CHAPTER III

SOLUTION IN MODAL CO-ORDINATES FOR PLANAR FRAMES

3.1 Transformation to Modal Co-ordinates

Lukkunaprasit & Widartawan (13) proposed the modal co-ordinate transformation method for solving the incremental equations of motion by transforming the system co-ordinates to modal co-ordinates. This technique can be applied effectively together with the substructuring technique because of the linear property of the left hand side of the incremental equations of motion, Eqs. (41).

The displacements of the master nodes of the structure is assumed to contain predominantly the lowest p eigen modes, i.e.

$$\{u_m\}_{n\times 1} = [\phi]_{n\times p}\{x\}_{p\times 1} \quad ; \; p \leqslant n \tag{47}$$

where {x} is the vector of generalized co-ordinates or modal co-ordinates, n is the total number of degrees-of-freedom of the master nodes, and $\{\phi\}$ is the matrix whose columns contain the lowest p eigenvectors of the elastic undamped free vibration of the reduced system.

In view of Eq. (47) and the associated orthogonality condition :

$$\{\phi_i\}^T[M]\{\phi_j\} = 0 \qquad \text{for } i \neq j \tag{48a}$$

$$\{\phi_i\}^T[K]\{\phi_j\} = 0 \qquad \text{for } i \neq j \tag{48b}$$

equations (41), with the non-linear term neglected, can be transformed to the following equations :

$$\Delta\ddot{x}_j + \omega_j^2 \, \Delta x_j = \Delta F_j \quad ; \quad j = 1, 2, \ldots, p \tag{49}$$

where

$$\Delta F_j = \frac{\{\phi_j\}^T(\{^{k+1} R^*\} - \{^k h^*\})}{\{\phi_j\}^T [M^*] \{\phi_j\}} - {}^k\ddot{x}_j \tag{50}$$

in which ω_j is the angular frequency corresponding to the j^{th} mode of free vibration.

Eqs. (49) are the incremental equations of motion for the reduced system which are uncoupled with respect to the generalized coordinates. These equations are step-by-step integrated by means of the familiar Newmark's average acceleration method (16).

3.2 Formulation in Terms of Stress Resultants For Planar Frames

Planar frames composed of straight prismatic beams made of a homogeneous material are considered in this study. The Bernoulli's hypothesis is assumed with the consequence that shear deformation is neglected. A standard beam element is used which assumes a linear axial displacement and a cubic transverse displacement field (20). The approximate formulation given by Lukkunaprasit (12) is followed to obtain the nonlinear constitutive relations in terms of the stress resultants.

By using the standard procedure in the finite element method (20) and the kinematic assumptions, one can relate the axial strain, ε, to the nodal point displacements through the transformation matrix [B], i.e.

$$\{\varepsilon(s,\xi)\} = [B\ (s,\xi)]\{u\} \tag{51}$$

where s and ξ are the coordinates along the longitudinal centroidal axis and normal axis, respectively. This strain consists of two parts, one is the axial deformation of the centroidal axis and the other is the contribution of rotation of the section due to bending, which is linear in ξ. Thus, $[B\ (s,\xi)]$ can be separated as follows :

$$[B\ (s,\xi)] = [B_1(s)] + \xi\ [B_2(s)] \tag{52}$$

The axial force, N, and moment, M, at a section are defined, respectively, as

$$N = \int_A \sigma\ dA \tag{53a}$$

$$M = \int_A \sigma \cdot \xi\ dA \tag{53b}$$

Taking the time derivatives of Eqs. (53) and substituting Eqs. (3) and (52) into the resulting equations leads to

$$\dot{M} = EI\ [B_2]\{\dot{u}\} - \frac{E}{\tau\sigma_o} \int_A \left|\frac{\sigma}{\sigma_o}\right|^{n-1} \sigma\ \xi\ dA \tag{54a}$$

$$\dot{N} = EA\ [B_1]\{\dot{u}\} - \frac{E}{\tau\sigma_o} \int_A \left|\frac{\sigma}{\sigma_o}\right|^{n-1} \sigma\ dA \tag{54b}$$

where I and A are the moment of inertia and the area of the section, respectively. In arriving at Eq. (54) the integrals involving $\int_A (\text{constant})\xi dA$ vanish due to the fact that the ξ-co-ordinates are measured from the centroidal axis of the cross section.

By virtue of Eqs. (8), (52) and (53), the element internal force vector, $\{h\}_e$ can be written as

$$\{h\}_e = \int_L [B_1]^T N \, ds + \int_L [B_2]^T M \, ds \tag{55}$$

in which L is the length of the element.

