CHAPTER III

QUASI-GROUP HYPERGRAPHS

3.1 Hypergraph Induced by the Quasi-group

Let $(Q,\circ)$ be a quasi-group and $\mathcal{A}$ be any set of $(\gamma-1)$-subsets of Q, where $\gamma > 2$. We shall say that $\mathcal{A}$ is admissible if for each A in $\mathcal{A}$, each a in A and each q in Q, there exists $B_{a,q}$ in $\mathcal{A}$ sucht that

$$(\{q\} \cup q \circ A) - \{q \circ a\} = (q \circ a) \circ B_{a,q}$$

Note that the empty set is an admissible set. For each admissible set $\mathcal{A}$, we define $\mathcal{E}_{\mathcal{A}}$ by

$$\mathcal{E}_{\mathcal{A}} = \{\{q\} \cup q \circ A \mid q \in Q \text{ and } A \in \mathcal{A}\}.$$

3.1.1 Lemma Let $(Q,\circ)$ be a quasi-group and $\mathcal{A}$ be an admissible set of $(\gamma-1)$-subsets of Q. Then for each A in $\mathcal{A}$ and each q in Q, q does not belong to $q \circ A$.

Proof Let $(Q,\circ)$ be a quasi-group and $\mathcal{A}$ be an admissible set of $(\gamma-1)$-subsets of Q. Let A belong to $\mathcal{A}$ and q belong to Q. Suppose that q belongs to $q \circ A$. Then $\{q\} \cup q \circ A = q \circ A$. Choose an a in A. Hence there exists $B_{a,q}$ in $\mathcal{A}$ such that $(\{q\} \cup q \circ A) - \{q \circ a\} = (q \circ a) \circ B_{a,q}$. Note that

$$\begin{aligned} |(\{q\} \cup q \circ A) - \{q \circ a\}| &= |q \circ A - \{q \circ a\}| \\ &= |q \circ A| - |\{q \circ a\}| \\ &= (\gamma-1)-1 \\ &= \gamma-2 \end{aligned}$$

But

$$|(q \circ a) \circ B_a| = \gamma - 1$$

Hence we have a contradiction. Therefore q does not belong to $q \circ A$. #

3.1.2 Proposition Let $(Q,\circ)$ be a quasi-group and $\mathcal{A}$ be an admissible set of $(\gamma-1)$-subsets of Q. Then each set in $\mathcal{E}_{\mathcal{A}}$ consists of γ elements.

Proof Let $(Q,\circ)$ be a quasi-group and $\mathcal{A}$ be an admissible set of $(\gamma-1)$-subsets of Q. Let E belong to $\mathcal{E}_{\mathcal{A}}$. Then $E = \{q\} \cup q \circ A$ for some q in Q and some A in $\mathcal{A}$. By lemma 3.1.1, q does not belong to $q \circ A$. Hence

$$\begin{aligned} |\{q\} \cup q \circ A| &= |\{q\}| + |q \circ A| \\ &= 1 + (\gamma - 1) \\ &= \gamma \end{aligned}$$

Therefore E consists of γ elements. #

Let $(Q,\circ)$ be a finite quasi-group and $\mathcal{A}$ be an admissible set of $(\gamma-1)$-subsets of Q. Then $(Q, \mathcal{E}_{\mathcal{A}})$ will be called a hypergraph induced by the quasi-group $(Q,\circ)$.*

3.2 Quasi-group Hypergraphs

Let $H = (V,\mathcal{E})$ be a hypergraph of rank $\gamma \geq 2$. H will be called a quasi-group hypergraph if there exists a binary operation $\circ$ on V such that $(V,\circ)$ is a quasi-group and there exists an admissible set $\mathcal{A}$ of $(\gamma-1)$-subsets of V such that $\mathcal{E} = \mathcal{E}_{\mathcal{A}}$.

* An example is given in the appendix.

3.2.1 Lemma Let $(V, \mathcal{E}_{\mathcal{A}})$ be a quasi-group hypergraph. Then the following hold :

(1) For each E in $\mathcal{E}_{\mathcal{A}}$ and each v in E, there exists A in $\mathcal{A}$ such that $E - \{v\} = v \circ A$.

