CHAPTER IV

SOME NECESSARY CONDITIONS

In this chapter we prove some necessary conditions for a given family of hypergraphs to be realizable. These conditions provide easy way of proving non-existence of realizations. Sections 4.1. and 4.2 give necessary conditions concerning neighbourhood hypergraphs and edges sizes of hypergraphs in the given family. Section 4.3 gives a necessary condition for a full family of Γ-injections to be compatible. This condition can be used in proving non-existence of any compatible full family of Γ-injections, which is equivalent to non-existence of any realization. Illustrations are given in section 4.4.

4.1 Necessary Condition Involving Neighbourhood Hypergraphs.

4.1.1 Proposition. Let $\Gamma = (K_v)_{v \in I}$, where $K_v = (W_v, \mathcal{F}_v)$ for all v in I, be a family of hypergraphs. Let $H = (V, \mathcal{E})$ be a realization of Γ. For each v in I, let α_v be any isomorphism from K_v to vH. Then for every v in I, we have

$$(1) \quad \alpha^{-1}_{\alpha_v(w)}(v) \in W_{\alpha_v(w)}$$

and

$$(2) \quad wK_v \cong \alpha^{-1}_{\alpha_v(w)}(v)K_{\alpha_v(w)} ,$$

for all w in W_v.

<u>Proof.</u> Let $\Gamma = (K_v)_{v \in I}$, where $K_v = (W_v, \mathcal{F}_v)$ for all v in I, be a family of hypergraphs. Let $H = (V, \mathcal{E})$ be a realization of Γ . For each v in I, let α_v be any isomorphism from K_v to vH.

Let v be any element in I. Let w be any element in W_v. Since α_v is an isomorphism from K_v to vH, hence $\alpha_v(w)$ is a vertex in vH and, by Proposition 3.1.2 ,

$$wK_v \cong \alpha_v(w)vH.$$

Hence, by Proposition 3.1.1 , v is a vertex in $\alpha_v(w)H$ and

$$\alpha_v(w)vH = v\alpha_v(w)H.$$

Since $\alpha_{\alpha_v(w)}$ is an isomorphism from $K_{\alpha_v(w)}$ to $\alpha_v(w)H$, hence $\alpha^{-1}_{\alpha_v(w)}$ is an isomorphism from $\alpha_v(w)H$ to $K_{\alpha_v(w)}$. Hence $\alpha^{-1}_{\alpha_v(w)}(v)$ is a vertex in $K_{\alpha_v(w)}$, i.e.

$$\alpha^{-1}_{\alpha_v(w)}(v) \in W_{\alpha_v(w)} .$$

Hence, by Proposition 3.1.2 ,

$$v\alpha_v(w)H \cong \alpha^{-1}_{\alpha_v(w)}(v)K_{\alpha_v(w)} .$$

Hence

$$\begin{aligned} wK_v &\cong \alpha_v(w)vH, \\ &= v\alpha_v(w)H, \end{aligned}$$

$$\cong \alpha^{-1}_{\alpha_v(w)}(v)K_{\alpha_v(w)} .$$

Therefore $\quad wK_v \quad \cong \alpha^{-1}_{\alpha_v(w)}(v)K_{\alpha_v(w)}$. $\qquad$ #

4.1.2 <u>Theorem</u>. Let $\Gamma = (K_v)_{v \in I}$, where $K_v = (W_v, \mathcal{F}_v)$ for all v in I, be a family of hypergraphs. For each v in I and any non-empty subset T of W_v, let

$$\mathcal{C}(v,T) = \{ u/u \in I-\{v\} \text{ and } wK_u \cong tK_v \text{ for some } w \text{ in } W_u \text{ and some } t \text{ in } T\}.$$

If Γ is realizable, then $|\mathcal{C}(v,T)| \geqslant |T|$ for all v in I and all non-empty subsets T of W_v.

<u>Proof.</u> Let $\Gamma = (K_v)_{v \in I}$, where $K_v = (W_v, \mathcal{F}_v)$ for all v in I, be a family of hypergraphs. For each v in I and any non-empty subset T of W_v, let

$$\mathcal{C}(v,T) = \{ u/u \in I-\{v\} \text{ and } wK_u \cong tK_v \text{ for some } w \text{ in } W_u \text{ and some } t \text{ in } T\}.$$

Suppose that Γ is realizable. Let $H = (I, \mathcal{E})$ be a realization of Γ such that $vH \cong K_v$ for all v in I. For each v in I, let α_v be an isomorphism from K_v to vH.

