CHAPTER IV

A FIXED POINT THEOREM FOR MAPPINGS WITH A NONEXPANSIVE ITERATE

In this chapter, we obtain conditions sufficient to guarantee existence of fixed points for mapping T such that T^N is nonexpansive, for some integer $N \geq 1$.

We first give the following definitions:

4.1 <u>Definition.</u> A normed vector space X is said to be <u>uniformly convex</u> if, to each $\varepsilon > 0$, corresponds $\delta(\varepsilon) > 0$ such that

$$\|x - y\| < \varepsilon$$

whenever $\|x\| = 1$, $\|y\| = 1$ and $\|\frac{1}{2}(x+y)\| > 1 - \delta(\varepsilon)$.

This definition is a geometric property of the unit sphere of the space: if the midpoint of a line segment with end points on the surface of the sphere approaches the surface, then the end points must come closer together.

Obviously, every uniformly convex is also convex, but the converse is not true.

Figure I

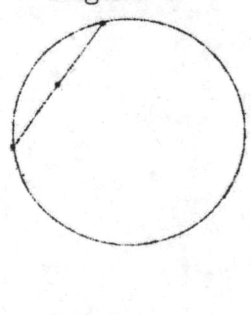
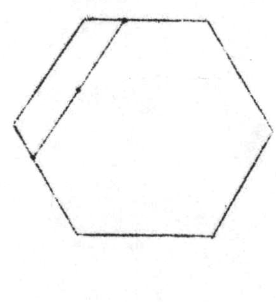


Figure II



For examples in $\mathbb{R}^2$, the Figure I is a uniformly convex, but the Figure II is not a uniformly convex, although it is a convex set.

4.2 Definition. For $S \subset X$, we denote the diameter of S by diam.(S). A point $x \in S$ is a diametral point of S provided

$$\sup\left\{ \|x-y\| : y \in S \right\} = \text{diam.}(S).$$

Every boundary point of a closed ball, or each of the four vertices of a rectangular in $\mathbb{R}^2$, is the simple example of diametral point of the closed ball or the rectangular, respectively.

4.3 Definition. A convex subset $K \subset X$ is said to have normal structure if, for each bounded convex subset H of K which contains more than one point, there is some point $x \in H$ which is not a diametral point of H, i.e., there exists $x \in H$ such that

$$\sup\left\{ \|x-y\| : y \in H \right\} < \text{diam.}(H).$$

It follows from Theorem 4.1 of [1] that, every uniformly convex Banach space has normal structure.

4.4 Definition. A self-mapping T of a metric space (X,d) is said to be quasi-nonexpansive provided T has at least one fixed point in X, and if $p \in X$ is any fixed point of T, then

$$d(T(x), p) \leq d(x,p)$$

holds for all $x \in X$.

From this definition, it is clear that, a nonexpansive mapping $T : X \longrightarrow X$ which has at least one fixed point in X is a quasi-nonexpansive mapping, since if p is a fixed point of T, then

$$d(T(x),p) = d(T(x),T(p)) \leq d(x,p)$$

holds for all $x \in X$.

4.5 Definition. A point x is called an interior point of a set X if x has a neighborhood $O(x) \subset X$.

4.6 Definition. Let E be either real or complex vector space. By (x,y) we mean the real open interval $tx + (1-t)y \; ; \; 0 < t < 1$. Suppose that M is a subset of E. A point z of M is called an extreme (extremal) point of M, if it belongs to no open interval (x,y) in M.

For example, in three dimensional Euclidean space, every point on the surface of a solid sphere is an extremal point, and the eight vertices of a solid cube are extremal points of the cube.

4.7 Definition. A normed space X is said to be strictly convex if, every boundary point of the closed unit ball K is an extremal point, or equivalently, if $\|x\| = \|y\| = 1$ and $x \neq y$, then $\|\frac{1}{2}(x+y)\| < 1$.

4.8 Remark. Every uniformly convex normed vector space is strictly convex. Every finite-dimensional strictly convex space is uniformly convex.

For the proof of this remark see e.g. [2] on page 23.

