CHAPTER V

A QUASI-GROUP HYPERGRAPH WITH PRECRIBED NEIGHBOURHOOD STRUCTURES

In this chapter we define quasi-group hypergraphs and discuss the problem on realizability of the given family of hypergraphs by a quasi-group hypergraph .

5.1 Quasi-group Hypergraphs.

A <u>quasi-group</u> is an ordered pair $(Q,*)$, where Q is a non-empty set and $*$ is a binary operation on Q such that for any p,q in Q, there exist unique elements x and y such that

$$q * x = p$$

and

$$y * q = p .$$

In what follows we shall consider only finite quasi-groups, i.e. a quasi-group $(Q,*)$ such that Q is a finite set. The number of elements of Q will be called the <u>order</u> of quasi-group $(Q,*)$. A <u>group</u> is a quasi-group $(G,*)$ which satisfies the associative law, for every a,b,c in G, $(a*b)*c = a*(b*c)$.

For any subset A of Q and any q in Q, the set $\{q*a / a \in A\}$ will be denoted by $q*A$. Here, and in the sequal, r will be denoted a positive integer greater than 1 . For any set $\mathcal{A}$ of (r-1)-subsets of Q, $\mathcal{A}$ will be said to be <u>admissible</u> if for each A in $\mathcal{A}$, each a in A and each q in Q, there exists $B_{a,q}$ in $\mathcal{A}$ such that

$$(\{q\} \cup q*A) - \{q*a\} = (q*a)*B_{a,q} .$$

Note that the empty set is an admissible set. In the sequal, any admissible set $\mathcal{A}$ we mean a non-empty admissible set. For each admissible set $\mathcal{A}$ of (r-1)-subsets of Q, we define $\mathcal{E}_{\mathcal{A}}$ by

$$\mathcal{E}_{\mathcal{A}} = \{\{q\} \cup q*A \;/\; q \in Q \text{ and } A \in \mathcal{A}\}.$$

It can be verified that

$$|\{q\} \cup q*A| = r$$

for all q in Q and A in $\mathcal{A}$. Hence $\mathcal{E}_{\mathcal{A}}$ is a set of r-subsets of Q and $\cup\mathcal{E}_{\mathcal{A}} = Q$. Therefore $(Q, \mathcal{E}_{\mathcal{A}})$ is an r-uniform hypergraph. The hypergraph $H = (Q, \mathcal{E}_{\mathcal{A}})$ will be called the <u>hypergraph induced by the quasi-group</u> $(Q,*)$ <u>and the admissible set</u> $\mathcal{A}$. In the remaining of this chapter, any hypergraph we mean an r-uniform hypergraph, and when we write $H = (Q, \mathcal{E}_{\mathcal{A}})$, we mean that Q has a binary operation $*$ such that $(Q,*)$ is a quasi-group and $\mathcal{A}$ is an admissible set of (r-1)-subsets of Q. A hypergraph $H = (V, \mathcal{E})$ will be said to be a <u>quasi-group hypergraph</u> if there exists a binary operation $*$ on V such that $(V,*)$ is a quasi-group and there exists an admissible set

$\mathcal{A}$ of (r-1)-subsets of V such that $\mathcal{E} = \mathcal{E}_{\mathcal{A}}$. If a binary operation * can be choosen such that (V,*) is a group, H will be called a group hypergraph.

The concept of quasi-group hypergraphs was introduced in [4]. Our notations differ from those in [4]. vV, $v\mathcal{E}$ and vH are denoted in [4] by V_v, $\mathcal{E}_v$ and H_v respectively. Proofs of the following propositions can be found in [4].

5.1.1 Proposition. Let $H = (V,\mathcal{E})$ be a quasi-group hypergraph. If a hypergraph H' isomorphic to H, then H' is a quasi-group hypergraph.

