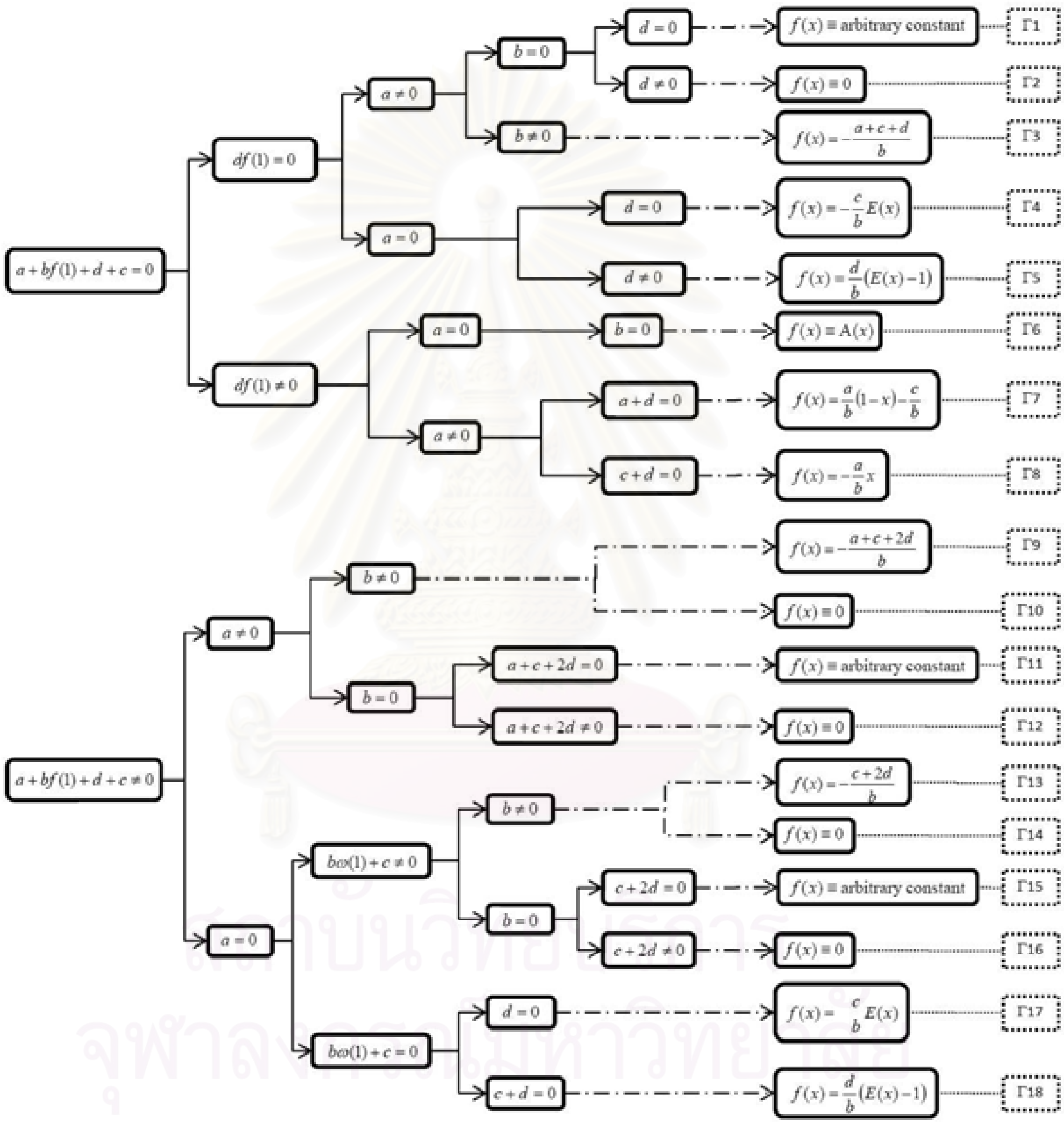
then all possible solution functions are



Theorem 4.1.2. *Let* a, b, $c\ (\neq 0)$, $d \in \mathbb{C}$. *Assume that* $f : \mathbb{R}^+ \to \mathbb{C}$ *satisfies the universal Cauchy functional equation. If* $a + bf(1) + d = 0$, *then all possible solution functions, where* $K = -\frac{df(1)}{c}$, *are*



Theorem 4.1.3. *Let* a, b, $c\,(\neq 0)$, $d \in \mathbb{C}$. *Assume that* $f : \mathbb{R}^+ \to \mathbb{C}$ *is bounded over* $(0, 1]$ *and satisfies the universal Cauchy functional equation. If* $a + bf(1) + d \neq 0$, *then all possible solution functions, where* $\omega(1) = -\frac{cf(1)(bf(1)+c+2d)}{(bf(1)+d)(bf(1)+c+d)}$, *are*



Our objective here is to investigate the above-mentioned two kinds of dependence relations among the four forms of the Cauchy's function equation displayed in Theorems 4.1.1, 4.1.2 and 4.1.3.

