CHAPTER IV

GENERAL SOLUTION OF $f(x\,y) + f(xy^{-1}) = 2f(x)f(y\theta)$ ON GROUPS.

4.1 Introduction

In [1], Kannappan determines all the functions f on a group G into the complex plane $\mathbb{C}$ satisfying the functional equation

(C) $$f(xy) + f(xy^{-1}) = 2f(x)f(y),$$

and an additional condition

(A) $$f(xyz) = f(xzy), \text{ for every } x, y, z \text{ in } G.$$

In this chapter, we use the method of [1] to obtain all functions f from a group G into $\mathbb{C}$ satisfying the functional equation

(S) $$f(xy) + f(xy^{-1}) = 2f(x)f(y\theta),$$

where θ is any fixed element of G, and the additional condition (A).

Observe that when θ is the identity of G, the equation (S) becomes (C). Hence the study of (C) is a special case of the study of (S).

Note that when G is commutative, the condition (A) is automatically satisfied. When G is the additive group of complex (or real) numbers, the equation (C) becomes

(C′) $$f(x+y) + f(x-y) = 2f(x)f(y),$$

which is an identity for cosine. Because of this, the equation (C) is known as the cosine equation.

In this case, if we take θ to be $\frac{\pi}{2}$, then the equation (S) becomes

(S$'$) $$f(x + y) + f(x - y) = 2f(x)f(y + \frac{\pi}{2}),$$

which is an identity for sine. So that the equation (S) includes as its special cases, the cosine and the sine functional equations.

4.2 Some Lemmas

4.2.1 Lemma Let G be an arbitrary group and let f be a complex-valued function satisfying (S) and (A) on G, not identically zero. Then we have

(4.2.1.1) $$f(\theta) = 1$$

(4.2.1.2) $$f(y^2\theta) = 2f(y\theta)^2 - 1$$

(4.2.1.3) $$f(x^2\theta^{-1}) + f(y^2\theta) = 2f(xy)f(xy^{-1})$$

(4.2.1.4) $$f(x^2\theta^{-1}) = 2f(x)^2 - 1$$

(4.2.1.5) $$\left[f(xy) - f(x)f(\theta y)\right]^2 = \left[f(x)^2 - 1\right]\left[f(\theta y)^2 - 1\right]$$

Proof Let x be an element of G.

Replacing y by e, the identity of G, in (S), we have

$$f(xe) + f(xe^{-1}) = 2f(x)f(e\theta).$$

Thus $$2f(x) = 2f(x)f(\theta).$$

Since f is not identically zero, hence we have

$$f(\theta) = 1 \qquad (4.2.1.1)$$

Replacing x by $y\theta$ in (S), we have

$$f(y\theta y) + f(y\theta y^{-1}) = 2f(y\theta)f(y\theta),$$

$$f(yy\theta) + f(yy^{-1}\theta) = 2f(y\theta)^2,$$

$$f(y^2\theta) + f(\theta) = 2f(y\theta)^2.$$

Thus $$f(y^2\theta) = 2f(y\theta)^2 - 1 \qquad (4.2.1.2)$$

Replacing x and y by xy and $xy^{-1}\theta^{-1}$, respectively, in (S), we get

$$f(xy\,xy^{-1}\theta^{-1}) + f(\,xy\theta y\,x^{-1}) = 2f(xy)f(xy^{-1}),$$

$$f(xyy^{-1}\theta^{-1}x) + f(xx^{-1}y\theta y) = 2f(xy)f(xy^{-1}),$$

$$f(x\theta^{-1}\,x) + f(y\theta y) = 2f(xy)f(xy^{-1}),$$

$$f(x^2\theta^{-1}) + f(y^2\theta) = 2f(xy)f(xy^{-1}) \qquad (4.2.1.3)$$

Replacing y by $x\theta^{-1}$ in (S), we have

$$f(xx\,\theta^{-1}) + f(x\theta x^{-1}) = 2f(x)\,f(x\theta^{-1}\theta),$$

$$f(x^2\theta^{-1}) + f(\theta) = 2f(x)^2,$$

thus $$f(x^2\theta^{-1}) = 2f(x)^2 - 1, \qquad (4.2.1.4)$$

Now using (S), (4.2.1.2), (4.2.1.3) and (4.2.1.4), we find that

$$\left[f(xy) - f(xy^{-1})\right]^2 = \left[f(xy) + f(xy^{-1})\right]^2 - 4\,f(xy)f(xy^{-1}),$$

