

Chapter 4

Collective Excitation of the Non-Interacting Bosons System

4.1 Experimental Background

After the realization of Bose-Einstein condensation (BEC) in dilute atomic vapors [1,2,3], the dynamical aspects of the system have been studied. Technically, the excitations are generated in response to small time-dependent perturbations of the trapping potential. There are various procedures and consequently various types of excitations. The first experiments have been performed by modulating the trapping potential and resulted in the observation of center-of-mass oscillations (also called sloshing modes) and shape oscillations. The MIT group [9] observed the shape oscillation of the trapped Bose condensate as shown in Fig. 4.1. This is a low-lying collective excitation of the condensate. The experimental setup for creating Bose condensates can be described briefly as follows. After the sodium atoms were optically cooled and trapped, they were transferred into a magnetic trap where they were further cooled by RF-induced evaporation. The condensate obtained in this way was confined in a cloverleaf magnetic trap which had cylindrical symmetry with trapping frequencies of 19 Hz axially and 250 Hz radially. The condensate was then excited by a time-dependent modulation of the trapping potential. The bias magnetic field was used to excite the shape oscillation; a weak oscillation of the radial width was also detected.

In the simplest case of a non-interacting ideal gas in an isotropic harmonic potential, it was found that all modes of oscillation have frequencies which are

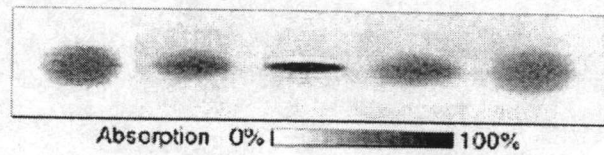


Figure 4.1: Shape oscillation of a Bose-Einstein condensate [9].

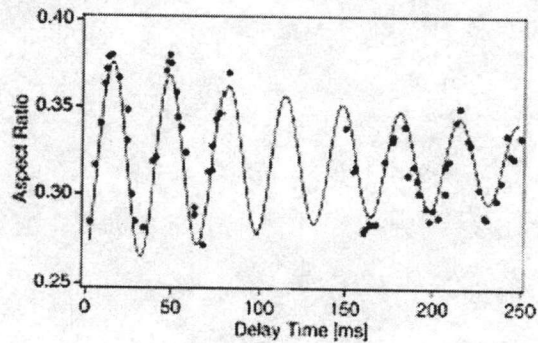


Figure 4.2: Oscillation at 30 Hz. Damping of the harmonic oscillations was observed with decay time of 250 ms [9].

integer multiples of the harmonic trapping frequency. In a homogeneous weakly interacting Bose condensate, the lowest frequency excitations of the density are phonons propagating at the speed of sound. The frequency is independent of the number of atoms in the condensate since the dependences on the sound velocity and on the size of the condensate exactly cancel (see [9] for details). It was also found that the damping of the oscillation as shown in Fig. 4.2 is probably due to the thermal interaction between the condensate and the thermal gas.

From the damping of the shape oscillation the lifetime of this quasi-particle can be determined. For the system with nearly pure condensate $T \approx 0$ the damping due to thermal excitation can be neglected. Studying of this low lying excitation is fundamental for a better understanding of dynamical aspects

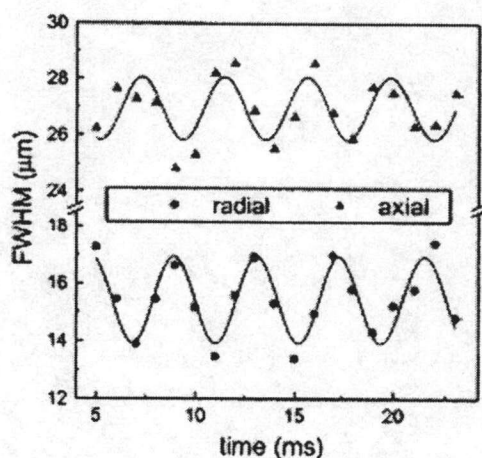


Figure 4.3: Zero-temperature excitation data from Jin et al. [10]. The oscillation of the condensate in radial and axial widths are shown as 180° out of phase.

of the condensate system. A similar experiment had been done by JILA group [10]. They found the relation between the width of the condensate and time as shown in Fig. 4.5, and also found the oscillation of the shape of the condensate. They claimed that the interaction energy determines the excitation spectrum of the condensate.

After these two experiments, many theoretical explanations had been done [25,26]. Elementary excitations (also called quasiparticles or normal modes) of the Bose-Einstein condensate are solutions of the linearized Gross-Pitaevskii equation. The coherent excitation of many quasiparticles leads to collective oscillations or density modulations (also called sound) of the trapped atomic cloud. The frequencies of these excitations agreed well with theoretical calculations [25,26]. The oscillations were damped by interactions between the collective modes and thermal excitations. For an ideal gas the frequencies of the excitations induced are simply multiples of the trap frequencies; for an interacting conden-

sate, however, deviations from these frequencies are expected and were indeed observed in these experiments.

Although this phenomenon is successfully described within the framework of Gross-Pitaevskii mean field theory, it is interesting to devise the many-body path integral method to study this system.

4.2 Path Integral Approach

In this section, we consider the time-evolution of the prepared state represented by $\rho_\nu(\bar{x}, \beta | \bar{x}, 0)$ where $\bar{x} \equiv \{x_1, x_2, \dots, x_N\}$, the density matrix of bosons in a harmonic trap with the trap frequency ν . We first consider the single-particle density matrix, then the generalization to the many-body density matrix will follow naturally. We can write the density matrix in the formal way as

$$\rho_w^1(\beta) = \sum_n |n\rangle p_n(\beta) \langle n| \quad (4.1)$$

where the weighting factor $p_n(\beta) = e^{-\beta E_n} / Z_1(\beta)$, the single-particle partition function $Z_1(\beta) = \sum_n e^{-\beta E_n}$, and $|n\rangle$ denotes the eigenstate with energy E_n of the oscillator of frequency ν . Let $U_w(t, 0)$ denote the time-evolution operator of the harmonic oscillator of frequency w . The time-dependent density matrix is then