The integrals in Eqs. (54) can be approximated by replacing $\left|\frac{\sigma}{\sigma_o}\right|$ by the yield function in the stress resultants, $f_y(\frac{N}{N_o}, \frac{M}{M_o})$(12). Thus, Eqs. (54) become

$$\dot{M} = EI[B_2]\{\dot{u}\} - \frac{E}{\tau\sigma_o} \left(f_y\left(\frac{N}{N_o}, \frac{M}{M_o}\right)\right)^{n-1} M \tag{56a}$$

$$\dot{N} = EA[B_1]\{\dot{u}\} - \frac{E}{\tau\sigma_o} \left(f_y\left(\frac{N}{N_o}, \frac{M}{M_o}\right)\right)^{n-1} N \tag{56b}$$

where the plastic axial force, N_o, and the plastic moment, M_o, are defined as

$$N_o = A\sigma_o \tag{57a}$$

$$M_o = Z\sigma_o \tag{57b}$$

in which Z is the plastic section modulus. For a rectangular section whose depth is d, the yield function is (14)

$$f_y\left(\frac{N}{N_o}, \frac{M}{M_o}\right) = \left(\frac{N}{N_o}\right)^2 + \left|\frac{M}{M_o}\right| = 1 \tag{58}$$

For a wide flange section the following yield condition is assumed (14)

$$f_y\left(\frac{N}{N_o}, \frac{M}{M_o}\right) = \left|\frac{M}{M_o}\right| = 1 \text{ when } 0 < \left|\frac{N}{N_o}\right| < 0.15 \tag{59a}$$

and

$$f_y\left(\frac{N}{N_o}, \frac{M}{M_o}\right) = \left|\frac{N}{N_o}\right| + \left|\frac{M}{1.18\, M_o}\right| = 1 \text{ when } \left|\frac{N}{N_o}\right| \geq 0.15 \tag{59b}$$

The integrals in Eqs. (56) are carried out by the implicit second order Rung-Kutta method. The details of integration technique can be seen in Reference (12). Then the integrated axial force and moment are substituted into Eq. (55) to obtain the required element internal force.

3.3 Programming Technique in Evaluating the Reduced Internal Static Forces

Here the equations for the reduced internal resisting force vector, $\{h^*\}$, of each substructure are written again for reference, i.e.,

$$\{h^*\} = \{h_m\} - [K_{ms}][K_{ss}]^{-1}\{hs\}$$

$$= \{h_m\} - [T]^T \{h_s\} \quad (60)$$

where $$[T] = [K_{ss}]^{-1} [K_{sm}] \quad (61)$$

In programming inversion of $[K_{ss}]$ is not required by using the following procedure. While the stiffness matrix of each substructure is partially condensed by the Gauss Elimination technique we obtain the displacement transformation matrix, $[T]$, relating the slave and master displacements, i.e.

$$\{u_s\} = -[K_{ss}]^{-1} [K_{sm}] \{u_m\} \quad (62)$$

The transformation matrix, $[T]$, is stored in the disc area. After that when the nodal internal force vector of each substructure is to be evaluated, this transformation matrix is read into core and is columnwise multiplied by the corresponding slave nodal internal force vector, $\{h_s\}$. The result is substracted from the corresponding master nodal internal force vector, $\{h_m\}$, to obtain the reduced internal static force vector of each substurcture.

3.4 Convergence Criteria

Belytschko, et al. (3) proposed the following discrete energy error criterion for dynamic analyses :

$$\left|\{\Delta u\}^T(\{^{k+1}\ \delta R\} + \{^k\ \delta R\})\right| < \varepsilon_1(^kU + {}^kT)\ ;\ k \geqslant 2 \tag{63}$$

where ^kU and ^kT are, respectively, the internal and kinetic energies of the structural system. We can write this criterion for the reduced system as follows :

$$\left|\{\Delta u_m\}^T(\{^{k+1}\ \delta R^*\} + \{^k\ \delta R^*\})\right| < \varepsilon_1(^kU^* + {}^kT^*)\ ;\ k \geqslant 2 \tag{64}$$

where

$$^kU^* = {}^{k-1}U^* + \frac{1}{2}\ (\{^ku\} - \{^{k-1}u\})^T(\{^{k-1}\ h^*\} + \{^k\ h^*\}) \tag{65}$$

$$^kT^* = \frac{1}{2}\ \{^k\dot{u}_m\}^T[M^*]\{^k\dot{u}_m\} \tag{66}$$

ε_1 = energy criterion factor, taken as 10^{-6} in this study

For static problems the displacement criterion that

$$(\{\delta u\}^T\ \{\delta u\})^{\frac{1}{2}} < \varepsilon_2(\{u\}^T\ \{u\})^{\frac{1}{2}} \tag{67}$$

is imposed for convergence.

Here ε_2 is a convergence tolerance, taken as 10^{-6} in this study and $(\{u\}^T\{u\})^{\frac{1}{2}}$ is the norm of the nodal displacements.