CHAPTER II

AN ELEMENTARY MATHEMATICAL MODEL FOR THE TURBULENT DIFFUSION OF SMOKE FROM A CONTINUOUS POINT SOURCE

First divide the XZ-plane into squares and time into discrete steps. Each step in time, assume one particle is emitted from a fixed square, called the source, at the origin. At each step, assume that two types of movement occur : first the wind blows the particle one square in the x-direction, and second turbulence may cause the particle to move one square up or down in the z-direction, or the particle may remain with the same ordinate. It is assumed that the movement of the particle in the z-direction is random with fixed transition probabilities.

Random numbers	Rules
1, 2	up
3, 4, 5, 6	none (remaining with the same ordinate)
7, 8	down

(Numbers 0 and 9 are ignored).

Table 2.1

If for each step, we draw a separate random number to indicate the vertical movement of each partical according to the rules in table 2.1, then the probability of moving up is equal to $\frac{1}{4}$, the probability of staying with the same ordinate is equal to $\frac{1}{2}$, and the probability of moving down is equal to $\frac{1}{4}$.

						$\frac{1}{1024}$	
					$\frac{1}{256}$	$\frac{10}{1024}$	
				$\frac{1}{64}$	$\frac{8}{256}$	$\frac{45}{1024}$	
			$\frac{1}{16}$	$\frac{6}{64}$	$\frac{28}{256}$	$\frac{120}{1024}$	
		$\frac{1}{4}$	$\frac{4}{16}$	$\frac{15}{64}$	$\frac{56}{256}$	$\frac{210}{1024}$	
Source	1	$\frac{2}{4}$	$\frac{6}{16}$	$\frac{20}{64}$	$\frac{70}{256}$	$\frac{252}{1024}$	
		$\frac{1}{4}$	$\frac{4}{16}$	$\frac{15}{64}$	$\frac{56}{256}$	$\frac{210}{1024}$	
			$\frac{1}{16}$	$\frac{6}{64}$	$\frac{28}{256}$	$\frac{120}{1024}$	
				$\frac{1}{64}$	$\frac{8}{256}$	$\frac{45}{1024}$	
					$\frac{1}{256}$	$\frac{10}{1024}$	
						$\frac{1}{1024}$	

Table 2.2

Hence for each square there is a number which gives the probability that it contains a particle.

Examples of the results obtained by following this procedure are shown in figures 2.1 (a),(b) and (c).