$$\rho_w^1(t, \beta) = \sum_n U_w(t, 0) |n\rangle p_n(\beta) \langle n| U_w^\dagger(t, 0) \quad (4.2)$$

and its coordinate representation of the time-dependent density matrix is

$$\begin{aligned} \rho_\beta(\bar{x}, t | \bar{x}') &= \langle \bar{x} | \rho_\nu^1(t, \beta) | \bar{x}' \rangle = \langle \bar{x} | U_w(t, 0) \rho_\nu^1(\beta) U_w^\dagger(t, 0) | \bar{x}' \rangle \\ &= \int \int d^N x'' d^N x''' \langle \bar{x} | U_w(t, 0) | \bar{x}'' \rangle \langle \bar{x}'' | \rho_\nu^1(\beta) | \bar{x}''' \rangle \langle \bar{x}''' | U_w^\dagger(t, 0) | \bar{x}' \rangle \\ &= \int \int d^N x'' d^N x''' K_w(\bar{x}, it | \bar{x}'') \rho_\nu(\bar{x}'', \beta | \bar{x}''') K_w(\bar{x}''', -it | \bar{x}'). \end{aligned} \quad (4.3)$$

Above $K_w(\bar{x}, it|\bar{x}'') = \langle x|U_w(t, 0)|x''\rangle$ is represented as the path integral (in real time) of the system with the Hamiltonian of harmonic oscillator of frequency w and $\rho_\nu(\bar{x}'', \beta|\bar{x}''') = \langle \bar{x}''|\rho_\nu^1(\beta)|\bar{x}'''\rangle = \langle \bar{x}''|e^{-\beta H_\nu}|\bar{x}'''\rangle$ is the equilibrium density matrix at temperature T related to β by $\beta = 1/(k_B T)$.

4.2.1 The Time-Dependent Density Matrix (TDDM)

To simulate the phenomenon described in the Section 4.1, we start by looking at the time-evolution of the system of particles confined in a harmonic potential under the changing of the frequency of the trap from ν to w . For simplicity we assume the isotropy of the potential ($w_x = w_y = w_z$). The generalization to the anisotropic case is not so difficult. From Eq. (4.3), the time-dependent density matrix is

$$\rho_\beta(x, t|x') = \int \int K_w(x, it|x'') \rho_\nu(x'', \beta|x''') K_w(x''', -it|x') dx''' dx''. \quad (4.4)$$

Note that we work in one dimension for convenience. The generalization to the three dimensional case is obvious. Using the well-known results, we find

$$\rho_\nu(x'', \beta|x') = \left(\frac{\nu}{2\pi \sinh \nu\beta} \right)^{\frac{1}{2}} \exp \left(-\frac{\nu}{2} \frac{((x'')^2 + (x')^2) \cosh \nu\beta - 2x'x''}{\sinh \nu\beta} \right), \quad (4.5)$$

$$K_w(x'', it|x') = \left(\frac{1}{2i\pi \sin wt} \frac{w}{\sin wt} \right)^{\frac{1}{2}} \exp \left(-\frac{w}{2} \frac{((x'')^2 + (x')^2) \cos wt - 2x'x''}{i \sin wt} \right). \quad (4.6)$$

Then the density matrix can be calculated:

$$\begin{aligned} \rho_\beta(x, t|x') &= \left(\frac{1}{8\pi^3} \frac{\nu}{\sinh \nu\beta} \frac{w^2}{\sin^2 wt} \right)^{\frac{1}{2}} \int \int dx''' dx'' \\ &\times \left(\exp \left(\frac{wi}{2} \frac{(x^2 + (x'')^2) \cos wt - 2xx''}{\sin wt} \right) \right. \\ &\times \exp \left(-\frac{\nu}{2} \frac{((x''')^2 + (x'')^2) \cosh \nu\beta - 2x'''x''}{\sinh \nu\beta} \right) \\ &\left. \times \exp \left(-\frac{wi}{2} \frac{((x''')^2 + (x')^2) (\cos wt) - 2x'''x'}{\sin wt} \right) \right). \quad (4.7) \end{aligned}$$

The integral over dx''' can be evaluated with the result

$$\begin{aligned}
\rho_\beta(x, t|x') &= \left(\frac{1}{8\pi^3} \frac{\nu w^2}{\sin wt} \right)^{\frac{1}{2}} \left(\frac{2\pi}{(\nu \sin wt \cosh \nu\beta + wi \cos wt \sinh \nu\beta)} \right)^{\frac{1}{2}} \\
&\times \exp \left(i \frac{w \cos wt}{2 \sin wt} (x^2 - (x')^2) \right) \\
&\times \int dx'' \exp \left[-\frac{1}{2} \frac{(\nu \cosh \nu\beta \sin wt - wi \cos wt \sinh \nu\beta)}{\sin wt \sinh \nu\beta} (x'')^2 \right. \\
&\quad \left. - \frac{wi}{\sin wt} x x'' + \frac{\sin wt \sinh \nu\beta}{2(\nu \cosh \nu\beta \sin wt + wi \cos wt \sinh \nu\beta)} \right. \\
&\quad \left. \times \left(\frac{\nu^2 (x'')^2}{\sinh^2 \nu\beta} - \frac{w^2 (x')^2}{\sin^2 wt} + \frac{2\nu wi x' x''}{\sinh \nu\beta \sin wt} \right) \right]. \quad (4.8)
\end{aligned}$$

After evaluating the integral over dx'' we obtain the final form of the time-dependent density matrix:

$$\begin{aligned}
\rho_\beta(x, t|x') &= \left(\frac{\nu w^2}{2\pi \sinh \nu\beta (\nu^2 \sin^2 wt + w^2 \cos^2 wt)} \right)^{\frac{1}{2}} \\
&\times \exp \left[-\frac{w}{2(\nu^2 \sin^2 wt + w^2 \cos^2 wt) \sinh \nu\beta} \{-2\nu w x x' \right. \\
&\quad \left. + [i(\nu^2 - w^2) \sinh \nu\beta \cos wt \sin wt + w\nu \cosh \nu\beta] (x')^2 \right. \\
&\quad \left. + [-i(\nu^2 - w^2) \sinh \nu\beta \cos wt \sin wt + \nu w \cosh \nu\beta] x^2 \right] \quad (4.9)
\end{aligned}$$

