CHAPTER I
PRELIMINARIES

1.1 Random Variables and Various Modes of Convergence

A probability space is a measure space $(\Omega, \mathfrak{A}, P)$ in which P is a positive measure such that $P(\Omega) = 1$. The set Ω will be refered to as a sample space. The elements of $\mathfrak{A}$ are called events. For any event A, the value P(A) is called the probability of A.

A function X from a probability space $(\Omega, \mathfrak{A}, P)$ to the set of complex numbers $\mathbb{C}$ is said to be a complex-valued random variable if for every Borel set B in $\mathbb{C}$, $X^{-1}[B]$ belongs to $\mathfrak{A}$. In the case that X is real-valued, we say that it is a real-valued random variable, or simply a random variable. We note that the composition between a Borel function and a complex-valued random variable is also a complex-valued random variable.

We will use the notations $P(X \leq x)$, $P(X \geq x)$ and $P(|X| \geq x)$ to denote $P(\{\omega | X(\omega) \leq x\})$, $P(\{\omega | X(\omega) \geq x\})$ and $P(\{\omega \mid |X(\omega)| \geq x\})$ respectively.

We define the expectation of a complex-valued random variable X to be

$$\int_{\Omega} X dP$$

provided that the integral $\int_{\Omega} X \, dP$ exists. It will be denoted by E[X].

Proposition 1.1.1 ([3], p.174) Let $X_1, X_2, X_3, \ldots, X_n$ be random variables. Then

$$E[X_1 + X_2 + \dots + X_n] = \sum_{j=1}^{n} E[X_j],$$

provided that the sums on the right hand side is meaningful.

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and $\mathcal{Y}$ be a topological space. Let X, $X_1, X_2, X_3, \dots, X_n$ be measurable functions from Ω to $\mathcal{Y}$. We will write

$$X_n \longrightarrow X \quad \text{a.e. } [\mu]$$

if (X_n) converges to X almost everywhere with respect to μ. In the case $\Omega = \mathbb{R}^{(k)}$ and μ is the Lebesgue measure, we simply write

$$X_n \longrightarrow X \quad \text{a.e.}.$$

From now on, we shall assume that all our complex-valued random variables, including real-valued random variables, are defined on a common probability space $(\Omega, \mathcal{A}, P)$.

A sequence (X_n) of complex-valued random variables converges in probability to a complex-valued random variable X if (X_n) converges in measure to X. In this case we write

$$X_n \xrightarrow{P} X.$$

In the case that $E[|X|]$ and $E[|X_n|]$, n = 1, 2, 3,..., are finite. We say that (X_n) converges in the mean to X and write

$$X_n \xrightarrow{m} X$$

if $E[|X_n - X|] \to 0$.

The following theorems are known properties of convergence in probability and convergence in the mean.

Theorem 1.1.2. ([9], P.63) Let $X, X_1, X_2, X_3, \dots$, be complex-valued random variables.

(i) $X_n \xrightarrow{P} X$ if and only if for every subsequence (X_{n_k}) of (X_n) contains a subsequence $(X_{n_{k_r}})$ such that $X_{n_{k_r}} \longrightarrow X$ a.e.[P].

(ii) If $X_n \xrightarrow{P} X$ and $g : \mathbb{C} \to \mathbb{C}$ is continuous, then $g(X_n) \xrightarrow{P} g(X)$.

Theorem 1.1.3. ([12], P.201) Let $X, X_1, X_2, X_3, \ldots$, and $Y, Y_1, Y_2, Y_3, \ldots$, be complex-valued random variables. If $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$ then $X_n + Y_n \xrightarrow{P} X + Y$.

Theorem 1.1.4. ([9], P.46) Let $X, Y, X_1, X_2, X_3, \ldots$, be complex-valued random variables.

(i) If $X_n \xrightarrow{P} X$ and $X_n \xrightarrow{P} Y$, then $X = Y$ a.e. [P].

(ii) If $X_n \xrightarrow{P} X$, then for every subsequence (X_{n_k}) of (X_n) $X_{n_k} \xrightarrow{P} X$.

Theorem 1.1.5. ([9], P.49) Let $X, X_1, X_2, X_3, \ldots$, be complex-valued random variables.

(i) If $X_n \xrightarrow{m} X$ then $X_n \xrightarrow{P} X$.

(ii) If $X_n \xrightarrow{P} X$ and there exists a complex-valued random variable Y such that $E[|Y|] < \infty$ and for each n, $|X_n| \le |Y|$ a.e. [P] then $X_n \xrightarrow{m} X$.

