CHAPTER II

FINITE ELEMENT FORMULATION

2.1 Constitutive Relation for An Elastic-Viscoplastic Material

In this study an elastic-viscoplastic model employed by Lukkunaprasit and Kelly (12) to describe the elastic-plastic behavior in numerical computation was adopted.

For small uniaxial deformation the total strain rate, $\dot{\varepsilon}$, consists of an elastic part $\dot{\varepsilon}^e$ and a viscoplastic part $\dot{\varepsilon}^p$, i.e.,

$$\dot{\varepsilon} = \dot{\varepsilon}^e + \dot{\varepsilon}^p \qquad (1)$$

The elastic deformation is given by Hooke's Law and the viscoplastic strain rate is assumed to be an exponential function of the actual stress σ, i.e.

$$\dot{\varepsilon}^e = \frac{\dot{\sigma}}{E} \qquad (2a)$$

and

$$\dot{\varepsilon}^p = \frac{\sigma}{\tau\sigma_o} \left|\frac{\sigma}{\sigma_o}\right|^{n-1} \qquad (2b)$$

where E is modulus of elasticity, τ, n, and σ_o are material constants. σ_o is usually taken as the static yield stress.

In view of Eqs. (2a), (2b) and (1) we obtain the following constitutive model :

$$\frac{\dot{\sigma}}{E} = \dot{\varepsilon} - \frac{\sigma}{\tau\sigma_o}\left|\frac{\sigma}{\sigma_o}\right|^{n-1} \qquad (3)$$

The constitutive relation, Eq. (3), can be generalized to multiaxial deformation (10) and take the general form :

$$\{\dot{\sigma}\} = [D]\{\dot{\varepsilon}\} - \{\dot{\sigma}\}^{(n)} \tag{4}$$

where $\{\dot{\sigma}\}$ and $\{\dot{\varepsilon}\}$ are the vectors of the stresses and strains, respectively, $[D]$ is a matrix of elasticity constants and $\{\dot{\sigma}\}^{(n)}$ represents the nonlinear part due to viscoplastic flow.

A complete elastic-viscoplastic stress strain response can be obtained from any prescribed strain rate history by integrating Eq. (3) numerically, without the need to identify the state (elastic or elastic-viscoplastic) of the element nor to check the loading or unloading process.

In a constant strain rate loading the stress approaches an asymptotic value, σ_d, defined as the dynamic yield stress which is given by

$$\left| \frac{\sigma_d}{\sigma_o} \right| = |\tau\dot{\varepsilon}|^{\frac{1}{n}} \tag{5}$$

Eq. (5) can be used to determine the material constants n and τ from dynamic load tests. As n tends to infinity, σ_d approaches the static yield stress, σ_o. Thus the constitutive model (3) can be utilized to approximate an elastic-perfectly plastic material by setting a large value for n and has been applied in several studies by Lukkunaprasit, et al. (12, 13).

2.2 Incremental Equations of Motion of an Elastic-Viscoplastic System

The incremental method which is a convenient technique for solving nonlinear problems is considered in this section. The constitutive relation described in section 2.1 will be introduced into the incremental equations of motion to obtain the equations for an elastic-viscoplastic material.

The well known equations of motion for a finite element model, when damping is neglected, is (20)

$$[M]\{\ddot{u}\} + \{h\} = \{R\} \tag{6}$$

where $[M]$ is the mass matrix, $\{u\}$ is the nodal displacement vector, $\{h\}$ is the nodal internal force vector and $\{R\}$ is the external nodal force vector. The mass matrix and the nodal internal force vector are given, respectively, by

$$[M] = \sum^{\text{elements}} \int_V \rho [N]^T [N] \, dv \tag{7}$$

and

$$\{h\} = \sum^{\text{elements}} \int_V [B]^T \{\sigma\} \, dv \tag{8}$$

in which ρ is the mass density, $[N]$ is the matrix of displacement shape functions, v is the volume of each element and $[B]$ relates the internal strain to the nodal displacements, i.e.