(2) For each w in V, $V_w = \{w \circ a \mid a \in \cup \mathcal{A}\}$.

(3) For each u in V, The function $\psi_u : V_u \to \cup \mathcal{A}$ define by $\psi_u (u \circ a) = a$ for all $u \circ a$ belongs to V_u is a one-to-one correspondence

Proof Let $(V, \mathcal{E}_{\mathcal{A}})$ be a quasi-group hypergraph. Let E belong to $\mathcal{E}_{\mathcal{A}}$ and v belong to E. Then there exists v_o in E and A_o in $\mathcal{A}$ such that $E = \{v_o\} \cup v_o \circ A_o$. Hence, by lemma 3.1.1, $E - \{v_o\} = v_o \circ A_o$

Case I If $v = v_o$, let $A = A_o$. Then

$$E - \{v\} = v \circ A.$$

Case II If $v \neq v_o$, then v belongs to $E - \{v_o\} = v_o \circ A_o$. Hence $v = v_o \circ a$ for some a in A_o. Then there exists B_{a,v_o} in $\mathcal{A}$ such that $(\{v_o\} \cup v_o \circ A_o) - \{v_o \circ a\} = (v_o \circ a) \circ B_{a,v_o}$ Therefore, by letting $A = B_{a,v_o}$, we see that

$$E - \{v\} = v \circ A$$

Hence (1) holds.

To prove (2), let w be any element of V. Observe that if B belongs to $\mathcal{E}_w$, then $B = E - \{w\}$ for some E in $\mathcal{E}_{\mathcal{A}}$ such that w belongs to E. Hence, by (1), we have $B = w \circ A$ for some A in $\mathcal{A}$. Therefore B belongs to $\{w \circ A \mid A \in \mathcal{A}\}$. Hence we have $\mathcal{E}_w \subseteq \{w \circ A \mid A \in \mathcal{A}\}$. Conversely, for each A in $\mathcal{A}$, $E = \{w\} \cup w \circ A$ belongs to $\mathcal{E}_{\mathcal{A}}$ and w belongs to E. Hence $w \circ A = E - \{w\}$ belongs to $\mathcal{E}_w$. Therefore $\{w \circ A \mid A \in \mathcal{A}\} \subseteq \mathcal{E}_w$. Hence

$$\mathcal{E}_w = \{w \circ A \mid A \in \mathcal{A}\}$$

Observe that

$$\begin{aligned} V_w &= \cup \mathcal{E}_w \\ &= \cup \{w \circ A \mid A \in \mathcal{A}\} \\ &= \{w \circ a \mid a \in \cup \mathcal{A}\} \end{aligned}$$

Hence (2) holds.

To prove (3), let u be any element of V. Hence, by (2), $V_u = \{u \circ a \mid a \in \cup \mathcal{A}\}$. Since V is a quasi-group, to each element v of V_u there exist unique a_v in $\cup \mathcal{A}$ such that $u \circ a_v = v$. Hence the equation $\psi_u(u \circ a) = a$ defines a function on V_u into $\cup \mathcal{A}$. Let $u \circ a_1$, $u \circ a_2$ be any elements in V_u such that $\psi_u(u \circ a_1) = \psi_u(u \circ a_2)$. Hence $a_1 = a_2$. Therefore $u \circ a_1 = u \circ a_2$. Hence ψ_u is one-to-one. For any a in $\cup \mathcal{A}$ we see that $\psi_u(u \circ a) = a$. Hence ψ_u is onto. Therefore (3) holds.

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3.2.2 Proposition Let $(V, \mathcal{E}_{\mathcal{A}})$ be a quasi-group hypergraph. Then there exist a system $(\psi_{uv})_{u,v \in V}$ such that each ψ_{uv} is an isomorphism from H_u onto H_v and for every u, v, v' in V if $v \neq v'$ then $\psi_{uv}(w) \neq \psi_{uv'}(w)$ for all w in V_u.