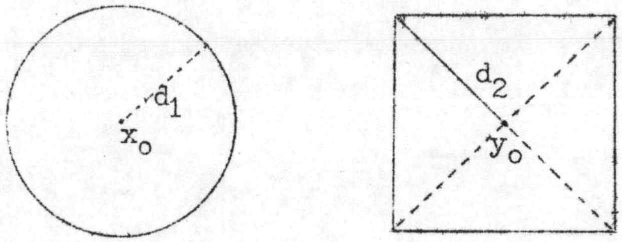
4.9 Notations. Let F denote a nonempty, bounded, closed and convex subset of the Banach space X. Let

$$r_x(F) = \sup\{\|x-y\| : y \in F\}$$

$$r(F) = \inf\{r_x(F) : x \in F\}$$

$$F_c = \{x \in F : r_x(F) = r(F)\}.$$

For examples in $\mathbb{R}^2$, let E and F denote the closed disc and rectangular, respectively.



From the Figure, x_o is the center of E and y_o is the intersection point of the two diagonals of F. Then we have

$$r_{x_o}(E) \quad = \quad \sup\{|x_o - x| : x \in E\} = d_1$$

and,

$$r_x(E) \quad \geq \quad d_1$$

for all $x \in E$. Then

$$r(E) \quad = \quad \inf\{r_x(E) : x \in E\} = d_1 .$$

Hence,

$$E_c = \{x_o\} .$$

And also,

$$r_{y_o}(F) \quad = \quad \sup\{|y_o - y| : y \in F\} = d_2$$

and,

$$r_y(F) \quad \geq \quad d_2$$

for all $y \in F$. Then

$$r(F) \quad = \quad \inf\{r_y(F) : y \in F\} = d_2 .$$

Hence,

$$F_c = \{y_o\} .$$

Our principal results are the following theorems:

4.10 Theorem. Let K be a nonempty, closed, convex subset of a Banach space X, and suppose $T : K \longrightarrow K$ has the property that, for some integer $N \geq 1$, T^N is the identity mapping on K. Suppose further that there is a constant k satisfying

$$N^{-2}\left[(N-1)(N-2)k^2 + 2(N-1)k\right] < 1 \qquad \text{.... (i)}$$

such that $\|T^j(x)-T^j(y)\| \leq k\|x-y\|$ for all $x, y \in K$, $1 \leq j \leq N-1$. Then the mapping T has a fixed point in K.

The next theorem, we obtain conditions sufficient to guarantee existence of fixed points for mapping T such that T^N is nonexpansive for some integer $N \geq 1$.

4.11 Theorem. Let X be a reflexive Banach space which has strictly convex norm and suppose K is a nonempty, bounded, closed and convex subset of X which possesses normal structure. Suppose the mapping $T : K \longrightarrow K$ has the property that for some integer $N \geq 1$, T^N is nonexpansive, and suppose further that there is a constant k satisfying the condition (i) of Theorem 4.10, such that $\|T^j(x)-T^j(y)\| \leq k\|x-y\|$ for all $x, y \in K$, $1 \leq j \leq N-1$. Then T has a fixed point in K.

Before proving the above two theorems, we establish the following theorem and lemmas:

4.12 Theorem. (Šmulian Theorem) A Banach space X is reflexive if and only if every bounded descending sequence (transfinite) of nonempty closed convex subsets of X have a nonempty intersection.

For the proof of this theorem see e.g. [13].

4.13 Lemma. If X is a reflexive Banach space and F is a nonempty bounded, closed and convex subset of X, then F_c is nonempty, closed and convex.

Proof. For each n = 1, 2, ... , let

$$F(x,n) = \{y \in F : \|x-y\| \leq r(F) + {}^1/n\}, \text{ and let}$$

$$C_n = \bigcap_{x \in F} F(x,n).$$

We first show that F(x,n) is a convex set. Select any $x_1 \in F(x,n)$ and $x_2 \in F(x,n)$. Then

$$\|x-x_1\| \leq r(F) + {}^1/n \text{ and } \|x-x_2\| \leq r(F) + {}^1/n.$$

With t such that $0 \leq t \leq 1$, we have

$$\begin{aligned}
\|(tx_1 + (1-t)x_2) - x\| &= \|tx_1 + x_2 - tx_2 - x\| \\
&= \|tx_1 - tx + tx + x_2 - tx_2 - x\| \\
&= \|t(x_1 - x) + (1-t)(x_2 - x)\| \\
&\leq t\|x_1 - x\| + (1-t)\|x_2 - x\| \\
&\leq t\left[r(F) + {}^1/n\right] + (1-t)\left[r(F) + {}^1/n\right] \\
&= r(F) + {}^1/n.
\end{aligned}$$

This proves the convexity of F(x,n). Moreover, F(x,n) is closed. By using the fact that the intersection of closed convex sets is closed and convex, we have that C_n is closed and convex.