5.1.2 Proposition. Let $H = (V, \mathcal{E}_{\mathcal{A}})$ be a hypergraph induced by a quasi-group (V,*) and an admissible set $\mathcal{A}$ of (r-1)-subsets of V. Then the followings hold:

(1) For each E in $\mathcal{E}_{\mathcal{A}}$ and each v in E, there exists A in $\mathcal{A}$ such that $E-\{v\} = v*A$.

(2) For each v in V, $vV = \{v*a / a \in \cup\mathcal{A}\}$.

(3) For each v in V, the function $\psi_v : vV \to \cup\mathcal{A}$ defined by

$$\psi_v(v*a) = a$$

for all v*a in vV is a one-to-one correspondence.

5.1.3 Proposition. Let $H = (V,\mathcal{E})$ be a quasi-group hypergraph. Then there exists a system $(\psi_{uv})_{u,v \in V}$ such that each ψ_{uv} is an isomorphism from uH to vH and for every u,v,v' in V if $v \neq v'$ then $\psi_{uv}(a) \neq \psi_{uv'}(a)$ for all a in uV.

In our work we shall also need the following:

5.1.4 <u>Proposition.</u> Let $H = (V, \mathcal{E}_{\mathcal{A}})$ be a hypergraph induced by a quasi-group $(V,*)$ and an admissible set $\mathcal{A}$ of (r-1)-subsets of V. Then

$$vH \cong (\cup\mathcal{A}, \mathcal{A})$$

for all v in V.

<u>Proof.</u> Let v be any element in V. From Proposition 5.1.2 , we know that

$$vV = \{v*a \ / \ a \in \cup\mathcal{A}\}$$

and the function $\psi_v : vV \to \cup\mathcal{A}$, definedby

$$\psi_v(v*a) = a$$

for all $v*a$ in vV, is a one-to-one coresspondence.

We shall show that the function ψ_v is an isomorphism from vH to $(\cup\mathcal{A}, \mathcal{A})$. Let S be any subset of V. Suppose that S belongs to $v\mathcal{E}_{\mathcal{A}}$. Hence

$$S = E-\{v\}$$

for some E in $\mathcal{E}_{\mathcal{A}}$ such that v belongs to E and $E-\{v\} \neq \emptyset$. By Proposition 5.1.2 , we have

$$E-\{v\} = v*A$$

for some A in $\mathcal{A}$. Hence

$$\begin{aligned} \psi_v[S] &= \psi_v[E-\{v\}] , \\ &= \psi_v[v*A] , \\ &= \{\psi_v(v*a)/a \in A\} , \\ &= A . \end{aligned}$$

The last equality follows from the definition of ψ_v. Hence $\psi_v[S]$ belongs to $\mathcal{A}$. Therefore

$$S \in v\mathcal{E}_{\mathcal{A}} \Longrightarrow \psi_v[S] \in \mathcal{A} \quad . \quad ....(1)$$

Let $S \subseteq V$ be such that $\psi_v[S]$ belongs to $\mathcal{A}$. Hence

$$\psi_v[S] = A$$

for some A in $\mathcal{A}$. By the definition of ψ_v, we have

$$\psi_v[v*A] = A .$$

Hence

$$\psi_v[S] = \psi_v[v*A] .$$

Since ψ_v is one-to-one, hence $S = v*A$. Since $\{v\} \cup v*A$ belongs to $\mathcal{E}_{\mathcal{A}}$, hence $v*A$ belongs to $v\mathcal{E}_{\mathcal{A}}$. Hence S belongs to $v\mathcal{E}_{\mathcal{A}}$.

Therefore

$$\psi_v[S] \in \mathcal{A} \Rightarrow S \in v\mathcal{E}_{\mathcal{A}} \quad . \quad ....(2)$$

From (1) and (2), we see that ψ_v is an isomorphism from vH to $(\cup\mathcal{A},\mathcal{A})$. Therefore $vH \cong (\cup\mathcal{A},\mathcal{A})$. #

5.1.5 Corollary. Let $H = (V,\mathcal{E})$ be a quasi-group hypergraph. Then

$$uH \cong vH$$

for all u, v in V.