1.2 Distribution Functions and Characteristic Functions.

A function F from R to R is said to be a distribution function if it is non-decreasing, right-continuous, $F(-\infty) = 0$ and $F(+\infty) = 1$.

For any random variable X, the function $F : \mathbb{R} \to \mathbb{R}$ defined by

$$F(x) = P(X \le x)$$

is a distribution function. It is the distribution function of the random variable X.

Theorem 1.2.1 ([3], p.57) In order that a function F is a distribution function of a random variable it is necessary and sufficient that F is non-decreasing, right-continuous, $F(-\infty) = 0$ and $F(+\infty) = 1$.

Proposition 1.2.2 ([9], p.28) Let X be a random variable with the distribution function F. If E[X] exists, then

$$E[X] = \int_{-\infty}^{\infty} x dF(x).$$

The expectation of a random variable X is also known as the mean. The expectation of $(X - E[X])^2$ is known as the variance of X and it denoted by $\sigma^2(X)$. Note that mean or variance of a random variable may be infinite.

Let F be a distribution function. The function $\varphi : \mathbb{R} \longrightarrow \mathbb{C}$ defined by

$$\varphi(t) = \int_{-\infty}^{\infty} e^{itx} d F(x)$$

is called the characteristic function of the distribution function F. If F is the distribution function of a random variable X, then φ is also called the characteristic function of X.

Proposition 1.2.3 ([8], p.45) For any characteristic function φ, we have

(i) $\varphi(0) = 1$

(ii) $|\varphi(t)| \leq 1$ for every t

(iii) φ is uniformly continuous on $\mathbb{R}$.

Proposition 1.2.4 ([11], p.45)

(i) The product of two characteristic functions is a characteristic function.

(ii) If φ is a characteristic function, then $|\varphi|^2$ is also a characteristic function.

Theorem 1.2.5 (Bochner's Theorem, [11], p.62) A function $\varphi:\mathbb{R} \to \mathbb{C}$ is a characteristic function if and only if the following hold.

(i) $\varphi(0) = 1$

(ii) φ is continuous

(iii) for any positive integer m, the sum

$$\sum_{i=1}^{m}\sum_{j=1}^{m}\varphi(t_i - t_j)c_i\bar{c}_j$$

is real and non-negative for any real numbers $t_1, t_2 \ldots, t_m$ and any complex numbers $c_1, c_2, \ldots, c_m$.

Proposition 1.2.6 ([6], p.477) Let (F_n) be a sequence of distribution functions and let (φ_n) be a sequence of corresponding characteristic functions. Let (p_n) be a sequence of non-negative numbers such that

$\sum_{k=1}^{\infty} p_k = 1$. Then the function

$$F(x) = \sum_{k=1}^{\infty} p_k F_k(x)$$

is a distribution function and the function

$$\varphi(t) = \sum_{k=1}^{\infty} p_k \varphi_k(t)$$

is the characteristic function of F.

Any random variables $X_1, X_2, X_3,..., X_n$ are called <u>independent</u> if

$$P(\bigcap_{j=1}^{n}\{\omega|X_j(\omega)\le x_j\}) = \prod_{j=1}^{n}P(X_j\le x_j)$$

holds for every real numbers $x_1, x_2,...,x_n$.

A sequence of random variables (X_n) is said to be a <u>sequence of independent random variables</u> if $X_{i_1},X_{i_2},X_{i_3},...,X_{i_n}$ are independent for all distinct $i_1, i_2, i_3,...,i_n$.

<u>Theorem 1.2.7</u>. ([3], p.188,191) Let $X_1, X_2, X_3,..., X_n$ be random variables with the characteristic functions $\varphi_1, \varphi_2, \varphi_3,..., \varphi_n$ respectively. Assume that $X_1, X_2, X_3,..., X_n$ are independent. Let

$$Y_n = X_1+X_2+X_3+...+X_n.$$

Then the following hold .

(i) The characteristic function φ of Y_n is given by

$$\varphi(t) = \varphi_1(t).\varphi_2(t).\varphi_3(t).... \varphi_n(t).$$

(ii) $$\sigma^2(Y_n) = \sigma^2(X_1)+\sigma^2(X_2)+\sigma^2(X_3)...+\sigma^2(X_n).$$

<u>Theorem 1.2.8</u>. ([8], P.48) Let F be a distribution function and φ be its characteristic function. If x_1 and x_2 are continuity points of F, then

$$F(x_2) - F(x_1) = \frac{1}{2\pi}\lim_{c\to\infty}\int_{-c}^{c}\left(\frac{e^{-itx_1}-e^{-itx_2}}{it}\right)\varphi(t)dt.$$

Remark 1.2.9 It follows from the above theorem that a distrubution function is uniquely determined by its charateristic function ([8], p.50).