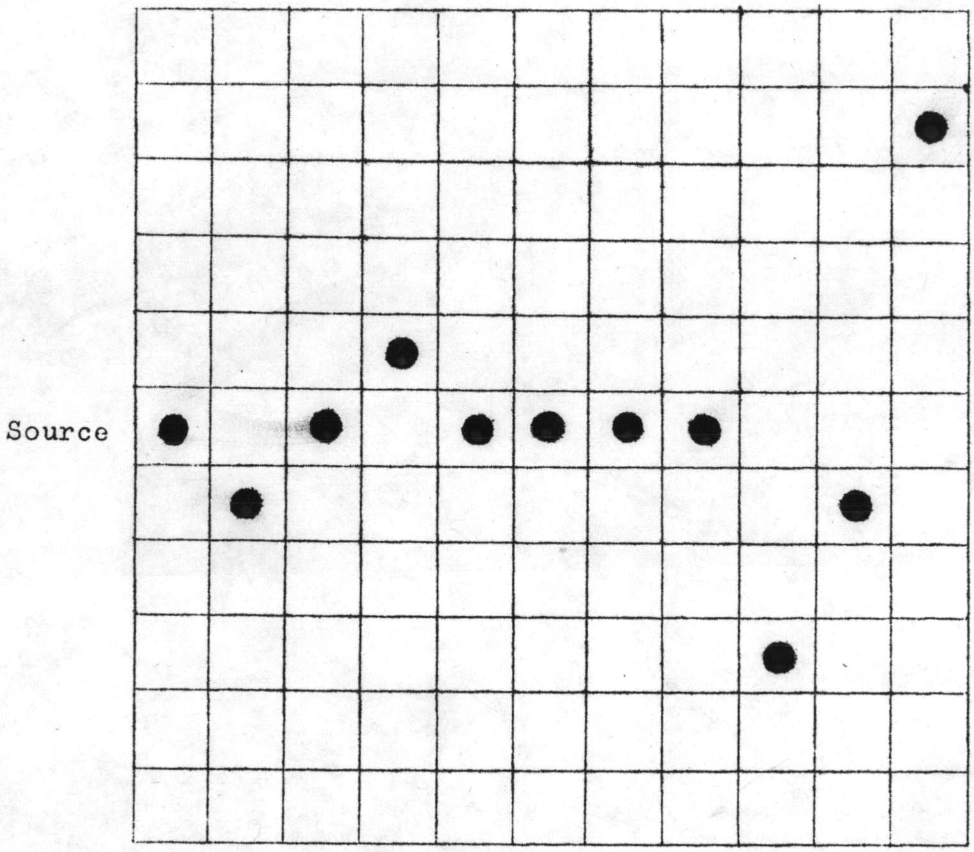

Figure 2.1 (a)

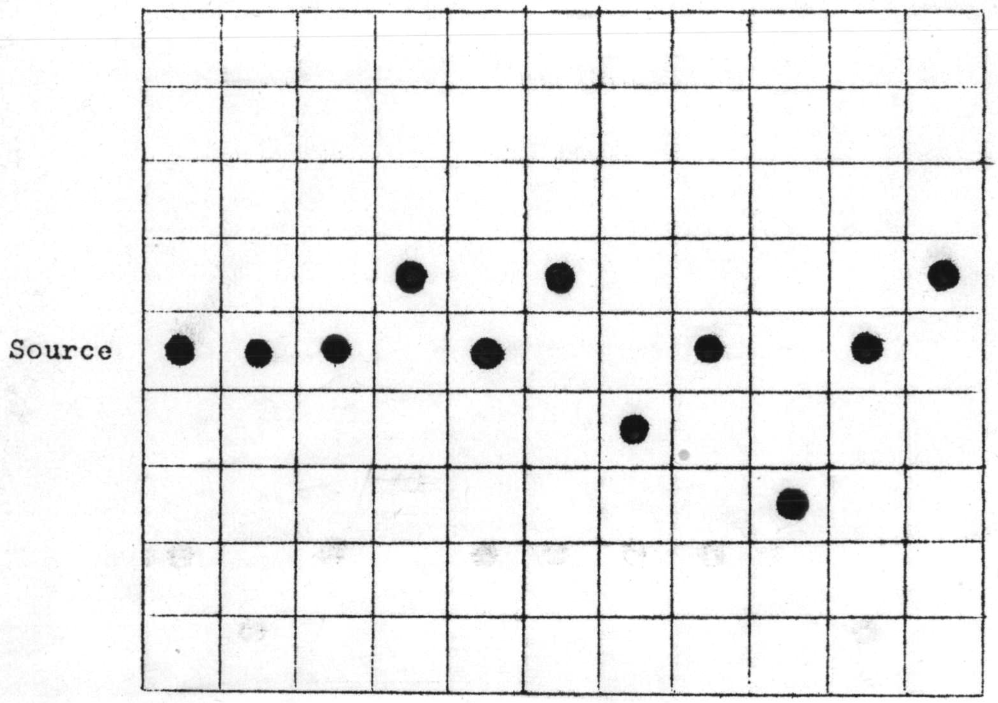

Figure 2.1 (b)