We now discuss the properties of this TDDM. The partition function can be found immediately as

$$Z_1(\beta) = \int \rho_\beta(x, t|x) dx = \frac{1}{2 \sinh \frac{1}{2} \nu\beta} \quad (4.10)$$

and the thus density of a distinguishable particle in three dimensions is

$$\begin{aligned}
n(\vec{r}, t) &= \frac{\rho_\beta(\vec{r}, t|\vec{r})}{Z_1(\beta)} \\
&= \left(\frac{\nu w^2 \tanh \frac{1}{2} \nu\beta}{\pi (\nu^2 \sin^2 wt + w^2 \cos^2 wt)} \right)^{\frac{3}{2}} e^{\left(-\vec{r}^2 \frac{\nu w^2 \tanh \frac{1}{2} \nu\beta}{(\nu^2 \sin^2 wt + w^2 \cos^2 wt)} \right)} \quad (4.11)
\end{aligned}$$

Note that the density is time dependent while the partition function is not. Moreover, we find that our TDDM has the semi-group property hence the sum over

all permutations can be applied to this system. For the non-interacting particle system, we can easily see that the particle density is independent of the number of the particles. This is because of the distinguishable nature of the particles in the system. We shall see later that this is not the case of the indistinguishable particle system where each particle is aware of the existence of other particles hence the density is N -dependent. To calculate the density we need to know the partition function first.

4.2.2 The Partition Function for Indistinguishable Particles

We know from Chapter 2 that to find the partition function for a system of indistinguishable particles we need to sum over all permutations:

$$\mathbb{Z}(N) = \frac{1}{N!} \sum_{M_1 M_2 \dots M_N} \left(\frac{N!}{\prod_{\gamma} M_{\gamma}! \gamma^{M_{\gamma}}} \right) \prod_{\gamma} Z_1(\gamma\beta)^{M_{\gamma}}. \quad (4.12)$$

The sum $\sum_{M_1 M_2 \dots M_N}$ is subject to the constraint $\sum_{\gamma} M_{\gamma} \gamma = N$. Notice that $\mathbb{Z}(N)$ is a canonical partition function of an N -particle system and $Z_1(\gamma\beta)$ is a single-particle partition function with imaginary time interval $\gamma\beta$ which comes from a cycle of length γ . As previously pointed out in Chapter 2, we need to remove the constraint by introducing the grand canonical partition function

$$\begin{aligned} \Xi(u) &= \sum_{N=0}^{\infty} \mathbb{Z}(N) u^N \\ &= \exp \left[\sum_{\gamma=1}^{\infty} \frac{Z_1(\gamma\beta) u^{\gamma}}{\gamma} \right]. \end{aligned} \quad (4.13)$$

In fact, this quantity has physical meaning as a partition function for a system in which the particle number is not conserved but here, we use it as a generating

function for finding the canonical partition function as

$$\mathbb{Z}(N) = \frac{1}{N!} \left. \frac{d^N}{du^N} \Xi(u) \right|_{u=0}. \quad (4.14)$$

This formula leads to the recursion relation for the canonical partition function.

We shall show some detailed calculation as follows.

Consider the derivative,

$$\frac{d}{du} \Xi(u) = \Xi(u) \sum_{\gamma=0}^{\infty} Z_1((\gamma+1)\beta) u^\gamma. \quad (4.15)$$

Then the N^{th} derivative can be calculated by using the binomial theorem

$$\begin{aligned} \frac{d^N}{du^N} \Xi(u) &= \frac{d^{N-1}}{du^{N-1}} \left(\frac{d}{du} \Xi(u) \right) \\ &= \sum_{m=0}^{N-1} \binom{N-1}{m} \left(\frac{d^{N-1-m}}{du^{N-1-m}} \Xi(u) \right) \left(\frac{d^m}{du^m} \sum_{\gamma=0}^{\infty} Z_1((\gamma+1)\beta) u^\gamma \right) \\ &= \sum_{m=0}^{N-1} \binom{N-1}{m} \left(\frac{d^{N-1-m}}{du^{N-1-m}} \Xi(u) \right) \sum_{\gamma=0}^{\infty} \frac{\gamma! u^{\gamma-m}}{(\gamma-m)!} Z_1((\gamma+1)\beta), \end{aligned} \quad (4.16)$$

where $\binom{N-1}{m} = \frac{(N-1)!}{(N-1-m)!m!}$, and from Eq. (4.14) we see that

$$\left. \frac{d^{N-1-m}}{du^{N-1-m}} \Xi(u) \right|_{u=0} = (N-1-m)! \mathbb{Z}(N-1-m).$$

When we set $u = 0$ in Eq. (4.16) only the terms with $m = \gamma$ survive and after redefining $\gamma + 1 \rightarrow \gamma$ we find

$$\mathbb{Z}(N) = \frac{1}{N} \sum_{\gamma=1}^{N-1} \mathbb{Z}(N-\gamma) Z_1(\gamma\beta). \quad (4.17)$$

Thus for a system of non-interacting particles or the interaction can be decoupled as in the model of Brosens et al. [7,8] the canonical partition function can be found by this recursion relation provided that we know the single-particle partition function. However this formula is not useful if we want to evaluate the partition function numerically.

