CHAPTER III

MAIN RESULT

Let X_j, $j = 1, 2, \ldots, n$ be independent integral-valued random variables. Let

$$S_n = X_1 + X_2 + \ldots + X_n .$$

In this chapter, we obtain approximations for $P(k_1 \leq S_n \leq k_2)$ in term of normal probabilities with certain types of correction terms. Since, by Theorem 2.1, $P(k_1 \leq S_n \leq k_2)$ can be written in terms of

$$R(T) = \frac{1}{2\pi} \int_0^{\pi} \frac{|\varphi_{S_n}(t)| \sin(T\sqrt{B_n}\, t - \alpha(t))dt}{\sin\frac{t}{2}} ,$$

so it suffices to obtain approximations for R(T). This is done in Theorem 3.11.

<u>Lemma 3.1.</u> Let X be a random variable. If X has moment of order p where p is a nonnegative integer. Then the following holds :

$$(3.1.1)\ldots \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-y)^p dF_X(x)dF_X(y) = \sum_{m=o}^{p} (-1)^{p-m}\binom{p}{m} E(X^m)E(X^{p-m}).$$

<u>Proof</u> :

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-y)^p dF_X(x)dF_X(y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sum_{m=o}^{p} \binom{p}{m} x^m(-y)^{p-m} dF_X(x)dF_X(y) ,$$

$$= \sum_{m=o}^{p} (-1)^{p-m}\binom{p}{m} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^m y^{p-m} dF_X(x)dF_X(y),$$

$$= \sum_{m=o}^{p} (-1)^{p-m} \binom{p}{m} \int_{-\infty}^{\infty} x^m d\,F_X(x) \int_{-\infty}^{\infty} y^{p-m} d\,F_X(y),$$

$$= \sum_{m=o}^{p} (-1)^{p-m} \binom{p}{m} E(X^m)E(X^{p-m}).$$

#

Lemma 3.2. Let X be a random variable with finite variance. Then the following hold :

(3.2.1)... $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-y)^2 d\,F_X(x)d\,F_X(y) = 2\sigma^2(X)$$

and

(3.2.2)... $$2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sin^2((x-y)\tfrac{t}{2})d\,F_X(x)dF_X(y) \leq t^2\sigma^2(X)$$

for $t \in \mathbb{R}$.

Proof : By taking p = 2 in (3.1.1), we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-y)^2 d\,F_X(x)d\,F_X(y) = \sum_{m=o}^{2} (-1)^{2-m} \binom{2}{m} E(X^m)E(X^{2-m}),$$

$$= 2\sigma^2(X),$$

ie. (3.2.1) holds.

Since $|\sin\theta| \leq |\theta|$, we have

$$0 \leq \sin^2((x-y)\tfrac{t}{2}) \leq (x-y)^2 \tfrac{t^2}{4} .$$

Hence

$$2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sin^2((x-y)\tfrac{t}{2})d\,F_X(x)d\,F_X(y) \leq \frac{t^2}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-y)^2 d\,F_X(x)d\,F_X(y),$$

$$= t^2\sigma^2(X).$$

#

Lemma 3.3. Let X be a random variable with finite variance. If $t^2 < \frac{1}{\sigma^2(X)+1}$, then

$$(3.3.1)\ldots \quad \log|\varphi_X(t)| = \frac{-1}{2} \sum_{p=1}^{\infty} \frac{2^p}{p} \left(\int_{-\infty}^{\infty} \int_{\infty}^{\infty} \sin^2((x-y)\tfrac{t}{2}) d\,F_X(x) d\,F_X(y) \right)^p.$$

Proof : Since $|\varphi_X(t)|^2$ is real and

$$|\varphi_X(t)|^2 = \varphi_X(t)\overline{\varphi_X(t)} ,$$

$$= \int_{-\infty}^{\infty} e^{itx}\, d\,F_X(x) \int_{-\infty}^{\infty} e^{-ity}\, d\,F_X(y) ,$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{it(x-y)}\, d\,F_X(x) d\,F_X(y) ,$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cos((x-y)t) d\,F_X(x) d\,F_X(y) + i \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sin((x-y)t) d\,F_X(x) dF_X(y),$$

so we have

$$|\varphi_X(t)|^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cos((x-y)t) d\,F_X(x) d\,F_X(y) ,$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (1-2\sin^2((x-y)\tfrac{t}{2})) d\,F_X(x) d\,F_X(y) ,$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\,F_X(x) d\,F_X(y) - 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sin^2((x-y)\tfrac{t}{2}) d\,F_X(x) d\,F_X(y) ,$$

$$= 1 - 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sin^2((x-y)\tfrac{t}{2}) d\,F_X(x) d\,F_X(y).$$

Since $t^2 < \frac{1}{\sigma^2(x)+1}$, by (3.2.2), we have

$0 \le 2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sin^2((x-y)\frac{t}{2}) d F_X(x) d F_X(y) < 1$. Hence $\log|\varphi_X(t)|$ is meaningful

and

$$\log|\varphi_X(t)| = \frac{1}{2} \log(1 - 2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sin^2((x-y)\frac{t}{2}) d F_X(x) d F_X(y)) ,$$

$$= -\frac{1}{2} \sum_{p=1}^{\infty} \frac{1}{p} (2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sin^2((x-y)\frac{t}{2}) d F_X(x) d F_X(y))^p ,$$

$$= -\frac{1}{2} \sum_{p=1}^{\infty} \frac{2^p}{p} (\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sin^2((x-y)\frac{t}{2}) d F_X(x) d F_X(y))^p .$$

#

For convenience, in dealing with a certain type of polynomials and their derivatives, we introduce the following notation.

Notation. For any polynomial function p(t) and any positive integer i, we use $p_{[i]}$ to denote the polynomial function defined by

$$p_{[i]}(t) = (p(t))^i.$$

Lemma 3.4. Let f(t), g(t) and h(t) be polynomial functions defined by

$$f(t) = a_o t^j + a_1 t^{j+1} + \ldots + a_p t^{j+p} + a_{p+1} t^{j+p+1},$$

$$g(t) = a_o t^j + a_1 t^{j+1} + \ldots + a_p t^{j+p} + b t^{j+p+1}$$

and

$$h(t) = |a_o| t^j + |a_1| t^{j+1} + \ldots + |a_p| t^{j+p} |b| t^{j+p+1} .$$

where $j \ge 0$ and $p \ge 0$.

Assume that $|a_{p+1}| \le |b|$, then for any positive integers i, q_o

such that $ij \le q_o < ij + p + 1$, we have

$$(3.4.1)\ldots \qquad \sum_{q=0}^{q_o} \frac{1}{q!}\, g_{[i]}^{(q)}(0)\,t^q - \sum_{q=q_o+1}^{i(j+p+1)} \frac{1}{q!}\, h_{[i]}^{(q)}(0)\,t^{q_o+1} \le f_{[i]}(t)$$

for all $t \in (0,1)$ and

$$(3.4.2)\ldots \qquad f_{[i]}(t) \le \sum_{q=0}^{q_o} \frac{1}{q!}\, g_{[i]}^{(q)}(0)\,f^q + \sum_{q=q_o+1}^{i(j+p+1)} \frac{1}{q!}\, h_{[i]}^{(q)}(0)\,t^{q_o+1}$$

for all $t \in (0,1)$.

<u>Proof</u> : Let i and q_o be any positive integers such that $ij \le q_o < ij + p + 1$.

Note that $f(t)$, $g(t)$ and $h(t)$ are polynomials, they can be express as their Taylor's polynomials. In doing so, we have

$$f_{[i]}(t) = \sum_{q=0}^{i(j+p+1)} \frac{1}{q!}\, f_{[i]}^{(q)}(0)\,t^q ,$$

$$g_{[i]}(t) = \sum_{q=0}^{i(j+p+1)} \frac{1}{q!}\, g_{[i]}^{(q)}(0)\,t^q ,$$

and

$$h_{[i]}(t) = \sum_{q=0}^{i(j+p+1)} \frac{1}{q!}\, h_{[i]}^{(q)}(0)\,t^q .$$

On the other hand, by using multinomial expansions, we have

$$(3.4.3)\ldots\ f_{[i]}(t) = \sum_{n_o+n_1+\ldots+n_{p+1}=i} \binom{i}{n_o, n_1, \ldots, n_{p+1}} a_o^{n_o} a_1^{n_1} \cdots a_{p+1}^{n_{p+1}} (t^j)^{n_o} (t^{j+1})^{n_1} \cdots (t^{j+p+1})^{n_{p+1}} ,$$

$$(3.4.4)\ldots \quad g_{[i]}(t) = \sum_{n_o+n_1+\ldots+n_{p+1}=i} \binom{i}{n_o,n_1,\ldots,n_{p+1}} a_o^{n_o} a_1^{n_1} \ldots$$

$$a_p^{n_p} b^{n_{p+1}} (t^j)^{n_o} (t^{j+1})^{n_1} \ldots (t^{j+p+1})^{n_{p+1}}$$

and

$$(3.4.5)\ldots \quad h_{[i]}(t) = \sum_{n_o+n_1+\ldots+n_{p+1}=i} \binom{i}{n_o,n_1,\ldots,n_{p+1}} |a_o|^{n_o} |a_1|^{n_1} \ldots$$

$$|a_p|^{n_p} |b|^{n_{p+1}} (t^j)^{n_o} (t^{j+1})^{n_1} \ldots (t^{j+p+1})^{n_{p+1}} .$$

From (3.4.3) and (3.4.4), we see that $f_{[i]}(t)$ and $g_{[i]}(t)$ contain only terms of degree larger than or equal to ij. So we have

$f_{[i]}^{(q)}(0) = g_{[i]}^{(q)}(0) = 0$ for $0 \leq q \leq ij-1$. Therefore

$$\sum_{q=0}^{ij-1} \frac{1}{q!} f_{[i]}^{(q)}(0)\, t^q = \sum_{q=0}^{ij-1} \frac{1}{q!} g_{[i]}^{(q)}(0)\, t^q .$$

From (3.4.3), we observe that any term of $f_{[i]}(t)$ must be of the form

$$\binom{i}{n_o,n_1,\ldots,n_{p+1}} a_o^{n_o} a_1^{n_1} \ldots a_{p+1}^{n_{p+1}} t^{ij+(n_1+2n_2+\ldots+(p+1)n_{p+1})}$$

for some nonnegative integers $n_o,\ldots,n_p, n_{p+1}$ such that $n_o+n_1+\ldots+n_{p+1} = i$.

