INTRODUCTION

In this thesis, we study universal algebras with certain properties, namely, those with idempotent elements. Our primary concern is to consider the endomorphism semigroup of some classes of these algebras.

The relevant preliminaries concepts and results have been summarized in Chapter I. These are the basic notions of endomorphism semigroups of algebras, and the fundamentals of Categories and functors.

Chapter II is devoted to the study of the class of algebras of type < 2 >. It points out the connection between monoids and the category of semigroups, and then we get the result that any monoid will be isomorphic with the endomorphism semigroup of some semigroup.

In chapter III, we define the notions of idempotent algebras and entire algebras. We then characterize the endomorphism semigroup of certain classes of these algebras. These characterizations are the main results of this thesis.

In chapter IV, we prove some properties of unary algebras, since these algebras play such prominent roll in the preceding chapters.

1.2 Definition. Let $< A; F >$ be an algebra of type τ and B a non-void subset of A. $< B; F >$ is called a subalgebra of $< A; F >$ if and only if $b_0, \ldots, b_{n_\gamma - 1} \in B$ implies $f_\gamma(b_0, \ldots, b_{n_\gamma - 1}) \in B$, for all $f_\gamma \in F$.

1.3 Definition. Let $< A; F^A >$ and $< B; F^B >$ be two algebras belonging to the same similarity class $K(\tau)$. A mapping $\psi : A \to B$ such that

$$f_\gamma^A(a_0, \ldots, a_{n_\gamma - 1})\psi = f_\gamma^B(a_0\psi, \ldots, a_{n_\gamma - 1}\psi)$$

for all $\gamma < O(\tau)$, $a_0, \ldots, a_{n_\gamma - 1} \in A$ is called a homomorphism of $< A; F^A >$ into $< B; F^B >$.

A homomorphism $\psi : A \to B$ is called an isomorphism between the algebras $< A; F^A >$ and $< B; F^B >$ if it is 1 - 1 and onto.

A homomorphism ψ of an algebra $< A; F >$ into itself is called an endomorphism of $< A; F >$.

Denote the set of endomorphisms of $< A; F >$ by $E(A; F)$.

1.4 Lemma. $< E(A; F); \circ >$ is a semigroup, where $\circ$ is the composition of mapping, and ε, the identity mapping, is the unit element of this semigroup. This semigroup is called the endomorphism semigroup of $< A; F >$.

1.5 Lemma. Suppose $\psi : A \to B$ is a homomorphism of $< A; F^A >$ into $< B; F^B >$. Then $< A\psi; F^B >$ is a subalgebra of $< B; F^B >$.

1.6 Definition. Let ψ be an endomorphism of the algebra $< A; F >$. If ψ is also 1 - 1 and onto, then ψ is called an automorphism.

Let Aut (A; F) denote the set of all automorphisms of < A; F >.

1.7 Lemma. < Aut (A; F), o > is a group and it is called the automorphism group of < A; F >.

Now, we recall some concepts of semigroups.

1.8 Definitions. A nonempty set S with an operation • such that $(a\cdot b)\cdot c = a\cdot(b\cdot c)$ for all a, b, c in S is called a semigroup.

An element e in S is called an identity of S if $e\cdot x = x\cdot e = x$ for all $x \in S$.

An element 0 in S is called zero of S if $0\cdot x = x\cdot 0 = 0$ for all x in S.

An element a in S is a right annihilator of S if $x\cdot a = a$ for all $x \in S$.

1.9 Definitions. Let I be a nonempty subset of a semigroup S. Then I is an ideal of S if $s \in S$, $x \in I$ implies $s\cdot x$, $x\cdot s \in I$.

A semigroup S is simple if S is its only ideal.

1.10 Definitions. Let S be a semigroup. A mapping $\lambda : S \to S$ $[\rho : S \to S]$ is a left [right] translation if $\lambda(xy) = (\lambda x)y$ $[(xy)\rho = x(y\rho)]$ for all $x, y \in S$.

Let $s \in S$. Then the mapping $\lambda_s [\rho_s]$ defined by $\lambda_s(x) = s\cdot x$ $[x\rho_s = x\cdot s]$ for all $x \in S$ is a left [right] translation, and it is called the inner left [right] translation induced by s.

A left translation λ and a right translation ρ are permutable if $(\lambda x)\rho = \lambda(x\rho)$ for all $x \in S$.

1.11 <u>Definition</u>. A semigroup S is called <u>globally idempotent</u> if $S^2 = S$. A monoid is globally idempotent.

1.12 <u>Lemma</u>. Let S be a globally idempotent semigroup. Then every left translation is permutable with every right translation.

Next, we will review the definitions of Category and Functor.

1.13 <u>Definition</u>. A <u>category $\mathcal{A}$</u> consists of a collection of objects, $Ob(\mathcal{A})$; and for two objects $A, B \in Ob(\mathcal{A})$, a set [A, B] called the set of morphisms of A into B; and for three objects $A, B, C \in Ob(\mathcal{A})$, a law of composition

$$[B, C] \times [A, B] \longrightarrow [A, C]$$

satisfying the following axioms :-

<u>CAT 1</u>. The sets [A, B] and [A', B'] are disjoint unless A = A' and B = B', in which case they are equal.