Let v be any element in I and T be any non-empty subset of W_v. Note that $vI \subseteq I-\{v\}$. Since α_v is an isomorphism from K_v to vH, hence $\alpha_v[W_v] = vI$. Therefore $\alpha_v[W_v] \subseteq I-\{v\}$.

Let t be any element in T. Hence $\alpha_v(t)$ belongs to I-{v}. Since $T \subseteq W_v$, t is in W_v. Hence, by Proposition 4.1.1 ,

$$\alpha^{-1}_{\alpha_v(t)}(v) \in W_{\alpha_v(t)}$$

and

$$\alpha^{-1}_{\alpha_v(t)}(v) K_{\alpha_v(t)} \cong tK_v \text{ .}$$

Hence $\alpha_v(t)$ belongs to $\mathcal{C}(v,T)$. Therefore $\alpha_v[T] \subseteq \mathcal{C}(v,T)$.

Hence

$$|\alpha_v[T]| \leqslant |\mathcal{C}(v,T)| .$$

Since α_v is one-to-one, hence $|T| = |\alpha_v[T]|$.

Therefore $|T| \leqslant |\mathcal{C}(v,T)|$. #

4.2 Necessary Condition Involving Edges Sizes.

4.2.1 Proposition. Let $H = (V, \mathcal{E})$ be a hypergraph. Then $\sum_{v \in V} \sum_{\substack{vE \in v\mathcal{E} \\ |vE| = r}} |vE|$ is divisible by r+1, r = 1,2,3,... .

Proof. Let $H = (V, \mathcal{E})$ be a hypergraph. Let r be any positive integer. For each v in V and each vE in $v\mathcal{E}$ such that $|vE| = r$, let

$$\mathcal{L}(v,vE) = \{ (u,v,vE)/u \in vE \} .$$

For each E in $\mathcal{E}$ such that $|E| = r+1$, let

$$\mathcal{S}(E) = \{(u,v,E-\{v\})/v \in E \text{ and } u \in E-\{v\}\} .$$

Observe that

(1) $v \neq v' \Rightarrow \mathcal{L}(v,vE) \cap \mathcal{L}(v',v'E) = \emptyset$,

(2) $vE \neq vE' \Rightarrow \mathcal{L}(v,vE) \cap \mathcal{L}(v,vE') = \emptyset$,

(3) $E \neq E' \Rightarrow \mathcal{S}(E) \cap \mathcal{S}(E') = \emptyset$,

(4) $\displaystyle\bigcup_{v \in V} \bigcup_{\substack{vE \in v\mathcal{E} \\ |vE|=r}} \mathcal{L}(v,vE) = \bigcup_{\substack{E \in \mathcal{E} \\ |E|=r+1}} \mathcal{S}(E)$,

(5) $|\mathcal{L}(v,vE)| = |vE|$

and

(6) $|\mathcal{S}(E)| = r(r+1)$.

Hence

$$\sum_{v \in V} \sum_{\substack{vE \in v\mathcal{E} \\ |vE|=r}} |vE| = \sum_{v \in V} \sum_{\substack{vE \in v\mathcal{E} \\ |vE|=r}} |\mathcal{L}(v,vE)| ,$$

$$= \Bigg| \bigcup_{v \in V} \bigcup_{\substack{vE \in v\mathcal{E} \\ |vE|=r}} \mathcal{L}(v,vE) \Bigg| ,$$

$$= \Bigg| \bigcup_{\substack{E \in \mathcal{E} \\ |E|=r+1}} \mathcal{S}(E) \Bigg| ,$$

$$= \sum_{\substack{E \in \mathcal{E} \\ |E|=r+1}} |\mathcal{S}(E)| ,$$

$$= \sum_{\substack{E \in \mathcal{E} \\ |E|=r+1}} r(r+1) ,$$

$$= (r+1) \sum_{\substack{E \in \mathcal{E} \\ |E|=r+1}} r .$$

Hence $\sum_{v \in V} \sum_{\substack{vE \in v\mathcal{E} \\ |vE| = r}} |vE|$ is divisible by r+1. #

4.2.2 Theorem. Let $\Gamma = (K_v)_{v \in I}$, where $K_v = (W_v, \mathcal{F}_v)$ for all v in I, be a family of hypergraphs. If Γ is realizable, then $\sum_{v \in I} \sum_{\substack{F_v \in \mathcal{F}_v \\ |F_v| = r}} |F_v|$ is divisible by r+1, r = 1,2,3,... .