Proof Let $(V, \mathcal{E}_{\mathcal{A}})$ be a quasi-group hypergraph. For every u, v in V, we define $\psi_{uv} : V_u \to V_v$ by

$$\psi_{uv}(w) = v \circ \psi_u(w) \text{ for all } w \text{ in } V_u ,$$

where ψ_u is as in lemma 3.2.1.

Since ψ_u is a one-to-one correspondence and V is a quasi-group, hence ψ_{uv} is a one-to-one correspondence.

Next we shall show that ψ_{uv} is an isomorphism from H_u onto H_v. Let B belong to $\mathcal{E}_u$. Then $\{u\} \cup B$ belongs to $\mathcal{E}_{\mathcal{A}}$. By lemma 3.2.1, there exists A in $\mathcal{A}$ such that $(\{u\} \cup B) - \{u\} = u \circ A$. But $(\{u\} \cup B) - \{u\} = B$, so that $B = u \circ A$. Since A belongs to $\mathcal{A}$, $\{v\} \cup v \circ A$ belongs to $\mathcal{E}_{\mathcal{A}}$, Hence $v \circ A$ belongs to $\mathcal{E}_v$

Observe that

$$\begin{aligned} \psi_{uv}(B) &= \psi_{uv}(u \circ A) \\ &= v \circ \psi_u(u \circ A) \\ &= v \circ A \end{aligned}$$

Hence $\psi_{uv}(B)$ belongs to $\mathcal{E}_v$. Conversely, let $\psi_{uv}(B)$ belong to $\mathcal{E}_v$. Then $\{v\} \cup \psi_{uv}(B)$ belongs to $\mathcal{E}_{\mathcal{A}}$. Hence, by lemma 3.2.1, there exists A in $\mathcal{A}$ such that $(\{v\} \cup \psi_{uv}(B)) - \{v\} = v \circ A$. But $(\{v\} \cup \psi_{uv}(B)) - \{v\} = \psi_{uv}(B)$, so that $v \circ A = \psi_{uv}(B)$. Observe that

$$\begin{aligned} \psi_{uv}(u \circ A) &= v \circ \psi_u(u \circ A) \\ &= v \circ A \\ &= \psi_{uv}(B) \end{aligned}$$

Hence $u \circ A = B$. Since A belongs to $\mathcal{A}$, therefore $\{u\} \cup u \circ A$ belongs to $\mathcal{E}_{\mathcal{A}}$. Hence $u \circ A$ belongs to $\mathcal{E}_u$ i.e. B belongs to $\mathcal{E}_u$. Therefore ψ_{uv} is an isomorphism from H_u onto H_v. Finally we shall show that for every u, v, v' in V if $v \neq v'$ then $\psi_{uv}(w) \neq \psi_{uv}(w)$ for all w in V_u.

Let u, v, v' belong to V. Assume that $v \neq v'$. Then $v \circ \psi_u(w) \neq v' \circ \psi_u(w)$ for all w in V_u. But $\psi_{uv}(w) = v \circ \psi_u(w)$ and $\psi_{uv'}(w) = v' \circ \psi_u(w)$, for all w in V_u. Hence

$$\psi_{uv}(w) \neq \psi_{uv'}(w) \quad \text{for all } w \text{ in } V_u.$$

Therefore there exists a system $(\psi_{uv})_{u,v \in V}$ such that each ψ_{uv} is an isomorphism from H_u onto H_v and for every u,v,v' in V if $v \neq v'$ then $\psi_{uv}(w) \neq \psi_{uv'}(w)$ for all w in V_u.

3.3 Regular and Specially regular Hypergraphs

Let $H = (V, \mathcal{E})$ be a hypergraph. We say that H is regular if for every u, v in V, there exists an isomorphism ψ_{uv} from H_u onto H_v. In addition, if a system $(\psi_{uv})_{u,v \in V}$ can be chosen such that for each u, v, v' in V if $v \neq v'$ then $\psi_{uv}(w) \neq \psi_{uv'}(w)$ for all w in V_u, we say that H is specially regular. We note that proposition 3.2.2 asserts that every quasi-group hypergraph is specially regular.