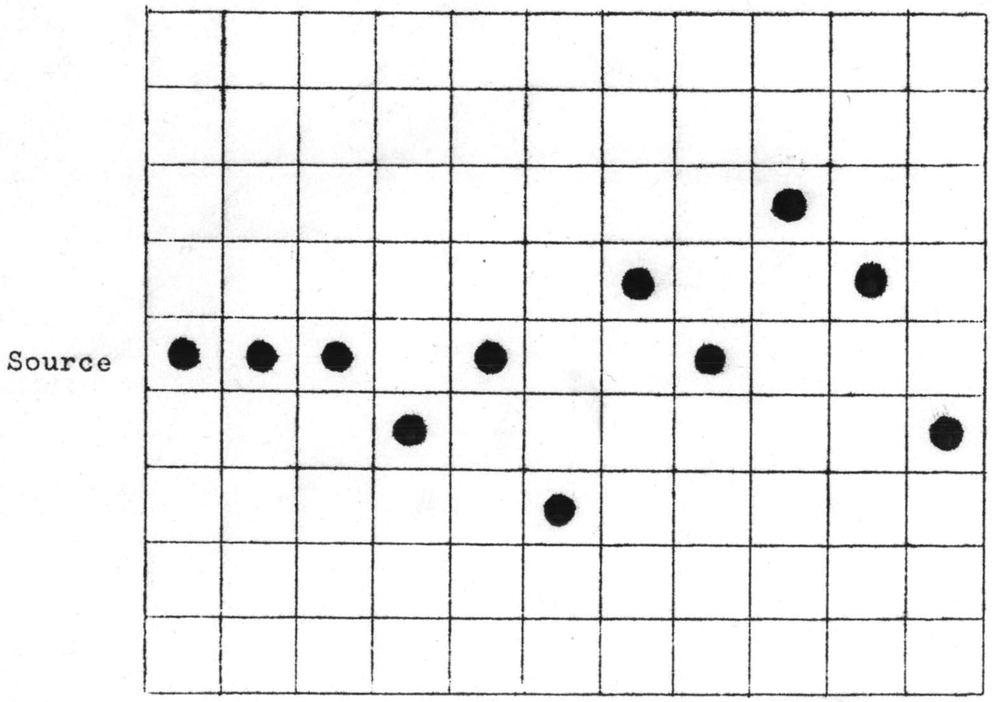

Figure 2.1 (c)

Particles may occur in squares whose coordinates are defined as shown in figure 2.2 with the probabilities in table 2.2