Therefore $n_{p+1} = 0$ for any term of $f_{[i]}(t)$ of degree smaller than $ij+p+1$. So the term of $f_{[i]}(t)$ of degree smaller than $ij+p+1$ must be of the form

$$\binom{i}{n_0,n_1,\ldots,n_p} a_o^{n_o} a_1^{n_1} \ldots a_p^{n_p} t^{ij+(n_1+2n_2+\ldots+pn_p)} .$$

015315

Similarly, we can show that for the term of $g_{[i]}[t]$ of degree smaller than $ij+p+1$ must be of the form

$$\binom{i}{n_o, n_1, \ldots, n_p} a_o^{n_o} a_1^{n_1} \ldots a_p^{n_p} t^{ij+(n_1+2n_2+\ldots+pn_p)}.$$

Hence for any $q < ij+p+1$, terms of degree q of $f_{[i]}(t)$ and $g_{[i]}(t)$ are equal.

Since $q_o < ij+p+1$, we have

$$(3.4.6)\ldots \quad \sum_{q=0}^{q_o} \frac{1}{q!} f_{[i]}^{(q)}(0)\, t^q = \sum_{q=0}^{q_o} \frac{1}{q!} g_{[i]}^{(q)}(0)\, t^q.$$

From (3.4.3), (3.4.5) and $t > 0$, we observe that for any nonnegative integer q, the term of degree q of $f_{[i]}(t)$ must be less than or equal to the term of degree q of $h_{[i]}(t)$ and must be larger than or equal to the term of degree q of $-h_{[i]}(t)$.

Hence

$$(3.4.7)\ldots \quad -\sum_{q=q_o+1}^{i(j+p+1)} \frac{1}{q!} h_{[i]}^{(q)}(0)\, t^q \leq \sum_{q=q_o+1}^{i(j+p+1)} \frac{1}{q!} f_{[i]}^{(q)}(0)\, t^q$$

$$\leq \sum_{q=q_o+1}^{i(j+p+1)} \frac{1}{q!} h_{[i]}^{(q)}(0)\, t^q.$$

From (3.4.6) and (3.4.7), we have

$$\sum_{q=0}^{q_o} \frac{1}{q!} g_{[i]}^{(q)}(0)\, t^q - \sum_{q=q_o+1}^{i(j+p+1)} \frac{1}{q!} h_{[i]}^{(q)}(0)\, t^q \leq f_{[i]}(t) \leq \sum_{q=0}^{q_o} \frac{1}{q!} g_{[i]}^{(q)}(0)\, t^q$$

$$+ \sum_{q=q_o+1}^{i(j+p+1)} \frac{1}{q!} h_{[i]}^{(q)}(0)\, t^q .$$

Since $0 < t < 1$ and $h_{[i]}(t) \geq 0$, so we have (3,4.1) and (3.4.2).

#

Lemma 3.5. Let X be a random variable with moment of order $2p_o + 2$ where p_o is a positive integer. Then there exist constants $\bar{c}_1(X)$, $\bar{c}_2(X), \ldots, \bar{c}_{p_o+1}(X)$ such that

i) $$\sum_{m=1}^{p_o} \bar{c}_m(X)\, t^{2m} - \bar{c}_{p_o+1}(X)\, t^{2p_o+2} \leq -\frac{1}{2} \sum_{p=1}^{p_o} \frac{2^p}{p} \left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sin^2((x-y)\tfrac{t}{2})\, dF_X(x)\, dF_X(y) \right]^p \leq \sum_{m=1}^{p_o} \bar{c}_m(X)\, t^{2m} + \bar{c}_{p_o+1}(X)\, t^{2p_o+2}$$

for all $t \in (0,1)$ and

ii) $\bar{c}_{p_o+1}(X) \geq 0$.

Proof : For $p = 1, 2, \ldots, p_o+1$, let

$$g_p(X) = \sum_{m=0}^{2p} \binom{2p}{m} (-1)^{2p-m} E(X^m) E(X^{2p-m}) ,$$

$$\tilde{H}_{X,p}(t) = \left[\sum_{m=1}^{p_o+1} \frac{(-1)^{m-1}}{2(2m)!} g_m(X)\, t^{2m} \right]^p$$

and

$$H_{X,p}(t) = \left[\sum_{m=1}^{p_o+1} \frac{1}{2(2m)!} g_m(X)\, t^{2m} \right]^p .$$

By using Taylor's formula for $\sin^2\theta$, we have

$$\sin^2((x-y)\tfrac{t}{2}) = \sum_{m=1}^{p_o} \frac{(-1)^{m-1}}{2(2m)!} (x-y)^{2m} t^{2m} + \frac{(-1)^{p_o}}{2(2p_o+2)!} (x-y)^{2p_o+2} \cos((x-y)t_o)\, t^{2p_o+2} ,$$

for some t_o.

So, $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sin^2((x-y)\frac{t}{2})d\,F_X(x)d\,F_X(y)$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\sum_{m=1}^{p_o} \frac{(-1)^{m-1}}{2(2m)!}(x-y)^{2m} t^{2m} + \frac{(-1)^{p_o}}{2(2p_o+2)!}(x-y)^{2p_o+2}\cos((x-y)t_o)t^{2p_o+2}\right]$$

$d\,F_X(x)d\,F_X(y)$,

$$= \sum_{m=1}^{p_o} \frac{(-1)^{m-1}}{2(2m)!} t^{2m}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-y)^{2m}d\,F_X(x)d\,F_X(y) + \frac{(-1)^{p_o}}{2(2p_o+2)!} t^{2p_o+2}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-y)^{2p_o+2} \cos((x-y)t_o)d\,F_X(x)d\,F_X(y) .$$

Since $g_p(X) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-y)^{2p}d\,F_X(x)d\,F_X(y)$, we have

$$(3.5.1)\ldots \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sin^2((x-y)\frac{t}{2})d\,F_X(x)d\,F_X(y) = \sum_{m=1}^{p_o} \frac{(-1)^{m-1}}{2(2m)!} g_m(X)t^{2m}$$

$$+ \frac{(-1)^{p_o}t^{2p_o+2}}{2(2p_o+2)!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-y)^{2p_o+2} \cos((x-y)t_o)d\,F_X(x)d\,F_X(y).$$

Let $f(t) = \sum_{m=1}^{p_o} \frac{(-1)^{m-1}}{2(2m)!} g_m(X)t^{2m} + \frac{(-1)^{p_o}t^{2p_o+2}}{2(2p_o+2)!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-y)^{2p_o+2}$

$$\cos((x-y)t_o)d\,F_X(x)d\,F_X(y) ,$$

$$g(t) = \sum_{m=1}^{p_o} \frac{(-1)^{m-1}}{2(2m)!} g_m(X)t^{2m} + \frac{(-1)^{p_o}}{2(2p_0+2)!} g_{p_o+1}(X)t^{2p_o+2}$$

and

$$h(t) = \sum_{m=1}^{p_o+1} \frac{1}{2(2m)!} g_m(X)t^{2m}.$$

Since $g_m(X) \geq 0$ for $m = 1, 2, \ldots, p_o+1$ and

$$\left| \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-y)^{2p_o+2} \cos((x-y)\frac{t_o}{2})\, d F_X(x)\, d F_X(y) \right| \leq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-y)^{2p_o+2}\, d F_X(x)\, d F_X(y),$$

$$= g_{p_o+1}(X),$$

by Lemma 3.4, we have, for each positive integer i,

$$\sum_{q=1}^{p_o} \frac{1}{(2q)!} g^{(2q)}_{[i]}(0)\, t^{2q} - \left[\sum_{q=p_o+1}^{i(p_o+1)} \frac{1}{(2q)!} h^{(2q)}_{[i]}(0)\right] t^{2p_o+2} \leq f_{[i]}(t)$$

$$\leq \sum_{q=1}^{p_o} \frac{1}{(2q)!} g^{(2q)}_{[i]}(0)\, t^{2q} + \left[\sum_{q=p_o+1}^{i(p_o+1)} \frac{1}{(2q)!} h^{(2q)}_{[i]}(0)\right] t^{2p_o+2}.$$

Observe that,

$$g_{[p]}(t) = \tilde{H}_{X,p}(t),$$

$$h_{[p]}(t) = H_{X,p}(t)$$

and by (3.5.1),

$$f(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sin^2((x-y)\frac{t}{2})\, d F_X(x)\, d F_X(y).$$

Hence we have

$$\sum_{q=1}^{p_o} \frac{1}{(2q)!} \tilde{H}^{(2q)}_{X,p}(0)\, t^{2q} - \sum_{q=p_o+1}^{p(p_o+1)} \frac{1}{(2q)!} H^{(2q)}_{X,p}(0)\, t^{2p_o+2}$$

$$\leq \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sin^2((x-y)\frac{t}{2})\, d F_X(x)\, d F_X(y)\right]^p$$

$$\leq \sum_{q=1}^{p_o} \frac{1}{(2q)!} \tilde{H}^{(2q)}_{X,p}(0)\, t^{2q} + \left[\sum_{q=p_o+1}^{p(p_o+1)} \frac{1}{(2q)!} H^{(2q)}_{X,p}(0)\right] t^{2p_o+2}.$$

Therefore

$$\sum_{q=1}^{p_o}\left(-\frac{1}{2}\sum_{p=1}^{p_o}\frac{2^p}{p(2q)!}\tilde{H}_{X,p}^{(2q)}(0)\right)t^{2q}-\left[\sum_{p=1}^{p_o}\frac{1}{2}\sum_{q=p_o+1}^{p(p_o+1)}\frac{2^p}{p(2q)!}H_{X,p}^{(2q)}(0)\right]t^{2p_o+2}$$

$$\leq -\frac{1}{2}\sum_{p=1}^{p_o}\frac{2^p}{p}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sin^2\left((x-y)\frac{t}{2}\right)d\,F_X(x)\,d\,F_X(y)\right]^p$$

$$\leq \sum_{q=1}^{p_o}\left(-\frac{1}{2}\sum_{p=1}^{p_o}\frac{2^p}{p(2q)!}\tilde{H}_{X,p}^{(2q)}(0)\right)t^{2q}+\left[\sum_{p=1}^{p_o}\frac{1}{2}\sum_{q=p_o+1}^{p(p_o+1)}\frac{2^p}{p(2q)!}H_{X,p}^{(2q)}(0)\right]t^{2p_o+2}.$$

By choosing $\bar{c}_q(X)$, $q = 1, 2, \ldots, p_o+1$ as follows:

$$\bar{c}_q(X) = -\frac{1}{2}\sum_{p=1}^{p_o}\frac{2^p}{p(2q)!}\tilde{H}_{X,p}^{(2q)}(0)\ ,\quad q = 1, 2, \ldots, p_o,$$

$$\bar{c}_{p_o+1}(X) = \frac{1}{2}\sum_{p=1}^{p_o}\sum_{q=p_o+1}^{p(p_o+1)}\frac{2^p}{p(2q)!}H_{X,p}^{(2q)}(0)\ ,$$

we have

$$\sum_{q=1}^{p_o}\bar{c}_q(X)t^{2q}-\bar{c}_{p_o+1}(X)t^{2p_o+2}\leq -\frac{1}{2}\sum_{p=1}^{p_o}\frac{2^p}{2}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sin^2\left((x-y)\frac{t}{2}\right)d\,F_X(x)\,d\,F_X(y)\right]^p$$

$$\leq \sum_{q=1}^{p_o}\bar{c}_q(X)\,t^{2q}+\bar{c}_{p_o+1}(X)\,t^{2p_o+2}$$

and $\bar{c}_{p_o+1}(X) \geq 0$.