Proof. Let $H = (V,\mathcal{E})$ be a quasi-group hypergraph. Hence there exists a binary operation $*$ such that $(V,*)$ is a quasi-group and there exists an admissible set $\mathcal{A}$ of (r-1)-subsets of V such that $\mathcal{E} = \mathcal{E}_{\mathcal{A}}$. Hence

$$H = (V, \mathcal{E}_{\mathcal{A}})$$

is a hypergraph induced by the quasi-group $(V,*)$ and an admissible set $\mathcal{A}$ of (r-1)-subsets of V. Hence, by Proposition 5.1.4, for any u,v in V, we have

$$\begin{aligned} uH &\cong (\cup\mathcal{A},\mathcal{A}), \\ &\cong vH . \end{aligned}$$

#

5.2 $(Q,*)$-realizable.

In this chapter we are interested in the followings problem: Given a family $\Gamma = (K_v)_{v \in I}$ of hypergraphs. Find a quasi-group hypergraph $H = (V, \mathcal{E})$ which is a realization of Γ.

First, we observe that for $H = (V, \mathcal{E})$ to be a quasi-group hypergraph, it is necessary that

$$uH \cong vH$$

for all u,v in V. Hence it is necessary that we have

$$K_u \cong K_v$$

for all u, v in I, i.e. all the hypergraphs in Γ are isomorphic. The above problem can be reformulated as follows: Given a hypergraph K and a non-empty set I, find a quasi-group hypergraph $H = (I, \mathcal{E})$ such that

$$vH \cong K$$

for all v in I. A variant of this is the followings problem: Given an (r-1)-uniform hypergraph K and a quasi-group $(I,*)$, find a hypergraph $H = (I, \mathcal{E}_{\mathcal{A}})$ induced by $(I,*)$ and an admissible set $\mathcal{A}$ of (r-1)-subsets of I such that

$$vH \cong K$$

for all v in I. In this section we shall be concerned with this problem.

Let K be an (r-1)-uniform hypergraph and $(Q,*)$ be a quasi-group. We say that K is <u>$(Q,*)$-realizable</u> if there exists an admissible set $\mathcal{A}$ of (r-1)-subsets of Q such that $H = (Q, \mathcal{E}_{\mathcal{A}})$ has the property that

$$qH \cong K$$

for all q in Q.

5.2.1 Theorem Let K be an (r-1)-uniform hypergraph and (Q,*) be a quasi-group. Then K is (Q,*)-realizable if and only if there exists an admissible set of (r-1)-subsets of Q such that $K \cong (\cup\mathcal{A}, \mathcal{A})$.

Proof. Let K be an (r-1)-uniform hypergraph and (Q,*) be a quasi-group. Assume that K is (Q,*)-realizable. Hence there exists an admissible set $\mathcal{A}$ of (r-1)-subsets of Q such that H = $(Q, \mathcal{E}_{\mathcal{A}})$ has the property that

$$qH \cong K$$

for all q in Q. By Proposition 5.1.4., we have

$$qH \cong (\cup\mathcal{A}, \mathcal{A})$$

for all q in Q. Hence $K \cong (\cup\mathcal{A}, \mathcal{A})$.

Now, assume that there exists an admissible set $\mathcal{A}$ of (r-1)-subsets of Q such that $K \cong (\cup\mathcal{A}, \mathcal{A})$. Let H = $(Q, \mathcal{E}_{\mathcal{A}})$ be a quasi-group hypergraph induced by the quasi-group (Q,*) and an admissible set $\mathcal{A}$. By Proposition 5.1.4., we have

$$qH \cong (\cup\mathcal{A}, \mathcal{A})$$

for all q in Q. Hence

$$qH \cong K$$

for all q in Q. Hence K is $(Q,*)$-realizable. #

The following examples illustrate how Theorem 5.2.1 can be applied.