Proof. Let $\Gamma = (K_v)_{v \in I}$, where $K_v = (W_v, \mathcal{F}_v)$ for all v in I, be a family of hypergraphs.

Suppose that Γ is realizable. Let $H = (I, \mathcal{E})$ be a realization of Γ such that $vH \cong K_v$ for all v in I. Let r be any positive integer. For any v in I, vH and K_v are isomorphic, hence

$$\sum_{\substack{F_v \in \mathcal{F}_v \\ |F_v| = r}} |F_v| = \sum_{\substack{vE \in v\mathcal{E} \\ |vE| = r}} |vE| .$$

Hence

$$\sum_{v \in I} \sum_{\substack{F_v \in \mathcal{F}_v \\ |F_v| = r}} |F_v| = \sum_{v \in I} \sum_{\substack{vE \in v\mathcal{E} \\ |vE| = r}} |vE| .$$

By Proposition 4.2.1 , $\sum_{v \in I} \sum_{\substack{vE \in v\mathcal{E} \\ |vE| = r}} |vE|$ is divisible by r+1.

Hence $\sum_{v \in I} \sum_{\substack{F_v \in \mathcal{F}_v \\ |F_v| = r}} |F_v|$ is divisible by r+1. #

4.3 Necessary Condition for a Full Family of Γ-injection to be Compatible.

Let $H = (V, \mathcal{E})$ be a hypergraph. Another useful concept about hypergraph associating to each vertex of a hypergraph is the followings. For each vertex v in V, let

$$\mathcal{E}^{v} = \{ E \in \mathcal{E} / v \notin E \}$$

and

$$V^{v} = \bigcup \mathcal{E}^{v} .$$

The hypergraph $(V^{v}, \mathcal{E}^{v})$ will be denoted by $H \setminus v$. In this section, for any hypergraph $H = (V, \mathcal{E})$ we denote $|\mathcal{E}|$, the number of edges in H, by q(H).

4.3.1 Proposition. Let $H = (V, \mathcal{E})$ be a hypergraph. Let v be any vertex in H. Then $q(H \setminus v) = q(H) - d_H(v)$.

Proof. Let $H = (V, \mathcal{E})$ be a hypergraph. Let v be any vertex in H. Observe that

$$\begin{aligned} q(H \setminus v) &= |\mathcal{E}^{v}|, \\ &= |\{E \in \mathcal{E} / v \notin E\}|, \\ &= |\mathcal{E} - \{E \in \mathcal{E} / v \in E\}|, \\ &= |\mathcal{E}| - |\{E \in \mathcal{E} / v \in E\}|, \end{aligned}$$

$= q(H) - d_H(v).$

Hence $q(H \setminus v) = q(H) - d_H(v).$ #

4.3.2 Proposition. Let $H = (V, \mathcal{E})$ and $H' = (V', \mathcal{E}')$ be hypergraphs. Let α be an isomorphism from H to H'. Let v be any vertex in H. Then $H \setminus v \cong H' \setminus \alpha(v)$.

Proof. Let α be an isomorphism from $H = (V, \mathcal{E})$ to $H' = (V', \mathcal{E}')$. Let v be any vertex in H. Let ρ be the restriction of α to V^v. A straightforward verification shows that ρ is an isomorphism from $H \setminus v$ to $H' \setminus \alpha(v)$. Hence $H \setminus v \cong H' \setminus \alpha(v)$. #

4.3.3 Proposition. Let $H = (V, \mathcal{E})$ be a hypergraph. Let v be any vertex in H. Then $\Gamma_v = ((uH) \setminus v)_{u \in I - \{v\}}$ is realizable.

Proof. Let $H = (V, \mathcal{E})$ be a hypergraph and v be any vertex in in H. Let

$$\mathcal{E}(v) = \mathcal{E}^v \cup \{\{u\} / u \in V - \{v\}\} .$$

Hence $\bigcup \mathcal{E}(v) = V - \{v\}$. Therefore $H(v) = (V - \{v\}, \mathcal{E}(v))$ is a hypergraph.