$$= \left[f(xy) + f(xy^{-1})\right]^2 - 2\left[f(x^2\theta^{-1}) + f(y^2\theta)\right],$$

$$= \left[f(xy) + f(xy^{-1})\right]^2 - 2\left[2\,f(x)^2 - 1 + 2f(y\theta)^2 - 1\right]$$

$$= \left[f(xy) + f(xy^{-1})\right]^2 - 4\left[f(x)^2 + f(y\theta)^2 - 1\right],$$

$$= 4\left[f(x)^2\,f(y\theta)^2 - f(x)^2 - f(y\theta)^2 + 1\right],$$

$$= 4\left[f(x^2)-1\right]\left[f(y\theta)^2-1\right].$$

Consequently, we obtain $f(xy)-f(xy^{-1}) = 2(\left[f(x)^2-1\right]\left[f(y\theta)^2-1\right])^{\frac{1}{2}}$.

Adding this equation to (S), we get

$$f(xy) = f(x)f(y\theta) + \left(\left[f(x)^2 - 1\right]\left[f(y\theta)^2 - 1\right] \right)^{\frac{1}{2}}.$$

So we have

$$\left[f(xy) - f(x)f(y\theta)\right]^2 = \left[f(x)^2 - 1\right]\left[f(y\theta)^2 - 1\right],$$

$$\left[f(xy) - f(x)f(ey\theta)\right]^2 = \left[f(x)^2 - 1\right]\left[f(ey\theta)^2 - 1\right],$$

$$\left[f(xy) - f(x)f(e\theta y)\right]^2 = \left[f(x)^2 - 1\right]\left[f(e\theta y)^2 - 1\right],$$

$$\left[f(xy) - f(x)f(\theta y)\right]^2 = \left[f(x)^2 - 1\right]\left[f(\theta y)^2 - 1\right]. \quad (4.2.1.5)$$

4.2.2 Lemma Let G be any group. If f is a complex-valued function not identically zero on G with the properties that

(1) f satisfies (S) on G,

(2) $f(G) \subseteq \{1, -1\}$,

(3) f satisfies (A) on G.

Then f has the form

(B) $$f(x) = \frac{h(\theta^{-1} x) + h(x^{-1}\theta)}{2}, \text{ for all } x \text{ in } G,$$

where h is a homomorphism of G into the multiplicative group of nonzero complex numbers, $\mathbb{C}^*$.

Proof Let $h(x) = f(\theta x)$.

Replacing x by θx in (4.2.1.5), we have

$$\left[f(\theta xy) - f(\theta x)f(\theta y)\right]^2 = \left[f(\theta x)^2 - 1\right]\left[f(\theta y)^2 - 1\right] \quad (4.2.2.1)$$

Since $f(x)^2 \equiv 1$ for all x in G,

(4.2.2.1) shows that $f(\theta x y) = f(\theta x)f(\theta y)$ (4.2.2.2)

Hence $h(xy) = f(\theta xy) = f(\theta x)f(\theta y) = h(x)h(y)$.

Therefore h is a homomorphism.

Since $h(x)^2 = f(\theta x)^2 = 1$ for all x in G.

Hence $$h(\theta^{-1}x)^2 = 1 .$$

Therefore, we have $h(\theta^{-1}x) = h(\theta^{-1}x)^{-1}$.

So that,

$$\begin{aligned} f(x) &= f(\theta\theta^{-1}x), \\ &= h(\theta^{-1}x), \\ &= \frac{h(\theta^{-1}x) + h(\theta^{-1}x)}{2}, \\ &= \frac{h(\theta^{-1}x) + h(\theta^{-1}x)^{-1}}{2}, \\ &= \frac{h(\theta^{-1}x) + h(x^{-1}\theta)}{2} . \end{aligned}$$

4.3 The Main Theorem

4.3.1 Theorem Let G be an arbitrary group. Any complex-valued function f not identically zero on G satisfies

(S) $$f(xy) + f(xy^{-1}) = 2f(x)f(y\theta)$$

and

(A) $$f(xyz) = f(xzy),$$

for every x, y, z in G, if and only if there exists a homomorphism h from G into $\mathbb{C}^*$ such that

(H) $$f(x) = \frac{h(\theta^{-1}x) + h(x^{-1}\theta)}{2},$$

for all x in G.

Proof Let f be a solution of (S) on G.