5.2.2 Example. Let $Q = \{0,1,2,3,4\}$. Let $*$ be defined on Q by

$$x*y \; = \; \text{the least non-negative residue modulo 5 of } x+y,$$

i.e. $(Q,*)$ is the cyclic group of order 5. Let

$$K = (\{1,2,3,4\},\{\{1,2\},\{2,3\},\{3,4\}\}).$$

We shall determine whether K is $(Q,*)$-realizable. First, we determine all the admissible sets of 2-subsets of Q. By the method described in Appendix 1 , we find that all the admissible sets of 2-subsets of Q are

$$\mathcal{A}_1 = \{\{1,2\},\{1,4\},\{3,4\}\},$$

$$\mathcal{A}_2 = \{\{1,3\},\{2,4\},\{2,3\}\}$$

and

$$\mathcal{A}_3 = \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}.$$

We have

$$(\cup\mathcal{A}_1, \mathcal{A}_1) = (\{1,2,3,4\},\{\{1,2\},\{1,4\},\{3,4\}\}),$$

$$(\cup\mathcal{A}_2, \mathcal{A}_2) = (\{1,2,3,4\},\{\{1,3\},\{2,4\},\{2,3\}\})$$

and

$$(\cup\mathcal{A}_3, \mathcal{A}_3) = (\{1,2,3,4\},\{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}).$$

Let $\psi : \{1,2,3,4\} \to \{1,2,3,4\}$ be defined by $\psi(1) = 3$, $\psi(2) = 4$, $\psi(3) = 1$ and $\psi(4) = 2$. It can be verified that ψ is an isomorphism from K to $(\cup \mathcal{A}_1, \mathcal{A}_1)$. Hence $K \cong (\cup \mathcal{A}_1, \mathcal{A}_1)$. Hence, by Theorem 5.2.1 , K is $(Q,*)$-realizable.

5.2.3 Example. Let $(Q,*)$ be asgiven in Example 5.2.2. Let

$$K = (\{1,2,3,4\},\{\{1,2\},\{1,4\},\{2,3\},\{3,4\}\}).$$

To determine whether K is $(Q,*)$-realizable, we look for $\mathcal{A}_i$, $i = 1,2,3$, (as given in Example 5.2.2.) such that $K \cong (\cup \mathcal{A}_i, \mathcal{A}_i)$. It turns out that K is not isomorphic to any of $(\cup \mathcal{A}_i, \mathcal{A}_i)$, $i = 1,2,3$. This can be seen by observing that the number of edges of K differs from those of $(\cup \mathcal{A}_i, \mathcal{A}_i)$, $i = 1,2,3$. Hence, by Theorem 5.2.1 , K is not $(Q,*)$-realizable.

5.3 Quasi-group Realizations.

Let K be a hypergraph and I be a finite non-empty set. We say that K is I-quasi-group (I-group) realizable if there exists a binary operation $*$ on I such that $(I,*)$ is a quasi-group (group) and K is $(I,*)$-realizable.

In this section, we are interested in the following problem: Given a hypergraph K and a finite non-empty set I, determine whether K is I-quasi-group (I-group) realizable.

The problem concerning I-quasi-group realizability can be answered by our results in Chapter 3 and Proposition 5.1.3. We illustrate this by an example:

5.3.1 <u>Example.</u> Let $I = \{1,2,3,4,5,6\}$ and

$$K = (\{1,2,3,4,5\},\{\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{1,5\}\}).$$

We shall show that K is not I-quasi-group realizable. Let H be H^{A_1} of example 3.4.2 , i.e.

$$H = (\{1,2,3,4,5,6\},\{\{1,2,3\},\{1,3,4\},\{1,4,5\},\{1,5,6\},\{1,2,6\},$$
$$\{2,3,5\},\{2,4,5\},\{2,4,6\},\{3,4,6\},\{3,5,6\}\}).$$