Let u be any element in $V - \{v\}$. Observe that, if u is in $\bigcup \mathcal{E}^v$, then

$$(u\mathcal{E})^v = \{uE / uE \in u\mathcal{E} \text{ and } v \notin uE\},$$

$$= \{E - \{u\} / E \in \mathcal{E}, u \in E, E - \{u\} \neq \emptyset \text{ and } v \notin E - \{u\}\} ,$$

$= \{E-\{u\} / E \in \mathcal{E},\ u \in E,\ E-\{u\} \neq \emptyset \text{ and } v \notin E\},$

$= \{E-\{u\} / E \in \mathcal{E}^v,\ u \in E \text{ and } E-\{u\} \neq \emptyset\},$

$= \{E-\{u\} / E \in \mathcal{E}(v),\ u \in E \text{ and } E-\{u\} \neq \emptyset\},$

$= u(\mathcal{E}(v)).$

If u is not in $\bigcup \mathcal{E}^v$, then $u(\mathcal{E}(v)) = \emptyset = (u\mathcal{E})^v$. Hence $(u\mathcal{E})^v = u(\mathcal{E}(v))$. By the above observation, we have

$$\begin{aligned}(uH) \setminus v &= (\bigcup (u\mathcal{E})^v,\ (u\mathcal{E})^v), \\ &= (\bigcup u(\mathcal{E}(v)),\ u(\mathcal{E}(v))) \\ &= u(H(v)).\end{aligned}$$

Hence $(uH) \setminus v = u(H(v))$ for all u in $V-\{v\}$. Hence the hypergraph H(v) is a realization of the family $\Gamma_v = ((uH) \setminus v)_{u \in I-\{v\}}$. Therefore Γ_v is realizable. #

4.3.4 Theorem. Let $\Gamma = (K_v)_{v \in I}$ be a family of hypergraphs. If $A = (\alpha_v)_{v \in I}$ is a compatible full family of Γ-injections, then for any v in I the family $\Gamma_v^* = (K_u \setminus \alpha_u^{-1}(v))_{u \in I-\{v\}}$ is realizable.

Proof. Let $\Gamma = (K_v)_{v \in I}$ be a family of hypergraphs. Suppose that $A = (\alpha_v)_{v \in I}$ is a compatible full family of Γ-injections. By Proposition 3.3.3., $H^A = (I, \mathcal{E}^A)$ is a realization of Γ and each α_v is an isomorphism from K_v to vH^A.

Let v be any element in I. Let u be any element in I-{v}. If u is in $\alpha_v[W_v]$, then u is a vertex in vH^A. Hence, by Proposition 3.1.1 , v is a vertex in uH^A. Note that α_u^{-1} is an isomorphism from uH to K_u. Hence, by Proposition 4.3.2 , $(uH) \setminus v \cong K_u \setminus \alpha_u^{-1}(v)$.

If u is not in $\alpha_v[W_v]$, then u is not a vertex in vH^A. Hence, by Proposition 3.1.1 , v is not a vertex in uH^A. Therefore $(uH^A) \setminus v = uH^A$. Since α_u^{-1} is an isomorphism from uH^A to K_u , hence $\alpha_u^{-1}(v)$ is not a vertex in K_u. Therefore $K_u \setminus \alpha_u^{-1}(v) = K_u$. Since K_u and uH^A are isomorphic, hence $(uH) \setminus v \cong K_u \setminus \alpha_u^{-1}(v)$.

Hence, we have

$$K_u \setminus \alpha_u^{-1}(v) \cong (uH) \setminus v$$

for all u in I-{v}. By Proposition 4.3.3 , $\Gamma_v = ((uH) \setminus v)_{u \in I-\{v\}}$ has a realization. This realization is also a realization of $\Gamma_v^* = (K_u \setminus \alpha_u^{-1}(v))_{u \in I-\{v\}}$, i.e. Γ_v^* is realizable. #

4.4 Examples.

In this section we apply our necessary conditions obtained in Theorems 4.1.2 , 4.2.2 and 4.3.4 to prove the non-existence of realizations of given families of hypergraphs. Examples 4.4.1 and 4.4.3 show that the necessary conditions in Theorems 4.1.2 and 4.2.2 are independent. However, they are not sufficient. This is shown by Example 4.4.5.