3.3.1 Proposition Let $H = (V, \mathcal{E})$ be a hypergraph. If there exists u in V such that for each v in V, there exists an isomorphism ψ_{uv} from H_u onto H_v such that for every v, v' in V if $v \neq v'$ then $\psi_{uv}(w) \neq \psi_{uv'}(w)$ for all w in V_u. Then H is specially regular.

Proof Let $H = (V, \mathcal{E})$ be a hypergraph. Assume that there exists u_o in V such that for each v in V, there exists an isomorphism $\psi_{u_o v}$ from H_{u_o} onto H_v such that for every v, v' in V if $v \neq v'$ then $\psi_{u_o v}(w) \neq \psi_{u_o v'}(w)$ for all w in V_{u_o}.

For every v, v' in V, define $\psi_{vv'}: V_v \to V_{v'}$ by

$$\psi_{vv'}(w) = \psi_{u_o v'}(\psi^{-1}_{u_o v}(w)) \quad \text{for all } w \text{ in } V_v.$$

Since $\psi_{u_o v'}$ and $\psi^{-1}_{u_o v}$ are isomorphism, hence $\psi_{vv'}$ is an isomorphism from H_v onto $H_{v'}$.

Next we shall show that for every u, v, v' in V if $v \neq v'$ then $\psi_{uv}(w) \neq \psi_{uv'}(w)$ for all w in V_u. Let u, v, v' belong to V. Assume that $v \neq v'$. Then $\psi_{u_o v}(\psi^{-1}_{u_o u}(w)) \neq \psi_{u_o v'}(\psi^{-1}_{u_o u}(w))$ for all w in V_u. Hence

$$\psi_{uv}(w) \neq \psi_{uv'}(w) \quad \text{for all } w \text{ in } V_u.$$

Therefore H is a specially regular hypergraph.

3.3.2 <u>Lemma</u> Let $H = (V, \mathcal{E})$ be a specially regular hypergraph. Then there exist one-to-one function P_v, v belongs to V, from V onto V such that

(1) $\forall v, v' \in V \; (v \neq v' \to \forall w \in V \; (P_v(w) \neq P_{v'}(w)))$

(2) $\forall v, v' \in V \; \forall A \subseteq V \; (P_v(A) \in \mathcal{E}_v \leftrightarrow P_{v'}(A) \in \mathcal{E}_{v'})$

<u>Proof</u> Let $H = (V, \mathcal{E})$ be a specially regular hypergraph. Choose an element u_o in V and let it be fixed. For each v in V, let $\psi_{u_o v}$ be an isomorphism from H_{u_o} onto H_v such that for every v, v' in V if $v \neq v'$ then $\psi_{u_o v}(w) \neq \psi_{u_o v'}(w)$ for all w in V_{u_o}. For convinience, let $W = N_{\overline{(H)}_2}(u_o)$. Since H is specially regular hypergraph, hence, by remark 2.5.1, $W = N_{\overline{(H)}_2}(u_o) = V - (V_{u_o} \cup \{u_o\})$. By proposition 2.5.2, $\overline{(H)}_2$ is regular. Therefore for each v in V, $|W| = |N_{\overline{(H)}_2}(v)|$. Hence, by proposition 2.2.2, for each v in V we

can associate a one-to-one function Π_v from W onto $N_{\overline{(H)}_2}(v)$ such that for every v, v' in V if $v \neq v'$ then $\Pi_v(w) \neq \Pi_{v'}(w)$ for all w in W. For each v in V, we define $P_v: V \to V$ by

$$P_v(w) = \begin{cases} \psi_{u_o v}(w) & \text{if } w \in V_{u_o}, \\ \Pi_v(w) & \text{if } w \in W, \\ v & \text{if } w = u_o \end{cases}$$

From the definitions of $\psi_{u_o v}$ and Π_v we see that they are one-to-one functions on distinct sets not containing u_o onto disjoint sets not containing v. Hence P_v is a one-to-one correspondence

To show (1), let v, v' be any distinct elements of V. Let w be any element of V.