It is shown there that H is the unique, up to isomorphism, hypergraph such that $vH \cong K$ for all v in I. Hence to show that K is not I-quasi-group realizable it suffices to show that H is not a quasi-group hypergraph. To show, by Proposition 5.1.3 , that H is not a quasi-group hypergraph we must show that there does not exist any system $(\psi_{uv})_{u,v \in I}$ such that each ψ_{uv} is an isomorphism from uH to vH and for every u,v,v' in I if $v \neq v'$ then $\psi_{uv}(a) \neq \psi_{uv'}(a)$ for all a in vI. It suffices to show that there does not exist any system $(\psi_{1v})_{v \in I}$ such that each ψ_{1v} is an isomorphism from 1H to vH and for every v,v' in I if $v \neq v'$ then $\psi_{1v}(a) \neq \psi_{1v'}(a)$ for all a in 1I. Note that

$1H = (\{2,3,4,5,6\},\{\{2,3\},\{3,4\},\{4,5\},\{5,6\},\{2,6\}\}),$

$2H = (\{1,3,4,5,6\},\{\{1,3\},\{1,6\},\{3,5\},\{4,5\},\{4,6\}\}),$

$3H = (\{1,2,4,5,6\},\{\{1,2\},\{1,4\},\{2,5\},\{4,6\},\{5,6\}\}),$

$4H = (\{1,2,3,5,6\},\{\{1,3\},\{1,5\},\{2,5\},\{2,6\},\{3,6\}\}),$

$5H = (\{1,2,3,4,6\},\{\{1,4\},\{1,6\},\{2,3\},\{2,4\},\{3,6\}\})$

and

$6H = (\{1,2,3,4,5\},\{\{1,5\},\{1,2\},\{2,4\},\{3,5\},\{3,4\}\}).$

Suppose that there exists a system $S = (\psi_{1v})_{v \in I}$ such that each ψ_{1v} is an isomorphism from 1H to vH and for every v, v' in I if v = v' then $\psi_{1v}(a) \neq \psi_{1v'}(a)$ for all a in 1I, 1I = {2,3,4,5,6}.

There are 10 isomorphisms from 1H to vH, v = 1,2,3,4,5,6. They are designated by ψ^k_{1v}, k = 1,2,...,10, v = 1,2,3,4,5,6, and are given in the followings tables:

Table 14.

x	$\psi^1_{11}(x)$	$\psi^2_{11}(x)$	$\psi^3_{11}(x)$	$\psi^4_{11}(x)$	$\psi^5_{11}(x)$	$\psi^6_{11}(x)$	$\psi^7_{11}(x)$	$\psi^8_{11}(x)$	$\psi^9_{11}(x)$	$\psi^{10}_{11}(x)$
2	2	2	3	3	4	4	5	5	6	6
3	3	6	4	2	5	3	6	4	2	5
4	4	5	5	6	6	2	2	3	3	4
5	5	4	6	5	2	6	3	2	4	3
6	6	3	2	4	3	5	4	6	5	2

Table 15.

x	$\psi_{12}^{1}(x)$	$\psi_{12}^{2}(x)$	$\psi_{12}^{3}(x)$	$\psi_{12}^{4}(x)$	$\psi_{12}^{5}(x)$	$\psi_{12}^{6}(x)$	$\psi_{12}^{7}(x)$	$\psi_{12}^{8}(x)$	$\psi_{12}^{9}(x)$	$\psi_{12}^{10}(x)$
2	1	1	3	3	4	4	5	5	6	6
3	3	6	5	1	5	6	3	4	1	4
4	5	4	4	6	3	1	1	6	3	5
5	4	5	6	4	1	3	6	1	5	3
6	6	3	1	5	6	5	4	3	4	1

Table 16.

x	$\psi_{13}^{1}(x)$	$\psi_{13}^{2}(x)$	$\psi_{13}^{3}(x)$	$\psi_{13}^{4}(x)$	$\psi_{13}^{5}(x)$	$\psi_{13}^{6}(x)$	$\psi_{13}^{7}(x)$	$\psi_{13}^{8}(x)$	$\psi_{13}^{9}(x)$	$\psi_{13}^{10}(x)$
2	1	1	2	2	4	4	5	5	6	6
3	2	4	1	5	1	6	2	6	4	5
4	5	6	4	6	2	5	1	4	1	2
5	6	5	6	4	5	2	4	1	2	1
6	4	2	5	1	6	1	6	2	5	4