#

<u>Lemma 3.6</u>. Let X be a random variable with moment of order $2p_o+2$ where p_o is a positive integer. Then there exist constants $c_1(X)$, $c_2(X), \ldots, c_{p_o+1}(X)$ such that

i) $\sum_{m=1}^{p_o} c_m(X)t^{2m} - c_{p_o+1}(X)t^{2p_o+2} \leq \log|\varphi_X(t)| \leq \sum_{m=1}^{p_o} c_m(X)t^{2m} + c_{p_0+1}(X)t^{2p_o+2}$

for $t^2 < \dfrac{1}{2\sigma^2(X)+1}$ and

ii) $c_1(X) = -\frac{1}{2}\sigma^2(X)$.

<u>Proof</u> : Observe that

$$-\frac{1}{2}\sum_{p=p_o+1}^{\infty} \frac{2^p}{p}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sin^2((x-y)\tfrac{t}{2})d\,F_X(x)d\,F_X(y)\right]^p$$

$$\geq -\frac{1}{2}\sum_{p=p_o+1}^{\infty}\left[2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sin^2((x-y)\tfrac{t}{2})d\,F_X(x)d\,F_X(y)\right]^p,$$

$$= -\frac{1}{2}\frac{\left[2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sin^2((x-y)\tfrac{t}{2})d\,F_X(x)d\,F_X(y)\right]^{p_0+1}}{1-2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sin^2((x-y)\tfrac{t}{2})d\,F_X(x)d\,F_X(y)},$$

$$\geq -(\sigma^2(X)t^2)^{p_o+1},$$

$$= -\sigma^{2p_o+2}(X)\,t^{2p_o+2},$$

where the first equality and the second inequality follow from (3.2.2) and the fact that $t^2 < \dfrac{1}{2\sigma^2(X)+1}$.

Hence we have

$$-\frac{1}{2}\sum_{p=p_o+1}^{\infty} \frac{2^p}{p}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sin^2((x-y)\tfrac{t}{2})d\,F_X(x)d\,F_X(y)\right]^p \geq -\sigma^{2p_o+2}(X)t^{2p_o+2}.$$

Therefore

หอสมุดกลาง สถาบันวิทยบริการ
จุฬาลงกรณ์มหาวิทยาลัย

$$(3.6.1)\ldots \quad -\sigma^{2p_o+2}(X)t^{2p_o+2} \leq -\frac{1}{2}\sum_{p=p_o+1}^{\infty} \frac{2^p}{p}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sin^2(x-y)\frac{t}{2})\, dF_X(x)\,dF_X(y)\right]^p \leq \sigma^{2p_o+2}(X)t^{2p_o+2}.$$

Let $\bar{c}_1(X), \bar{c}_2(X), \ldots, \bar{c}_{p_o+1}(X)$ be constants in our proof of Lemma 3.4 and let $c_p(X) = \bar{c}_p(X)$ for $p = 1, 2, \ldots, p_o$ and $c_{p_o+1}(X) = \bar{c}_{p_o+1} + \sigma^{2p_o+2}(X)$.

By Lemma 3.3, Lemma 3.5(i) and (3.6.1), we have

$$\sum_{m=1}^{p_o} c_m(X)t^{2m} - c_{p_o+1}(X)t^{2p_o+2} \leq \log\left|\varphi_X(t)\right| \leq \sum_{m=1}^{p_o} c_m(X)t^{2m} + c_{p_o+1}(X)t^{2p_o+2}.$$

To show (ii), we note that

$$c_1(X) = \frac{-1}{2}\sum_{p=1}^{p_o} \frac{2^p}{2p}\tilde{H}^{(2)}_{X,p}(0)$$

where

$$\tilde{H}^{(2)}_{X,p}(0) = \begin{cases} \frac{1}{2}g_1(X) & \text{if } p = 1, \\ 0 & \text{if } p > 1. \end{cases}$$

Hence $c_1(X) = -\frac{1}{4}g_1(X)$. So, by (3.2.1), we have $c_1(X) = -\frac{1}{2}\sigma^2(X)$.

#

<u>Theorem 3.7.</u> Let X_j, $j = 1,2,\ldots,n$ be independent random variables with moment of order $2p_o+2$ where p_o is a positive integer. Let $c_p(X_j)$, $j = 1, 2, \ldots, n$, $p = 1, 2, \ldots, p_o+1$ be constants in Lemma 3.6.

We define $\sigma^2_{max} = \max\{\sigma^2(X_j) \mid j = 1, 2, \ldots, n\} + 1$, $\gamma = \dfrac{1}{2\sqrt[4]{n}\sqrt{\sigma^2_{max}}}$,

$$K_p = \sum_{j=1}^{n} c_p(X_j),$$

$$K = \sum_{p=1}^{p_o+1} |K_p| ,$$

$$\tilde{G}(t) = \sum_{p=2}^{p_o+1} K_p t^{2p},$$

$$G(t) = \sum_{p=2}^{p_o+1} |K_p| \, t^{2p} ,$$

$$h_1(t) = \begin{cases} 0 & \text{if } p_o = 1 , \\ \sum_{p=1}^{p_o} \frac{1}{p!} \sum_{q=2}^{p+p_0-1} \frac{1}{(2q)!} \tilde{G}^{(2q)}_{[p]}(0)\, t^{2q} & \text{if } p_o > 1 \end{cases}$$

and

$$h_2(t) = \begin{cases} K_2 e^{\frac{K}{16n}} t^4 & \text{if } p_o = 1 , \\ \sum_{p=1}^{p_o} \frac{1}{p!} \left[\sum_{q=p+p_0}^{p(2p_0+2)} \frac{1}{(2q)!} G^{(2q)}_{[p]}(0) \right] t^{2(p+p_o)} + \frac{1}{(p_o+1)!} e^{\frac{K}{16n}} K^{p_o+1} t^{4(p_o+1)} & \end{cases}$$

if $p_o > 1$,

then

$$\left| |\varphi_{S_n}(t)| - (1 + h_1(t))e^{-\frac{1}{2}B_n t^2} \right| \leq h_2(t) e^{-\frac{1}{2}B_n t^2}$$

for all $t \in (0, \Upsilon)$.

<u>Proof</u> : Since, by Lemma 3.6(ii), $c_1(X_j) = -\frac{1}{2}\sigma^2(X_j)$, we have $K_1 = -\frac{1}{2}B_n$.

<u>Case $p_o = 1$.</u> By Lemma 3.6(i), we have

$$-\frac{1}{2}B_n t^2 - K_2 t^4 \leq \log|\varphi_{S_n}(t)| \leq -\frac{1}{2}B_n t^2 + K_2 t^4 ,$$

and $K_2 \geq 0$. So we have

$$e^{-\frac{1}{2}B_n t^2 - K_2 t^4} \leq |\varphi_{S_n}(t)| \leq e^{-\frac{1}{2}B_n t^2 + K_2 t^4},$$

$$(e^{-K_2 t^4} - 1)e^{-\frac{1}{2}B_n t^2} \leq |\varphi_{S_n}(t)| - e^{-\frac{1}{2}B_n t^2} \leq (e^{K_2 t^4} - 1)e^{-\frac{1}{2}B_n t^2}.$$

Note that for $x > 0$, $e^x - 1 \leq xe^x$ and $e^{-x} - 1 \geq -x$. Hence

$$-K_2 t^4 e^{-\frac{1}{2}B_n t^2} \leq |\varphi_{S_n}(t)| - e^{-\frac{1}{2}B_n t^2} \leq K_2 t^4 e^{K_2 t^4} e^{-\frac{1}{2}B_n t^2}.$$

Since $t^4 < \frac{1}{16n}$,

$$-K_2 e^{\frac{K_2}{16n}} t^4 e^{-\frac{1}{2}B_n t^2} \leq |\varphi_{S_n}(t)| - e^{-\frac{1}{2}B_n t^2} \leq K_2 e^{\frac{K_2}{16n}} t^4 e^{-\frac{1}{2}B_n t^2}.$$

Hence $$\left| |\varphi_{S_n}(t)| - e^{-\frac{1}{2}B_n t^2}(1 + h_1(t)) \right| \leq h_2(t) e^{-\frac{1}{2}B_n t^2},$$

where $$h_1(t) = 0$$

and $$h_2(t) = K_2 e^{\frac{K_2}{16n}} t^4.$$

Case $p_o > 1$. Since $X_1, X_2, \ldots, X_n$ are independent, by Lemma 3.6(i), we have

$$-\frac{1}{2}B_n t^2 + \sum_{m=2}^{p_o} K_m t^{2m} - K_{p_o+1} t^{2p_o+2} \leq \log|\varphi_{S_n}(t)| \leq -\frac{1}{2}B_n t^2 + \sum_{m=2}^{p_o} K_m t^{2m} + K_{p_o+1} t^{2p_o+2},$$

$$e^{-\frac{1}{2}B_n t^2 + \sum_{m=2}^{p_o} K_m t^{2m} - K_{p_o+1} t^{2p_o+2}} \leq |\varphi_{S_n}(t)| \leq e^{-\frac{1}{2}B_n t^2 + \sum_{m=2}^{p_o} K_m t^{2m} + K_{p_o+1} t^{2p_o+2}},$$

$$(3.7.1)\ldots\quad \left(e^{\sum_{m=2}^{p_o} K_m t^{2m} - K_{p_o+1} t^{2p_o+2}} - 1\right) e^{-\frac{1}{2}B_n t^2} \le |\varphi_{s_n}(t)| - e^{-\frac{1}{2}B_n t^2}$$

$$\le \left(e^{\sum_{m=2}^{p_o} K_m t^{2m} + K_{p_o+1} t^{2p_o+2}} - 1\right) e^{-\frac{1}{2}B_n t^2}.$$

Observe that for $x \varepsilon (-\frac{K}{16n}, \frac{K}{16n})$,

$$(3.7.2)\ldots\quad 1 + \sum_{p=1}^{p_o} \frac{x^p}{p!} - \frac{|x|^{p_o+1}}{(p_o+1)!} e^{\frac{K}{16n}} \le e^x \le 1 + \sum_{p=1}^{p_o} \frac{x^p}{p!} + \frac{|x|^{p_o+1}}{(p_o+1)!} e^{\frac{K}{16n}}.$$

Since $t^4 < \frac{1}{16n}$, we have $\left| \sum_{m=2}^{p_o} K_m t^{2m} \pm K_{p_o+1} t^{2p_o+2} \right| < \frac{K}{16n}$.