We next show that C_n is nonempty. Since $r(F) + 1/n > r(F)$ for all n and since $r(F) = \inf\{r_x(F) : x \in F\}$, there exists $x_n \in F$ such that

$$r_{x_n}(F) \leq r(F) + 1/n.$$

Then for any $x \in F$, we have

$$\|x-x_n\| \leq r_{x_n}(F) \leq r(F) + 1/n,$$

implies that $x_n \in F(x,n)$ for all $x \in F$, i.e., $x_n \in C_n$.

Moreover, if $n_1 > n_2$, then $r(F) + 1/n_1 < r(F) + 1/n_2$, that is

$$F(x,n_1) \subset F(x,n_2)$$

for all $x \in F$. Then for any $y \in C_{n_1}$ we have $y \in F(x,n_1)$ for all $x \in F$. Hence,

$$y \in F(x,n_2)$$

for all $x \in F$, i.e., $y \in C_{n_2}$.

Hence $\{C_n\}$ is a decreasing sequence of nonempty closed and convex sets.

We claim that $F_c = \bigcap_{n=1}^{\infty} C_n$. In fact, for every $y \in F_c$ we have

$$\begin{aligned} r_y(F) &= \sup\{\|y-x\| : x \in F\} \\ &= r(F). \end{aligned}$$

It follows that for any $x \in F$,

$$\|y-x\| < r(F) + 1/n$$

for all n, or equivalently,

$$y \in F(x,n)$$

for all n and for all $x \in F$, i.e.,

$$y \in \bigcap_{x \in F} F(x,n) = C_n$$

for all n. Hence $y \in \bigcap_{n=1}^{\infty} C_n$.

Finally, by theorem 4.12 we have that F_c is nonempty closed and convex. This proves the lemma.

4.14 Lemma. Let F be a closed convex subset of X which contains more than one point. If F has normal structure, then

$$\text{diam.}(F_c) < \text{diam.}(F),$$

where diam.(A) denotes the diameter of a set A.

Proof. By the definition of normal structure, F contains at least one nondiametral point x. Then

$$\sup\left\{\|x-y\| : y \in F\right\} < \text{diam.}(F),$$

and hence,

$$r_x(F) < \text{diam.}(F). \qquad \ldots . \text{(i)}$$

If z and w are any two points of F_c, then

$$\|z-w\| \leq \sup\left\{\|z-y\| : y \in F\right\} = r_z(F) = r(F). \quad \ldots . \text{(ii)}$$

Hence, by (i) and (ii) we get

$$\begin{aligned} \text{diam.}(F_c) &= \sup\left\{\|z-w\| : z,w \in F_c\right\} \\ &\leq r(F) \\ &\leq r_x(F) < \text{diam.}(F). \end{aligned}$$

Then the lemma is proved completely.

4.15 Lemma. Let K be a nonempty, bounded, closed and convex subset of a reflexive Banach space X, and suppose that K has normal structure. If T is a nonexpansive self-mapping of K, then T has a fixed point.

Proof. Let $\mathfrak{F}$ denote the collection of all nonempty closed and convex subsets of K, each of which is mapped into itself by T. We have $\mathfrak{F}$ is nonempty, since $K \in \mathfrak{F}$.

Apply Šmulian Theorem (Theorem 4.12), for every chain in a partially ordered sets in $\mathfrak{F}$ has a nonempty intersection. Then by Zorn's lemma $\mathfrak{F}$ has a minimal element which denote by F.

We complete the proof by showing that F consists of a single point. Let $x \in F_c$. Then

$$\|T(x)-T(y)\| \leq \|x-y\| \leq r_x(F) = r(F)$$

for all $y \in F$. Hence

$$T(F) \subset U = \{w \in X : \|T(x)-w\| \leq r(F)\}$$

We claim that $T(F \cap U) \subset (F \cap U)$. In fact, let z be any element of $T(F \cap U)$, then there exists $x_o \in F \cap U$ such that $T(x_o) = z$, and since $T(F) \subset U$ and $T(F) \subset F$, $T(x_o) \in U$ and $T(x_o) \in F$, i.e.,

$$z \in (F \cap U).$$

By the minimality of F implies that $F \cap U = F$, that is $F \subset U$. Hence $T(x) \in F_c$, since

$$r_{T(x)}(F) = \sup\{\|T(x)-y\| : y \in F\},$$

then by the definition of U and $F \subset U$ we get

$$r_{T(x)}(F) \leq r(F) \qquad \ldots . (i)$$

and clearly,

$$r(F) \leq r_{T(x)}(F). \qquad \ldots . (ii)$$

The inequalities (i) and (ii) implies that

$$r(F) = r_{T(x)}(F) ,$$

i.e., $T(x) \in F_c$.