Lemma 4.2.2 is the present theorem if $f(G) \subset \{1, -1\}$. Suppose that there is an x_0 in G such that

$$f(\theta x_0)^2 \neq 1 \qquad (4.3.1.1)$$

Let $\alpha = f(\theta\, x_0)$ and β be a square root of $(\alpha^2 - 1)$.

That is, $$\alpha^2 - 1 = \beta^2 \qquad (4.3.1.2)$$

We now define

$$h(x) = f(\theta x) + \frac{1}{\beta}\left[f(\theta x x_0) - f(\theta x) f(\theta x_0)\right],$$ for all x in G. It follows that

$$h(x) = \frac{1}{\beta}\left[f(\theta x x_0) + (\beta - \alpha)\, f(\theta x)\right] \qquad (4.3.1.3)$$

Further, utilizing (4.2.1.5), (4.3.1.2) and (4.3.1.3), we have

$$\left[h(x) - f(\theta\, x)\right]^2 = \frac{1}{\beta^2}\left[f(\theta x x_0) - f(\theta\, x)\, f(\theta\, x_0)\right]^2,$$

$$= \frac{1}{\beta^2}\left[f(\theta x)^2 - 1\right]\left[f(\theta x_0)^2 - 1\right],$$

$$= \frac{\alpha^2 - 1}{\beta^2}\left[f(\theta x)^2 - 1\right],$$

$$= f(\theta x)^2 - 1.$$

Therefore, we obtain

$$h(x)^2 - 2h(x)\, f(\theta x) + 1 = 0 \qquad (4.3.1.4)$$

From (4.3.1.4) we conclude that $h(x) \neq 0$ for any x,

moreover $$f(\theta\, x) = \frac{h(x)^2 + 1}{2h(x)},$$

$$= \frac{h(x) + h(x)^{-1}}{2}.$$

Replacing x by $\theta^{-1} x$, we have

$$f(x) = \frac{h(\theta^{-1}x) + h(\theta^{-1}x)^{-1}}{2}.$$

It remains only to prove that h defined by (4.3.1.3) is a homomorphism, that is, $h(x\,y) = h(x)\,h(y)$, for every x, y in G.

$$2f(\theta x x_0)\,f(\theta y x_0) = 2f(\theta x x_0)\,f(e\theta y x_0),$$
$$= 2f(\theta x x_0)\,f(ey\,x_0\theta),$$
$$= 2f(\theta x x_0)\,f(y\,x_0\theta).$$

From (S), we have

$$2f(\theta x x_0)\,f(\theta y x_0) = f(\theta x x_0\,y x_0) + f(\theta x x_0\,(y x_0)^{-1}),$$
$$= f(\theta x x_0\,y x_0) + f(\theta x x_0 x_0^{-1} y^{-1}),$$
$$= f(\theta x\,y x_0^2) + f(\theta\,x y^{-1}),$$
$$= \left[2f(\theta\,x\,y x_0)f(x_0\theta) - f(\theta\,xy)\right] + \left[2f(\theta x)\,f(y\theta) - f(\theta\,xy)\right],$$
$$= \left[2f(\theta\,xyx_0)\,f(e\,x_0\theta) - f(\theta\,xy)\right] + \left[2f(\theta\,x)f(ey\theta) - f(\theta\,xy)\right],$$
$$= \left[2f(\theta\,xy\,x_0)\,f(e\theta\,x_0) - f(\theta\,xy)\right] + \left[2f(\theta\,x)f(e\theta y) - f(\theta\,x\,y)\right],$$
$$= \left[2f(\theta\,xy\,x_0)f(\theta\,x_0) - f(\theta\,xy)\right] + \left[2f(\theta\,x)f(\theta y) - f(\theta\,xy)\right],$$
$$= \left[2\left[f(\theta\,xyx_0)f(\theta\,x_0) + f(\theta x)f(\theta y) - f(\theta\,xy)\right]\right] \qquad (4.3.1.5)$$