4.2.3 Recursion Relation for Partition Function

The recurrence relation in Eq. (4.17) can be written in our case in terms of a single-partition function of the harmonic oscillator in three dimensions as

$$\begin{aligned} \mathbb{Z}(N) &= \frac{1}{N} \sum_{\gamma=1}^N \mathbb{Z}(N-\gamma) \frac{1}{(2 \sinh \frac{1}{2} \gamma \nu \beta)^3} \\ &= \frac{1}{N} \sum_{m=0}^{N-1} \frac{b^{\frac{3}{2}(N-m)}}{(1-b^{(N-m)})^3} \mathbb{Z}(m), \end{aligned} \quad (4.18)$$

where $b = e^{-\beta \nu}$ (in SI unit this factor is $e^{-\hbar \beta \nu}$). In general, the factor $\frac{b^{\frac{3}{2}(N-m)}}{(1-b^{(N-m)})^3}$ is very small. For example, at low temperature and moderate number of particles with $b = 0.75$ and $N = 1000$, this factor is of the order 10^{-188} which is almost impossible for numerical calculation. So we define a quantity $\varrho(N)$, by

$$\varrho(N) = \frac{\mathbb{Z}(N)}{\mathbb{Z}(N-1)} b^{-\frac{3}{2}}, \quad (4.19)$$

in order to single out the small terms. Then we find

$$\begin{aligned} \frac{\mathbb{Z}(N)}{\mathbb{Z}(m)} &= \frac{\mathbb{Z}(N)}{\mathbb{Z}(N-1)} \frac{\mathbb{Z}(N-1)}{\mathbb{Z}(N-2)} \cdots \frac{\mathbb{Z}(m+1)}{\mathbb{Z}(m)} \\ &= \varrho(N) b^{\frac{3}{2}} \varrho(N-1) b^{\frac{3}{2}} \cdots \varrho(m+1) b^{\frac{3}{2}} \\ &= b^{\frac{3}{2}(N-m)} \prod_{j=m+1}^N \varrho(j), \end{aligned} \quad (4.20)$$

and from Eq. (4.19) we can find $\varrho(1)$ as

$$\varrho(1) = b^{-\frac{3}{2}} \frac{\mathbb{Z}(1)}{\mathbb{Z}(0)} = b^{-\frac{3}{2}} \frac{b^{\frac{3}{2}}}{(1-b)^3} = \frac{1}{(1-b)^3}. \quad (4.21)$$

The next ϱ can be successively obtained from Eq. (4.20) and Eq. (4.18), for example,

$$\begin{aligned} \varrho(2) &= b^{-\frac{3}{2}} \frac{\mathbb{Z}(2)}{\mathbb{Z}(1)} = \frac{b^{-\frac{3}{2}}}{\mathbb{Z}(1)} \frac{1}{2} \sum_{m=0}^1 \frac{b^{\frac{3}{2}(2-m)}}{(1-b^{(2-m)})^3} \mathbb{Z}(m) \\ &= \frac{1}{2} \left[\frac{1}{(1-b^2)^3} \frac{1}{\varrho(1)} + \frac{1}{(1-b)^3} \right] \end{aligned} \quad (4.22)$$

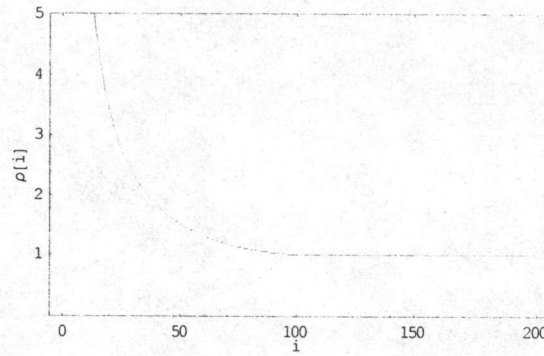


Figure 4.4: The behaviour of the function $\varrho(i)$ with i being the number of particles.

$$\begin{aligned}\varrho(3) &= b^{-\frac{3}{2}} \frac{\mathbb{Z}(3)}{\mathbb{Z}(2)} = \frac{b^{-\frac{3}{2}}}{\mathbb{Z}_I(2)} \frac{1}{3} \sum_{m=0}^2 \frac{b^{\frac{3}{2}(3-m)}}{(1-b^{3-m})^3} \mathbb{Z}(m) \\ &= \frac{1}{3} \left[\left(\frac{1}{(1-b^{(3)})^3} \varrho(1) + \frac{1}{(1-b^{(2)})^3} \right) \frac{1}{\varrho(2)} + \frac{1}{(1-b)^3} \right].\end{aligned}\quad (4.23)$$

Defining

$$\xi(j) \equiv \frac{1}{(1-b^j)^3}, \quad (4.24)$$

by deduction, we obtain the formula

$$\varrho(N) = \frac{1}{N} \left(\xi(1) + \sum_{l=2}^{N-1} \xi(l) \prod_{k=N+1-l}^{N-1} \frac{1}{\varrho(k)} \right). \quad (4.25)$$

The character of this function can be shown graphically in Fig. 4.4. Then the partition function can be calculated numerically. However, the aim of this chapter is to find the density of indistinguishable particles which will be shown in the next subsection.

4.2.4 Density of Indistinguishable Particles

If we want to compare our theory with the experimental results then the quantity of interest is the density distribution of the particles. In this subsection, we

calculate this quantity for a system of indistinguishable particles. Recall that the density of an N -particle system is

$$n(\vec{r}, t) = \frac{1}{N} \sum_{i=1}^N \langle \delta^3(\vec{r} - \vec{r}_i) \rangle = \frac{1}{(2\pi)^3} \int d^3q e^{-i\vec{q}\cdot\vec{r}} n_{\vec{q}} \quad (4.26)$$

where $n_{\vec{q}} = \frac{1}{N} \sum_{i=1}^N \langle e^{i\vec{q}\cdot\vec{r}_i} \rangle$. When the particles are indistinguishable, the permutation sum must be applied in the same way as in [7,8]. Note that this case is simpler than the case of distinguishable particles because there is no center of mass coordinates. The explicit form of $n_{\vec{q}}$ is

$$n_{\vec{q}} = \frac{1}{N} \sum_{i=1}^N \frac{1}{N!} \sum_P \frac{1}{\mathbb{Z}(N)} \int d\vec{r}_1 d\vec{r}_2 \cdots d\vec{r}_N \prod_{j=1}^N \rho_{\beta}(P\vec{r}_j, t | \vec{r}_j, 0) e^{i\vec{q}\cdot\vec{r}_i}. \quad (4.27)$$