$$\{\varepsilon\} = [B] \{u\} \tag{9}$$

By writing the equations of motion of the structure in two adjacent states "k" and "k+1" at times "t" and "t+Δt", respectively, and substracting one arrives at the following incremental equations of motion:

$$[M]\{\Delta\ddot{u}\} + \{\Delta h\} = \{\Delta R\} \tag{10}$$

where Δ denotes the change between states "k" and "k+1". In view of Eqs. (8), (4) and (9), the following expression for {Δh} can be derived (12):

$$\{\Delta h\} = [K]\{\Delta u\} - \{\Delta h\}^{(n)} \tag{11}$$

where $$[K] = \sum^{\text{elements}} \int_v [B]^T [D] \; [B] \; dv \tag{12}$$

and $$\{\Delta h\}^{(n)} = \sum^{\text{elements}} \int_v [B]^T \; \Delta\{\dot{\bar{\sigma}}\}^{(n)} \; \Delta t \; dv \tag{13}$$

Here $\{\dot{\bar{\sigma}}\}^{(n)}$ is evaluated at some stress state within the time interval Δt. Substituting Eq. (11) into Eq. (10) results in the incremental equations of motion

$$\begin{aligned} [M]\{\Delta\ddot{u}\} + [K]\{\Delta u\} &= \{\Delta R\} + \{\Delta h\}^{(n)} \\ &= \{^{k+1}R\} - \{^{k}R\} + \{\Delta h\}^{(n)} \\ &= \{^{k+1}R\} - [M]\{^{k}\ddot{u}\} - \{^{k}h\} + \{\Delta h\}^{(n)} \end{aligned} \tag{14}$$

It should be observed that the differential operator appearing on the left hand side of Eq. (14) is the same as in a linear system so that the substructure technique and the modal coordinate transformation technique can be applied effectively (13). The nonlinearity in material behavior is included in the term $\{\Delta h\}^{(n)}$ which is known as a pseudo-force.

2.3 Incremental Equations of Motion For the Substructured System

In the previous section we described the incremental equations of motion for an elastic-viscoplastic system. To obtain the solution of these equations for a large structural system it would require a large size computer, which may not be available. In this case we may consider the structure as consisting of substructures each of which will be small enough to be handled by the available computer.

Because of the elastic stiffness matrix appearing in the incremental equations of motion, Eq. (14), we may use the same transformation matrix between the slave and master degrees-of-freedom (d.o.f.) in every iteration step.

The principle of virtual work can now be used to obtain the equations of motion for the reduced system.

Consider a deformable body in motion under a set of external nodal forces. Suppose that the body is subjected to a kinematically compatible virtual velocity field, $\{\delta\dot{u}\}$. Then the total rate of internal virtual work done, $\delta\dot{U}$, is

$$\delta\dot{U} = \sum^{\text{substructures}} \left[\int_{v_s} \{\delta\dot{\varepsilon}\}^T\{\sigma\}\, dv + \int_{v_s} \{\delta\dot{u}\}^T \rho[N]^T[N]\{\ddot{u}\}\, dv\right] \tag{15}$$

in which $\{\delta\dot{\varepsilon}\}$ is the virtual strain rate vector and v_s is the volume of each substructure. The rate of virtual work done by the external loads, $\delta\dot{W}_E$, is

$$\delta\dot{W}_E = \sum^{\text{substructures}} \{\delta\dot{u}\}^T\{R\} \tag{16}$$

According to the principle of virtual work, the rate of total virtual work done is equal to zero. Hence, in view of expressions (15) and (16) we have

$$\sum^{\text{substructures}} \left[\int_{v_s}\{\delta\dot{\varepsilon}\}^T\{\sigma\}\,dv + \int_{v_s}\{\delta\dot{u}\}^T\rho\,[N]^T[N]\{\ddot{u}\}\,dv - \{\delta\dot{u}\}^T\{R\}\right] = 0 \tag{17}$$