Then F_c is mapped into itself by T. By Lemma 4.13, F_c is nonempty closed and convex subset of K, i.e., $F_c \in \mathfrak{F}$.

Suppose that diam.$(F) > 0$. Then by Lemma 4.14, F_c is properly contained in F. Since this contradicts the minimality of F, diam.$(F) = 0$ and then F consists of a single point, this proves the lemma.

4.16 Lemma. Let K be a convex subset of a Banach space and suppose $T : K \longrightarrow K$ has the property that, for some integer $N \geq 2$, T^N is the identity mapping on K. Suppose further that there is a constant k with $k < (N-1)/(N-2)$ such that $\| T^j(x)-T^j(y) \| \leq k \| x-y \|$ for all $x,y \in K$, $1 \leq j \leq N-1$. If the mapping $G : K \longrightarrow K$ is defined by

$$G(x) = N^{-1}(x + T(x) + \ldots + T^{N-1}(x)), \quad x \in K,$$

then the mappings T and G have the same fixed points.

Proof. Clearly, $T(x) = x$ implies $G(x) = x$, since

$$G(x) = N^{-1}(Nx) = x.$$

Suppose conversely that $G(x) = x$. From the definition of G we obtain

$$x = N^{-1}(x + T(x) + \dots + T^{N-1}(x))$$

$$x - N^{-1}x = N^{-1}(T(x) + T^2(x) + \dots + T^{N-1}(x)).$$

Then

$$x\left(\frac{N-1}{N}\right) = N^{-1}(T(x) + T^2(x) + \dots + T^{N-1}(x))$$

$$x = (N-1)^{-1}(T(x) + T^2(x) + \dots + T^{N-1}(x)).$$

This, with the Lipschitz condition on T^i yields, for $1 \leq i \leq N-1$,

$$\begin{aligned}
\|x - T^i(x)\| &= \|(N-1)^{-1}(T(x) + T^2(x) + \dots + T^{N-1}(x)) - T^i(x)\| \\
&= (N-1)^{-1}\|(T(x) + \dots + T^{N-1}(x)) - (N-1)T^i(x)\| \\
&\leq (N-1)^{-1}\Big[\|T(x)-T^i(x)\| + \dots + \|T^{i-1}(x)-T^i(x)\| + \\
&\qquad \|T^{i+1}(x)-T^i(x)\| + \dots + \|T^{N-1}(x)-T^i(x)\|\Big] \\
&= (N-1)^{-1}\Big[\|T(x)-T(T^{i-1}(x))\| + \dots + \|T^{i-1}(x)-T^{i-1}(T(x))\| \\
&\qquad + \|T^{i+1}(T^{N-1}(x))-T^{i+1}(x)\| + \dots + \|T^{N-1}(T^{i+1}(x))-T^{N-1}(x)\|\Big] \\
&\leq k(N-1)^{-1}\Big[\|x-T^{i-1}(x)\| + \dots + \|x-T(x)\| + \\
&\qquad \|T^{N-1}(x)-x\| + \dots + \|T^{i+1}(x)-x\|\Big].
\end{aligned}$$

Hence,

$$\begin{aligned}
\|x-T^i(x)\| \leq k(N-1)^{-1}\Big[&\|x-T(x)\| + \|x-T^2(x)\| + \dots + \|x-T^{i-1}(x)\| + \\
&\|x-T^{i+1}(x)\| + \dots + \|x-T^{N-1}(x)\|\Big].
\end{aligned}$$

Since $\| x-T^i(x) \|$ does not appear in the last summand,

$$\sum_{i=1}^{N-1} \|x-T^i(x)\| \leq k(N-1)^{-1}\Big[(N-2)\|x-T(x)\| + (N-2)\|x-T^2(x)\| + \dots + (N-2)\|x-T^{N-1}(x)\|\Big]$$

$$= k(N-1)^{-1}(N-2)\sum_{i=1}^{N-1}\|x-T^i(x)\|.$$

Since $k < (N-1)(N-2)^{-1}$,

$$\sum_{i=1}^{N-1}\|x-T^i(x)\| < (N-1)(N-2)^{-1}(N-1)^{-1}(N-2)\sum_{i=1}^{N-1}\|x-T^i(x)\|$$

$$= \sum_{i=1}^{N-1}\|x-T^i(x)\|,$$

which implies that,

$$T(x) = x.$$

Hence the lemma is proved.