Case I If w belongs to V_{u_o}, then $\psi_{u_o v}(w) \neq \psi_{u_o v'}(w)$. But $P_v(w) = \psi_{u_o v}(w)$ and $P_{v'}(w) = \psi_{u_o v'}(w)$. Hence $P_v(w) \neq P_{v'}(w)$.

Case II If w belongs to W, then $\Pi_v(w) \neq \Pi_{v'}(w)$. But $P_v(w) = \Pi_v(w)$ and $P_{v'}(w) = \Pi_{v'}(w)$. Hence $P_v(w) \neq P_{v'}(w)$.

Case III If $w = u_o$, then $P_{v'}(w) = v'$ and $P_v(w) = v$. Hence $P_v(w) \neq P_{v'}(w)$.

Therefore for every v, v', w in V, $P_v(w) \neq P_{v'}(w)$ when $v \neq v'$. Hence (1) holds

To prove (2), let v, v' be any elements of V and A be any subset of V. First assume that $P_{v'}(A)$ belongs to $\mathcal{E}_{v'}$. Therefore $P_{v'}(A) \subseteq V_{v'}$. Hence we have $P^{-1}_{v'}(P_{v'}(A)) \subseteq P^{-1}_{v'}(V_{v'})$. But $P^{-1}_{v'}(P_{v'}(A)) = A$ and

$P_{v'}^{-1}(V_{v'}) = \psi_{u_o v'}^{-1}(V_{v'}) = V_{u_o}$. Hence $A \subseteq V_{u_o}$. Since $P_{v'}$ and $\psi_{u_o v'}$ are identical on V_{u_o}, hence $P_{v'}(A) = \psi_{u_o v'}(A)$. By the same argument we see that $P_v(A) = \psi_{u_o v}(A)$. Since $\psi_{u_o v'}(A) = P_{v'}(A)$ belongs to $\mathcal{E}_{v'}$, hence A belongs to $\mathcal{E}_{u_o}$. Therefore $P_v(A) = \psi_{u_o v}(A)$ belongs to $\mathcal{E}_v$. Similarly, we can show that $P_{v'}(A)$ belongs to $\mathcal{E}_{v'}$ if $P_v(A)$ belongs to $\mathcal{E}_v$.

Hence (2) holds.

Therefore there exist one-to-one function P_v, v belongs to V, from V onto V such that

(1) $\forall v, v' \in V \; (v \neq v' \rightarrow \forall w \in V (P_v(w) \neq P_{v'}(w)))$

(2) $\forall v, v' \in V \; \forall A \subseteq V \; (P_v(A) \in \mathcal{E}_v \leftrightarrow P_{v'}(A) \in \mathcal{E}_{v'})$.

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3.3.3 Proposition A hypergraph of rank at least 2 is a quasi-group hypergraph if it is specially regular.

Proof Let $(V, \mathcal{E})$ be a specially regular hypergraph. Hence, by lemma 3.3.2, there exist one-to-one function P_v, v belongs to V, from V onto V such that

(1) $\forall v, v' \in V \; (v \neq v' \rightarrow \forall w \in V \; (P_v(w) \neq P_{v'}(w)))$

(2) $\forall v, v' \in V \; \forall A \subseteq V \; (P_v(A) \in \mathcal{E}_v \leftrightarrow P_{v'}(A) \in \mathcal{E}_{v'})$

We define a binary operation $\circ$ on V by

$$u \circ v = P_u(v) \quad \text{for all } u, v \text{ in } V.$$

First we shall show that $(V, \circ)$ is a quasi-group. Let u, v belong to V. Since P_u is onto, hence $P_u(s) = v$ for some s in V. Therefore there exists s in V such that $u \circ s = v$. Suppose s, t in V are such that $u \circ s = v = u \circ t$. Then $P_u(s) = P_u(t)$. Since P_u is one-to-one, hence $s = t$. Therefore there exists unique s belongs to V such that $u \circ s = v$. Since $P_w(u)$, w belongs to V, are distinct, hence $\{P_w(u) \mid w \in V\} = V$. Therefore there exists s' in V such that $P_{s'}(u) = v$, i.e. $s' \circ u = v$. Suppose s', t' in V such that $s' \circ u = v = t' \circ u$. Then $P_{s'}(u) = P_{t'}(u)$. Hence $s' = t'$. Therefore there exists unique s' in V such that $s' \circ u = v$.