The above equation can be decomposed into permutation cycles; as in the computation of the partition function but in this case there are two types of the permutation cycles, one is the cycle that contains $e^{i\vec{q}\cdot\vec{r}_i}$ and the others are those without this factor. If the factor $e^{i\vec{q}\cdot\vec{r}_i}$ appears in a cycle of length α and there are M_{α} cycles of length α , then there are αM_{α} ways of putting this factor into a cycle. After summing over i and performing the integration these αM_{α} terms give the same result. Thus the sum over particle numbers $\sum_{i=1}^N$ is changed to sum over cycle lengths $\sum_{\alpha=1}^N \alpha M_{\alpha}$. Then we can write the Fourier transform of the density as

$$n_{\vec{q}} = \frac{1}{N\mathbb{Z}(N)} \sum_{M_1 \dots M_N} \sum_{\alpha=1}^N \left(\alpha M_{\alpha} \mathcal{K}_{\alpha}(\vec{q}) \frac{1}{M_{\alpha}! \alpha^{M_{\alpha}}} \mathcal{K}_{\alpha}^{M_{\alpha}-1} \right) \prod_{\gamma \neq \alpha} \frac{1}{M_{\gamma}! \gamma^{M_{\gamma}}} \mathcal{K}_{\gamma}^{M_{\gamma}} \quad (4.28)$$

where the term in the brackets corresponds to a cycle which contains the factor $e^{i\vec{q}\cdot\vec{r}}$. The cycle containing this factor is

$$\mathcal{K}_{\gamma}(\vec{q}) = \int d\vec{r}_{\gamma+1} \cdots \int d\vec{r}_1 \delta(\vec{r}_{\gamma+1} - \vec{r}_1) e^{i\vec{q}\cdot\vec{r}_{\alpha}} \prod_{j=1}^{\gamma} \rho_{\beta}(\vec{r}_{j+1}, t | \vec{r}_j, 0). \quad (4.29)$$

and those do not contain the factor can be written as

$$\mathcal{K}_\gamma = \int d\vec{r}_{\gamma+1} \cdots \int d\vec{r}_1 \delta(\vec{r}_{\gamma+1} - \vec{r}_1) \prod_{j=1}^{\gamma} \rho_\beta(\vec{r}_{j+1}, t | \vec{r}_j, 0), \quad (4.30)$$

Since the single-particle TDDM satisfies the semigroup property, then performing integration is equivalent to replacing the imaginary time β by $\gamma\beta$ for the cycle of length γ . Thus we can evaluate Eq. (4.30)

$$\mathcal{K}_\gamma = \left(\frac{1}{2 \sinh \frac{\gamma\nu\beta}{2}} \right)^3. \quad (4.31)$$

We can see that this is simply a partition function of single particle. For $\mathcal{K}_\gamma(\vec{q})$, we obtain

$$\mathcal{K}_\gamma(\vec{q}) = \left(\frac{1}{2 \sinh \frac{\gamma\nu\beta}{2}} \right)^3 \exp \left[-\vec{q}^2 \frac{(\nu^2 \sin^2 wt + w^2 \cos^2 wt)}{4\nu w^2 \tanh \frac{\gamma\nu\beta}{2}} \right]. \quad (4.32)$$

Collecting the above results, one finds

$$n_{\vec{q}} = \frac{1}{N\mathbb{Z}(N)} \sum_{M_1 \cdots M_N} \sum_{\alpha} \alpha M_\alpha \exp \left[-\vec{q}^2 \frac{(\nu^2 \sin^2 wt + w^2 \cos^2 wt)}{4\nu w^2 \tanh \frac{\alpha\nu\beta}{2}} \right] \quad (4.33)$$

$$\times \prod_{\gamma=1}^N \frac{1}{M_\gamma! \gamma^{M_\gamma}} \left(\frac{1}{2 \sinh \frac{\gamma\nu\beta}{2}} \right)^{3M_\gamma}. \quad (4.34)$$

To remove the constraint $\sum_{\gamma} \gamma M_\gamma = N$, as in Chapter 2, we introduce the generating function for the grand canonical ensemble as

$$\mathcal{G}_1(u, \mathbf{q}) = \sum_{N=0}^{\infty} [\mathbb{Z}(N) N n_{\vec{q}}] u^N \quad (4.35)$$

which can be simplified as follows. From Eq. (4.28) we write

$$\begin{aligned}
\mathcal{G}_1(u, \vec{q}) &= \sum_{N=0}^{\infty} \sum_{M_1 \dots M_N} \sum_{\alpha=1}^N \left(\alpha M_\alpha \mathcal{K}_\alpha(\vec{q}) \frac{u^{\alpha M_\alpha}}{M_\alpha! \alpha^{M_\alpha}} \mathcal{K}_\alpha^{M_\alpha-1} \right) \prod_{\gamma \neq \alpha} \frac{(u^\gamma \mathcal{K}_\gamma / \gamma)^{M_\gamma}}{M_\gamma!} \\
&= \sum_{M_1=0}^{\infty} \dots \sum_{M_N=0}^{\infty} \sum_{\alpha=1}^{\infty} \left(u^\alpha \mathcal{K}_\alpha(\vec{q}) \frac{(u^\alpha \mathcal{K}_\alpha / \alpha)^{M_\alpha-1}}{(M_\alpha-1)!} \right) \prod_{\gamma \neq \alpha} \frac{(u^\gamma \mathcal{K}_\gamma / \gamma)^{M_\gamma}}{M_\gamma!} \\
&= \sum_{\alpha=1}^{\infty} \left(u^\alpha \mathcal{K}_\alpha(\vec{q}) \sum_{M_\alpha=1}^{\infty} \frac{(u^\alpha \mathcal{K}_\alpha / \alpha)^{M_\alpha-1}}{(M_\alpha-1)!} \right) \prod_{\gamma \neq \alpha} \sum_{M_\gamma=0}^{\infty} \frac{(u^\gamma \mathcal{K}_\gamma / \gamma)^{M_\gamma}}{M_\gamma!} \\
&= \sum_{\alpha=1}^{\infty} u^\alpha \mathcal{K}_\alpha(\vec{q}) \prod_{\gamma=1}^N \sum_{M_\gamma=0}^{\infty} \frac{(u^\gamma \mathcal{K}_\gamma / \gamma)^{M_\gamma}}{M_\gamma!}.
\end{aligned}$$