4.17 Lemma. Let E be a strictly convex normed space, and if $\|x + y\| = \|x\| + \|y\|$ and $y \neq 0$, then $x = ky$ for some nonnegative k.

Proof. Suppose that $\|x + y\| = \|x\| + \|y\|$ for $x, y \in E$, and if $\|y\| \geq \|x\|$ and $x, y \neq 0$, then we let

$$x_o = \frac{x}{\|x\|} \text{ and } y_o = \frac{y}{\|y\|}.$$

Then $\|x_o\| = \|y_o\| = 1$, and

$$\left\|\tfrac{1}{2}(x_o + y_o)\right\| = \tfrac{1}{2}\left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|$$

$$\geq \tfrac{1}{2}\left[\left\|\frac{x}{\|x\|} + \frac{y}{\|x\|}\right\| - \left\|\frac{y}{\|x\|} - \frac{y}{\|y\|}\right\|\right]$$

$$= \frac{1}{2}\left[\frac{\|x\| + \|y\|}{\|x\|} - \|y\|\left(\frac{1}{\|x\|} - \frac{1}{\|y\|}\right)\right]$$

$$= \frac{1}{2}\left[1 + \frac{\|y\|}{\|x\|} - \frac{\|y\|}{\|x\|} + 1\right] = 1$$

Then by definition of strictly convex we have that $x_o = y_o$, i.e.,

$$\frac{x}{\|x\|} = \frac{y}{\|y\|}.$$

Thus $x = ky$ where $k = \frac{\|x\|}{\|y\|}$. Hence the lemma is proved completely.

4.18 Lemma. If C is a closed convex subset of a strictly convex normed vector space, and $T : C \longrightarrow C$ is quasi-nonexpansive, then

$$F(T) = \{p : p \in C \text{ and } T(p) = p\}$$

is a nonempty closed and convex set on which T is continuous.

Proof. It follows immediately from the definition of quasi-nonexpansiveness that $F(T) \neq \emptyset$, and that T is continuous at each $p \in F(T)$, since for any given $\varepsilon > 0$, choose $\delta = \varepsilon$, whenever $\|x-p\| < \delta$ implies that $\|T(x) - T(p)\| = \|x - p\| < \varepsilon$.

Suppose F(T) is not closed. Then there exists a limit point x of F(T) such that $x \notin F(T)$.

Since C is closed and $F(T) \subset C$, $\overline{F(T)} \subset C$, implies that $x \in C$. Since $x \notin F(T)$, $T(x) \neq x$. Let

$$r = (\tfrac{1}{4}) \|T(x) - x\| > 0.$$

Since x is a limit point of F(T), there exists $y \in F(T)$ such that

$$\|x - y\| < r.$$

Since T is quasi-nonexpansive mapping,

$$\| T(x) - y\| \leq \|x - y\| < r.$$

Hence, we get

$$4r = \|T(x) - x\| \leq \|T(x) - y\| + \|y - x\| < 2r.$$

This is a contradiction, and thus F(T) is closed.

Next, to prove that F(T) is convex. Suppose $a, b \in F(T)$, $a \neq b$ and $0 < t < 1$. Since C is convex, $c = (1-t)a + tb \in C$. .....(1)

Since T is quasi-nonexpansive,

$$\|T(c) - a\| \leq \|c - a\| \text{ and } \|T(c) - b\| \leq \|c - b\|. \quad .....(2)$$

From (1), we get

$$c - a = t(b - a), \text{ and}$$

$$c - b = (1-t)(a - b). \quad .....(3)$$

It follows from (2) and (3) that,

$$\begin{aligned} \|b - a\| &\leq \|b - T(c)\| + \|T(c) - a\| \\ &\leq \|c - b\| + \|c - a\| \\ &= t\|b - a\| + (1-t)\|a - b\| \\ &= \|b - a\|. \end{aligned}$$

Hence,

$$\begin{aligned} \|b - T(c)\| + \|T(c) - a\| &= \|b - a\| \\ &= \|(b - T(c)) + (T(c) - a)\|. \quad ......(4) \end{aligned}$$

If $(b - T(c)) = 0$, then by (2), (3) and (4) we have that

$$\|T(c) - a\| = \|b - a\| \leq \|c - a\| = t\|b - a\|.$$

Hence $1 \leq t$ which is not true. Similarly $(T(c) - a) \neq 0$.