For the rest of the chapter, only solution functions which are *complex-valued functions with domain* $\mathbb{R}^+$ *and bounded over the interval* $(0,1]$ are considered.

4.2 Alien and weakly alien relations

In this section, alien and weakly alien relations among the solutions of the four versions of Cauchy's equation are determined.

Theorem 4.2.1. *For distinct* $i, j \in \{1,2,3,4\}$*, each pair* C_i, C_j*, except for the pair* C_3 *and* C_4*, is both alien and weakly alien relative to* $(\mathbb{R}^+, \mathbb{C})$.

Proof. There are six pairs of equations to be considered and we proceed to treat each one of them separately.

1. C_1 **and** C_2

The functional equation $C_1 + C_2$ takes the form

$$f(xy) - f(x)f(y) + f(x+y) - f(x) - f(y) = 0, \tag{4.1}$$

which corresponds to the universal Cauchy functional equation (3.1) with $a = c = 1$ and $b = d = -1$.

If $f \equiv k$ is a constant solution function, then $0 = k - k^2 + k - k - k = -k^2$, i.e.,

$$f \equiv k = 0. \tag{4.2}$$

Since $c \neq 0$, those solution functions in Theorem 4.1.1 are untenable.

For solution functions in Theorem 4.1.2, we refer to Figure 2.

The possibilities (B1) and (B2) yield the zero function.

The possibility (B3) is ruled out because $d = -1 \neq 0$.

The possibility (B4) is ruled out since $a + c + d = 1 \neq 0$.

The possibilities (B5), (B6) and (B7) are ruled out since $b = -1 \neq 0$.

For solution functions in Theorem 4.1.3, we refer to Figure 3.

The possibilities (Γ1), (Γ2), (Γ11) and (Γ12) are ruled out since $b = -1 \neq 0$.

The possibility (Γ3) yields $f(x) = -1$, contradicting (4.2).

The possibilities (Γ4), (Γ5), (Γ6) and (Γ13) up to (Γ18) are ruled out since $a = 1 \neq 0$.

The possibilities (Γ7) and (Γ8) yield $f(x) = x$, which satisfies (4.1).

The possibilities (Γ9) and (Γ10) yield $f \equiv 0$.

To sum up, each solution of (4.1) is of the form $f \equiv 0$ or $f(x) = x$. Since both are also solutions of C_1 and C_2, we conclude that C_1 and C_2 are alien relative to $(\mathbb{R}^+, \mathbb{C})$, and automatically, C_1 and C_2 are weakly alien relative to $(\mathbb{R}^+, \mathbb{C})$.

2. C_2 and C_3

The functional equation $C_2 + C_3$ takes the form

$$f(xy) - 2f(x)f(y) + f(x+y) = 0, \tag{4.3}$$

which corresponds to the universal Cauchy functional equation (3.1) with $a = c = 1$, $b = -2$ and $d = 0$.

If $f \equiv k$ is a constant function, then $0 = k - 2k^2 + k = 2k(1-k)$, i.e.,

$$f \equiv k = 0 \text{ or } f \equiv k = 1.$$

All solutions from Theorem 4.1.1 are ruled out since $c \neq 0$.

For solution functions in Theorem 4.1.2, we refer to Figure 2.

The possibilities (B1), (B2) and (B3) yield $f \equiv 0$.

The possibility (B4) is untenable because $a + c + d = 2 \neq 0$.

The possibilities (B5), (B6) and (B7) are untenable because $b = -2 \neq 0$.

For solution functions in Theorem 4.1.3, we refer to Figure 3,

The possibilities ($\Gamma1$), ($\Gamma2$), ($\Gamma11$) and ($\Gamma12$) are not possible since $b = -2 \neq 0$.

The possibilities ($\Gamma3$) and ($\Gamma9$) yield $f(x) = 1$.

The possibilities ($\Gamma4$), ($\Gamma5$) and ($\Gamma13$) up to ($\Gamma18$) are not possible since $a = 1 \neq 0$.

The possibilities ($\Gamma6$), ($\Gamma7$) and ($\Gamma8$) are ruled out since $d = 0$.

The possibility ($\Gamma10$) yields the zero function.