<u>CAT 2</u>. For each $A \in Ob(\mathcal{A})$, there is a morphism $1_A \in [A, A]$ such that for any $\gamma \in [B, A]$, $\beta \in [A, B]$, $1_A o \gamma = \gamma$ and $\beta o 1_A = \beta$ for all $B \in Ob(\mathcal{A})$.

<u>CAT 3</u>. Given $\alpha \in [A, B]$, $\beta \in [B, C]$ and $\gamma \in [C, D]$, then $(\gamma o \beta) o \alpha = \gamma o (\beta o \alpha)$ for all $A, B, C, D \in Ob(\mathcal{A})$.

<u>Examples of categories</u>.

(1) Let Ens be the category whose objects are sets, and whose morphisms are maps between sets. Then Ens is called the <u>category of sets</u>.

(2) Let Grp be the category of groups; i.e. the category whose objects are groups, and whose morphisms are group-homomorphisms.

(3) Let $\mathcal{S}$ be the category of semigroups; i.e. the category whose objects are semigroups, and whose morphisms are semigroup-homomorphisms.

1.14 Definition. Let $\mathfrak{A}$ and $\mathfrak{B}$ be categories. A (covariant) functor F of $\mathfrak{A}$ into $\mathfrak{B}$ is an assignment of an object $F(A) \in Ob(\mathfrak{B})$ to each object $A \in Ob(\mathfrak{A})$ and a morphism $F(\alpha) \in [F(A), F(A')]$ in $\mathfrak{B}$ to each morphism $\alpha \in [A, A']$ in $\mathfrak{A}$, subject to the following axioms :

FUN 1. For each $A \in Ob(\mathfrak{A})$, we have $F(1_A) = 1_{F(A)}$.

FUN 2. If $\alpha \in [A, B]$ and $\beta \in [B, C]$ then $F(\beta o \alpha) = F(\beta) o F(\alpha)$.

Examples of functors.

(1) Let $\mathfrak{A}$ be a category. Define $1_{\mathfrak{A}} : \mathfrak{A} \to \mathfrak{A}$ by

$$1_{\mathfrak{A}}(A) = A \qquad (A \in \mathfrak{A}),$$

and

$$1_{\mathfrak{A}}(\alpha) = \alpha \qquad (\alpha \in [A, A'], A, A' \in Ob(\mathfrak{A})).$$

Then $1_{\mathfrak{A}}$ is a functor called the identity functor on $\mathfrak{A}$.

(2) Let $\mathfrak{A}$ be a category, and A be a fixed object in $\mathfrak{A}$. Define $M_A : \mathfrak{A} \longrightarrow Ens$ by

$$M_A(X) = [A, X] \qquad (X \in Ob(\mathfrak{A})),$$

and for every $X, X' \in Ob(\mathfrak{A})$, $\psi \in [X, X']$,

$$M_A(\psi) : [A, X] \longrightarrow [A, X']$$

by the rule

$$g \longrightarrow \psi o g \qquad\qquad (g \in [A, X]).$$

Then M_A is a functor.

1.15 Definition. Let S and T be (convariant) functors of a category $\mathcal{A}$ into a category $\mathcal{B}$. A natural transformation η from S to T (and we write $\eta : S \longrightarrow T$.) is an assignment to every object $A \in Ob(\mathcal{A})$, a morphism $\eta_A \in [S(A), T(A)]$ in $\mathcal{B}$ such that for every $\alpha \in [A, A']$, for all A, A' in $\mathcal{A}$, the following diagram

$$\begin{array}{ccc} S(A) & \xrightarrow{\eta_A} & T(A) \\ {\scriptstyle S(\alpha)}\downarrow & & \downarrow{\scriptstyle T(\alpha)} \\ S(A') & \xrightarrow[\eta_{A'}]{} & T(A') \end{array}$$

commutes.

Note. Let S, T, U be functors of $\mathcal{A}$ into $\mathcal{B}$. Let $\eta : S \rightarrow T$ and $\rho : T \rightarrow U$ be natural transformations from S to T and T to U, respectively. Then we have a composition $\rho o \eta \; : S \longrightarrow U$ defined by

$$(\rho o \eta)_A \quad = \quad \rho_A o \eta_A$$

and for any functor T we have the identity transformation $1_T : T \longrightarrow T$ such that

$$(1_T)_A \quad = \quad 1_{T(A)}$$

for all $A \in Ob(\mathcal{A})$.

If $\mathcal{A}$ is a small category (i.e. Ob ($\mathcal{A}$) is a set) then $\mathcal{B}^{\mathcal{A}}$ is a category whose $Ob(\mathcal{B}^{\mathcal{A}})$ is the set of all functors of $\mathcal{A}$ into $\mathcal{B}$ and for any functors S, T of $\mathcal{B}^{\mathcal{A}}$, [S,T] is the set of natural transformations from S to T. We call $\mathcal{B}^{\mathcal{A}}$ a functor category.

1.16 <u>Definitions</u>. Let T be a functor of $\mathfrak{A}$ into $\mathfrak{B}$. Then T is called <u>faithful</u> if T is one-one on morphisms, T is called <u>full</u> if for every pair of objects A, $B \in \mathrm{Ob}(\mathfrak{A})$, the function

$T : [A, B] \longrightarrow [T(A), T(B)]$ is onto.

A functor will be called a <u>full embedding</u> if it is full, faithful and one-one on objects.