Consider now each term of Eq. (17) separately. By virtue of the constitutive relation, Eq. (4) and the strain-displacement transformation equation, Eq. (9), one can express the first term in Eq. (17) as follows :

$$\sum^{\text{substructures}} \int_v\{\delta\dot{\varepsilon}\}^T\{\sigma\}\,dv = \sum^{\text{substructures}} \int_{v_s}\{\delta\dot{u}\}^T[B]^T([D]\{\varepsilon\} - \int_t\{\dot{\sigma}\}^{(n)}dt)\,dv$$

$$= \sum^{\text{substructures}} \int_{v_s}\{\delta\dot{u}\}^T([B]^T[D][B]\{u\} - \int_t[B]^T\{\dot{\sigma}\}^{(n)}dt)\,dv \tag{18}$$

Within each substructure, the slave nodal displacements are assumed to be related to the master nodal values as in the standard static condensation procedure. Thus, if the stiffness matrix of each substructures is partitioned in accordance with the master and slave degrees-of-freedom as follows :

$$[K] = \left[\begin{array}{c|c} [K_{mm}] & [K_{ms}] \\ \hline [K_{sm}] & [K_{ss}] \end{array}\right] \tag{19}$$

Then

$$\{u_s\} = -\left[K_{ss}\right]^{-1}\left[K_{sm}\right]\{u_m\} \tag{20}$$

Here the subscripts m and s are used to designate master and slave coordinates, respectively.

In view of Eq. (20) the nodal displacements of the structure can be expressed in the master displacements as

$$\{u\} = [A]\{u_m\} \tag{21}$$

where

$$\{u\} = \left\{\begin{matrix}\{u_m\} \\ \hline \{u_s\}\end{matrix}\right\} \tag{22}$$

$$[A] = \left\{\begin{matrix}[I] \\ \hline -\left[K_{ss}\right]^{-1}\left[K_{sm}\right]\end{matrix}\right\} \tag{23}$$

and $[I]$ = Identity matrix

By virtue of Eq. (21) with the displacements replaced by the velocities, one can rewrite Eq. (18) as

$$\sum^{\text{substructures}} \int_{v_s} \{\delta\dot{\varepsilon}\}^T\{\sigma\}dv = \sum^{\text{substructures}} \{\delta\dot{u}_m\}^T\left([K]^*\{u_m\}-\{h^*\}^{(n)}\right) \tag{24}$$

in which

$$[K^*] = [A]^T[K][A]$$

$$= \left[K_{mm}\right] - \left[K_{ms}\right]\left[K_{ss}\right]^{-1}\left[K_{sm}\right] \tag{25}$$

$$[K] = \sum^{\text{elements}} \int_{v_s} [B]^T[D]\ [B]\ dv \tag{26}$$

and

$$\{h^*\}^{(n)} = \sum^{\text{elements}} \int_{v_s} [A]^T[B]^T\{\dot{\bar{\sigma}}\}^{(n)}\Delta t\ dv \tag{27}$$

The matrix $[K]$ can be identified as the uncondensed stiffness matrix of the substructure, while $[K^*]$ is the corresponding reduced stiffness matrix. In computation, $[K^*]$ is obtained by the Gauss Elimination method instead of the direct matrix operation which involves inversion of $[K_{ss}]$.

The second term of Eq. (17) is related to the kinetic energy and will be expressed in terms of the master coordinates by employing the simplified mass condensation scheme introduced by Lukkunaprasit and Alam (11). In this procedure the slave nodal velocities are assumed to be interpolated from the master nodal velocities through some transformation matrix $\{\eta\}$, i.e.,

$$\{\dot{u}_s\} = [\eta]\{\dot{u}_m\} \tag{28}$$

If the transformation matrix $[\eta]$ is taken the same as in the static condensation process in Eq. (20), then we have the consistent transformation scheme. However a simple method will be employed in which an approximation of $\{\eta\}$ for each substructure is assumed. The form of this transformation matrix for a planar frame is given in the next section.