Since the space is strictly convex, by lemma 4.17 and the equation (4) there exists $k \geq 0$ such that $T(c) - a = k(b - T(c))$. Then

$$T(c) - a = kb - kT(c)$$

$$T(c)(1+k) = a + kb.$$

Then

$$T(c) = \left(\frac{1}{1+k}\right)a + \left(\frac{k}{1+k}\right)b$$

$$= (1-s)a + (sb)$$

where $s = \dfrac{k}{1+k} \geq 0$. Thus

$$T(c) - a = s(b - a).$$

It follows from (2) and (3) that,

$$s\,\|b - a\| = \|T(c) - a\| \leq \|c - a\| = t\,\|b - a\|$$

which gives, $s \leq t$.

Similarly; $$T(c) - b = (1-s)a + sb - b$$

$$= (1-s)a - (1-s)b$$

$$= (1-s)(a-b) \text{ , and so}$$

$$(1-s)\,\|a - b\| = \|T(c) - b\|$$

$$\leq \|c - b\|$$

$$= (1-t)\,\|a-b\|$$

which gives, $(1-s) \leq (1-t)$, implies that $s \geq t$.

Thus $s = t$, and so

$$T(c) = (1-t)a + tb = c.$$

Hence $c \in F(T)$, i.e., $(1-t)a + tb \in F(T)$; $0 \leq t \leq 1$.

The lemma is proved.

<u>Proof.</u> (of Theorem 4.10) Define the mapping $G : K \longrightarrow K$ by

$$G(x) = N^{-1}(x + T(x) + \dots + T^{N-1}(x)) \quad ; \quad x \in K.$$

For $j = 0, 1, 2, \dots,$ let

$$\delta_j = \sup\left\{ \| G^j(x) - T^i G^j(x) \| \; : \; i = 1, 2, \dots, N-1 \right\}.$$

Then

$$\begin{aligned} \| G^j(x) - G^{j+1}(x) \| &= \| G^j(x) - G(G^j(x)) \| \\ &= \left\| G^j(x) - N^{-1}\left[G^j(x) + TG^j(x) + \dots + T^{N-1}G^j(x) \right] \right\| \\ &= \left\| N^{-1}\left[NG^j(x) \right] - N^{-1}\left[G^j(x) + TG^j(x) + \dots + T^{N-1}G^j(x) \right] \right\| \\ &\leq N^{-1}\left[\| G^j(x) - TG^j(x) \| + \| G^j(x) - T^2G^j(x) \| + \dots + \| G^j(x) - T^{N-1}G^j(x) \| \right] \\ &= N^{-1}\sum_{i=1}^{N-1} \| G^j(x) - T^iG^j(x) \| \\ &= (N-1)N^{-1}\delta_j. \qquad \dots . (1) \end{aligned}$$

Next we compare δ_j with δ_{j-1}. For $j = 1, 2, \dots$, and $1 \leq i \leq N-1$,

$$\begin{aligned} \| G^j(x) - T^iG^j(x) \| &= \| G(G^{j-1}(x)) - T^iG^j(x) \| \\ &= \left\| N^{-1}\left[G^{j-1}(x) + TG^{j-1}(x) + \dots + T^{N-1}G^{j-1}(x) \right] - T^iG^j(x) \right\| \\ &\leq N^{-1}\Big[\| G^{j-1}(x) - T^iG^j(x) \| + \| TG^{j-1}(x) - T^iG^j(x) \| + \\ &\qquad \dots + \| T^{N-1}G^{j-1}(x) - T^iG^j(x) \| \Big], \end{aligned}$$

i.e.,

$$\| G^j(x) - T^iG^j(x) \| \leq N^{-1}\sum_{p=0}^{N-1} \| T^pG^{j-1}(x) - T^iG^j(x) \|. \qquad \dots . (2)$$