using the explicit forms of $\mathcal{K}_\gamma(\vec{q})$ and \mathcal{K}_γ , we obtain

$$\begin{aligned}
\mathcal{G}_1(u, \vec{q}) &= \Xi(u) \sum_{\gamma=1}^{\infty} \exp \left[-\vec{q}^2 \frac{(\nu^2 \sin^2 wt + w^2 \cos^2 wt)}{4\nu w^2} \coth \frac{\gamma\nu\beta}{2} \right] \\
&\quad \times \left(\frac{u^\gamma}{(2 \sinh \frac{\gamma\nu\beta}{2})^3} \right)
\end{aligned} \tag{4.36}$$

where

$$\begin{aligned}
\Xi(u) &= \sum_{M=0}^{\infty} \prod_{\gamma=1}^{\infty} \frac{1}{M_\gamma!} \left(\frac{u^\gamma}{\gamma (2 \sinh \frac{\gamma\nu\beta}{2})^3} \right)^{M_\gamma} \\
&= \exp \left[\sum_{\gamma=1}^{\infty} \frac{u^\gamma}{\gamma (2 \sinh \frac{\gamma\nu\beta}{2})^3} \right]
\end{aligned} \tag{4.37}$$

is the generating function for the partition function of N identical particles as found in the previous subsection. The Fourier transform of the density can be found from this generating function as

$$\frac{1}{NZ(N) N!} \frac{d^N}{du^N} \mathcal{G}_1(u, \vec{q}) \Big|_{u=0} = \frac{1}{NZ(N) N!} \frac{d^N}{du^N} \sum_{k=0}^{\infty} [Z(k) k n_{\vec{q}}] u^k \Big|_{u=0} = n_{\vec{q}}. \tag{4.38}$$

Thus

$$n_{\vec{q}} = \frac{1}{N\mathbb{Z}(N)N!} \frac{d^N}{du^N} \left[\Xi(u) \sum_{\gamma=1}^{\infty} \exp \left[-\vec{q}^2 \frac{(\nu^2 \sin^2 wt + w^2 \cos^2 wt)}{4\nu w^2} \coth \frac{\gamma\nu\beta}{2} \right] \times \left(\frac{u^\gamma}{(2 \sinh \frac{\gamma\nu\beta}{2})^3} \right) \right]_{u=0}. \quad (4.39)$$

Using the binomial theorem as in the case of finding the recursion relation for the partition function, we find

$$n_{\vec{q}} = \frac{1}{N} \sum_{\gamma=1}^N \frac{\exp \left[-\vec{q}^2 \frac{(\nu^2 \sin^2 wt + w^2 \cos^2 wt)}{4\nu w^2} \coth \frac{\gamma\nu\beta}{2} \right] \mathbb{Z}(N-\gamma)}{(2 \sinh \frac{\gamma\nu\beta}{2})^3 \mathbb{Z}(N)}. \quad (4.40)$$

The spatial density distribution can be found by

$$\begin{aligned} n(\vec{r}, t) &= \frac{1}{N} \sum_{\gamma=1}^N \frac{1}{(2 \sinh \frac{\gamma\nu\beta}{2})^3} \frac{\mathbb{Z}(N-\gamma)}{\mathbb{Z}(N)} \\ &\times \int \frac{d^3 q}{(2\pi)^3} \exp \left[-\vec{q}^2 \frac{(\nu^2 \sin^2 wt + w^2 \cos^2 wt)}{4\nu w^2} \coth \frac{\gamma\nu\beta}{2} - i\vec{q} \cdot \vec{r} \right] \\ &= \frac{1}{N} \sum_{\gamma=1}^N \frac{1}{(2 \sinh \frac{\gamma\nu\beta}{2})^3} \frac{\mathbb{Z}(N-\gamma)}{\mathbb{Z}(N)} \left(\frac{\nu w^2 \tanh \frac{\gamma\nu\beta}{2}}{\pi (\nu^2 \sin^2 wt + w^2 \cos^2 wt)} \right)^{\frac{3}{2}} \\ &\times \exp \left[-\vec{r}^2 \frac{\nu w^2 \tanh \frac{\gamma\nu\beta}{2}}{(\nu^2 \sin^2 wt + w^2 \cos^2 wt)} \right] \\ &\equiv \frac{1}{N} \sum_{\gamma=1}^N \frac{\mathbb{Z}(N-\gamma)}{\mathbb{Z}(N)} \frac{e^{-\frac{\gamma\nu\beta}{2}} \exp[-w\mathbb{A}(t, \gamma\beta) \vec{r}^2]}{(1 - e^{-\gamma\nu\beta})^3} \left(\frac{w}{\pi} \mathbb{A}(t, \gamma\beta) \right)^{\frac{3}{2}} \quad (4.41) \end{aligned}$$

where

$$\mathbb{A}(t, \gamma\beta) = \frac{\nu w \tanh \frac{\nu\beta}{2}}{(\nu^2 \sin^2 wt + w^2 \cos^2 wt)}. \quad (4.42)$$

We have defined $\mathbb{A}(t, \gamma\beta)$ to be a dimensionless quantity. Note that if we set $\nu = w$, which means there is no perturbation of the condensate, the factor $\mathbb{A}(t, \gamma\beta)$ becomes a time-independent quantity and the problem reduces to that of the static case. We now see that the result resembles the one from the work

by Brosens et al. [7,8] except the modification of the factor $\mathbb{A}(t, \gamma\beta)$. Since the formula Eq. (4.41) is not useful for numerical calculation, we apply the following technique.