4.4.1 <u>Example.</u> Let $I = \{1,2,3,\ldots,8\}$. For each v in I, let $K_v = (W_v, \mathcal{F}_v)$, where

$$W_1 = \{1,2,3,4\},$$

$$\mathcal{F}_1 = \{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}\};$$

$$W_2 = W_3 = W_4 = W_5 = \{1,2,3,4\},$$

$$\mathcal{F}_2 = \mathcal{F}_3 = \mathcal{F}_4 = \mathcal{F}_5 = \{\{1,2,3\},\{1,3,4\}\};$$

$$W_6 = W_7 = W_8 = \{1,2\},$$

$$\mathcal{F}_6 = \mathcal{F}_7 = \mathcal{F}_8 = \{\{1,2\}\}.$$

Let $\Gamma = (K_v)_{v \in I}$. We shall show that Γ is not realizable. Observe that for each u in $I-\{1\}$ there does not exist any w in W_u such that $wK_u \cong 1K_1$. Hence

$$\mathcal{C}(1,\{1\}) = \{u/u \in I-\{1\} \text{ and } wK_u \cong 1K_1 \text{ for some } w \text{ in } W_u\},$$
$$= \emptyset .$$

Therefore $|\mathcal{C}(1,\{1\})| = 0$. Hence $|\mathcal{C}(1,\{1\})| < |\{1\}|$. Hence, by Theorem 4.1.2 ; Γ is not realizable.

4.4.2 <u>Note.</u> In the above example, we have

$$\sum_{v \in I} \sum_{\substack{F_v \in \mathcal{F}_v \\ |F_v| = r}} |F_v| = \begin{cases} 36 & \text{when } r = 3, \\ 6 & \text{when } r = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, for each r, this sum is divisible by $r+1$. Therefore, this example shows that the necessary condition of Theorem 4.1.2 is independent from Theorem 4.2.2.

4.4.3 Example. Let $I = \{1,2,3,4\}$. For each v in I, let $K_v = (W_v, \mathcal{F}_v)$, where

$$W_1 = W_2 = W_3 = W_4 = \{1,2\},$$
$$\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3 = \mathcal{F}_4 = \{\{1,2\}\}.$$

Let $\Gamma = (K_v)_{v \in I}$. We shall show that Γ is not realizable. Observe that

$$\sum_{v \in I} \sum_{\substack{F_v \in \mathcal{F}_v \\ |F_v| = 2}} |F_v| = 8$$

is not divisible by 3. Hence, by Theorem 4.2.2 , Γ is not realizable.

4.4.4 Note. For the given family Γ in the above example, it can be verified that

$$|\mathcal{C}(v,T)| = |\{u / u \in I - \{v\} \text{ and } wK_u \cong tK_v \text{ for some } w \text{ in } W_u \text{ and some } t \text{ in } T\}|,$$
$$= |I - \{v\}| ,$$
$$= 3 ,$$

and $|T| \leqslant 2$.

Hence $|\mathcal{C}(v,T)| \geqslant |T|$ for all v in I and all non-empty subsets T of W_v. Therefore, this example shows that the necessary condition of Theorem 4.2.2 is independent from that of Theorem 4.1.2.

4.4.5 Example. Let $I = \{1,2,3,4,5\}$. For each v in I, let $K_v = (W_v, \mathcal{F}_v)$, where

$$W_1 = W_2 = W_3 = W_4 = W_5 = \{1,2,3\},$$
$$\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3 = \mathcal{F}_4 = \mathcal{F}_5 = \{\{1,2\},\{2,3\},\{1,3\}\}.$$

Let $\Gamma = (K_v)_{v \in I}$. We shall show that Γ is not realizable. Hence, by Theorem 3.3.4 , it suffices to show that there does not exist a compatible full family of Γ-injections.

Suppose that there exists a compatible full family of Γ-injections $A = (\alpha_v)_{v \in I}$. Hence, by Theorem 3.3.3 , $H^A = (I, \mathcal{E}^A)$ is a realization of Γ and each α_v is an isomorphism from K_v to vH.

Fix $v = 1$. Let u be any element in $I-\{1\}$. Note that $\alpha_1[W_1] \subseteq I-\{1\}$.