To sum up, the only two solutions of (4.3) are the zero function and $f \equiv 1$, so C_2 and C_3 are alien and weakly alien relative to $(\mathbb{R}^+, \mathbb{C})$.

3. C_3 and C_4.

The functional equation $C_3 + C_4$ is

$$f(xy) - f(x)f(y) + f(x+y) - f(x) - f(y) = 0, \tag{4.4}$$

which corresponds to the universal Cauchy functional equation (3.1) with $a = c = 1$ and $b = d = -1$.

Note that (4.4) is the same as (4.1), so the only two solutions are the zero function and the identity function. But $f(x) = x$ does not satisfy $f(x+y) = f(x)f(y)$, so C_3 and C_4 are not alien relative to $(\mathbb{R}^+, \mathbb{C})$. Moreover, C_3 and C_4 are not weakly alien relative to $(\mathbb{R}^+, \mathbb{C})$.

The remaining three cases, which we omit their analogous proofs, are simpler as the zero function is the only solution. $\square$

The functional equations E_1 and E_2 can be alien without $-E_1$ and E_2 being so. This is seen by considering for example from the constant function $f \equiv 2$ which is a solution of

$$-C_1 + C_2 : -f(x+y) + f(x) + f(y) + f(xy) - f(x)f(y) = 0.$$

Here $f \equiv 2$ does not satisfy C_1 and C_2. Thus C_1 and C_2 are alien relative to $(\mathbb{R}^+, \mathbb{C})$, but $-C_1$ and C_2 are not.

This remark leads us to consider constant multiples of the four versions of the Cauchy functional equation in the next section.

4.3 Multiple of functional equations

In this section, we investigate whether any pair taken from pC_1, qC_2, rC_3 and sC_4, where $p, q, r, s \in \mathbb{C} \setminus \{0\}$, are pairwise alien or weakly alien. The analysis is similar to that of Theorem 4.2.1 and we shall give detailed proofs only for some of them.

Proposition 4.3.1. *Let $p, q \in \mathbb{C} \setminus \{0\}$. If $p = q$, then pC_1 and qC_2 are both alien and weakly alien relative to $(\mathbb{R}^+, \mathbb{C})$. If $p \neq q$, the functional equations pC_1 and qC_2 are neither alien nor weakly alien relative to $(\mathbb{R}^+, \mathbb{C})$.*

Proof. The functional equation $pC_1 + qC_2$ is

$$qf(xy) - qf(x)f(y) + pf(x+y) - pf(x) - pf(y) = 0, \tag{4.5}$$

which corresponds to the universal Cauchy's functional equation with $a = q$, $b = -q$, $c = p$ and $d = -p$.

If $f \equiv k$ is a constant solution function, then $0 = qk - qk^2 + pk - 2pk = k(q - qk - p)$, and so

$$f \equiv k = 0 \text{ or } f \equiv k = \frac{q-p}{q} \neq 1. \tag{4.6}$$

The solutions from Theorem 4.1.1 are untenable as $c = p \neq 0$.

Next we treat the solutions from Theorem 4.1.2 displayed in Figure 2.

The possibility (B1) yields the zero function.

The possibility (B2) yields $f(x) = \frac{c-a}{b} = \frac{q-p}{q}$.

The possibility (B3) is ruled out because $d = -p \neq 0$.

The possibility (B4) is ruled out since $a + c + d = q \neq 0$.

The possibilities (B5), (B6) and (B7) are ruled out since $b = -q \neq 0$.

Next consider the solutions from Theorem 4.1.3 in Figure 3.

The possibilities (Γ1), (Γ2), (Γ11) and (Γ12) are not possible since $b = -q \neq 0$.

The possibility (Γ3) yields $f(x) = -\frac{a+c+d}{b} = -\frac{q+p-p}{-q} = 1$, contradicting (4.6).

The possibilities (Γ4), (Γ5), (Γ6) and (Γ13) up to (Γ18) are ruled out since $a = q \neq 0$.

The possibility (Γ7) yields $f(x) = \frac{a}{b}(1-x) - \frac{c}{b} = x - 1 + \frac{p}{q}$; if $0 = a + d = q - p$, then $f(x) = x$.

The possibility (Γ8) gives $f(x) = -\frac{a}{b}x = x$ which satisfies (4.5).

The possibility (Γ9) yields $f(x) = -\frac{a+c+2d}{b} = \frac{q-p}{q}$.

The possibility (Γ10) yields $f \equiv 0$.