Let $F, F_1, F_2, F_3,\ldots,$ be bounded non-decreasing functions. A sequence (F_n) converges weakly to F if

(i) for every continuity point x of F, $F_n(x) \to F(x)$, and

(ii) $F_n(\pm\infty) \longrightarrow F(\pm\infty)$.

We will write

$$F_n \xrightarrow{w} F$$

if (F_n) converges weakly to F. Note that the weak limit of the sequence (F_n), if it exists, is unique. In the following theorems we state some facts of weak convergence which be used in our work.

Theorem 1.2.10 (Helly's Theorem, [9], P.133) Let (F_n) be a sequence of uniformly bounded, non-decreasing, right-continuous functions. Then (F_n) contains a subsequence which converges weakly to a bounded, non-decreasing, right-continuous function.

Let $\mathfrak{M}$ be the set of bounded, non-decreasing, right-continuous functions M from R into $[0,\infty)$ which vanish at $-\infty$. The function L defined for any $M_1, M_2 \in \mathfrak{M}$ by

$$L(M_1,M_2) = \inf_{h\geq 0} \{h \mid M_1(x-h)-h \leq M_2(x) \leq M_1(x+h)+h \text{ for every } x\}$$

is complete metric on $\mathfrak{M}$. ([8],p.39)

In the following corollary, it follows from Theorem 1.2.10 and the fact that the elements in $\mathfrak{M}$ vanish at $-\infty$.

Corollary 1.2.11 Let (M_n) be a uniformly bounded sequence of elements in $\mathfrak{M}$. Then it contains a subsequence which converges weakly to an element in $\mathfrak{M}$.

Theorem 1.2.12 ([8], p.39) Let $M, M_1, M_2, M_3, \ldots$, be elements of $\mathfrak{M}$. Then the following statements are equivalent.

(i) $M_n \xrightarrow{w} M$.

(ii) For every bounded continuous function g on R,

$$\int_{-\infty}^{\infty} g(x)dM_n(x) \longrightarrow \int_{-\infty}^{\infty} g(x)dM(x),$$

(iii) $L(M_n, M) \to 0$.

In the following, we summarize facts concerning weak convergence of distribution functions needed in our work.

Theorem 1.2.13 ([13], p.15) Let (F_n) and (φ_n) be sequences of distribution functions and their characteristic functions. Let F be a distribution function with the characteristic function φ. If $F_n \xrightarrow{w} F$, then (φ_n) converges to φ uniformly in an arbitrary finite interval.

Theorem1.2.14 ([13], p.15) Let (F_n) and (φ_n) be sequences of distribution functions and their characteristic functions. Let φ be a complex-valued function which is continuous at 0. If (φ_n) converges to φ for every t, then there exists a distribution function F such that $F_n \xrightarrow{w} F$ and the characteristic function of F is φ.

1.3) Infinitely Divisible Distribution Functions.

A disl ribution function F with the characteristic function φ is said to be infinitely divisible if for every natural number n, there exists a characteristic functions φ_n such that for every t,

$$\varphi(t) = \{\varphi_n(t)\}^n.$$

The characteristic function of any infintely divisible distribution function is also said to be infinitely divisible. A random variable is said to be infinitely divisible if its distribution function is infinitely divisible.

Theorem 1.3.1 ([7], p.305) If φ is an infinitely divisible characteristic function, then for every t, $\varphi(t) \neq 0$.

Proposition 1.3.2 ([11], p.81) If φ is an infinitely divisible characteristic function, then $|\varphi|^2$ is also infinitely divisible characteristic function.

Theorem 1.3.3 ([7], p.307) In order that a disbribution function F with finite variance is infinitely divisible it is necessary and sufficient that there exist the constant μ and a non-decreasing function of bounded variation K such that the logarithm of its characteristic function φ is given by

$$\text{(1)} \qquad \mathrm{Log}\varphi(t) = i\mu t + \int_{-\infty}^{\infty} f(t,x)\,d\,K(x),$$

where

$$f(t,x) = \begin{cases} (e^{itx} - 1 - itx)\dfrac{1}{x^2} & \text{if } x \neq 0, \\ -\dfrac{t^2}{2} & \text{if } x = 0. \end{cases}$$

In the sequel, $f(t,x)$ always denote this function. The formula (1) is known as Kolmogorov formula.