Therefore, by (3.7.2), we have

$$(3.7.3)\ldots\quad \sum_{p=1}^{p_o} \frac{1}{p!} \left[\sum_{m=2}^{p_o} K_m t^{2m} - K_{p_o+1} t^{2p_o+2} \right]^p$$

$$- \frac{e^{\frac{K}{16n}}}{(p_o+1)!} \left| \sum_{m=2}^{p_o} K_m t^{2m} - K_{p_o+1} t^{2p_o+2} \right|^{p_o+1} \le e^{\sum_{m=2}^{p_o} K_m t^{2m} - K_{p_o+1} t^{2p_o+2}} - 1$$

and

$$(3.7.4)\ldots\quad e^{\sum_{m=2}^{p_o} K_m t^{2m} + K_{p_o+1} t^{2p_o+2}} - 1 \le \sum_{p=1}^{p_o} \frac{1}{p!} \left[\sum_{m=2}^{p_o} K_m t^{2m} + K_{p_o+1} t^{2p_o+2} \right]^p$$

$$+ \frac{e^{\frac{K}{16n}}}{(p_o+1)!} \left| \sum_{m=2}^{p_o} K_m t^{2m} + K_{p_o+1} t^{2p_o+2} \right|^{p_o+1}.$$

By Lemma 3.4, if we let

$$f(t) = \sum_{m=2}^{p_o} K_m t^{2m} - K_{p_o+1} t^{2p_o+2} ,$$

$$v(t) = \sum_{m=2}^{p_o} K_m t^{2m} + K_{p_o+1} t^{2p_o+2} ,$$

$$g(t) = \tilde{G}(t)$$

and

$$h(t) = G(t) ,$$

then for $p \leq p_o$, we have

$$(3.7.5)\ldots \quad \left[\sum_{m=2}^{p_o} K_m t^{2m} - K_{p_o+1} t^{2p_o+2}\right]^p$$

$$\geq \sum_{q=1}^{p+p_o-1} \frac{1}{(2q)!} \tilde{G}_{[p]}^{(2q)}(0)\, t^{2q} - \left[\sum_{q=p+p_o}^{p(p_o+1)} \frac{1}{(2q)!} G_{[p]}^{(2q)}(0)\right] t^{2(p_o+p)}$$

and

$$(3.7.6)\ldots \quad \left[\sum_{m=2}^{p_o} K_m t^{2m} + K_{p_o+1} t^{2p_o+2}\right]^p$$

$$\leq \sum_{q=1}^{p+p_o-1} \frac{1}{(2q)!} \tilde{G}_{[p]}^{(2q)}(0)\, t^{2q} + \left[\sum_{q=p+p_o}^{p(p_o+1)} \frac{1}{(2q)!} G_{[p]}^{(2q)}(0)\right] t^{2(p_o+p)} .$$

From (3.7.3) and (3.7.5),

$$e^{\sum_{m=2}^{p_o} K_m t^{2m} - K_{p_o+1} t^{2p_o+2}} - 1 \geq \sum_{p=1}^{p_o} \frac{1}{p!} \sum_{q=1}^{p+p_o-1} \frac{1}{(2q)!} \tilde{G}_{[p]}^{(2q)}(0)\, t^{2q}$$

$$- \sum_{p=1}^{p_o} \frac{1}{p!} \left[\sum_{q=p+p_o}^{p(p_o+1)} \frac{1}{(2q)!} G_{[p]}^{(2q)}(0)\right] t^{2(p_o+p)}$$

$$- \frac{e^{\frac{K}{16n}}}{(p_o+1)!} \left| \sum_{m=2}^{p_o} K_m t^{2m} - K_{p_o+1} t^{2p_o+2} \right|^{p_o+1}$$

Since $t^2 < 1$, $\left| \sum_{m=2}^{p_o} K_m t^{2m} - K_{p_o+1} t^{2p_o+2} \right|^{p_o+1} \leq K^{p_o+1} t^{4(p_o+1)}$.

Hence we have

$$(3.7.7)\ldots \quad e^{\sum_{m=2}^{p_o} K_m t^{2m} - K_{p_o+1} t^{2p_o+2}} - 1$$

$$\geq \sum_{p=1}^{p_o} \frac{1}{p!} \sum_{q=1}^{p+p_o-1} \frac{1}{(2q)!} \tilde{G}^{(2q)}_{[p]}(0)\, t^{2q}$$

$$- \sum_{p=1}^{p_o} \frac{1}{p!} \left[\sum_{q=p+p_o}^{p(p_o+1)} \frac{1}{(2q)!} G^{(2q)}_{[p]}(0) \right] t^{2(p+p_o)} - \frac{e^{\frac{K}{16n}}}{(p_o+1)!} K^{p_o+1} t^{4(p_o+1)}$$

$$= h_1(t) - h_2(t).$$

Similarly, from (3.7.4) and (3.7.6), we can show that

$$(3.7.8)\ldots \quad e^{\sum_{m=2}^{p_o} K_m t^{2m} - K_{p_o+1} t^{2p_o+2}} - 1 \leq h_1(t) + h_2(t)$$

From (3.7.1), (3.7.7) and (3.7.8) we have

$$(h_1(t) - h_2(t))\, e^{-\frac{1}{2} B_n t^2} \leq |\varphi_{S_n}(t)| - e^{-\frac{1}{2} B_n t^2} \leq (h_1(t) + h_2(t))\, e^{-\frac{1}{2} B_n t^2}$$

Hence

$$-h_2(t)\, e^{-\frac{1}{2} B_n t^2} \leq |\varphi_{S_n}(t)| - (1 + h_1(t))\, e^{-\frac{1}{2} B_n t^2} \leq h_2(t)\, e^{-\frac{1}{2} B_n t^2}$$

Since $h_2(t) \geq 0$,

$$\left| |\varphi_{S_n}(t)| - (1 + h_1(t))\, e^{-\frac{1}{2}B_n t^2} \right| \leq h_2(t)\, e^{-\frac{1}{2}B_n t^2}.$$

#

Theorem 3.8 Let X_j, $j = 1, 2, \ldots, n$ be independent random variables and S be a real number such that $0 < S < 1$. Let

i) each $\theta_{x_j}(t)$ has $(2p_o+2)$ - th derivative on $(-S, S)$ and there exists an a such that $\left|\theta_{X_j}^{(2p_o+2)}(t)\right| \leq a$ on $(-S, S)$ and

ii) each $\theta_{x_j}(t)$ has $(2p_o+1)$ - th continuous derivative on $[-S, S]$.

Let

$$\tilde{F}(t) = \sum_{m=1}^{p_o} \sum_{j=1}^{n} \theta_{x_j}^{(2m+1)}(0) \frac{t^{2m+1}}{(2m+1)!} + n\,a \frac{t^{2p_o+2}}{(2p_o+2)!},$$

$$F(t) = \sum_{m=1}^{p_o} \sum_{j=1}^{n} \left|\theta_{x_j}^{(2m+1)}(0)\right| \frac{t^{2m+1}}{(2m+1)!} + n\,a \frac{t^{2p_o+2}}{(2p_o+2)!},$$

$$h_3(t) = \sum_{i=1}^{p_o} \frac{(-1)^i}{(2i)!} \sum_{q=2i}^{2i+p_o-1} \frac{1}{(2q)!} \tilde{F}_{[2i]}^{(2q)}(0)\, t^{2q},$$

$$h_4(t) = \sum_{i=1}^{p_o} \frac{(-1)^{i-1}}{(2i-1)!} \sum_{q=2i-1}^{2i+p_o-1} \frac{1}{(2q-1)!} \tilde{F}_{[2i-1]}^{(2q-1)}(0)\, t^{(2q-1)}$$

and

$$h_5(t) = \sum_{i=1}^{p_o} \frac{1}{(2i)!} \left[\sum_{q=4i+2p_o}^{2i(2p_o+2)} \frac{1}{q!} F_{[2i]}^{(q)}(0) \right] t^{4i+2p_o}$$

$$+ \sum_{i=1}^{p_o} \frac{1}{(2i-1)!} \left[\sum_{q=4i+2p_o-2}^{(2i-1)(2p_o+2)} \frac{1}{q!} F_{[2i-1]}^{(q)}(0) \right] t^{4i+2p_o-2}$$

$$+ \frac{2}{(2p_o+1)!} (F_{[1]}(1))^{2p_o+1} t^{6p_o+3}.$$

If $t \varepsilon (0, S)$, then

$$\left| \sin(T\sqrt{B_n}t - \alpha(t)) - \sin(T\sqrt{B_n}t)(1 + h_3(t)) + \cos(T\sqrt{B_n}t)h_4(t) \right| < h_5(t).$$

<u>Proof</u> : Observe that for $j = 1, 2, \ldots, n$,

$$\varphi_{X_j}(t) = \int_{-\infty}^{\infty} e^{ixt} dF_X(x),$$

$$= \int_{-\infty}^{\infty} \cos xt \, dF_X(x) + i \int_{-\infty}^{\infty} \sin xt \, dF_X(x),$$

so we have $\theta_{X_j}(t) = \arctan \left(\frac{\int_{-\infty}^{\infty} \sin xt \, F_X(x)}{\int_{-\infty}^{\infty} \cos xt \, F_X(x)} \right)$ and $\theta_{X_j}(t)$ is odd.