To sum up, all the solutions of (4.5) are $f \equiv 0$, $f \equiv \frac{q-p}{q}$, $f(x) = x$ and $f(x) = x - 1 + \frac{p}{q}$. All of them are also solution of pC_1 and qC_2 if $p = q$. Thus, pC_1 and qC_2 are alien relative to $(\mathbb{R}^+, \mathbb{C})$ if $p = q$. On the other hand, if $p \neq q$, the above conclusion shows that pC_1 and qC_2 are neither alien nor weakly alien relative to $(\mathbb{R}^+, \mathbb{C})$. $\square$

Proposition 4.3.2. *Let $p, r \in \mathbb{C} \setminus \{0\}$. Then pC_1 and rC_3 are alien if and only if $p = r$, while pC_1 and rC_3 are always weakly alien.*

Proof. The functional equation $pC_1 + rC_3$ reads

$$-rf(x)f(y) + (p + r)f(x + y) - pf(x) - pf(y) = 0, \tag{4.7}$$

which corresponds to the universal Cauchy functional equation with $a = 0$, $b = -r$, $c = p + r$ and $d = -p$.

If $f \equiv k$ is a constant function, then $0 = -rk^2 + (p+r)k - 2pk = k(-rk + r - p)$, i.e.,

$$f \equiv k = 0 \text{ or } f \equiv k = \frac{r-p}{r} \neq 1. \tag{4.8}$$

We distinguish two cases whether $c(= p + r) = 0$.

Case $c = p + r = 0$. The solutions arising from Theorems 4.1.2 and 4.1.3 are untenable. We treat next the solutions from the Theorem 4.1.1 individually.

The possibilities (A1) and (A2) are ruled out since $b = -r \neq 0$.

The possibility (A3) yields $f(x) = -\frac{a+2d}{b} = 2$.

The possibility (A4) yields the zero function.

The possibilities (A5), (A6) and (A7) are ruled out since $d = -p \neq 0$.

The possibilities (A8) and (A9) are ruled out since $a + d = -p \neq 0$.

Case $c = p + r \neq 0$.

The solutions in Theorem 4.1.1 are untenable since $c = p + r \neq 0$. Next, we consider the solutions in Theorem 4.1.2 individually.

The possibility (B1) yields the zero function.

The possibility (B2) gives $f(x) = \frac{c-a}{b} = -\frac{p+r}{r}$, which is a contradiction.

The possibility (B3) is ruled out since $d = -p \neq 0$.

The possibility (B4) is ruled out since $a + c + d = 0 + p + r - p = r \neq 0$.

The possibilities (B5), (B6) and (B7) are ruled out since $b = -r \neq 0$.

Finally for the solutions in Theorem 4.1.3, again we treat them individually.

The possibilities $(\Gamma 1), (\Gamma 2), (\Gamma 6), (\Gamma 15)$ and $(\Gamma 16)$ are ruled out since $b = -r \neq 0$.

The possibility $(\Gamma 3)$ yields $f(x) = -\frac{a+c+d}{b} = -\frac{0+p+r-p}{-r} = 1$, which is not valid.

The possibilities $(\Gamma 4)$ and $(\Gamma 17)$ are ruled out since $d = -p \neq 0$.

The possibilities ($\Gamma5$) and ($\Gamma18$) are ruled out since $c + d = r \neq 0$.

The possibilities ($\Gamma7$) up to ($\Gamma12$) are ruled out since $a = 0$.

The possibility ($\Gamma13$) yields $f(x) = -\frac{c+2d}{b} = \frac{r-p}{r}$.

The possibility ($\Gamma14$) yields $f \equiv 0$.

Thus, the only solutions of (4.7) are the zero function, $f \equiv 2$ and $f \equiv \frac{r-p}{r}$. Consequently, pC_1 and rC_3 are weakly alien relative to $(\mathbb{R}^+, \mathbb{C})$. Since the only constant solution of pC_1 is the zero function and those of rC_3 are $f \equiv 0$ and $f \equiv 1$, we deduce that pC_1 and rC_3 are alien if and only if $p = r$. □

Proposition 4.3.3. *Let $q, r \in \mathbb{C} \setminus \{0\}$. Then qC_2 and rC_3 are alien and weakly alien relative to $(\mathbb{R}^+, \mathbb{C})$ if $q + r \neq 0$. Moreover, if $q + r = 0$, they are weakly alien but not alien.*

Proof. The functional equation $qC_2 + rC_3$ reads

$$qf(xy) - (q + r)f(x)f(y) + rf(x + y) = 0, \tag{4.9}$$

which corresponds to the universal Cauchy functional equation with $a = q$, $b = -(q + r)$, $c = r$ and $d = 0$.