Again using (S), we get

I15492J58

$$2\left[f(\theta xx_0)f(\theta y) + f(\theta y x_0)f(\theta x)\right] = 2\left[f(\theta xx_0)f(e\theta y) + f(\theta y x_0)f(e\theta x)\right],$$

$$= 2\left[f(\theta xx_0)f(ey\theta) + f(\theta y x_0)f(e x \theta)\right],$$

$$= 2\left[f(\theta xx_0)f(y\theta) + f(\theta y x_0)f(x\theta)\right],$$

$$= f(\theta xx_0 y) + f(\theta xx_0 y^{-1}) + f(\theta y x_0 x) + f(\theta y x_0 x^{-1}),$$

$$= f(\theta xx_0 y) + f(\theta x_0 y^{-1} x) + f(\theta x y x_0) + f(\theta x_0 x^{-1} y),$$

$$= 2f(\theta xx_0 y) + f(\theta x_0 x y^{-1}) + f(\theta x_0 y x^{-1}),$$

$$= 2\left[f(\theta xx_0 y) + f(\theta x_0)f(x y^{-1}\theta)\right],$$

$$= 2\left[f(\theta xx_0 y) + f(\theta x_0)f(e x y^{-1}\theta)\right],$$

$$= 2\left[f(\theta xx_0 y) + f(\theta x_0)f(e\theta x y^{-1})\right],$$

$$= 2\left[f(\theta xx_0 y) + f(\theta x_0)f(\theta x y^{-1})\right],$$

$$= 2\left[f(\theta xx_0 y) + \alpha f(\theta x y^{-1})\right],$$

$$= 2\left[f(\theta xx_0 y) + \alpha\left\{2f(\theta x)f(y\theta) - f(\theta x y)\right\}\right],$$

$$= 2\left[f(\theta xx_0 y) + \alpha\left\{2f(\theta x)f(\theta y) - f(\theta x y)\right\}\right] \quad (4.3.1.6).$$

In view of (4.3.1.3), (4.3.1.5), (4.3.1.6) and (4.3.1.2), we obtain

$$h(x)h(y) = \frac{1}{\beta^2}\left[f(\theta xx_0) + (\beta - \alpha)f(\theta x)\right]\left[f(\theta yx_0) + (\beta - \alpha)f(\theta y)\right],$$

$$= \frac{1}{\beta^2}\Big[f(\theta xx_0)f(\theta yx_0) + (\beta - \alpha)\left\{f(\theta x)f(\theta yx_0) + f(\theta y)f(\theta xx_0)\right\} + (\beta - \alpha)^2 f(\theta x)f(\theta y)\Big],$$

$$= \frac{1}{\beta^2}\Big[f(\theta x y x_0)f(\theta x_0) + f(\theta x)f(\theta y) - f(\theta x y) + (\beta - \alpha)\left\{f(\theta xx_0 y) + \alpha\left\{2f(\theta x)f(\theta y) - f(\theta x y)\right\}\right\} + (\beta - \alpha)^2 f(\theta x)f(\theta y)\Big],$$

$$= \frac{1}{\beta^2}\left[\beta\, f(\theta x x_0 y) + (\beta^2 - (\alpha^2 - 1)) f(\theta x) f(\theta y) - (1 + \alpha\beta - \alpha^2) f(\theta x y)\right],$$

$$= \frac{1}{\beta^2}\left[\beta\, f(\theta x x_0 y) - (\alpha\beta - \beta^2)\, f(\theta x y)\right],$$

$$= \frac{1}{\beta}\left[f(\theta x y x_0) + (\beta - \alpha)\, f(\theta x y)\right],$$

$$= h(x y)\,.$$

Hence h is a homomorphism.

Then $$f(x) = \frac{h(\theta^{-1}x) + h(\theta^{-1}x)^{-1}}{2},$$

$$= \frac{h(\theta^{-1}x) + h(x^{-1}\theta)}{2}.$$

Conversely, assume that

$$f(x) = \frac{h(\theta^{-1}x) + h(x^{-1}\theta)}{2},$$

for all x in G and h is a homomorphism from G to $\mathbb{C}^*$.

Hence, $$f(xy) + f(xy^{-1}) = \frac{h(\theta^{-1}xy) + h((xy)^{-1}\theta)}{2} + \frac{h(\theta^{-1}xy^{-1}) + h((xy^{-1})^{-1}\theta)}{2},$$

$$= \frac{h(\theta^{-1}xy) + h(y^{-1}x^{-1}\theta) + h(\theta^{-1}xy^{-1}) + h(yx^{-1}\theta)}{2},$$

$$= \frac{1}{2}\Big[h(\theta^{-1})h(x)h(y) + h(y^{-1})h(x^{-1})h(\theta) + h(\theta^{-1})h(x)h(y^{-1}) + h(y)h(x^{-1})h(\theta)\Big],$$