There are (N-1) terms in the last summand of the term

$$\| T^pG^{j-1}(x)-T^iG^j(x) \|$$

with $p \neq i$. For fixed p, we have

$$\| T^pG^{j-1}(x)-T^iG^j(x) \| \leq k \| T^sG^{j-1}(x)-G^j(x) \| \quad . . . . (3)$$

for an appropriate integer s, $1 \leq s \leq N-1$. (Note that if $p > i$, then $s = p-i$ while if $p < i$, then $s = N+p-i$). Since

$$\| T^sG^{j-1}(x)-G^j(x) \| = \| T^sG^{j-1}(x)-G(G^{j-1}(x)) \|$$

$$= \| T^sG^{j-1}(x)-N^{-1}\left[G^{j-1}(x)+TG^{j-1}(x)+ \ldots + T^{N-1}G^{j-1}(x)\right] \|,$$

i.e.,

$$\| T^sG^{j-1}(x)-G^j(x) \| \leq N^{-1}\sum_{p=0}^{N-1} \| T^sG^{j-1}(x)-T^pG^{j-1}(x) \|, \quad . . . . (4)$$

and since

$$\| T^sG^{j-1}(x)-T^pG^{j-1}(x) \| \leq k. \delta_{j-1}.$$

(In fact $\| T^sG^{j-1}(x)-T^pG^{j-1}(x) \| \leq k. \| T^iG^{j-1}(x)-G^{j-1}(x) \|$, where $1 \leq i \leq N-1$, and if $s > p$, then $i = s-p$, and if $s < p$, then $i = N+s-p$.)

From (4), we get

$$\| T^sG^{j-1}(x)-G^j(x) \| = N^{-1}\Big[\| T^sG^{j-1}(x)-G^{j-1}(x) \| + \| T^sG^{j-1}(x)-TG^{j-1}(x) \|$$

$$+ \ldots + \| T^sG^{j-1}(x)-T^{s-1}G^{j-1}(x) \| +$$

$$\| T^sG^{j-1}(x)-T^{s+1}G^{j-1}(x) \| + \| T^sG^{j-1}(x)-T^{s+2}G^{j-1}(x) \|$$

$$+ \dots + \left\| T^s G^{j-1}(x) - T^{N-1} G^{j-1}(x) \right\|$$

for $1 \leq s \leq N-1$. Then

$$\left\| T^s G^{j-1}(x) - G^j(x) \right\| \leq N^{-1} \left[\delta_{j-1} + (N-2)k . \delta_{j-1} \right]$$

$$= N^{-1} . \delta_{j-1} + (N-2)N^{-1} k . \delta_{j-1} \quad \dots . (5)$$

for $1 \leq s \leq N-1$.

Combining (5) with (1), (2) and (3) yields

$$\left\| G^j(x) - T^i G^j(x) \right\| \leq N^{-1}(N-1) \sup_{p \neq i} \left[\left\| T^p G^{j-1}(x) - T^i G^j(x) \right\| \right] +$$

$$N^{-1} \left\| T^i G^{j-1}(x) - T^i G^j(x) \right\|$$

$$\leq \left(\frac{N-1}{N}\right) k . \sup_{1 \leq s \leq N-1} \left\| T^s G^{j-1}(x) - G^j(x) \right\| +$$

$$\frac{1}{N} k \left\| G^{j-1}(x) - G^j(x) \right\|$$

$$\leq \left(\frac{N-1}{N}\right) k \left[\left(\frac{N-2}{N}\right) k . \delta_{j-1} + \frac{1}{N} . \delta_{j-1} \right] + \frac{1}{N} k \left(\frac{N-1}{N}\right) . \delta_{j-1} .$$

Therefore

$$\delta_j = \sup \left[\left\| G^j(x) - T^i G^j(x) \right\| ; \ i = 1, 2, \dots, N-1 \right]$$

$$\leq N^{-2} \left[(N-1)(N-2)k^2 + 2(N-1)k \right] . \delta_{j-1} .$$

Let $c = N^{-2} \left[(N-1)(N-2)k^2 + 2(N-1)k \right]$. Then $c < 1$, by the hypothesis. Hence $\delta_j < \delta_{j-1}$ for $j = 1, 2, 3, \dots$, and so

$$\left\| G^j(x) - G^{j+1}(x) \right\| \leq (N-1)N^{-1} . \delta_j$$

$$\leq (N-1)N^{-1} c . \delta_{j-1}$$

$$< c . \delta_{j-1} < c^2 . \delta_{j-2} < \dots < c^j . \delta_0 ;$$

i.e., $\| G^j(x)-G^{j+1}(x) \| < c^j\delta_0$ for $j = 1, 2, \ldots$.