Hence [see appendix]. For each $t \varepsilon (-S, S)$ and each $j \varepsilon \{1, 2, \ldots, n\}$, there is t_j such that

$$\theta_{X_j}(t) = \theta_{X_j}^{(1)}(0)t + \sum_{m=1}^{p_o} \theta_{X_j}^{(2m+1)}(0) \frac{t^{2m+1}}{(2m+1)!} + \theta_{X_j}^{(2p_o+2)}(t_j) \frac{t^{2p_o+2}}{(2p_o+2)!}.$$

Since $X_1, X_2, \ldots, X_n$ are independent,

$$\theta_{S_n}(t) = \sum_{j=1}^{n} \theta_{X_j}^{(1)}(0)\, t + \sum_{m=1}^{p_o} \sum_{j=1}^{n} \theta_{X_j}^{(2m+1)}(0) \frac{t^{2m+1}}{(2m+1)!}$$

$$+ \sum_{j=1}^{n} \theta_{X_j}^{(2p_o+2)}(t_j) \frac{t^{2p_o+2}}{(2p_o+2)!}.$$

Since $\theta_{X_j}^{(1)}(0) = E(X_j)$ and $\alpha(t) = \theta_{S_n}(t) - \sum_{j=1}^{n} E(X_j)$,

$$\alpha(t) = \sum_{m=1}^{p_o} \sum_{j=1}^{n} \theta_{X_j}^{(2m+1)}(0) \frac{t^{2m+1}}{(2m+1)!} + \sum_{j=1}^{n} \theta_{X_j}^{(2p_o+2)}(t_j) \frac{t^{2p_o+2}}{(2p_o+2)!}.$$

By using Taylor's formula expansion, there are α_o and α' such that

$$\cos(\alpha(t)) = \sum_{i=o}^{p_o} \frac{(-1)^i}{(2i)!} \alpha_{[2i]}(t) + \frac{1}{(2p_o+1)!} \sin\alpha_o \, \alpha_{[2p_o+1]}(t)$$

and

$$\sin(\alpha(t)) = \sum_{i=1}^{p_o} \frac{(-1)^{i-1}}{(2i-1)!} \alpha_{[2i-1]}(t) + \frac{1}{(2p_o+1)!} \cos\alpha' \, \alpha_{[2p_o+1]}(t).$$

Hence

$$\sin(T\sqrt{B_n}t) - \alpha(t)) = \sin(T\sqrt{B_n}t)\cos(\alpha(t)) - \cos(T\sqrt{B_n}t)\sin(\alpha(t)),$$

$$= \sin(T\sqrt{B_n}t) \left[\sum_{i=o}^{p_o} \frac{(-1)^i}{(2i)!} \alpha_{[2i]}(t) + \frac{1}{(2p_o+1)!} \sin\alpha_o \, \alpha_{[2p_o+1]}(t) \right]$$

$$- \cos(T\sqrt{B_n}t) \left[\sum_{i=1}^{p_o} \frac{(-1)^{i-1}}{(2i-1)!} \alpha_{[2i-1]}(t) + \frac{1}{(2p_o+1)!} \cos\alpha' \, \alpha_{[2p_o+1]}(t) \right],$$

$$\leq \sin(T\sqrt{B_n}t) \sum_{i=o}^{p_o} \frac{(-1)^i}{(2i)!} \alpha_{[2i]}(t) - \cos(T\sqrt{B_n}t) \sum_{i=1}^{p_o} \frac{(-1)^{i-1}}{(2i-1)!} \alpha_{[2i-1]}(t)$$

$$+ \frac{2}{(2p_o+1)!} \left| \alpha_{[2p_o+1]}(t) \right| ,$$

$$= \sin(T\sqrt{B_n}t) + \sin(T\sqrt{B_n}t) \sum_{i=1}^{p_o} \frac{(-1)^i}{(2i)!} \alpha_{[2i]}(t)$$

$$- \cos(T\sqrt{B_n}t) \sum_{i=1}^{p_o} \frac{(-1)^{i-1}}{(2i-1)!} \alpha_{[2i-1]}(t) + \frac{2}{(2p_o+1)!} \left|\alpha_{[2p_o+1]}(t)\right| .$$

Since $|\alpha(t)| \leq F(t)$ and $0 < t < 1$,

(3.8.1)... $\sin(T\sqrt{B_n}t - \alpha(t))$

$$\leq \sin(T\sqrt{B_n}t) + \sin(T\sqrt{B_n}t) \sum_{i=1}^{p_o} \frac{(-1)^{i}}{(2i)!} \alpha_{[2i]}(t)$$

$$- \cos(T\sqrt{B_n}t) \sum_{i=1}^{p_o} \frac{(-1)^{i-1}}{(2i-1)!} \alpha_{[2i-1]}(t)$$

$$+ \frac{2}{(2p_o+1)!} (F_{[1]}(1))^{2p_o+1} t^{6p_o+3} .$$

Similarly we can show that

(3.8.2)... $\sin(T\sqrt{B_n}t - \alpha(t))$

$$\geq \sin(T\sqrt{B_n}t) + \sin(T\sqrt{B_n}t) \sum_{i=1}^{p_o} \frac{(-1)^{i}}{(2i)!} \alpha_{[2i]}(t)$$

$$- \cos(T\sqrt{B_n}t) \sum_{i=1}^{p_o} \frac{(-1)^{i-1}}{(2i-1)!} \alpha_{[2i-1]}(t)$$

$$- \frac{2}{(2p_o+1)!} (F_{[1]}(1))^{2p_o+1} t^{6p_o+3} .$$

By Lemma 3.4, let $f(t) = \alpha(t)$, $g(t) = \tilde{F}(t)$ and $h(t) = F(t)$, we have

$$(3.8.3)\ldots \quad (-1)^i \alpha_{[2i]}(t) \le (-1)^i \sum_{q=o}^{4i+2p_o-1} \frac{1}{q!} \tilde{F}^{(q)}_{[2i]}(0)\, t^q + \left[\sum_{q=4i+2p_o}^{2i(2p_o+2)} \frac{1}{q!} F^{(q)}_{[2i]}(0)\right] t^{4i+2p_o}$$

and

$$(3.8.4)\ldots \quad (-1)^i \alpha_{[2i-1]}(t) \le (-1)^i \sum_{q=o}^{4i+2p_o-1} \frac{1}{q!} \tilde{F}^{(q)}_{[2i-1]}(0)\, t^q + \left[\sum_{q=4i+2p_o}^{(2i-1)(2p_o+2)} \frac{1}{q!} F^{(q)}_{[2i-1]}(0)\right] t^{4i+2p_o}.$$

From (3.8.1), (3.8.3) and (3.8.4), we have

$$(3.8.5)\ldots \quad \sin(T\sqrt{B_n}t) - \alpha(t))$$

$$\le \sin(T\sqrt{B_n}t) + \sin(T\sqrt{B_n}t) \sum_{i=1}^{p_o} \frac{(-1)^i}{(2i)!} \sum_{q=o}^{4i+2p_o-1} \frac{1}{q!} \tilde{F}^{(q)}_{[2i]}(0)\, t^q$$

$$- \cos(T\sqrt{B_n}t) \sum_{i=1}^{p_o} \frac{(-1)^{i-1}}{(2i-1)!} \sum_{q=o}^{4i+2p_o-3} \frac{1}{q!} \tilde{F}^{(q)}_{[2i-1]}(0)\, t^q$$

$$+ \sum_{i=1}^{p_o} \frac{1}{(2i)!} \left[\sum_{q=4i+2p_o}^{2i(2p_o+2)} \frac{1}{q!} F^{(q)}_{[2i]}(0)\right] t^{4i+2p_o}$$

$$+ \sum_{i=1}^{p_o} \frac{1}{(2i-1)!} \left[\sum_{q=4i+2p_o-2}^{(2i-1)(2p_o+2)} \frac{1}{q!} F^{(q)}_{[2i-1]}(0)\right] t^{4i+2p_o-2}$$

$$+ \frac{2}{(2p_o+1)!} (F_{[1]}(1))^{2p_o+1}\, t^{6p_o+3},$$

Since

$$\tilde{F}_{[2i]}(t) = \left[\sum_{m=1}^{p_o} \sum_{j=1}^{n} \theta_{X_j}^{(2m+1)}(0) \frac{t^{2m+1}}{(2m+1)!} + na \frac{t^{2p_o+2}}{(2p_o+2)!} \right]^{2i},$$

$$= t^{6i} \left[\sum_{m=1}^{p_o} \sum_{j=1}^{n} \theta_{X_j}^{(2m+1)}(0) \frac{t^{2m-2}}{(2m+1)!} + na \frac{t^{2p_o-1}}{(2p_o+2)!} \right]^{2i},$$

we observe that for a nonnegative interger q such that $0 \leq q < 4i$,

$\tilde{F}^{(q)}_{[2i]}(0) = 0$ and we see that all the coefficient of terms of odd degree

q such that $4i < q \leq 4i + 2p_o - 1$ are zero. Hence

$$\tilde{F}^{(q)}_{[2i]}(0) = 0$$

for all odd q such that $4i < q \leq 4i + 2p_o - 1$.

Therefore

$$\sum_{i=1}^{p_o} \frac{(-1)^i}{(2i)} \sum_{q=o}^{4i+2p_o-1} \frac{1}{q} \tilde{F}^{(q)}_{[2i]}(0)\, t^q = \sum_{i=1}^{p_o} \frac{(-1)^i}{(2i)!} \sum_{q=2i}^{2i+p_o-1} \frac{1}{(2q)!} \tilde{F}^{(q)}_{[2i]}(0)\, t^{2q},$$

$$= h_3(t).$$

By a similar reasoning, we see that

$$\sum_{i=1}^{p_o} \frac{(-1)^{i-1}}{(2i-1)!} \sum_{q=0}^{4i+2p_o-1} \frac{1}{q!} \tilde{F}^{(q)}_{[2i-1]}(0)\, t^q = \sum_{i=1}^{p_o} \frac{(-1)^{i-1}}{(2i-1)!} \sum_{q=2i}^{2i+p_o-1} \frac{1}{(2q-1)!} \tilde{F}^{(2q-1)}_{[2i-1]}(0)\, t^{2q-1},$$

$$= h_4(t).$$

Hence, by (3.8.5)

$$(3.8.6)\ldots \quad \sin(T\sqrt{B_n}t-\alpha(t)) \le \sin(T\sqrt{B_n}t)+\sin(T\sqrt{B_n}t)h_3(t) - \cos(T\sqrt{B_n}t)h_4(t)+h_5(t).$$

Similarly, by (3.8.2) and Lemma 3.4, we can show that

$$(3.8.7)\ldots \quad \sin(T\sqrt{B_n}t-\alpha(t)) \ge \sin(T\sqrt{B_n}t)+\sin(T\sqrt{B_n}t)h_3(t) - \cos(T\sqrt{B_n}t)h_4(t)-h_5(t).$$

From (3.8.6) and (3.8.7), we have

$$\left|\sin(T\sqrt{B_n}t-\alpha(t)) - \sin(T\sqrt{B_n}t)(1+h_3(t))+\cos(T\sqrt{B_n}t)h_4(t)\right| \le h_5(t).$$