In case u is in $\alpha_1[W_1]$. Hence u is a vertex in $1H^A$. Therefore, by Proposition 3.1.1 , 1 is a vertex in uH^A. Note that α_u^{-1} is an isomorphism from uH^A to K_u. Hence, by Proposition 4.3.2 , $(uH^A)\setminus 1 \cong K_u \setminus \alpha_u^{-1}(1)$. Hence

$$q(K_u \setminus \alpha_u^{-1}(1)) = q((uH^A)\setminus 1). \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

By Proposition 4.3.1 , we have

$$q((uH^A)\setminus 1) = q(uH^A) - d_{uH^A}(1). \quad \ldots\ldots\ldots\ldots\ldots(2)$$

Since uH^A and K_u are isomorphic, hence

$$q(uH^A) = q(K_u). \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Since 1 is a vertex in uH^A, hence by Proposition 3.1.1 ,

$$d_{uH^A}(1) = d_{1H^A}(u). \quad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Since α_1^{-1} is an isomorphism from $1H^A$ to K_1, hence by Proposition 2.3.1 ,

$$d_{1H^A}(u) = d_{K_1}(\alpha_1^{-1}(u)). \quad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Hence, by (1)-(5), we have

$$q(K_u \setminus \alpha_u^{-1}(1)) = q(K_u) - d_{K_1}(\alpha_1^{-1}(u)).$$

In case u is not in $\alpha_1[W_1]$. Hence u is not a vertex in $1H^A$. Therefore, by Proposition 3.1.1 , 1 is not a vertex in uH^A. Hence $\alpha_u^{-1}(1)$ is not a vertex in K_u. Hence $K_u \setminus \alpha_u^{-1}(1) = K_u$. Therefore

$$q(K_u \setminus \alpha_u^{-1}(1)) = q(K_u).$$

By Theorem 4.3.4 , the family $\Gamma_1^* = (K_u \setminus \alpha_u^{-1}(1))_{u \in I-\{1\}}$ is realizable. Let $H_1 = (I-\{1\}, \mathcal{E}_1)$ be a realization of Γ_1^* such that $u(H_1) \cong K_u \setminus \alpha_u^{-1}(1)$ for all u in $I-\{1\}$. Hence for each u in $\alpha_1[W_1]$,

$$\begin{aligned} d_{H_1}(u) &= q(u(H_1)), \\ &= q(K_u \setminus \alpha_u^{-1}(1)), \end{aligned}$$

$$= \quad q(K_u) - d_{K_1}(\alpha_1^{-1}(u)),$$

$$= \quad 3 - 2,$$

$$= \quad 1.$$

For u is not in $\alpha_1[W_1]$, hence

$$d_{H_1}(u) \quad = \quad q(u(H_1)),$$

$$= \quad q(K_u \setminus \alpha_u^{-1}(1)),$$

$$= \quad q(K_u),$$

$$= \quad 3.$$

Since $(d_{H_1}(u))_{u \in I-\{1\}}$ is the degree sequence of H_1 and $|\alpha_1[W_1]| = 3$, hence (3,1,1,1) is a degree sequence of H_1. Since each hypergraph in Γ_1^* is a 2-uniform hypergraph, hence H_1 is a 3-uniform hypergraph. Hence, by Proposition 2.3.2 , $|\mathcal{E}_1| = 2$. Hence, H_1 has a vertex of degree 3 but it has only 2 edges. Therefore we have a contradiction.

4.4.6 <u>Note</u>. For the family Γ given in the above example we have the followings:

$$(1) \quad \sum_{v \in I} \sum_{\substack{F_v \in \mathcal{F}_v \\ |F_v| = r}} |F_v| = \begin{cases} 30 & \text{when } r = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, for each r, the sum is divisible by r+1. Therefore Γ satisfies the necessary condition of Theorem 4.2.2.

(2) For every v in I and for any non-empty subset T of W_v

$$|\mathcal{C}(v,T)| = |\{ u/u \in I-\{v\} \text{ and } wK_u \cong tK_v \text{ for some } w \text{ in } W_u \text{ and some } t \text{ in } T\}|,$$

$$= |I-\{v\}| ,$$

$$= 4 ,$$

and $|T| \leqslant 3$.

Hence $|\mathcal{C}(v,T)| \geqslant |T|$ for all v in I and all non-empty subsets T of W_v. Therefore, the necessary condition in Theorem 4.1.2 holds.

This example shows that the necessary conditions in Theorem 4.1.2 and Theorem 4.2.2 are not sufficient.