and

$$2f(x)f(y\theta) = 2\left\{\frac{h(\theta^{-1}x) + h(x^{-1}\theta)}{2} \cdot \frac{h(\theta^{-1}y\theta) + h((y\theta)^{-1}\theta)}{2}\right\},$$

$$= \frac{1}{2}\left[h(\theta^{-1}x) + h(x^{-1}\theta)\right]\left[h(\theta^{-1}y\theta) + h(\theta^{-1}y^{-1}\theta)\right],$$

$$= \frac{1}{2}\Big[h(\theta^{-1}x)h(\theta^{-1}y\theta) + h(x^{-1}\theta)h(\theta^{-1}y\theta) + h(\theta^{-1}x)h(\theta^{-1}y^{-1}\theta) + h(x^{-1}\theta)h(\theta^{-1}y^{-1}\theta)\Big],$$

$$= \frac{1}{2}\Big[h(\theta^{-1})h(x)h(\theta^{-1})h(y)h(\theta) + h(x^{-1})h(\theta)h(\theta^{-1})h(y)h(\theta)$$

$$+ h(\theta^{-1})h(x)h(\theta^{-1})h(y^{-1})h(\theta) + h(x^{-1})h(\theta)h(\theta^{-1})h(y^{-1})h(\theta) \Big] ,$$

$$= \frac{1}{2}\Big[h(x)h(y)h(\theta^{-1})h(\theta\theta^{-1}) + h(x^{-1})h(y)h(\theta)h(\theta\theta^{-1})$$

$$+ h(x)h(y^{-1})h(\theta^{-1})h(\theta\theta^{-1}) + h(x^{-1})h(y^{-1})h(\theta)h(\theta\theta^{-1}) \Big] ,$$

$$= \frac{1}{2}\Big[h(x)h(y)h(\theta^{-1}) + h(x^{-1})h(y)h(\theta) + h(x)h(y^{-1})h(\theta^{-1})$$

$$+ h(x^{-1})h(y^{-1})h(\theta) \Big] .$$

Therefore, $f(x\,y) + f(x\,y^{-1}) = 2f(x)f(y\theta)$.

Also, for any x, y, z in G

$$f(x\,yz) = \frac{h(\theta^{-1}\,x\,yz) + h((xyz)^{-1}\theta)}{2} ,$$

$$= \frac{h(\theta^{-1}x\,yz) + h(z^{-1}y^{-1}\,x^{-1}\theta)}{2} ,$$

$$= \frac{h(\theta^{-1})h(x)h(y)h(z) + h(z^{-1})h(y^{-1})h(x^{-1})h(\theta)}{2} ,$$

and

$$f(xzy) = \frac{h(\theta^{-1}x\,zy) + h((xzy)^{-1}\theta)}{2} ,$$

$$= \frac{h(\theta^{-1}\,xzy) + h(y^{-1}z^{-1}\,x^{-1}\theta)}{2} ,$$

$$= \frac{h(\theta^{-1})h(x)h(z)h(y) + h(y^{-1})h(z^{-1})h(x^{-1})h(\theta)}{2} .$$

Therefore, $f(xyz) = f(xzy)$.

4.3.2 <u>Corollary</u> Let G be a commutative group. Then $f : G \longrightarrow \mathbb{C}$ not identically zero on G satisfies

(S) $\quad f(x\,y) + f(x\,y^{-1}) = 2f(x)f(y\theta)$, if and only if f is of the form

$$\text{(H)} \qquad f(x) = \frac{h(\theta^{-1}x) + h(x^{-1}\theta)}{2},$$

for all x in G, where h is a homomorphism from G to $\mathbb{C}^*$.

<u>Proof</u> Since G is a commutative group, hence the condition

$$\text{(A)} \qquad f(xyz) = f(xzy),$$

for every x ,y,z in G, holds for all functions $f : G \longrightarrow \mathbb{C}$.
Hence the class of all functions that satisfies (S) coincides with the class of all functions that satisfies (S) and (A).

4.3.3 <u>Corollary</u> Let G be any group. Then every solution of

$$f(xy) + f(xy^{-1}) = 2f(x)f(y)$$

satisfying the condition

$$f(xyz) = f(xzy),$$

for all x , y, z in G, has the form

$$f(x) = \frac{h(x) + h(x^{-1})}{2},$$

for all x in G, where h is a homomorphism of G into $\mathbb{C}^*$.

<u>Proof</u> This corollary is a special case of theorem 4.3.1 when $\theta = e$, the identity of G.