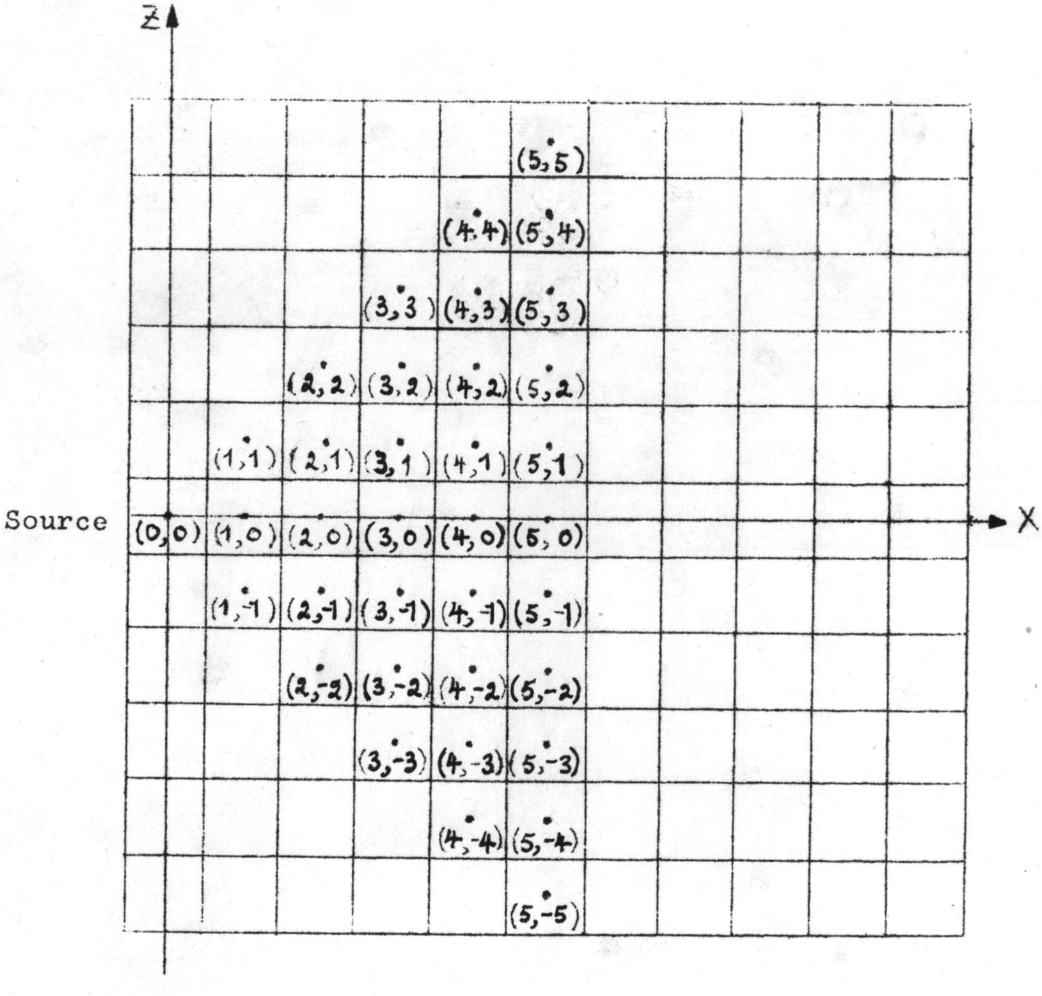

Figure 2.2

Following the example of Brown (1968), who discusses a similar model, we observe from table 2.2 that :

(i) The maximum values of the probability are on the **X**-axis; this resembles coning of smoke because the centre of the plume has the maximum concentration.

(ii) at equal distances (in z-direction) from the X-axis the probability values are the same, i.e. the plume is symmetrical about the **X**-axis.

(iii) at the point (x,z) the probability is ${}^{2x}C_r\ /2^{2x}$, where $r = x - z$.

(iv) the sum of the probabilities along a column is unity, as is obvious since it is the value of $\left(\frac{1}{4} + \frac{1}{2} + \frac{1}{4}\right)^x$, where $x = 0, 1, 2, \ldots$.

(v) the mean displacement of a particle from the X-axis is zero, i.e. $\bar{z} = 0$.

(vi) the mean-square displacement from the X-axis after x steps is $\frac{x}{2}$. For example, after 3 steps we have

$$\overline{z^2} = \frac{1}{64}\times 3^2 + \frac{6}{64}\times 2^2 + \frac{15}{64}\times 1^2 + \frac{20}{64}\times 0^2 + \frac{15}{64}\times(-1)^2 + \frac{6}{64}\times(-2)^2 + \frac{1}{64}\times(-3)^2.$$

$$= \frac{96}{64} = \frac{3}{2} = \frac{x}{2}.$$

This result is true for all x though a general proof requires the explicit expression for the probability $P_x(z)$ that after x steps the random particle is z steps from the X-axis, which from (iii) above is

$$P_x(z) = {}^{2x}C_r \left(\frac{1}{2}\right)^{2x} , \quad \text{where} \quad r = x - z ,$$

or

$$P_x(z) = \frac{(2x)!}{(x-z)!(x+z)!} \left(\frac{1}{2}\right)^{2x} . \quad \ldots\ldots\ldots\ldots(2.1)$$

We can show that the mean displacement $\bar{z}$ is equal to zero by using the formula :

$$\bar{z} = \sum_{z=-x}^{z=x} z \, P_x(z) .$$

From (2.1), we obtain

$$\bar{z} = \sum_{z=-x}^{z=x} \frac{z(2x)!}{(x-z)!(x+z)!} \left(\frac{1}{2}\right)^{2x}$$

$$= 0 ,$$

and we can show that the mean-square displacement $\overline{z^2}$ is equal to $\frac{x}{2}$ for x = 0, 1, 2,..., by mathematical induction.

If $x = 0$, then $$\overline{z^2} = \sum_{z=-x}^{z=x} z^2 P_x(z)$$

$$= \sum_{z=-x}^{z=x} \frac{z^2(2x)!}{(x-z)!(x+z)!} \left(\frac{1}{2}\right)^{2x}$$

$$= 2\sum_{z=1}^{z=x} \frac{z^2(2x)!}{(x-z)!(x+z)!} \left(\frac{1}{2}\right)^{2x}$$

$$= 0 = \frac{x}{2} .$$

If $x = 1$, then $$\overline{z^2} = 2\,\frac{2!}{0!2!}\left(\frac{1}{2}\right)^2 = \frac{1}{2} = \frac{x}{2} .$$