Table 17.

x	$\psi_{14}^{1}(x)$	$\psi_{14}^{2}(x)$	$\psi_{14}^{3}(x)$	$\psi_{14}^{4}(x)$	$\psi_{14}^{5}(x)$	$\psi_{14}^{6}(x)$	$\psi_{14}^{7}(x)$	$\psi_{14}^{8}(x)$	$\psi_{14}^{9}(x)$	$\psi_{14}^{10}(x)$
2	1	1	2	2	3	3	5	5	6	6
3	3	5	5	6	1	6	1	2	2	3
4	6	2	1	3	5	2	3	6	5	1
5	2	6	3	1	2	5	6	3	1	5
6	5	3	6	5	6	1	2	1	3	2

Table 18.

x	$\psi_{15}^1(x)$	$\psi_{15}^2(x)$	$\psi_{15}^3(x)$	$\psi_{15}^4(x)$	$\psi_{15}^5(x)$	$\psi_{15}^6(x)$	$\psi_{15}^7(x)$	$\psi_{15}^8(x)$	$\psi_{15}^9(x)$	$\psi_{15}^{10}(x)$
2	1	1	2	2	3	3	4	4	6	6
3	6	4	3	4	6	2	1	2	1	3
4	3	2	6	1	1	4	6	3	4	2
5	2	3	1	6	4	1	3	6	2	4
6	4	6	4	3	2	6	2	1	3	1

Table 19.

x	$\psi_{16}^1(x)$	$\psi_{16}^2(x)$	$\psi_{16}^3(x)$	$\psi_{16}^4(x)$	$\psi_{16}^5(x)$	$\psi_{16}^6(x)$	$\psi_{16}^7(x)$	$\psi_{16}^8(x)$	$\psi_{16}^9(x)$	$\psi_{16}^{10}(x)$
2	1	1	2	2	3	3	4	4	5	5
3	5	2	4	1	4	5	2	3	1	3
4	3	4	3	5	2	1	1	5	2	4
5	4	3	5	3	1	2	5	1	4	2
6	2	5	1	4	5	4	3	2	3	1

The isomorphism ψ_{11} in S must be one of the ψ_{11}^k, $k = 1,2,\ldots,10$. Assume that $\psi_{11} = \psi_{11}^1$, i.e. ψ_{11}^1 is in S. Observe that

$$\psi_{11}^1(3) = \psi_{12}^1(3) ,$$

$$\psi_{11}^1(4) = \psi_{12}^2(4) ,$$

$$\psi_{11}^1(4) = \psi_{12}^3(4) ,$$

$$\psi_{11}^{1}(6) = \psi_{12}^{5}(6) ,$$

$$\psi_{11}^{1}(3) = \psi_{12}^{7}(3) ,$$

$$\psi_{11}^{1}(5) = \psi_{12}^{9}(5) .$$

Hence ψ_{12}^{1}, ψ_{12}^{2}, ψ_{12}^{3}, ψ_{12}^{5}, ψ_{12}^{7} and ψ_{12}^{9} are not in S. Hence we can conclude the following

(2) ψ_{12} can be one of the following:

(2-1) ψ_{12}^{4} ,

(2-2) ψ_{12}^{6} ,

(2-3) ψ_{12}^{8} ,

(2-4) ψ_{12}^{10} .