#

Lemma 3.9. Let $X_1, X_2, \ldots, X_n$ be independent integral-valued random variables with finite variances. Let $\mathfrak{T}$ be as in Theorem 3.7. Then there exist constants $\hat{c}_1, \hat{c}_2, \ldots, \hat{c}_n$ such that

$$(3.9.1)\ldots \quad \text{i)} \quad |\varphi_{S_n}(t)| \le e^{\sum_{j=1}^{n}\hat{c}_j t^2}$$

for $t \in [0, \mathfrak{T}]$ and

$$(3.9.2)\ldots \quad \text{ii)} \quad \frac{1}{2}\left|\int_{\mathfrak{T}}^{\pi} \frac{|\varphi_{S_n}(t)|\sin(T\sqrt{B_n}t-\alpha(t))}{\sin\frac{t}{2}}\,dt\right| \le \frac{e^{-\sum_{j=1}^{n}\hat{c}_j\mathfrak{T}^2}}{4\sum_{j=1}^{n}\hat{c}_j\mathfrak{T}^2}.$$

Proof : i) Let $t \in [0, \pi]$. If there is j such that $|\varphi_{X_j}(t)| = 0$, then we choose $\hat{c}_1 = \hat{c}_2 = \ldots = \hat{c}_n = 0$. Suppose that for each j, $|\varphi_{X_j}(t)| > 0$. By (3.3.1), we have

$$\log|\varphi_{X_j}(t)| = -\frac{1}{2}\sum_{p=1}^{\infty}\frac{2^p}{p!}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sin^2((x-y)\tfrac{t}{2})\,dF_{X_j}(x)\,dF_{X_j}(y)\right]^p$$

Since X_j is an integral-valued random variable, we have

$$\log|\varphi_{X_j}(t)| = -\frac{1}{2}\sum_{p=1}^{\infty}\frac{2^p}{p!}\left[\sum_{k=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty}p_{X_j}(k)\,p_{X_j}(\ell)\sin^2((x-y)\tfrac{t}{2})\right]^p.$$

Hence

$$\log|\varphi_{X_j}(t)| \le -\sum_{k=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty}p_{X_j}(k)\,p_{X_j}(\ell)\sin^2((x-y)\tfrac{t}{2}),$$

$$\le -2\sum_{k=-\infty}^{\infty}p_{X_j}(k)\,p_{X_j}(k+1)\sin^2\frac{t}{2},$$

$$\le -\frac{2}{\pi^2}\sum_{k=-\infty}^{\infty}p_{X_j}(k)\,p_{X_j}(k+1)\,t^2$$

where the last inequality follows from the fact that $\sin\frac{t}{2} \ge \frac{t}{\pi}$ on $[0, \pi]$.

Let $\hat{c}_j = \frac{2}{\pi^2}\sum_{k=-\infty}^{\infty}p_{X_j}(k)\,p_{X_j}(k+1)$.

Hence

$$\log|\varphi_{S_n}(t)| \le -\sum_{j=1}^{n}c_j t^2,$$

$$|\varphi_{s_n}(t)| \leq e^{-\sum_{j=1}^{n} \hat{c}_j t^2} .$$

To prove (ii), we have

$$\frac{1}{2\pi}\left|\int_{\tau}^{\pi} \frac{|\varphi_{s_n}(t)| \sin(t\sqrt{B_n}t - \alpha(t))}{\sin\frac{t}{2}} dt\right|$$

$$\leq \frac{1}{2}\int_{\tau}^{\pi} \frac{|\varphi_{s_n}(t)|dt}{t} ,$$

$$\leq \frac{1}{2}\int_{\tau}^{\pi} \frac{e^{-\sum_{j=1}^{n} \hat{c}_j t^2}}{t} dt ,$$

$$\leq \frac{1}{2}\int_{\tau}^{\infty} \frac{e^{-\sum_{j=1}^{n} \hat{c}_j t^2}}{t} dt ,$$

$$\leq \frac{1}{4\sum_{j=1}^{n} \hat{c}_j \tau^2} e^{-\sum_{j=1}^{n} \hat{c}_j \tau^2} ,$$

where the second inequality follows from (i) and the last inequality follows from the fact that for $x > 0$, $\int_{x}^{\infty} \frac{e^{-t^2}}{t} dt \leq \frac{e^{-x^2}}{2x^2}$.

#

Lemma 3.10. For any positive integer p_o, there exist polynomial functions $A(t)$, $B(t)$ such that the following hold :

(i) $A(t)$ is of degree at most $2p_o - 2$,

(ii) $B(t)$ is of degree larger than $2p_o - 2$

and

$$\text{(iii)} \quad \left| \frac{1}{\sin\frac{t}{2}} - \frac{2}{t}(1 + A(t)) \right| \leq B(t)$$

for all $t \in (0, 1)$.

Proof :

Case $p_o = 1$. Since $\sin\frac{t}{2} = \frac{t}{2} - \frac{\cos t'}{48} t^3$ for some t',

we have
$$\left| \frac{1}{\sin\frac{t}{2}} - \frac{2}{t} \right| = 2 \left| \frac{\frac{t}{2} - \sin\frac{t}{2}}{t \sin\frac{t}{2}} \right| ,$$

$$\leq \frac{t^2}{24 \left|\sin\frac{t}{2}\right|} ,$$

$$\leq \frac{\pi t}{24} ,$$

where the last inequality follows from the fact that $\sin\frac{t}{2} \geq \frac{t}{\pi}$ on $[0, \pi]$

Hence

$$\left| \frac{1}{\sin\frac{t}{2}} - \frac{2}{t}(1 + A(t)) \right| \leq B(t)$$

where $A(t) = 0$ and $B(t) = \frac{\pi t}{24}$.

Case $p_o \geq 2$. Since $\sin\frac{t}{2} = \frac{t}{2}(1 + Z(t))$ where

$$Z(t) = \sum_{p=1}^{p_o - 1} \frac{(-1)^p}{(2p+1)!} \left(\frac{t}{2}\right)^{2p} + \frac{(-1)^{p_o}}{(2p_o+1)!} \cos t_o \left(\frac{t}{2}\right)^{2p_o}$$

for some t_o and $|Z(t)| < 1$ for $0 < t < 1$, we have

$$\frac{1}{\sin\frac{t}{2}} = \frac{2}{t(1+Z(t))} ,$$

$$= \frac{2}{t} \sum_{k=o}^{\infty} (-1)^k Z^k(t) ,$$

$$= \frac{2}{t} \left[\sum_{k=o}^{p_o-1} (-1)^k Z^k(t) + (-1)^{p_o} \frac{Z^{p_o}(t)}{1+Z(t)} \right] ,$$

$$= \frac{2}{t} + \frac{2}{t} \left[\sum_{k=1}^{p_o-1} (-1)^k Z^k(t) + (-1)^{p_o} \frac{Z^{p_o}(t)}{1+Z(t)} \right] ,$$

$$= \frac{2}{t} + \frac{2}{t} \sum_{k=1}^{p_o-1} (-1)^k \left[\sum_{p=1}^{p_o-1} \frac{(-1)^p}{(2p+1)!} \left(\frac{t}{2}\right)^{2p} + \frac{(-1)^{p_o}}{(2p_o+1)!} \cos t_o \left(\frac{t}{2}\right)^{2p_o} \right]^k$$

$$+ \frac{(-1)^{p_o}}{\sin\frac{t}{2}} \left[\sum_{p=1}^{p_o-1} \frac{(-1)^p}{(2p+1)!} \left(\frac{t}{2}\right)^{2p} + \frac{(-1)^{p_o}}{(2p_o+1)!} \cos t_o \left(\frac{t}{2}\right)^{2p_o} \right]^{p_o} .$$

Hence

$$(3.10.1)\ldots \quad \frac{1}{\sin\frac{t}{2}} \leq \frac{2}{t} + \frac{2}{t} \sum_{k=1}^{p_o-1} (-1)^k \left[\sum_{p=1}^{p_o-1} \frac{(-1)^p}{(2p+1)!} \left(\frac{t}{2}\right)^{2p} + \frac{(-1)^{p_o}}{(2p_o+1)!} \cos t_o \left(\frac{t}{2}\right)^{2p_o} \right]^k + \frac{1}{\left|\sin\frac{t}{2}\right|} \left[\sum_{p=1}^{p_o} \frac{1}{(2p+1)!} \left(\frac{t}{2}\right)^{2p} \right]^{p_o}$$

and

$$(3.10.2)\ldots \quad \frac{1}{\sin\frac{t}{2}} \geq \frac{2}{t} + \frac{2}{t} \sum_{k=1}^{p_o-1} (-1)^k \left[\sum_{p=1}^{p_o-1} \frac{(-1)^p}{(2p+1)!} \left(\frac{t}{2}\right)^{2p} + \frac{(-1)^{p_o}}{(2p_o+1)!} \cos t_o \left(\frac{t}{2}\right)^{2p_o} \right]^k$$

$$- \frac{1}{\left|\sin\frac{t}{2}\right|}\left[\sum_{p=1}^{p_o}\frac{1}{(2p+1)!}\left(\frac{t}{2}\right)^{2p}\right]^{p_o}.$$

Let

$$(3.10.3)\ldots\quad f(t) = \sum_{p=1}^{p_o-1}\frac{(-1)^p}{(2p+1)!}\left(\frac{t}{2}\right)^{2p} + \frac{(-1)^{p_o}}{(2p_o+1)!}\cos t_o\left(\frac{t}{2}\right)^{2p_o},$$

$$\tilde{D}(t) = \sum_{p=1}^{p_o}\frac{(-1)^p}{(2p+1)!}\left(\frac{t}{2}\right)^{2p}$$

and

$$D(t) = \sum_{p=1}^{p_o}\frac{1}{(2p+1)!}\left(\frac{t}{2}\right)^{2p},$$

By Lemma 3.4, for a positive integer i, we have

$$(-1)^i f_{[i]}(t) \leq (-1)^i\sum_{m=1}^{p_o-1}\frac{1}{(2m)!}\tilde{D}^{(2m)}_{[i]}(0)\left(\frac{t}{2}\right)^{2m} + \sum_{m=p_o}^{ip_o}\frac{1}{(2m)!}D^{(2m)}_{[i]}(0)\left(\frac{t}{2}\right)^{2p_o}.$$

By (3.10.3), we have

$$(-1)^i\left[\sum_{p=1}^{p_o-1}\frac{(-1)^p}{(2p+1)!}\left(\frac{t}{2}\right)^{2p} + \frac{(-1)^{p_o}}{(2p_o+1)!}\cos t_o\left(\frac{t}{2}\right)^{2p_o}\right]^i$$

$$\leq (-1)^i\sum_{m=1}^{p_o-1}\frac{1}{(2m)!}\tilde{D}^{(2m)}_{[i]}(0)\left(\frac{t}{2}\right)^{2m} + \sum_{m=p_o}^{ip_o}\frac{1}{(2m)!}D^{(2m)}_{[i]}(0)\left(\frac{t}{2}\right)^{2p_o}.$$