Then the sequence $\{G^n(x)\}$ is a Cauchy sequence. In fact, for any $p \geq 0$, we have

$$\begin{aligned} \| G^n(x)-G^{n+p}(x) \| &\leq \|G^n(x)-G^{n+1}(x)\| + \|G^{n+1}(x)-G^{n+2}(x)\| + \ldots + \\ &\quad \|G^{n+p-1}(x)-G^{n+p}(x)\| \\ &< c^n\cdot\delta_0 + c^{n+1}\cdot\delta_0 + \ldots + c^{n+p-1}\cdot\delta_0 \\ &= \delta_0(c^n + c^{n+1} + \ldots + c^{n+p-1}) \\ &= \delta_0 c^n(1 + c + c^2 + \ldots + c^{p-1}) \\ &< \delta_0 c^n\left(\frac{1}{1-c} \right). \end{aligned}$$

But the expression on the right can be made arbitrarily small for sufficiently large n, since $c < 1$.

Since K is complete (because K is a closed subset of a Banach space), there exists $z \in K$ such that

$$\lim_{n \to \infty} G^n(x) = z,$$

and then

$$\lim_{n \to \infty} G(G^{n-1}(x)) = z.$$

We next show that G is continuous on K.

Let $x_0 \in K$. Given $\varepsilon > 0$ and choose $\delta = \frac{\varepsilon N^{-1}}{1+Nk-k}$. Then whenever $\|x_0-x\| < \delta$ we have

$$\| T^i(x_0)-T^i(x) \| \leq k\|x_0-x\| < k\delta, \text{ for } 1 \leq i \leq N-1.$$

Hence

$$N^{-1}\Big[\|x_o-x\| + \|T(x_o)-T(x)\| + \ldots + \|T^{N-1}(x_o)-T^{N-1}(x)\|\Big]$$

$$< \quad N^{-1}\Big[\delta + (N-1)k\delta\Big],$$

implies that

$$N^{-1}\|(x_o+T(x_o)+ \ldots +T^{N-1}(x_o)) - (x+T(x)+ \ldots +T^{N-1}(x))\| < N^{-1}\Big[\delta+ (N-1)k\delta\Big],$$

and hence

$$\|G(x_o)-G(x)\| < N^{-1}\Big[\delta+ (N-1)k\delta\Big]$$

$$= \frac{\delta(1+Nk-k)}{N}$$

$$= \varepsilon .$$

Then G is continuous at x_o, and then on K. Thus

$$\lim_{n\to\infty} G(G^{n-1}(x)) = G \lim_{n\to\infty} G^{n-1}(x) = z.$$

Hence $G(z) = z$, i.e., the limit z of $G^n(x)$ is fixed under G.

If $N > 2$, then $k < \frac{(N-1)}{(N-2)}$, since if $k \geq \frac{(N-1)}{(N-2)}$, then

$$N^{-2}\Big[(N-1)(N-2)k^2+ 2(N-1)k\Big] \geq N^{-2}\Big[(N-1)(N-2)\frac{(N-1)^2}{(N-2)^2} + 2(N-1)\frac{(N-1)}{(N-2)}\Big]$$

$$= N^{-2}\Big[\frac{(N-1)^3}{(N-2)}+ 2\frac{(N-1)^2}{(N-2)}\Big]$$

$$= N^{-2}\Big[\frac{(N-1)^2(N+1)}{(N-2)}\Big]$$

$$= 1 + \frac{N^2- N + 1}{N^2(N-2)}$$

$$> 1 ,$$

which contradicts the hypothesis.

Then by Lemma 4.16 we have that z is also a fixed point of T.

For N = 2. Then

$$G(x) = 2^{-1}(x + T(x)).$$

Suppose z is any fixed point of G. Then

$$z = G(z) = 2^{-1}(z + T(z)),$$

implies that

$$T(z) = z.$$

Hence the theorem is proved completely.

Proof. (of Theorem 4.11) Since T^N is a nonexpansive mapping, by Lemma 4.15 we have that T^N has a nonempty fixed point set C in K.

Then T^N is also a quasi-nonexpansive mapping. By Lemma 4.18 we get C is a closed and convex set.

Let $x \in C$. Then $T^N(x) = x$, and so $T^{N+1}(x) = T(x)$, i.e., $T^N(T(x)) = T(x)$. Hence $T(x) \in C$, that is T is a self-mapping of C, i.e., $T : C \longrightarrow C$, and T^N is the identity mapping on C.

Thus the assumptions for Theorem 4.10 are satisfied for T on C yielding a fixed point for T in C.

Hence the theorem is proved completely.