Similarly for v = 3,4,5,6, it turns out that the following holds:

(3) ψ_{13} can be one of the following:

(3-1) ψ_{13}^{1} ,

(3-2) ψ_{13}^{6} ,

(3-3) ψ_{13}^{9} ,

(3-4) ψ_{13}^{10} ,

(4) ψ_{14} can be one of the following:

(4-1) ψ_{14}^{2} ,

(4-2) ψ_{14}^{7} ,

(4-3) ψ_{14}^{8} ,

(4-4) ψ_{14}^{9} ,

(5) ψ_{15} can be one of the following:

(5-1) ψ_{15}^{1} ,

(5-2) ψ_{15}^{5} ,

(5-3) ψ_{15}^{7} ,

(5-4) ψ_{15}^{8} ,

(6) ψ_{16} can be one of the following:

(6-1) ψ_{16}^{1} ,

(6-2) ψ_{16}^{5} ,

(6-3) ψ_{16}^{6} ,

(6-4) ψ_{16}^{9} .

From (2), note that if $\psi_{12} = \psi_{12}^4$ then $\psi_{12}^4(a) \neq \psi_{16}(a)$ for all a in $\{2,3,4,5,6\}$. Since

$$\psi_{12}^4(5) = \psi_{16}^1(5) ,$$

$$\psi_{12}^4(2) = \psi_{16}^5(2) ,$$

$$\psi_{12}^4(2) = \psi_{16}^6(2) ,$$

$$\psi_{12}^4(5) = \psi_{16}^9(5) ,$$

hence ψ_{16} can not be one of ψ_{16}^1, ψ_{16}^5, ψ_{16}^6 and ψ_{16}^9. Therefore we have a contradiction. Hence ψ_{12} can not be ψ_{12}^4.

If $\psi_{12} = \psi_{12}^6$, then $\psi_{12}^6(a) \neq \psi_{15}(a)$ for all a in $\{2,3,4,5,6\}$. Since

$$\psi_{12}^6(3) = \psi_{15}^1(3) ,$$

$$\psi_{12}^6(3) = \psi_{15}^5(3) ,$$

$$\psi_{12}^6(5) = \psi_{15}^7(5) ,$$

$$\psi_{12}^6(2) = \psi_{15}^8(2) ,$$

hence ψ_{15} can not be one of ψ_{15}^1, ψ_{15}^5, ψ_{15}^7 and ψ_{15}^8. Therefore we have a contradiction. Hence ψ_{12} can not be ψ_{12}^6.

If $\psi_{12} = \psi_{12}^8$, then $\psi_{12}^8(a) \neq \psi_{14}(a)$ for all a in $\{2,3,4,5,6\}$. Since

$$\psi_{12}^{8}(6) = \psi_{14}^{2}(6),$$

$$\psi_{12}^{8}(2) = \psi_{14}^{7}(2),$$

$$\psi_{12}^{8}(2) = \psi_{14}^{8}(2),$$

$$\psi_{12}^{8}(5) = \psi_{14}^{9}(5),$$

hence ψ_{14} can not be one of ψ_{14}^{2}, ψ_{14}^{7}, ψ_{14}^{8} and ψ_{14}^{9}. Therefore we have a contradiction. Hence ψ_{12} can not be ψ_{12}^{8}.

If $\psi_{12} = \psi_{12}^{10}$, then $\psi_{12}^{10}(a) \neq \psi_{13}(a)$ for all a in $\{2,3,4,5,6\}$.

Since

$$\psi_{12}^{10}(4) = \psi_{13}^{1}(4),$$

$$\psi_{12}^{10}(4) = \psi_{13}^{6}(4),$$

$$\psi_{12}^{10}(2) = \psi_{13}^{9}(2),$$

$$\psi_{12}^{10}(2) = \psi_{13}^{10}(2),$$

hence ψ_{13} can not be one of ψ_{13}^{1}, ψ_{13}^{6}, ψ_{13}^{9} and ψ_{13}^{10}. Therefore we have a contradiction. Hence ψ_{12} can not be ψ_{12}^{10}.

Hence ψ_{12} can not be one of ψ_{12}^{4}, ψ_{12}^{6}, ψ_{12}^{8} and ψ_{12}^{10}. Therefore we have a contradiction. Hence ψ_{11} can not be ψ_{11}^{1}.