Hence, by (3.10.1), we have

$$\frac{1}{\sin\frac{t}{2}} \leq \frac{2}{t} + \sum_{k=1}^{p_o-1}(-1)^k\sum_{m=1}^{p_o-1}\frac{1}{(2m)!}\tilde{D}^{(2m)}_{[k]}(0)\left(\frac{t}{2}\right)^{2m-1}$$

$$+\left[\sum_{k=1}^{p_o-1}\sum_{m=p_o}^{kp_o}\frac{1}{(2m)!}D_{[k]}^{(2m)}(0)\right](\tfrac{t}{2})^{2p_o-1} + \frac{1}{\left|\sin\frac{t}{2}\right|}\left[\sum_{p=1}^{p_o}\frac{1}{(2p+1)!}(\tfrac{t}{2})^{2p}\right]$$

Since $\sin\frac{t}{2} \geq \frac{t}{\pi}$ on $[0,\pi]$ and $t \in (0,1)$ we have

$$(3.10.4)\ldots\quad \frac{1}{\sin\frac{t}{2}} \leq \frac{2}{t} + \sum_{m=1}^{p_o-1}\sum_{k=1}^{p_o-1}\frac{(-1)^k}{(2m)!}\tilde{D}_{[k]}^{(2m)}(0)(\tfrac{t}{2})^{2m-1}$$

$$+\left[\sum_{k=1}^{p_o-1}\sum_{m=p_o}^{kp_o}\frac{1}{(2m)!}D_{[k]}^{(2m)}(0)\right](\tfrac{t}{2})^{2p_o-1}$$

$$+\frac{\pi}{2}\left[\sum_{p=1}^{p_o}\frac{1}{(2p+1)!}\right](\tfrac{t}{2})^{2p_o-1}.$$

Let $A(t) = \frac{1}{2}\sum_{m=1}^{p_o-1}\sum_{k=1}^{p_o-1}\frac{(-1)^k}{(2m)!}\tilde{D}_{[k]}^{(2m)}(0)(\tfrac{t}{2})^{2m}$

and

$$B(t) = \left[\sum_{k=1}^{p_o-1}\sum_{m=p_o}^{kp_o}\frac{1}{(2m)!}D_{[k]}^{(2m)}(0) + \frac{\pi}{2}\sum_{p=1}^{p_o}\frac{1}{(2p+1)!}\right](\tfrac{t}{2})^{2p_o-1}.$$

Here from (3.10.4), we have

$$\frac{1}{\sin\frac{t}{2}} \leq \frac{2}{t} + \frac{2A(t)}{t} + B(t).$$

Similarly, we can show that

$$\frac{1}{\sin\frac{t}{2}} \geq \frac{2}{t} + \frac{2A(t)}{t} - B(t).$$

So $\left|\frac{1}{\sin\frac{t}{2}} - \frac{2}{t}(1 + A(t))\right| \leq B(t).$ #

<u>Notation</u>. In stating the next theorem, which is our main theorem, we need various notations introduced in Lemma 3.5, Lemma 3.6, Theorem 3.7, Theorem 3.8 and Lemma 3.9. For convenience, they are listed below.

Let $X_1, X_2, \ldots, X_n$ be any random variables and p_o be any positive integer. Let

$$g_p(X_j) = \sum_{m=o}^{2p} \binom{2p}{m} (-1)^{2p-m} E(X_j^m)E(X_j^{2p-m}) \quad \text{for } p = 1, 2, \ldots, p_o+1 .$$

For a positive integer p, let

$$\tilde{H}_{X_j,p}(t) = \left[\sum_{m=1}^{p_o+1} \frac{(-1)^{m-1}}{2(2m)!} g_m(X_j)t^{2m} \right]^p ,$$

$$H_{X_j,p}(t) = \left[\sum_{m=1}^{p_o+1} \frac{1}{2(2m)!} g_m(X_j)t^{2m} \right]^p .$$

$$c_p(X_j) = -\frac{1}{2} \sum_{m=1}^{p_o} \frac{2^m}{m(2p)!} \tilde{H}^{(2p)}_{X_j,m}(0) \quad \text{for } p = 1, 2, \ldots, P_o ,$$

$$c_{p_o+1}(X_j) = \frac{1}{2} \sum_{p=1}^{p_o} \sum_{q=p_o+1}^{p(p_o+1)} \frac{2^p}{p(2q)!} H^{(2q)}_{X_j,p}(0) + \sigma^{2p_o+2}(X_j),$$

$$K_p = \sum_{j=1}^{n} c_p(X_j) \quad \text{for } p = 1, 2, \ldots, p_o+1$$

$$K = \sum_{p=1}^{p_o+1} | K_p | ,$$

$$\tilde{G}(t) = \sum_{p=2}^{p_o+1} K_p t^{2p} ,$$

$$G(t) = \sum_{p=2}^{p_o+1} |K_p| \, t^{2p} ,$$

$$h_1(t) = \begin{cases} 0 & \text{if } p_o = 1 , \\ \sum_{m=1}^{p_o} \frac{1}{m!} \sum_{q=2}^{m+p_o-1} \frac{1}{(2q)!} \tilde{G}_{[m]}^{(2q)}(0)\, t^{2q} & \text{if } p_o > 1 , \end{cases}$$

$$h_2(t) = \begin{cases} K_2 e^{\frac{K_2}{16n}} t^4 & \text{if } p_o = 1 , \\ \sum_{m=1}^{p_o} \frac{1}{m!} \left[\sum_{q=m+p_o}^{m(2p_o+2)} \frac{1}{(2q)!} \tilde{G}_{[m]}^{(2q)}(0) \right] t^{2(m+p_o)} & \\ + \frac{1}{(p_o+1)!} e^{\frac{K}{16n}} K^{p_o+1} t^{4(p_o+1)} & \text{if } P_o > 1 . \end{cases}$$

Let $\mathcal{T} = \dfrac{1}{2\sqrt[4]{n}\sqrt{\sigma^2 max}}$ where $\sigma^2_{max} = \max\{\sigma^2(X_j) \,|\, j = 1,2,\ldots,n\} + 1$.

If each $\theta_{X_j}(t)$, $j = 1, 2, \ldots , n$ has $(2p_o+2)$ - th derivative and $\left|\theta_{X_j}^{(2p_o+2)}(t)\right| \leq a$ on $(-\mathcal{T} , \mathcal{T})$, then we have

$$\tilde{F}(t) = \sum_{m=1}^{p_o} \sum_{j=1}^{n} \theta_{X_j}^{(2m+1)}(0) \frac{t^{2m+1}}{(2m+1)!} + n\,a \frac{t^{2p_o+2}}{(2p_o+2)!} ,$$

$$F(t) = \sum_{m=1}^{p_o} \sum_{j=1}^{n} \left|\theta_{X_j}^{(2m+1)}(0)\right| \frac{t^{2m+1}}{(2m+1)!} + n\,a \frac{t^{2p_o+2}}{(2p_o+2)!} ,$$

$$h_3(t) = \sum_{i=1}^{p_o} \frac{(-1)^i}{(2i)!} \sum_{q=2i}^{2i+p_o-1} \frac{1}{(2q)!} \tilde{F}_{[2i]}^{(2q)}(0)\, t^{2q} ,$$

$$h_4(t) = \sum_{i=1}^{p_o} \frac{(-1)^{i-1}}{(2i-1)!} \sum_{q=2i}^{2i+p_o-1} \frac{1}{(2q-1)!} \tilde{F}_{[2i-1]}^{(2q-1)}(0)\, t^{2q-1} ,$$

and

$$h_5(t) = \sum_{i=1}^{p_o} \frac{1}{(2i)!}\left[\sum_{q=4i+2p_o}^{2i(2p_o+2)} \frac{1}{q!} F_{[2i]}^{(q)}(0)\right] t^{4i+2p_o}$$

$$+ \sum_{i=1}^{p_o} \frac{1}{(2i-1)!}\left[\sum_{q=4i+2p_o-2}^{(2i-1)(2p_o+2)} \frac{1}{q!} F_{[2i-1]}^{(q)}(0)\right] t^{4i+2p_o-2}$$

$$+ \frac{2}{(2p_o+1)!} (F_{[1]}(1))^{2p_o+1} t^{6p_o+3} .$$

For each $j = 1, 2, \ldots, n$, let

$$\hat{c}_j = \frac{2}{\pi^2} \sum_{k=-\infty}^{\infty} p_{X_j}(k)\, p_{X_j}(k+1) ,$$

and

$$c = \sum_{j=1}^{n} \hat{c}_j .$$

<u>Theorem 3.11</u> Let X_j, $j = 1, 2, \ldots, n$ be independent integral-valued random variables with finite moments of order $2p_o+2$ where p_o is a positive integer. Assume that $B_n > 0$. Let $A(t)$, $B(t)$ be as define in Lemma 3.10. If

i) each $\theta_{X_j}(t)$ has $(2p_o+2)$-th derivative on $(-\mathfrak{T}, \mathfrak{T})$ and there exists an a such that $\left|\theta_{X_j}^{(2p_o+2)}(t)\right| \leq a$ on $(-\mathfrak{T}, \mathfrak{T})$ and

ii) each $\theta_{X_j}(t)$ has $(2p_o+1)$-th continuous derivative on $[-\mathfrak{T}, \mathfrak{T}]$,

then

$$R(T) = \frac{1}{\sqrt{2\pi}} \int_0^T e^{-\frac{t^2}{2}} dt + \frac{1}{\pi} \int_0^\infty e^{-\frac{1}{2} B_n t^2} \frac{\sin(T\sqrt{B_n}t) A(t)}{t} dt$$

$$+ \frac{1}{\pi} \int_0^\infty e^{-\frac{1}{2} B_n t^2} \sin(T\sqrt{B_n}t)\left(\frac{1}{t} + \frac{A(t)}{t}\right)(h_1(t) + h_1(t)h_3(t) + h_3(t))dt$$

$$- \frac{1}{\pi} \int_0^\infty e^{-\frac{1}{2} B_n t^2} \cos(T\sqrt{B_n}t)\left(\frac{1}{t} + \frac{A(t)}{t}\right)(h_4(t) + h_1(t)h_4(t))dt$$

$$+ \Delta ,$$

where

$$|\Delta| < \frac{e^{-\tau^2 c}}{\tau^2 c} + \frac{1}{2\pi} \int_0^\infty e^{-ct^2} B(t)dt + \frac{1}{\pi} \int_0^\infty e^{-\frac{1}{2} B_n t^2} h_2(t) \left|\frac{1}{t} + \frac{A(t)}{t}\right| dt$$

$$+ \frac{1}{\pi} \left| \int_0^\tau e^{-\frac{1}{2} B_n t^2} (1 + h_1(t))h_5(t)\left(\frac{1}{t} + \frac{A(t)}{t}\right)dt \right| +$$

$$\frac{1}{\pi} \int_\tau^\infty e^{-\frac{1}{2} B_n t^2} \left|(1 + h_1(t))(1 + h_3(t) + h_4(t))\left(\frac{1}{t} + \frac{A(t)}{t}\right)\right| dt .$$

Proof : By Lemma 3.10, there are polynomial functions $A(t)$, $B(t)$ such that

$$(3.11.1)\ldots \quad \frac{1}{2\pi} \int_0^\tau \frac{|\varphi_{S_n}(t)| \sin(T\sqrt{B_n}t - \alpha(t))}{\sin \frac{t}{2}} dt$$

$$= \frac{1}{\pi} \int_0^\tau |\varphi_{S_n}(t)| \sin(T\sqrt{B_n}t - \alpha)\left(\frac{1}{t} + \frac{A(t)}{t}\right)dt + \Delta_1$$

such that $\quad |\Delta_1| \leq \frac{1}{2\pi} \int_0^\tau |\varphi_{S_n}(t)| B(t) dt ,$

$$\leq \frac{1}{2\pi} \int_0^{\tau} e^{-ct^2} B(t)\, dt ,$$

$$\leq \frac{1}{2\pi} \int_0^{\infty} e^{-ct^2} B(t)\, dt$$

where the second inequality follows from (3.9.1).