If $x = 2$, then $$\overline{z^2} = 2\left[\frac{4!}{1!3!}\left(\frac{1}{2}\right)^4 + \frac{2^2\ 4!}{0!4!}\left(\frac{1}{2}\right)^4\right]$$

$$= 2\left[\frac{1}{4} + \frac{1}{4}\right] = 1 = \frac{x}{2} .$$

Assume that it is true for $x = k$,

i.e. $$\overline{z^2} = 2\sum_{z=1}^{z=x} \frac{z^2(2k)!}{(k-z)!(k+z)!}\left(\frac{1}{2}\right)^{2k} = \frac{k}{2} ,$$

or $$2\left[\sum_{z=1}^{z=k-1} \frac{z^2(2k)!}{(k-z)!(k+z)!}\left(\frac{1}{2}\right)^{2k} + k^2\left(\frac{1}{2}\right)^{2k}\right] = \frac{k}{2} .$$

Hence $$\sum_{z=1}^{z=k-1} \frac{z^2(2k)!}{(k-z)!(k+z)!}\left(\frac{1}{2}\right)^{2k} = \frac{k}{4} - k^2\left(\frac{1}{2}\right)^{2k} \quad \ldots\ldots\ldots \ (2.2)$$

We shall prove that it is also true for $x = k+1$, that is we have to prove that

$$\overline{z^2} = 2\sum_{z=1}^{z=k+1} \frac{z^2(2k+2)!}{(k-z+1)!(k+z+1)!}\left(\frac{1}{2}\right)^{2k+2} = \frac{k+1}{2}.$$

Proof : $$\overline{z^2} = 2\left[\sum_{z=1}^{z=k} \frac{z^2(2k+2)!}{(k-z+1)!(k+z+1)!}\left(\frac{1}{2}\right)^{2k+2} + \frac{(k+1)^2(2k+2)!}{0!(2k+2)!}\left(\frac{1}{2}\right)^{2k+2}\right]$$

$$= 2\left[\sum_{z=1}^{z=k} \frac{z^2(2k+2)!}{(k-z+1)!(k+z+1)!}\left(\frac{1}{2}\right)^{2k+2} + (k+1)^2\left(\frac{1}{2}\right)^{2k+2}\right]$$

Let $k+1 = h$, i.e. $k = h-1$; then we get

$$\overline{z^2} = 2\left[\sum_{z=1}^{z=h-1} \frac{z^2(2h)!}{(h-z)!(h+z)!}\left(\frac{1}{2}\right)^{2h} + h^2\left(\frac{1}{2}\right)^{2h}\right]$$

$$= 2\left[\frac{h}{4} - h^2\left(\frac{1}{2}\right)^{2h} + h^2\left(\frac{1}{2}\right)^{2h}\right], \quad \text{from (2.2)}$$

$$= \frac{h}{2}$$

$$= \frac{k+1}{2}.$$

Therefore, $\overline{z^2} = \frac{x}{2}$, for $x = 0, 1, 2, \ldots.$

We shall find a simple form of (2.1) which is useful when x is large because it is difficult to compute the value of factorials of large numbers. The method of Feller is used. (Ref.7, Volume 1, Chapter 7).

Let x tend to infinity, so that 2x also tends to infinity. As the number of steps x increases, we suppose that also the numbers (x-z) and (x+z) will increase, so that

$$2x \longrightarrow \infty \;, \quad x - z \longrightarrow \infty \;, \quad x + z \longrightarrow \infty \quad \text{........(2.3)}$$

Then we express the factorials in (2.1) by means of Stirling's formula

$$n! \simeq (2\pi n)^{1/2}\, n^n\, e^{-n} \;, \quad \text{as} \quad n \longrightarrow \infty \,.$$

From (2.1), we get

$$P_x(z) \simeq \left\{\frac{x}{\pi(x-z)(x+z)}\right\}^{1/2} \left(\frac{x}{x-z}\right)^{x-z} \left(\frac{x}{x+z}\right)^{x+z} \quad \text{............(2.4)}$$

The last two factors on the right are equal to unity for $z = 0$, and their product decreases as $|z|$ increases. Therefore, it is natural to rewrite (2.4) as follows

$$P_x(z) \simeq \left\{\frac{x}{\pi(x-z)(x+z)}\right\}^{1/2} \frac{1}{\left(1-\frac{z}{x}\right)^{x-z} \left(1+\frac{z}{x}\right)^{x+z}} \quad \text{............(2.5)}$$