By the same manner, we can show that ψ_{11} can not be one of ψ_{11}^k, k = 2,3,...,10. Therefore ψ_{11} can not be one of ψ_{11}^k, k = 1,2, 3,...,10. Hence we have a contradiction to the existence of a system S. Therefore the system S does not exist.

It is much easier to determine whether a given hypergraph K is I-group realizable than to determine whether it is I-quasi-group realizable. This is because for each I there are fewer group structures that can be defined on I. Besides, it is not difficult to determine all the admissible sets of a given group.

Suppose that a hypergraph $K = (V, \mathcal{E})$ and a finite non-empty set I are given. To determine whether K is I-group realizable, we do the following:

(1) Determine all non-isomorphic group structures on I.

(2) For each group structure on I, we determine all the admissible sets $\mathcal{A}$ such that $|\mathcal{A}| = |\mathcal{E}|$.

(3) For each $\mathcal{A}$ in (2) we determine whether $K \cong (\cup\mathcal{A}, \mathcal{A})$.

If $\mathcal{A}$ in (3) can be found, we know that K is I-group realizable, otherwise it is not.

We illustrate these by showing that K in the above example (Example 5.3.1) is not I-group realizable.

It is well-know that any group of order 6 must be of the form:

(1) $G_1 = (\{1,a,a^2,a^3,a^4,a^5\},*)$, where $a^6 = 1$ and $a^i * a^j = a^{i+j}$ for all $i,j = 1,2,3,4,5$; or

(2) $G_2 = (\{1,a,b,b^2,ab,ab^2\},\cdot)$, where $a^2 = 1 = b^3$ and $a^{-1}ba = b^2$.

Hence, if $(I,*)$ is a group, then $(I,*)$ must be isomorphic to exactly one of G_1 and G_2. Hence to show that K is not I-group realizable, it suffices to show that K is neither G_1-realizable nor G_2-realizable.

First, we shall show that K is not G_1-realizable. By the method described in Appendix I, we find that the followings are all the minimal admissible sets of 2-subsets of G_1;

$$\mathcal{A}_1 = \{\{a,a^2\},\{a,a^5\},\{a^4,a^5\}\},$$

$$\mathcal{A}_2 = \{\{a,a^3\},\{a^2,a^5\},\{a^3,a^5\}\},$$

$$\mathcal{A}_3 = \{\{a,a^4\},\{a^3,a^5\},\{a^2,a^3\}\}$$

and

$$\mathcal{A}_4 = \{\{a^2,a^4\}\}.$$

By Theorem 1 in Appendix I, any admissible set of 2-subsets of G_1 is a union of minimal admissible sets of 2-subsets of G_1. Hence, if $\mathcal{A}$ is any admissible set of 2-subsets of G_1, then $\mathcal{A}$ must have cardinality 1,3,4,6,7,9,10. Therefore it is not possible to find an admissible set $\mathcal{A}$ of 2-subsets of G_1 such that $K \cong (\cup\mathcal{A}, \mathcal{A})$.

Hence, by Theorem 5.2.1 , K is not G_1-realizable.

Next, we shall show that K is not G_2-realizable. We find that all the minimal admissible sets of 2 subsets of G_2 are

$$\mathcal{A}_1 = \{\{a,b\},\{a,ab\},\{b^2,ab\}\},$$

$$\mathcal{A}_2 = \{\{a,b^2\},\{a,ab^2\},\{b,ab^2\}\},$$

$$\mathcal{A}_3 = \{\{ab,ab^2\},\{ab,b\},\{ab^2,b^2\}\}$$

and

$$\mathcal{A}_4 = \{\{b, b^2\}\}.$$

By the same arguments, it can be seen that there does not exist any admissible set $\mathcal{A}$ of 2-subsets of G_1 such that K $\cong (\cup\mathcal{A}, \mathcal{A})$. Hence, by Theorem 5.2.1 , K is not G_2-realizable.