Hence $(V, \circ)$ is a quasi-group.

Next, we shall define an admissible set $\mathcal{A}$ such that $\mathcal{E} = \mathcal{E}_{\mathcal{A}}$.

Case I Assume that $\mathcal{E} = \emptyset$. Let $\mathcal{A} = \emptyset$. Then $\mathcal{E}_{\mathcal{A}} = \emptyset$. Therefore $\mathcal{E} = \mathcal{E}_{\mathcal{A}}$.

Case II Assume that $\mathcal{E} \neq \emptyset$. Let E belong to $\mathcal{E}$ and u belong to E. Hence $\mathcal{E}_u \neq \emptyset$. Let

$$\mathcal{A}_u = \{A \mid u \circ A \in \mathcal{E}_u\}$$

To see that $\mathcal{A}_u \neq \emptyset$, let B belong to $\mathcal{E}_u$. Since P_u is onto, hence there exists a subset A of V such that $P_u(A) = B$. Note that

$$\begin{aligned} u \circ A &= P_u(A) \\ &= B \end{aligned}$$

Therefore $u \circ A$ belongs to $\mathcal{E}_u$. So that A belongs to $\mathcal{A}_u$. Hence $\mathcal{A}_u \neq \emptyset$. Let

$$\mathcal{A} = \mathcal{A}_u$$

Before verifying that $\mathcal{A}$ is admissible we shall show that for any v in V, $\mathcal{A}_v = \{A \mid v \circ A \in \mathcal{E}_v\}$ coincides with $\mathcal{A}$. To do this let v be any element of V and put $\mathcal{A}_v = \{A \mid v \circ A \in \mathcal{E}_v\}$. Since $(V, \mathcal{E})$ is specially regular, hence $H_u \cong H_v$. Therefore $\mathcal{E}_v \neq \emptyset$. Hence, by the above argument, we also have $\mathcal{A}_v \neq \emptyset$. To see that $\mathcal{A}_v = \mathcal{A}$, let A belong to $\mathcal{A}_v$. Then $v \circ A$ belongs to $\mathcal{E}_v$. Observe that

$$P_v(A) = v \circ A$$

Therefore $P_v(A)$ belongs to $\mathcal{E}_v$. Hence, by (2), $P_u(A)$ belongs to $\mathcal{E}_u$. Note that

$$P_u(A) = u \circ A$$

Therefore $u \circ A$ belongs to $\mathcal{E}_u$. Hence A belongs to $\mathcal{A}_u$ Therefore $\mathcal{A}_v \subseteq \mathcal{A}_u$. Similarly, we can show that $\mathcal{A}_u \subseteq \mathcal{A}_v$. Hence

$$\mathcal{A}_v = \mathcal{A}_u = \mathcal{A}.$$

To verify that $\mathcal{A}$ is admissible, let A belong to $\mathcal{A}$, a belong to A and v belong to V. Since $\mathcal{A} = \mathcal{A}_v$, hence A belongs to $\mathcal{A}_v$. Therefore $v \circ A$ belongs to $\mathcal{E}_v$. Hence $\{v\} \cup v \circ A$ belongs to $\mathcal{E}$. Therefore $(\{v\} \cup v \circ A) - \{v \circ a\}$ belongs to $\mathcal{E}_{v \circ a}$. Since $P_{v \circ a}$ is onto, hence there exists a subset B of V such that $P_{v \circ a}(B) = (\{v\} \cup v \circ A) - \{v \circ a\}$. Choose $B_{a,v} = B$. Observe that

$$\begin{aligned} (v \circ a) \circ B_{a,v} &= P_{v \circ a}(B_{a,v}) \\ &= P_{v \circ a}(B) \\ &= (\{v\} \cup v \circ A) - \{v \circ a\}, \end{aligned}$$

which belongs to $\mathcal{E}_{v \circ a}$. Hence $B_{a,v}$ belongs to $\mathcal{A}$. Therefore $\mathcal{A}$ is an admissible set.