To evaluate the last fraction we use logarithms. In the interval $|z| \leqslant \frac{x}{2}$ we may use Taylor's expansion and find for the logarithm of the denominator

$$(x-z)\ln(1-\tfrac{z}{x})+(x+z)\ln(1+\tfrac{z}{x})$$

$$= (x-z)\left(-\frac{z}{x} - \frac{z^2}{2x^2} - \frac{z^3}{3x^3} - \dots\right) + (x+z)\left(\frac{z}{x} - \frac{z^2}{2x^2} + \frac{z^3}{3x^3} - \dots\right) \quad (2.6)$$

Reordering the terms according to powers of z, we get

$$\frac{z^2}{x} + \frac{z^4}{6x^3} + \;\ldots\ldots\ldots , \qquad \ldots\ldots\ldots\ldots\ldots(2.7)$$

Suppose that z increases with x in such a manner that

$$\frac{z^4}{x^3} \longrightarrow 0 \qquad \ldots\ldots\ldots\ldots\ldots(2.8)$$

(In this case also $\frac{z}{x} \longrightarrow 0$ so that (2.3) holds and the expansion (2.6) is justified.)

From (2.8) the terms within braces in (2.5) become $(\pi x)^{-1/2}$. The logarithm of the denominator in (2.5) is given by (2.7), but in view of (2.8) all terms except the first one may be neglected; the first term equals z^2/x . Combining these results, we have

$$P_x(z) \approx \frac{1}{\sqrt{\pi x}} \, e^{-z^2/x} \quad , \qquad \ldots\ldots\ldots\ldots\ldots(2.9)$$

which is known as a Gaussian distribution.

The mean displacement $\bar{z}$ is

$$\bar{z} = \int_{-\infty}^{\infty} z \, P(z) \, dz$$

$$= \frac{1}{\sqrt{\pi x}} \int_{-\infty}^{\infty} z \, e^{-z^2/x} \, dz$$

$$= 0 \; .$$

The mean-square displacement is

$$\overline{z^2} = \int_{-\infty}^{\infty} z^2 P(z)\,dz$$

$$= \frac{1}{\sqrt{\pi x}} \int_{-\infty}^{\infty} z^2 e^{-z^2/x} \, dz$$

$$= \frac{2}{\sqrt{\pi x}} \int_{0}^{\infty} z^2 e^{-z^2/x} \, dz$$

$$= \frac{2}{\sqrt{\pi x}} \times \frac{\Gamma(3/2)}{2(1/x)^{3/2}} {}^{*}$$

$$= \frac{1/2\,\Gamma(1/2)}{\sqrt{\pi}} \times x$$

$$= \frac{x}{2}$$

Using the same model for the XY-plane, where y denotes the horizontal displacement, we obtain the probability $P_x(y)$ that after x steps the particle is y steps from the X-axis to be

$$P_x(y) \simeq \frac{1}{\sqrt{\pi x}} e^{-y^2/x} \quad ,$$

which is similar to (2.9).

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* See appendix B.

Therefore, the product $P_x(y)\ P_x(z)$ of the independent probabilities is the concentration of particles at the point (x,y,z) in three dimensions. Let it be denoted by $\chi(x,y,z)$.

Hence $$\chi(x,y,z) = \frac{1}{\pi x}\, e^{-(y^2+z^2)/x} \qquad \text{.........(2.10)}$$

This solution is similar to the solution obtain from an equation in the book named " Micrometeorology " by O.G.Sutton; McGraw-Hill, 1953. Sutton derives this solution by making two steps: first the concentration for an instantaneous point source is calculated. Then, from this result, the concentration for a continuous point source is found. The following treatment is based on that of Sutton (pages 134 to 137).

Let a quantity of matter Q gm. be generated at t = 0 and allowed to diffuse. The differential equation is

$$\frac{\partial \chi}{\partial t} = \kappa \nabla^2 \chi = \frac{K}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \chi}{\partial r}\right), \qquad \text{...........(2.11)}$$

for spherical symmetry.

where χ, the concentration, is the density of suspended matter (gm.cm^{-3}), $r^2 = x^2 + y^2 + z^2$, with the origin at the point of generation and K is the diffusivity.