In view of Eq. (28) the vector of all nodal velocities is given by

$$\{\dot{u}\} = \begin{Bmatrix} [I] \\ \hline [\eta] \end{Bmatrix} \{\dot{u}_m\} \tag{29}$$

Introducing Eq. (29) into the second term of Eq. (17) yields

$$\sum^{\text{substructures}} \int_{V_s} \{\delta\dot{u}\}^T \rho [N]^T [N] \{\ddot{u}\}\, dv$$

$$= \sum^{\text{substructures}} \int_{V_s} \{\delta\dot{u}_m\}^T \left[[I]^T \,\vdots\, \{\eta\}^T \right] \rho [N]^T [N] \left\{ \begin{array}{c} [I] \\ \hline \{\eta\} \end{array} \right\} \{\ddot{u}_m\}\, dv$$

$$= \sum^{\text{substructures}} \{\delta\dot{u}_m\}^T [M^*] \{\ddot{u}_m\} \qquad (30)$$

in which $[M^*]$ is the reduced mass matrix of each substructure defined as

$$[M^*] = \left[[I]^T \,\vdots\, \{\eta\}^T \right] [M] \left\{ \begin{array}{c} [I] \\ \hline \{\eta\} \end{array} \right\} \qquad (31)$$

and $[M]$ is the mass matrix of each substructure defined in Eq. (7)

The third term of Eq. (17) becomes, upon substitution of Eq. (29),

$$\sum^{\text{substructures}} \{\delta\dot{u}\}^T \{R\} = \sum^{\text{substructures}} \{\delta\dot{u}_m\}^T \left[[I]^T \,\vdots\, \{\eta\}^T \right] \{R\}$$

$$= \sum^{\text{substructures}} \{\delta\dot{u}_m\}^T \{R^*\} \qquad (32)$$

where $\{R^*\}$ is the reduced external force vector of each substructure given by

$$\{R^*\} = \left[[I]^T \,\vdots\, \{\eta\}^T \right] \{R\} \qquad (33)$$

Finally, introducing Eqs. (24), (30) and (32) into Eq. (17) yields

$$\sum^{\text{substructures}} \{\delta\dot{u}_m\}^T \left[[K^*]\{u_m\} - \{h^*\}^{(n)} + [M^*]\{\ddot{u}_m\} - \{R^*\} \right] = 0 \qquad (34)$$

Since $\{\delta\dot{u}_m\}^T$ is any arbitrary virtual vector which is not a zero vector then Eq. (34) holds if and only if

$$[M^*]\{\ddot{u}_m\} + [K^*]\{u_m\} - \{h^*\}^{(n)} - \{R^*\} = 0 \qquad (35)$$

Eq. (35) is the system of equations of motion for the reduced system. The incremental form of the equations of motion can be readily shown to be

$$[M^*]\{\Delta\ddot{u}_m\} + [K^*]\{\Delta u_m\} = \{^{k+1}R^*\} - \{^k R^*\} + \{\Delta h^*\}^{(n)} \qquad (36)$$

Now from Eq. (35), $\{^k R^*\} = [M^*]\{^k\ddot{u}_m\} + [K^*]\{^k u_m\} - \{^k h^*\}^{(n)}$ (37)

The first term in Eq. (37) is the reduced internal inertia force vector and the last two terms are the reduced internal resisting nodal forces.

Using the principle of virtual work we can derive another simple form for the reduced internal resisting forces which includes both the linear and nonlinear parts. To do this we return once again to Eq. (17). The first term is reconsidered while the second and the third terms are as before. By using the standard displacement transformation matrix in Eq. (21), one can rewrite the first term of Eq. (17) as

$$\sum^{\text{substructures}} \int_{V_s} \{\delta\dot{\varepsilon}\}^T\{\sigma\}\, dv = \sum^{\text{substructures}} \int_{V_s} \{\delta\dot{u}\}[B]^T\{\sigma\}\, dv$$

$$= \sum^{\text{substructures}} \int_{V_s} \{\delta\dot{u}_m\}^T[A]^T[B]^T\{\sigma\}\, dv$$

$$= \sum^{\text{substructures}} \int_{V_s} \{\delta \dot{u}_m\}^T \left[[I]^T \;\vdots\; -[K_{ms}][K_{ss}]^{-1} \right] \begin{Bmatrix} [B_m]^T \\ \hline [B_s]^T \end{Bmatrix} \{\sigma\} \, dv$$

$$= \sum^{\text{substructures}} \int_{V_s} \{\delta \dot{u}_m\}^T ([B_m]^T - [K_{ms}][K_{ss}]^{-1}[B_s]^T) \{\sigma\} \, dv$$

$$= \sum^{\text{substructures}} \{\delta \dot{u}_m\}^T \{h^*\} \tag{38}$$

where $\{h^*\} = \{h_m\} - [K_{ms}][K_{ss}]^{-1}\{h_s\}$ (39)