Our trick is to find the recursion relation for the density of particles. The factor $\frac{\mathbb{Z}(N-\gamma)}{\mathbb{Z}(N)}$ appears in the density can be simplified as in the case of the partition function. Using Eq. (4.20)

$$\frac{\mathbb{Z}(N)}{\mathbb{Z}(k)} = b^{\frac{3}{2}(N-k)} \prod_{j=k+1}^N \varrho(j) \Rightarrow \frac{\mathbb{Z}(N-\gamma)}{\mathbb{Z}(N)} = b^{-\frac{3}{2}\gamma} \prod_{j=N-\gamma+1}^N \frac{1}{\varrho(j)}, \quad (4.43)$$

then

$$\begin{aligned} n(\vec{r}, t) &= \frac{1}{N} \sum_{\gamma=1}^N \frac{1}{(1-b\gamma)^3} \left(\frac{w}{\pi} \mathbb{A}(t, \gamma\beta) \right)^{\frac{3}{2}} \exp[-w\mathbb{A}(t, \gamma\beta) \vec{r}^2] \left(\prod_{j=N-\gamma+1}^N \frac{1}{\varrho(j)} \right) \\ &\equiv \frac{1}{N} \sum_{\gamma=1}^N a(\gamma) \left(\prod_{j=N-\gamma+1}^N \frac{1}{\varrho(j)} \right), \end{aligned} \quad (4.44)$$

where

$$\mathbb{A}(t, \gamma\beta) = \frac{vw \tanh \frac{\gamma v\beta}{2}}{(v^2 \sin^2 wt + w^2 \cos^2 wt)} \quad (4.45)$$

$$a(\gamma) = \frac{1}{(1-b\gamma)^3} \left(\frac{w}{\pi} \mathbb{A}(t, \gamma\beta) \right)^{\frac{3}{2}} \exp[-w\mathbb{A}(t, \gamma\beta) \vec{r}^2]. \quad (4.46)$$

We can write the density of particles explicitly as

$$n(\vec{r}, t) = \frac{1}{N} \frac{a(1) + \frac{a(2) + \frac{a(3) + \dots + \frac{a(N-1) + \frac{a(N)}{\rho(1)}}{\rho(2)}}{\rho(N-2)}}{\rho(N-1)}}{\rho(N)}. \quad (4.47)$$

With this formula we can calculate $n(\vec{r}, t)$ numerically. From this calculation we found that the frequency of oscillation is $2w$ which arises from the time-dependent factor

$$(\nu^2 \sin^2 wt + w^2 \cos^2 wt) = \frac{1}{2} (\nu^2 + w^2) + \frac{1}{2} (\nu^2 - w^2) \cos 2wt, \quad (4.48)$$

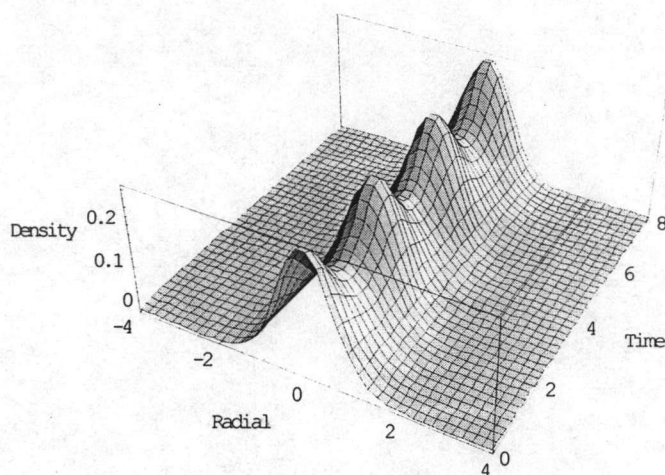


Figure 4.5: The breathing mode calculated from our method.

in accordance with the experimental results and other theoretical calculations. This is equivalent to lowest-lying collective mode of the excitations. The result is shown schematically below in Fig. 4.5 with the parameters $\nu = 1$, $w = 1.2$, $b = 0.75$ and $N = 1000$ (for this value of b corresponds to the temperature of the order 0.1 nK). The result is compared to the case of distinguishable particles in Fig. 4.6 which is calculated from Eq. (4.11) with the same set of parameters except the number of particles N which is meaningless in this case.

We can see that the particles accumulate at the center of the condensate making the width of the density very narrow compared to the case of distinguishable particles. Although our model contains no interaction, the symmetry properties of the density matrix make the system behaves as if it were an interacting one. This is well known as the “statistical potential” which is attractive for bosons and repulsive for fermions (for details discussions, see Huang [4]).

In the experiment the higher modes exist together with the decay of os-

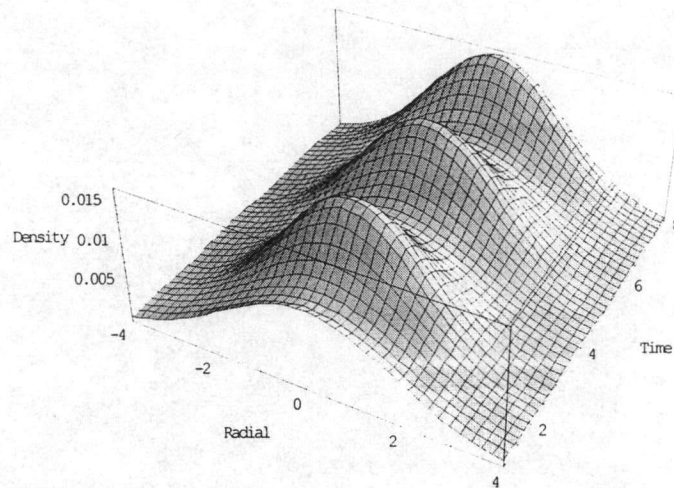


Figure 4.6: The probability density of the distinguishable particles.

cillation. As mentioned before, this damping is due to the thermal excitation and interaction between particles. In our model, the particles interact statistically and the condensate oscillates with no damping. This corresponds to the experimental result obtained by Jin et al.[10] which the no-damping oscillation at zero-temperature limits has been studied. However, it is interesting to study the system within our model by adding the interactions between particles which we shall discuss in the next section. Although we are not successful in the calculation of the analytic result but we present it for academic purpose.