The conditions are $\chi \longrightarrow 0$ as $t \longrightarrow 0$, $r > 0$

$$\chi \longrightarrow 0 \text{ as } t \longrightarrow \infty ,$$

together with the continuity condition,

$$\iiint_{-\infty}^{\infty} \chi \, dx \, dy \, dz = Q$$

which expresses the fact that matter is neither created nor destroyed during the diffusion process.

The solution quoted by Sutton may be obtained as follows. From (2.11),

$$\frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial \chi}{\partial r} \right) = \frac{1}{K} \frac{\partial \chi}{\partial t} .$$

Take the Laplace transform of both sides with respect to t :

$$\frac{1}{r^2} \frac{d}{dr}\left(r^2 \frac{d\overline{\chi}}{dr} \right) = \frac{1}{K} \left[s\overline{\chi}(r,s) - \chi(r,0) \right] ,$$

where, by the initial condition, $\chi(r,0) = 0$.

Then $$\frac{1}{r^2} \frac{d}{dr}\left(r^2 \frac{d\overline{\chi}}{dr} \right) = \frac{s}{K} \overline{\chi} .$$

Hence $$\overline{\chi}'' + \frac{2}{r} \overline{\chi}' - \frac{s}{K} \overline{\chi} = 0 ,$$ which is an ordinary differential equation of the second order in $\overline{\chi}$.

We can solve this by using the normal form given below.

If in the differential equation

$$y'' + P(x)y' + Q(x)y = 0 ,$$

we let $$y = v \exp\left\{-\frac{1}{2}\int P dx\right\},$$

we obtain the normal form,

$$v'' + Iv = 0,$$

where $$I = Q - \frac{1}{2}P' - \frac{1}{4}P^2 .$$

Therefore, let $$\bar{\chi} = v \exp\left\{-\frac{1}{2}\int P dr\right\},$$

where $P = \frac{2}{r}$ and $Q = -\frac{s}{K}$.

Then $$I = -\frac{s}{K} - \frac{1}{2}\left(-\frac{2}{r^2}\right) - \frac{1}{4}\left(\frac{4}{r^2}\right) = -\frac{s}{K} .$$

We obtain for our equation the normal form

$$v'' - \frac{s}{K} v = 0,$$

which has the solution

$$v = c_1 e^{\sqrt{s/K}\, r} + c_2 e^{-\sqrt{s/K}\, r},$$

where c_1, c_2 are constants.

To determine the solution of the original equation for $\bar{\chi}$ we calculate

$$e^{-\frac{1}{2}\int P dr} = e^{-\frac{1}{2}\cdot 2\int \frac{dr}{r}} = e^{-\ln r} = \frac{1}{r},$$

and therefore obtain $$\bar{\chi}(r,s) = \frac{1}{r}\left[c_1 e^{\sqrt{s/K}\, r} + c_2 e^{-\sqrt{s/K}\, r}\right] .$$

Since $\chi(r,t)$ is a bounded function of r for $t > 0$, by the boundary condition as $t \rightarrow \infty$, $\bar{\chi}(r,s)$ is also a bounded function.

i.e. $\lim_{r\rightarrow\infty} \bar{\chi}(r,s) \leqslant M$, where M = constant.

This implies that $c_1 = 0$.

Hence $$\bar{\chi}(r,s) = c_2 \frac{e^{-\sqrt{s/K}\,r}}{r},$$

and $$\chi(r,t) = \frac{c_2}{r}\mathcal{L}^{-1}\left\{e^{-\sqrt{s/K}\,r}\right\}$$
$$= \frac{c_2}{r} \times \frac{r}{2\sqrt{K}}\,\frac{e^{-r^2/4Kt}}{\sqrt{\pi t^3}}\ ^{*},$$
$$= \frac{c_2\, e^{-r^2/4Kt}}{2\sqrt{\pi K t^3}}$$

The value of c_2 is found by using the condition

$$\iiint_{-\infty}^{\infty} \chi \, dx\, dy\, dz = Q,$$

i.e. $$\int_0^{\infty} \chi(r,t)\,d\underline{r} = \int_0^{\infty} \chi\, 4\pi r^2 dr = Q.$$