Finally we shall show that $\mathcal{E} = \mathcal{E}_{\mathcal{A}}$. Let E belong to $\mathcal{E}$. Then for each v in E, $E - \{v\}$ belongs to $\mathcal{E}_v$. Since $E - \{v\}$ is a subset of V and P_v is onto, hence there exists a subset A_v of V such that $P_v(A_v) = E - \{v\}$. Observe that

$$\begin{aligned} v \circ A_v &= P_v(A_v) \\ &= E - \{v\} \end{aligned}$$

Therefore $v \circ A_v$ belongs to $\mathcal{E}_v$. Hence A_v belongs to $\mathcal{A}$. Therefore $\{v\} \cup v \circ A_v$ belongs to $\mathcal{E}_{\mathcal{A}}$, i.e. E belongs to $\mathcal{E}_{\mathcal{A}}$. Hence $\mathcal{E} \subseteq \mathcal{E}_{\mathcal{A}}$. Conversely, let E belong to $\mathcal{E}_{\mathcal{A}}$. Then there exists v in V and A in $\mathcal{A}$ such that $E = \{v\} \cup v \circ A$. Since A belongs to $\mathcal{A}_v$, $v \circ A$ belongs to $\mathcal{E}_v$. Therefore $\{v\} \cup v \circ A$ belongs to $\mathcal{E}$, i.e., E belongs to $\mathcal{E}$. Hence $\mathcal{E}_{\mathcal{A}} \subseteq \mathcal{E}$.

Therefore

$$\mathcal{E} = \mathcal{E}_{\mathcal{A}}$$

Hence $(V, \mathcal{E})$ is a quasi-group hypergraph #

We may now summarize proposition 3.2.2 and proposition 3.3.3 into the following:

3.3.4 <u>Theorem</u> A hypergraph of rank at least 2 is a quasi-group hypergraph if and only if it is specially regular.

3.4 Quasi-group graphs

Graphs are hypergraphs of rank 2. It is clear that our definition of regular graphs given in section 2.2 agrees with that of regular hypergraph in section 3.3, where we also introduce the concept of specially regularity. For graphs, i.e. hypergraphs of rank 2, it turns out the regularity and specially regularity are equivalent, i.e. we have the following :

3.4.1 Proposition Let $(V, \mathcal{E})$ be a hypergraph of rank 2. Then $(V, \mathcal{E})$ is specially regular if and only if it is regular.

Proof Let $(V, \mathcal{E})$ be a hypergraph of rank 2.
If $(V, \mathcal{E})$ is specially regular. Then it is clear that $(V, \mathcal{E})$ is regular. Conversely, assume that $(V, \mathcal{E})$ is regular of degree k. Let W be a set such that $|W| = k$. Hence, by proposition 2.2.2, for each v in V we can associate a one-to-one function π_v from W onto $N_G(v)$ such that

$$\forall u, v \in V \ (u \neq v \rightarrow \forall w \in W \ (\pi_u(w) \neq \pi_v(w)))$$

For every u, v in V, let

$$\psi_{uv} = \pi_v \circ \pi_u^{-1}$$

clearly, for every u, v in V, ψ_{uv} is an isomorphism from H_u onto H_v.
Let u, v, v' be any elements in V such that $v \neq v'$.
Let w be any element in V_u. Hence $\pi_u^{-1}(w)$ belongs to W. Therefore $\pi_v(\pi_u^{-1}(w)) \neq \pi_{v'}(\pi_u^{-1}(w))$, i.e., $\psi_{uv}(w) \neq \psi_{uv'}(w)$.
Hence $(V, \mathcal{E})$ is specially regular. #

From Theorem 3.3.4 and proposition 3.4.1, we have the following

3.4.2 Corollary A graph is quasi-group graph if and only if it is regular.