If $f \equiv k$ is a constant function solution, then

$$0 = qk - (q + r)k^2 + rk = k\{q - (q + r)k + r\}.$$

If $q + r \neq 0$, then

$$f \equiv k = 0 \text{ or } f \equiv k = 1. \tag{4.10}$$

If $q + r = 0$, then any arbitrary constant function can be a solution of (4.9).

The solutions from Theorem 4.1.1 are untenable because $c = r \neq 0$.

For the solutions from Theorem 4.1.2 (Figure 2), we distinguish two possible cases.

Case $b = -(q+r) \neq 0$.

The possibilities (B1) and (B3) yield $f \equiv 0$.

The possibility (B2) yields $f(x) = \frac{c-a}{b} = \frac{r-q}{-(q+r)}$, contradicting (4.10).

The possibility (B4) is ruled out since $a + c + d = q + r \neq 0$.

The possibilities (B5), (B6) and (B7) are ruled out since $a + d = q \neq 0$.

Case $b = -(q+r) = 0$.

The possibilities (B1) up to (B4) are ruled out since $b = -(q+r) = 0$.

The possibilities (B5) up to (B7) are ruled out since $a + d = q \neq 0$.

For the solutions from Theorem 4.1.3 (Figure 3), we again distinguish the two possible cases.

Case $b = -(q+r) \neq 0$.

The possibilities (Γ1) and (Γ2) are ruled out since $b = -(q+r) \neq 0$.

The possibilities (Γ3) and (Γ9) yield $f(x) = 1$.

The possibilities (Γ4),(Γ5),(Γ6), and (Γ13) up to (Γ18) are ruled out since $a = q \neq 0$.

The possibility (Γ7) is ruled out since $a + d = q \neq 0$.

The possibility (Γ8) is ruled out since $c + d = r \neq 0$.

The possibility (Γ10) yields the zero function.

The possibilities (Γ11) and (Γ12) are ruled out since $b = -(q+r) \neq 0$.

Case $b = -(q+r) = 0$.

The possibilities (Γ1) and (Γ11) yield any arbitrary constant solution.

The possibility (Γ2) yields the zero function.

The possibilities (Γ3), (Γ9) and (Γ10) are ruled out since $b = 0$.

The possibilities (Γ4),(Γ5),(Γ6), and (Γ13) up to (Γ18) are ruled out since $a = q \neq 0$.

The possibility (Γ7) is ruled out since $a + d = q \neq 0$.

The possibility (Γ8) is ruled out since $c + d = r \neq 0$.

The possibility (Γ12) is ruled out since $a + c + 2d = q + r = 0$.

If $q + r = 0$, then the only solutions of (4.9) are abitrary constants. But $f \equiv 0$ and $f \equiv 1$ are the only two constant functions satisfying qC_2 and rC_3. Thus, relative to $(\mathbb{R}^+, \mathbb{C})$, qC_2 and rC_3 are not alien, but weakly alien in this case.

If $q+r \neq 0$, then qC_2 and rC_3 are both alien and weakly alien relative $(\mathbb{R}^+, \mathbb{C})$. $\square$

The remaining cases can be proved in the same manner and we merely state them without proofs.

Proposition 4.3.4. *1. Let $p, s \in \mathbb{C} \setminus \{0\}$. Then pC_1 and sC_4 are alien and weakly alien relative to $(\mathbb{R}^+, \mathbb{C})$ if $p + s \neq 0$. If $p + s = 0$, they are weakly alien but not alien.*

2. Let $q, s \in \mathbb{C} \setminus \{0\}$. Then relative to $(\mathbb{R}^+, \mathbb{C})$, the functional equations qC_2 and sC_4 are weakly alien, while they are also alien if $q = s$.

3. Let $r, s \in \mathbb{C} \setminus \{0\}$. Then rC_3 and sC_4 are neither alien nor weakly alien relative to $(\mathbb{R}^+, \mathbb{C})$.

CHAPTER V

HEUVERS' EQUATION

5.1 Introduction

In 1999, Heuvers, [7], proved that the functional equation

$$f(x+y) - f(x) - f(y) = f\left(\frac{1}{x} + \frac{1}{y}\right) \tag{5.1}$$

and the logarithmic Cauchy functional equation, see e.g. Chapter 2 of [1],

$$f(xy) = f(x) + f(y) \tag{5.2}$$

are equivalent in the sense that any function $f : \mathbb{R}^+ \to \mathbb{R}$ which is a solution of one functional equation is also a solution of the other.

In 2005, [8], the functional equation

$$f(x+y) - f(xy) = f\left(\frac{1}{x} + \frac{1}{y}\right) \tag{5.3}$$

is added to this list of equivalent functional equations.