4.3 The Breathing Mode in Interacting Bose Gas

Our system is particles confined in a harmonic potential interacting via some kind of two-particle potential, for example, contact potential or s-wave scattering interaction, Morse potential, etc. What we are interested in is the evolution of

the system after abruptly changing the confining potential from ν to w . We have already found the density matrix and the particle density for the *non-interacting* particles under a harmonic potential, the number density of the system oscillates with frequency $2w$. Note that this is the zeroth order of the perturbation theory of our system. In the real experimental results the condensate oscillates and is damped down due to the interaction between the condensate and the thermal cloud. We would like to see how the *interaction* modifies the oscillation of the system. We also want to see how the *statistics* modifies the oscillation of the system, quantitatively. If we want to study this system within the path integral framework, we have to seek for the many-particle Lagrangian with interactions which the corresponding path integral can be decomposed into the product of single-particle density matrices (in order to be applicable for cyclic decomposition in permutation sum). The most well-known interacting model is the harmonic interacting model made by Brosens et al.[7,8] which we review briefly in Chapter 2. In this model, the particles interact with each others via the potential of the form

$$U \sum_{i=1}^N \sum_{j=1}^N (\vec{r}_i - \vec{r}_j)^2, \quad (4.49)$$

where U is the strength of interaction. With this form of interaction the density matrix of the system can be rewritten as

$$\rho_D(\vec{r}', \beta | \vec{r}, 0) = \frac{\rho(\sqrt{N}\vec{R}', \beta | \sqrt{N}\vec{R}, 0)}{\rho(\sqrt{N}\vec{R}', \beta | \sqrt{N}\vec{R}, 0)_w} \prod_{i=1}^N \rho(\vec{r}', \beta | \vec{r}, 0)_w, \quad (4.50)$$

with the introduction of the center of mass coordinates \vec{R} . Then the partition function is

$$\begin{aligned} \mathbb{Z}^{int} &= \int d^3 R \int \frac{d^3 k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{R}} \frac{\rho(\sqrt{N}\vec{R}', \beta | \sqrt{N}\vec{R}, 0)}{\rho(\sqrt{N}\vec{R}', \beta | \sqrt{N}\vec{R}, 0)} \Omega \prod_{i=1}^N \rho(\vec{r}', \beta | \vec{r}, 0)_w \\ &\times \sum_{M_1 M_2 \dots M_N} \prod_{\gamma} \frac{(\mathcal{K}_{\gamma}^{int}(\vec{k}))^{M_{\gamma}}}{M_{\gamma}! \gamma^{M_{\gamma}}}, \end{aligned} \quad (4.51)$$

where the superscript *int* denotes the interacting system and

$$\mathcal{K}_{\gamma}^{int}(\vec{k}) = \int d\vec{r}_{\gamma+1} \cdots \int d\vec{r}_1 \delta(\vec{r}_{\gamma+1} - \vec{r}_1) \prod_{j=1}^{\gamma} \rho(\vec{r}_{j+1}, \beta | \vec{r}_j, 0) e^{-i\vec{k}\cdot\vec{r}_j/N}. \quad (4.52)$$

Comparing this expression with Eq. (4.30), the extra factor is

$$\prod_{j=1}^{\gamma} e^{-i\vec{k}\cdot\vec{r}_j/N} = e^{-\int_0^{\gamma\beta} \vec{f}(\tau)\cdot\vec{r}(\tau)d\tau} \quad \text{with} \quad \vec{f}(\tau) = \frac{i\vec{k}}{N} \sum_{j=0}^{\gamma-1} \delta(\tau - \gamma\beta). \quad (4.53)$$

If $\rho(\vec{r}_{j+1}, \beta | \vec{r}_j, 0)$ is the propagator of a system with Lagrangian \mathcal{L} of the harmonic oscillator as in Chapter 2, then the full propagator in $\mathcal{K}_{\gamma}^{int}(\vec{k})$ is the propagator of a system with Lagrangian $\mathcal{L} + \int_0^{\gamma\beta} \vec{f}(\tau) \cdot \vec{r}(\tau) d\tau$. The exact solution of the path integral with this Lagrangian can be found in [21]. The partition function in Eq. (4.51) can be found in closed form. However, when we apply this calculation scheme to our case of the TDDM, i.e.,

$$\mathcal{K}_{\gamma}^{int}(\vec{k}, t) = \int d\vec{r}_{\gamma+1} \cdots \int d\vec{r}_1 \delta(\vec{r}_{\gamma+1} - \vec{r}_1) \prod_{j=1}^{\gamma} \rho_{\beta}(\vec{r}_{j+1}, t | \vec{r}_j, 0) e^{-i\vec{k}\cdot\vec{r}_j/N}, \quad (4.54)$$

we found that we cannot write the force term as in Eq. (4.53), hence no closed form for $\mathcal{K}_{\gamma}^{int}(\vec{k}, t)$. The problem stems from introducing the change of the trap frequency and this causes the time dependent factor $(\nu^2 \sin^2 wt + w^2 \cos^2 wt)$. When we add the force terms $e^{-i\vec{k}\cdot\vec{r}_j/N}$, we cannot find the propagator in closed form. In fact, this factor can be calculated numerically and then the permutation

sum can be done, so we leave it for future work. However, we can think intuitively that this harmonic model is nothing but a set of free oscillators with extra degree of freedom from the center of mass coordinate. Hence, we suspect that the harmonic model might not give the damping of the oscillation. The alternative Lagrangian with dissipation might be a good candidate.

In this chapter we have calculated the quasi-static excitation (breathing mode) of the Bose-Einstein condensation of Boson gas in a spherical magnetic trap by the method of many-body path integrals. We found that the non-interacting model gives the perpetual oscillation of the condensate while in the experiment the damping of the oscillation was found. Nevertheless, in the zero-temperature limit the damping is very small. It is generally accepted that the damping is due to the interaction between bosons. We find that the mathematical manipulation is formidable and that we cannot obtain the density matrix in a closed form