Theorem 1.3.4 ([9], p.246) For each infinitely divisible distribution function, the function K in Theorem 1.3.2 can be chosen to be right-continuous and $K(-\infty) = 0$. The function K in this theorem is unique.

Throughout this work, we assume that the function K in Kolmogorov formula has properties in the Theorem 1.3.4.

Theorem 1.3.5 ([8],p.85) Let X be an infinitely divisible random variable with finite variance. Let the constant μ and the function K be given in the Kolmogorov formula of the characteristic function of X. Then

(i) $E[X] = \mu$

(ii) $\text{var}(X) = K(+\infty)$.

Theorem 1.3.6 ([11],p.81) The product of a finite number of infinitely divisible characteristic functions is infinitely divisible.

Theorem 1.3.7 ([11],p.82) A characteristic function which is the limit of a sequence of infinitely divisible characteristic functions is infinitely divisible.

1.4 Kolmogorov Theorems.

In this section, we let (X_{nj}) be a double sequence of random variables with finite variances. Here, we assume that $j = 1, 2, 3,..., j_n$, $n = 1, 2, 3,...$. For each n and j, we let μ_{nj}, σ_{nj}^2 and F_{nj} be the expectation, variance and distrubution function of X_{nj} respectively.

In [8], Kolmogorov gives a necessary and sufficient condition for weak convergence of the disbribution functions of the sums

$$S_n = X_{n1} + X_{n2} + \ldots + X_{nj_n} - A_n$$

where (A_n) is a sequence of real numbers. There are two important convergence theorems (Theorem 1.4.1 and Theorem 1.4.2). In the first theorem (X_{nj}) must satisfy the following conditions .

(α) $(X_{nj} - \mu_{nj})$ is infinitesimal, i.e., for every $\varepsilon > 0$

$$\sup_{1 \le j \le j_n} P(|X_{nj} - \mu_{nj}| \ge \varepsilon) \to 0.$$

(β) There exists a real number c such that

$$\sum_{j=1}^{j_n} \sigma_{nj}^2 < c.$$

In order to prove the first theorem, Kolmogorov defines the <u>accompanying distribution function</u> of the sums

$$S_n = X_{n1} + X_{n2} + \ldots + X_{nj_n} - A_n.$$

to be the distribution function whose logarithm of its characteristic function is given by

$$\operatorname{Log}\psi_n(t) = -iA_n t + it \sum_{j=1}^{j_n} \mu_{nj} + \sum_{j=1}^{j_n} \int_{-\infty}^{\infty} (e^{itx} - 1)\, dF_{nj}(x + \mu_{nj}).$$

<u>Theorem 1.4.1</u> ([8], p.18) Assume that (X_{nj}) satisfies the conditions (α), (β) and for each n, $X_{n1}, X_{n2}, \ldots, X_{nj_n}$ are independent. Then there exists a sequence (A_n) of real numbers such that the distribution functions of the sums

$$S_n = X_{n1} + X_{n2} + \ldots + X_{nj_n} - A_n$$

converge weakly to a limit distribution function if and only if the accompanying distribution functions of S_n converge weakly to the same limit distribution function.

Theorem 1.4.2 ([8], p.100) Assume that (X_{nj}) satisfy the condition (α) and for each n, $X_{n1}, X_{n2}, \ldots, X_{nj_n}$ are independent. Then there exists a sequence (A_n) of real numbers such that

(i) the distribution functions of the sums

$$S_n = X_{n1} + X_{n2} + \ldots + X_{nj_n} - A_n$$

converge weakly to a limit distribution function F whose variance is σ^2, and

(ii) $$\sum_{j=1}^{j_n} \sigma_{nj}^2 \to \sigma^2,$$

if and only if there exists a function K in $\mathfrak{M}$ such that

(i') $K_{j_n}(u) \to K(u)$, for every continuity point u of K, and

(ii') $K_{j_n}(+\infty) \to K(+\infty)$

where

$$K_{j_n}(u) = \sum_{j=1}^{j_n} \int_{-\infty}^{u} x^2 \, dF_{nj}(x+\mu_{nj}).$$

The constants A_n may be chosen according to the formula

$$A_n = \sum_{j=1}^{j_n} \mu_{nj} - \mu$$

where μ is any real number. Logarithm of the characteristic function of the limit distribution function is given by

$$\text{Log}\varphi(t) = i\mu t + \int_{-\infty}^{\infty} f(t,x) d\, K(x).$$