It is thus natural to find out what functions $f : \mathbb{R}^+ \to \mathbb{C}$ are solutions of a more general functional equation of the form

$$af(x+y) + bf(x) + cf(y) = df\left(\frac{1}{x} + \frac{1}{y}\right), \tag{5.4}$$

where $a, b, c, d \in \mathbb{C}$. Our analysis is divided into two possible cases. The cases where either $(b \neq c)$ or $(b = c = 0)$ or $(b = c \neq 0$ and $d = 0)$ are quite simple and it is shown in the next section that there are only constant function or the additive function solutions. The remaining case except for the case $(b = c = -1$ and $a = d = 1)$ is dealt with in the last section. It is shown that under a continuity condition only constant function solutions are possible.

5.2 The cases without continuity condition

In this section, cases where continuity is not assumed are solved.

Theorem 5.2.1. *Assume that* $f : \mathbb{R}^+ \to \mathbb{C}$ *satisfies the functional equation* (5.4).
When $b \neq c$*,*

1. *if* $a + b + c - d \neq 0$*, then* $f \equiv 0$*;*
2. *if* $a + b + c - d = 0$*, then* f *is an arbitrarily constant function.*

When $b = c = 0$*,*

1. *if* $a \neq d$*, then* $f \equiv 0$*;*
2. *if* $a = d \neq 0$*, then* f *is an arbitrary constant function;*
3. *if* $a = d = 0$*, then there is no equation.*

When $b = c \neq 0$ *and* $d = 0$*,*

1. *if* $a = 0$*, then* $f \equiv 0$*;*
2. *if* $a + b = 0$*, then* f *is an additive function;*
3. *if* $a + b \neq d$ *and* $a + 2b = 0$*, then* f *is an arbitrary constant function;*

4. if $a+b\neq d$ *and* $a+2b\neq 0$*, then* $f\equiv 0$*.*

Proof. Interchanging x and y in (5.4), we get

$$af(y+x)+bf(y)+cf(x)=df\left(\frac{1}{y}+\frac{1}{x}\right). \tag{5.5}$$

Combining (5.4) and (5.5), we have

$$0=(b-c)\left(f(x)-f(y)\right).$$

If $b\neq c$, then f must be a constant function, say $f\equiv k$. Substituting into (5.4), we get

$$(a+b+c-d)k=0.$$

Thus, $f\equiv 0$ when $a+b+c-d\neq 0$ and f is any constant function provided $a+b+c-d=0$.

Next, we deal with the case $b=c=0$. The functional equation (5.4) reduces to

$$af(x+y)=df\left(\frac{1}{x}+\frac{1}{y}\right).$$

Replacing $\frac{1}{x}$ for x and $\frac{1}{y}$ for y leads to

$$af\left(\frac{1}{x}+\frac{1}{y}\right)=df(x+y).$$

Incorporating the last two equations yield

$$(a-d)\left(f(x+y)+f\left(\frac{1}{x}+\frac{1}{y}\right)\right)=0. \tag{5.6}$$

• If $a \neq d$, then

$$f(x+y) + f\left(\frac{1}{x} + \frac{1}{y}\right) = 0. \tag{5.7}$$

Putting $y = \frac{1}{x}$ in (5.7), we get

$$f\left(x + \frac{1}{x}\right) = 0.$$

Since the map $x \mapsto x + \frac{1}{x}$ takes $\mathbb{R}^+$ onto $[2, \infty)$, we have

$$f(x) = 0 \quad \text{for all } x \in [2, \infty).$$

Replacing x and y in (5.7) by $\frac{x}{2}$, we obtain

$$f(x) + f(4/x) = 0.$$

For each $x \in (0, 2)$, since $4/x > 2$, we get $f(x) = -f(4/x) = 0$, yielding f as the zero function.

• If $a = d$, the equation (5.6) becomes

$$af(x+y) = af(1/x + 1/y).$$

If $a = 0$, we have no functional equation. Assume then that $a \neq 0$ and divide through by a, we get

$$f(x+y) = f(1/x + 1/y). \tag{5.8}$$

Choose $x_1, y_1 \in \mathbb{R}^+$ such that $x_1 + y_1 = 1$ and let $\dfrac{1}{z} = \dfrac{1}{x_1} + \dfrac{1}{y_1}$. Thus,

$$f\left(\frac{1}{z}\right) = f\left(\frac{1}{x_1} + \frac{1}{y_1}\right) = f(x_1 + y_1) = f(1).$$

Since $z = x_1(1 - x_1) \leq 1/4$, we have

$$f(w) = f(1) \quad \text{for each } w \geq 4. \tag{5.9}$$

For $w \in [1, 4)$, invoking upon (5.8) and making use of (5.9) we have

$$f(w) = f\left(1/4 + (w - 1/4)\right) = f\left(4 + \frac{1}{(w - 1/4)}\right) = f(1). \tag{5.10}$$