By Theorem 3.7, there are polynomial functions h_1, h_2 such that

$$(3.11.2)\ldots \quad \frac{1}{\pi} \int_0^{\tau} |\varphi_{s_n}(t)| \sin(T\sqrt{B_n}t - \alpha(t))\left(\frac{1}{t} + \frac{A(t)}{t}\right) dt$$

$$= \frac{1}{\pi} \int_0^{\tau} e^{-\frac{1}{2}B_n t^2} (1 + h_1(t)) \sin(T\sqrt{B_n}t - \alpha(t))\left(\frac{1}{t} + \frac{A(t)}{t}\right) dt + \Delta_2$$

where $$|\Delta_2| \leq \frac{1}{\pi} \int_0^{\tau} e^{-\frac{1}{2}B_n t^2} h_2(t) \left|\left(\frac{1}{t} + \frac{A(t)}{t}\right)\right| dt ,$$

$$\leq \frac{1}{\pi} \int_0^{\infty} e^{-\frac{1}{2}B_n t^2} h_2(t) \left|\frac{1}{t} + \frac{A(t)}{t}\right| dt.$$

By Theorem 3.8, there are polynomial functions h_3, h_4 and h_5 such that

$$(3.11.3)\ldots \quad \frac{1}{\pi} \int_0^{\tau} e^{-\frac{1}{2}B_n t^2} (1 + h_1(t)) \sin(T\sqrt{B_n}t - \alpha(t))\left(\frac{1}{t} + \frac{A(t)}{t}\right) dt$$

$$= \frac{1}{\pi} \int_0^{\tau} e^{-\frac{1}{2}B_n t^2} (1 + h_1(t))(\sin(T\sqrt{B_n}t)(1 + h_3(t)) - \cos(T\sqrt{B_n}t)h_4(t))$$

$$\left(\frac{1}{t} + \frac{A(t)}{t}\right) dt + \Delta_3$$

where

$$|\Delta_3| \leq \frac{1}{\pi}\left|\int_0^{\delta} e^{-\frac{1}{2}B_n t^2}(1+h_1(t))h_5(t)\left(\frac{1}{t}+\frac{A(t)}{t}\right)dt\right| .$$

From (3.11.1), (3.11.2) and (3.11.3), we have

(3.11.4)... $$\frac{1}{\pi}\int_0^{\delta}\frac{|\varphi_{s_n}(t)|\sin(T\sqrt{B_n}t-\alpha(t))}{\sin\frac{t}{2}}\,dt$$

$$= \frac{1}{\pi}\int_0^{\delta} e^{\frac{1}{2}B_n t^2}\frac{\sin(T\sqrt{B_n}t)}{t}\,dt + \frac{1}{\pi}\int_0^{\delta} e^{\frac{1}{2}B_n t^2}\frac{\sin(T\sqrt{B_n}t)}{t}A(t)dt,$$

$$+ \frac{1}{\pi}\int_0^{\delta} e^{-\frac{1}{2}B_n t^2}\sin(T\sqrt{B_n}t)\left(\frac{1}{t}+\frac{A(t)}{t}\right)(h_1(t)+h_1(t)h_3(t)+h_3(t))dt,$$

$$- \frac{1}{\pi}\int_0^{\delta} e^{-\frac{1}{2}B_n t^2}\cos(T\sqrt{B_n}t)\left(\frac{1}{t}+\frac{A(t)}{t}\right)(h_4(t)+h_1(t)h_4(t))dt + \Delta_4,$$

where $$|\Delta_4| \leq \frac{1}{2\pi}\int_0^{\infty} e^{-ct^2}B(t)\,dt + \frac{1}{\pi}\int_0^{\infty} e^{-\frac{1}{2}B_n t^2}h_2(t)\left|\frac{1}{t}+\frac{A(t)}{t}\right|dt$$

$$+ \frac{1}{\pi}\left|\int_0^{\delta} e^{-\frac{1}{2}B_n t^2}(1+h_1(t))h_5(t)\left(\frac{1}{t}+\frac{A(t)}{t}\right)dt\right|$$

Since

$$R(T) = \frac{1}{2\pi}\int_0^{\delta}\frac{|\varphi_{s_n}(t)|\sin(T\sqrt{B_n}t-\alpha(t))}{\sin\frac{t}{2}}\,dt$$

$$+ \frac{1}{2\pi}\int_{\delta}^{\pi}\frac{|\varphi_{s_n}(t)|\sin(T\sqrt{B_n}t-\alpha(t))}{\sin\frac{1}{2}}\,dt ,$$

by 3.9.2, we have

$$R(T) = \frac{1}{2\pi} \int_0^{\varsigma} \frac{|\varphi_{s_n}(t)| \sin(T\sqrt{B_n}t - \alpha(t))}{\sin\frac{t}{2}} dt + \Delta_5$$

where

$$|\Delta_5| \leq \frac{e^{-c\varsigma^2}}{4c\varsigma^2} .$$

Hence by (3.11.4), we have

$$R(T) = \frac{1}{\pi} \int_0^{\varsigma} \frac{e^{-\frac{1}{2}B_n t^2} \sin(T\sqrt{B_n}t)}{t} dt + \frac{1}{\pi} \int_0^{\varsigma} \frac{e^{-\frac{1}{2}B_n t^2} \sin(T\sqrt{B_n}t) A(t)}{t} dt$$

$$+ \frac{1}{\pi} \int_0^{\varsigma} e^{-\frac{1}{2}B_n t^2} \sin(T\sqrt{B_n}t)\left(\frac{1}{t} + \frac{A(t)}{t}\right)(h_1(t) + h_1(t)h_3(t) + h_3(t))\, dt$$

$$- \frac{1}{\pi} \int_0^{\varsigma} e^{-\frac{1}{2}B_n t^2} \cos(T\sqrt{B_n}t)\left(\frac{1}{t} + \frac{A(t)}{t}\right)(h_4(t) + h_1(t)h_4(t))\, dt + \Delta_6$$

where $|\Delta_6| \leq |\Delta_4| + |\Delta_5|$.

Hence

$$R(T) = \frac{1}{\pi} \int_0^{\infty} \frac{e^{-\frac{1}{2}B_n t^2} \cdot \sin(T\sqrt{B_n}t)}{t} dt + \frac{1}{\pi} \int_0^{\infty} \frac{e^{-\frac{1}{2}B_n t^2} \sin(T\sqrt{B_n}t) A(t)}{t} dt$$

$$+ \frac{1}{\pi} \int_0^{\infty} e^{-\frac{1}{2}B_n t^2} \sin(T\sqrt{B_n}t)\left(\frac{1}{t} + \frac{A(t)}{t}\right)(h_1(t) + h_1(t)h_3(t) + h_3(t))\, dt$$

$$- \frac{1}{\pi} \int_0^{\infty} e^{-\frac{1}{2}B_n t^2} \cos(T\sqrt{B_n}t)\left(\frac{1}{t} + \frac{A(t)}{t}\right)(h_4(t) + h_1(t)h_4(t)))\, dt + \Delta_6 + \Delta_7,$$

where $|\Delta_7| \leq \frac{1}{\pi} \int_{\varsigma}^{\infty} e^{-\frac{1}{2}B_n t^2} \left| (1 + h_1(t))(1 + h_3(t) + h_4(t))\left(\frac{1}{t} + \frac{A(t)}{t}\right) \right| dt.$

Since $\int_0^\infty e^{-\frac{1}{2}B_n t^2} \frac{\sin(T\sqrt{B_n}t)}{t} dt = \sqrt{\frac{\pi}{2}} \int_0^T e^{-\frac{1}{2}B_n t^2} dt$,

$$R(T) = \frac{1}{\sqrt{2\pi}} \int_0^T e^{-\frac{1}{2}B_n t^2} dt + \frac{1}{\pi} \int_0^\infty e^{-\frac{1}{2}B_n t^2} \frac{\sin(T\sqrt{B_n}t)A(t)}{t} dt$$

$$+ \frac{1}{\pi} \int_0^\infty e^{-\frac{1}{2}B_n t^2} \sin(T\sqrt{B_n}t)\left(\frac{1}{t} + \frac{A(t)}{t}\right)(h_1(t) + h_1(t)h_3(t) + h_3(t))\, dt$$

$$- \frac{1}{\pi} \int_0^\infty e^{-\frac{1}{2}B_n t^2} \cos(T\sqrt{B_n}t)\left(\frac{1}{t} + \frac{A(t)}{t}\right)(h_4(t) + h_1(t)h_4(t))dt + \Delta$$

where $|\Delta| < \frac{e^{-\gamma^2 c}}{\gamma^2 c} + \frac{1}{2\pi} \int_0^\infty e^{-ct^2} B(t)\, dt + \frac{1}{\pi} \int_0^\infty e^{-\frac{1}{2}B_n t^2} h_2(t) \left|\frac{1}{t} + \frac{A(t)}{t}\right| dt$

$$+ \frac{1}{\pi} \int_0^\infty e^{-\frac{1}{2}B_n t^2} \left|(1 + h_1(t))h_5(t)\left(\frac{1}{t} + \frac{A(t)}{t}\right)\right| dt$$

$$+ \frac{1}{\pi} \int_\gamma^\infty e^{-\frac{1}{2}B_n t^2} \left|(1 + h_1(t))(1 + h_3(t) + h_4(t))\left(\frac{1}{t} + \frac{A(t)}{t}\right)\right| dt$$

#