Substituting Eqs. (38), (30) and (32) into Eq. (17) leads to, by virtue of the arbitrariness of $\{\delta \dot{u}_m\}$,

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$$[M^*]\{\ddot{u}_m\} + \{h^*\} = \{R^*\} \tag{40}$$

In view of Eq. (40), the incremental equations (36) become

$$[M^*]\{\Delta \ddot{u}_m\} + [K^*]\{\Delta u_m\} = \{^{k+1} R^*\} - [M^*]\{^k \ddot{u}_m\} - \{^k h^*\} + \{\Delta h^*\}^{(n)} \tag{41}$$

The vector $\{^k h^*\}$ in Eq. (40) can be identified as the reduced internal resisting nodal force vector. A programming technique for evaluating this vector is given in the next section. The reduced matrices of the whole system are obtained by assembling all individual reduced substructure matrices using the standard assembling process in the finite element method.

After each application of the iteration process, the unbalanced nodal forces may still exist because of the material nonlinearity and approximations used in the numerical computation. Iteration is then performed within the time step by applying the unbalanced force $\{^{k+1}\delta R^*\}$ to the system. In other words we solve

$$[M^*]\{\delta\ddot{u}_m\}+[K^*]\{\delta u_m\} = \{^{k+1}\delta R^*\} \tag{42}$$

in which

$$\{\delta u_m\} = \{^{i+1}u_m\} - \{^{i}u_m\} \tag{43}$$

and

$$\{^{k+1}\delta R^*\} = \{^{k+1}R^*\}-[M^*]\{^{i}\ddot{u}_m\}-\{^{i}h^*\} \tag{44}$$

In Eq. (42) the nonlinear term has been neglected during the iteration process. When convergence is reached so that $\{^{k+1}\delta R^*\}$ approaches zero, then the external forces balance the internal resisting forces and the inertia forces, which means that the equations of motion are satisfied within some small tolerance.

2.4 Simplified Displacement Transformation Matrix for Mass Condensation of a Planar Frame

In the previous section we employed a simplified condensation scheme to obtain a simplified mass matrix of each substructure by assuming some approximate velocity transformation within the substructure. Figure (1) shows a typical substructure "k" of a planar frame. The masses are assumed to be concentrated at each nodal point of the frame. Rotational masses are ignored. For a column line i-1 of the k^{th} substructure (see Fig. (1)), the velocities of the slave nodes $\begin{Bmatrix} \{\dot{u}_s\} \\ \hline \{\dot{v}_s\} \end{Bmatrix}_{i-1}$, will be expressed in terms of the master coordinates $[\{\dot{u}_m{}^i\}\ \{\dot{u}_m{}^1\}\ \{\dot{v}_m{}^i\}\ \{\dot{v}_m{}^1\}]^T$ by a linear velocity interpolation, i.e.,

$$\begin{Bmatrix} \{\dot{u}_s\} \\ \hline \{\dot{v}_s\} \end{Bmatrix}_{i-1} = \begin{Bmatrix} \{\eta\} & \{o\} \\ \{o\} & \{\eta\} \end{Bmatrix} \begin{Bmatrix} \{\dot{u}_m\} \\ \hline \{\dot{v}_m\} \end{Bmatrix} \qquad (45)$$

where $\dot{u}$ and $\dot{v}$ denote the horizontal and vertical velocities, respectively, and $\{\eta\}$ is the assumed velocity transformation matrix given by

$$\{\eta\} = \begin{Bmatrix} (1-\frac{t_1}{L_k}) & \frac{t_1}{L_k} \\ (1-\frac{t_2}{L_k}) & \frac{t_2}{L_k} \\ \vdots & \vdots \\ (1-\frac{t_\ell}{L_k}) & \frac{t_\ell}{L_k} \end{Bmatrix} \qquad (46)$$

in which ℓ is the number of slave coordinates in the column line i-1.