Using (5.8), for any $x \in \mathbb{R}^+$ we have

$$f(x) = f\left(\frac{x}{2} + \frac{x}{2}\right) = f\left(\frac{2}{x} + \frac{2}{x}\right) = f\left(\frac{4}{x}\right).$$

For each $w \in (0, 1)$, since $\frac{4}{w} > 4$, (5.9) implies that

$$f(w) = f\left(\frac{4}{w}\right) = f(1). \tag{5.11}$$

The required conclusion follows at once from (5.9), (5.10) and (5.11).

If $b = c \neq 0$ and $d = 0$, the equation (5.4) becomes

$$\left(\frac{a}{b}\right) f(x + y) + f(x) + f(y) = 0. \tag{5.12}$$

If $a = 0$, then f is only the zero function. If $a + b = 0$, then (5.12) is the usual Cauchy's functional equation on $\mathbb{R}^+$, so f is any additive function.

Next, we deal with the case $a + b \neq 0$. Let $x, y, z \in \mathbb{R}^+$. Using (5.12) two times, we have

$$\left(\frac{a}{b}\right)^2 f(x + y + z) = -\left(\frac{a}{b}\right) f(x) + f(y) + f(z),$$

and similarly,

$$\left(\frac{a}{b}\right)^2 f(x+y+z) = f(x) + f(z) - \left(\frac{a}{b}\right) f(y).$$

Thus

$$(a+b)\left(f(x) - f(y)\right) = 0.$$

Since $a+b \neq 0$, f must be a constant function, say $f \equiv k$. Substituting into (5.12), we get

$$(a+2b)k = 0.$$

Thus, $f \equiv 0$ when $a+2b \neq 0$, if not, f is any constant function. □

5.3 Continuous solution functions

In this case, we are able to solve the functional equation under a continuity condition.

Theorem 5.3.1. *Let $f : \mathbb{R}^+ \to \mathbb{C}$ be continuous function satisfying the functional equation*

$$\alpha f(x+y) + f(x) + f(y) = \beta f\left(\frac{1}{x} + \frac{1}{y}\right), \tag{5.13}$$

where $\alpha, \beta \in \mathbb{C}$. Then f is either the zero function or the constant function except for the case $\alpha = \beta \in \{0, -1\}$ and $\beta = 0$.

Proof. Since f is continuous, we adopt the notions

$$f(0^+) := \lim_{x \to 0^+} f(x), \quad f(\infty) := \lim_{x \to \infty} f(x),$$

when both values are allowed to be infinite. We break up our consideration into two cases.

Case I: $\alpha \neq -1$. The equation (5.13) yields

$$\alpha f(x+y) + f(y) = \beta f\left(\frac{1}{x}+\frac{1}{y}\right) - f(x). \tag{5.14}$$

For each $x \in \mathbb{R}^+$, $\lim_{y\to\infty}\left(\beta f\left(\frac{1}{x}+\frac{1}{y}\right) - f(x)\right)$ exists so (5.14) implies

$$\lim_{y\to\infty}\left(\alpha f(x+y) + f(y)\right) \text{ exists,}$$

so $f(\infty)$ is finite.

By (5.13) and $\lim_{y\to 0^+}\left(\alpha f(x+y) + f(x) - \beta f\left(\frac{1}{x}+\frac{1}{y}\right)\right)$ exists, $f(0^+)$ is finite and

$$f(x) = \frac{1}{\alpha+1}(\beta f(\infty) - f(0^+)) =: K, \text{ a constant function.}$$

Substituting this into the original functional equation (5.13), we have

$$(\alpha + 2 - \beta)K = 0.$$

Consequently, if $\alpha + 2 - \beta \neq 0$, then $f \equiv 0$, while if $\alpha + 2 - \beta = 0$, then f is an arbitrary constant function.

Case II: $\alpha = -1$. The equation (5.13) becomes

$$-f(x+y) + f(x) + f(y) = \beta f\left(\frac{1}{x}+\frac{1}{y}\right) \tag{5.15}$$

It is necessary now to distinguish the values of $f(0^+)$.

• **If** $f(0^+)$ **is finite**, then fix x and let $y \to 0^+$ in (5.15) to get $\beta f(\infty) = f(0^+)$. Since $\beta \neq 0$, we see that $f(\infty)$ is also finite. Replacing x and y by $\frac{1}{x}$ and $\frac{1}{y}$, respectively in

(5.15), we have

$$-f\left(\frac{1}{x}+\frac{1}{y}\right)+f\left(\frac{1}{x}\right)+f\left(\frac{1}{y}\right)=\beta f(x+y). \tag{5.16}$$

Fix x and let $y \to 0^+$ in (5.16), we obtain $-f(\infty)+f\left(\frac{1}{x}\right)+f(\infty)=\beta f(x)$, i.e.,

$$f\left(\frac{1}{x}\right)=\beta f(x). \tag{5.17}$$

Substituing y by $\frac{1}{x}$ in (5.15) and using (5.17), we get

$$\beta f\left(x+\frac{1}{x}\right)=-f\left(x+\frac{1}{x}\right)+f(x)+f\left(\frac{1}{x}\right)=-f\left(x+\frac{1}{x}\right)+f(x)+\beta f(x), \tag{5.18}$$

and so

$$(\beta+1)\left(f(x)-f\left(x+\frac{1}{x}\right)\right)=0.$$

Since $\alpha=-1$, by assumption, $\beta \neq -1$, so we get

$$f(x)=f\left(x+\frac{1}{x}\right).$$

Iterating this relation, we get

$$f(x)=f\left(x+\frac{1}{x}\right)=f\left(x+\frac{1}{x}+\frac{1}{x+\frac{1}{x}}\right)=\cdots=\lim_{x\to\infty}f(x)=f(\infty),$$

i.e., f is an arbitrary constant function.

• **If** $f(0^+)$ **is infinite,** then substituting x,y by $\frac{1}{x}$ and $\frac{1}{y}$ in (5.15), we get

$$f\left(\frac{1}{x}+\frac{1}{y}\right)=-\beta f(x+y)+f\left(\frac{1}{x}\right)+f\left(\frac{1}{y}\right). \tag{5.19}$$

Adding (5.15) and (5.19), we obtain

$$-(\beta+1)f(x+y)+f(x)+f\left(\frac{1}{x}\right)+f(y)+f\left(\frac{1}{y}\right)=(\beta+1)f\left(\frac{1}{x}+\frac{1}{y}\right). \quad (5.20)$$

On the other hand, substituting $y=\frac{1}{x}$ into (5.15), we get

$$f(x)+f\left(\frac{1}{x}\right)=(\beta+1)f\left(x+\frac{1}{x}\right). \quad (5.21)$$

Applying this relation to the last four terms on the left-hand-side of (5.20) and dividing through by $\beta+1\neq 0$ yields

$$f\left(\frac{1}{x}+\frac{1}{y}\right)=-f(x+y)+f\left(x+\frac{1}{x}\right)+f\left(y+\frac{1}{y}\right).$$

Replacing the first term on the right-hand-side using (5.15) twice to get

$$\begin{aligned} f(x)-f\left(x+\frac{1}{x}\right)+f(y)-f\left(y+\frac{1}{y}\right) &= (\beta-1)f\left(\frac{1}{x}+\frac{1}{y}\right) \\ &= \frac{\beta-1}{\beta}\left(-f(x+y)+f(x)+f(y)\right). \end{aligned}$$

Simplifying, we have

$$(1-\beta)f(x+y)=f(x)-\beta f\left(x+\frac{1}{x}\right)+f(y)-\beta f\left(y+\frac{1}{y}\right). \quad (5.22)$$

Abbreviating $\mathcal{F}(x):=f(x)-\beta f\left(x+\frac{1}{x}\right)$, the equation (5.22) reads

$$(1-\beta)f(x+y)=\mathcal{F}(x)+\mathcal{F}(y). \quad (5.23)$$

For a fixed $c \in \mathbb{R}^+$, choose $x, y \in \mathbb{R}^+$ such that $x + y = c$. Thus,

$$(1-\beta)f(c) = \mathcal{F}(x) + \mathcal{F}(c-x).$$

Assume that $\beta \neq 1$. Then

$$f(c) = \frac{1}{1-\beta}\left(\mathcal{F}(x) + \mathcal{F}(c-x)\right).$$

Since $\mathcal{F}(0^+)$ is finite, the above equation implies that $f(0^+)$ is finite, a contradiction. Thus $\beta = 1$. By (5.22),

$$f(x) + f(y) = \beta f\left(x + \frac{1}{x}\right) + \beta f\left(y + \frac{1}{y}\right).$$

Take $y = x$, we obtain

$$f\left(x + \frac{1}{x}\right) = f(x).$$

By the same agruments as at the end of the preceding case, we have

$$f(x) = \lim_{x\to\infty} f(x) = f(\infty),$$

contradicting the fact that $f(\infty) = f(0^+)$ is infinite. $\square$

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