Substituting for χ, we get

$$\frac{c_2 4\pi}{2\sqrt{\pi K t^3}}\int_0^{\infty} r^2 e^{-r^2/4Kt}\, dr = Q$$

or $$\frac{c_2\ 2\sqrt{\pi}}{\sqrt{Kt^3}}\ \frac{\Gamma(3/2)}{2\left(\frac{1}{4Kt}\right)^{3/2}}\ ^{**} = Q$$

Therefore $$c_2 = \frac{Q}{4\pi K}$$

Finally we obtain $$\chi(r,t) = \frac{Q}{8(\pi K t)^{3/2}} \exp\left(-\frac{r^2}{4Kt}\right)$$

or $$\chi(x,y,z,t) = \frac{Q}{8(\pi K t)^{3/2}} \exp\left(-\frac{x^2+y^2+z^2}{4Kt}\right) \quad (2.12)$$

If the origin is at the point (x', y', z'), the solution is

$$\chi(x,y,z,t) = \frac{Q}{8(\pi K t)^{3/2}} \exp\left[-\frac{(x-x')^2+(y-y')^2+(z-z')^2}{4Kt}\right].$$

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* See appendix C.   ** See appendix B

This is the solution for an instantaneous point source, if the medium is at rest. The solution for a source emitting continuously from t = 0 to t = t at the point (x',y',z') is obtained without difficulty by integrating the expression for an instantaneous point source with respect to time.

Hence, writing $r^2 = (x-x')^2 + (y-y')^2 + (z-z')^2$,

$$\chi(x,y,z,t) = \frac{Q}{8(\pi K)^{3/2}} \int_0^t \exp\left[-\frac{r^2}{4K(t-t')}\right]\frac{dt'}{(t-t')^{3/2}}$$

Let $y = \frac{r}{\sqrt{4K}(t-t')^{1/2}}$, $dy = \frac{1}{2}\frac{r\,dt'}{\sqrt{4K}(t-t')^{3/2}}$

With conditions : $y = r/\sqrt{4Kt}$ as $t' = 0$, and

$y = \infty$ as $t' = t$,

we have $\chi(x,y,z,t) = \frac{Q}{8(\pi K)^{3/2}} \cdot \frac{2\sqrt{4\pi K}}{2r}\left[\frac{2}{\sqrt{\pi}}\int_{r/\sqrt{4Kt}}^{\infty} e^{-y^2}\,dy\right]$

$$= \frac{Q}{4\pi K r}\ \mathrm{erfc}\ \frac{r}{\sqrt{4Kt}}$$

As $t \longrightarrow \infty$, this reduces to

$$\chi(x,y,z) = \frac{Q}{4\pi K r}\ \mathrm{erfc}\ (0) = \frac{Q}{4\pi K r} .$$

This corresponds to a source which is maintained indefinitely.

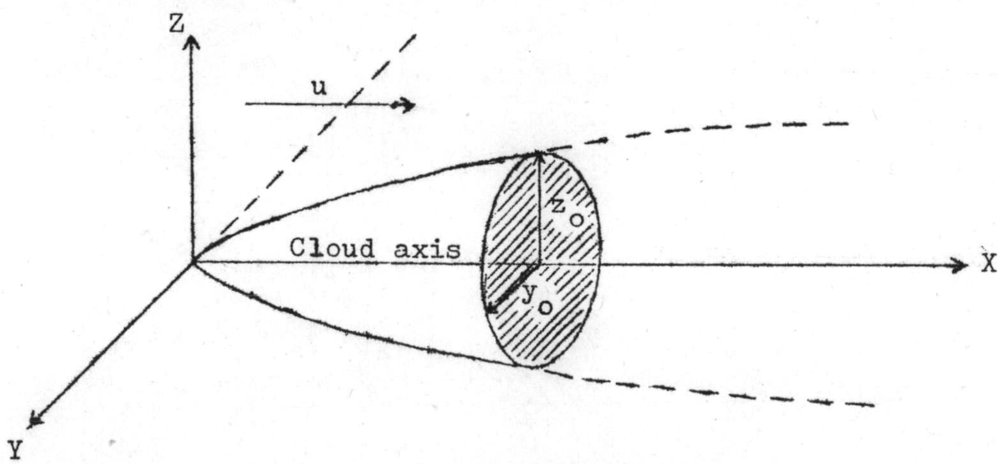

Figure 2.4

In the analysis which follows it is supposed that the wind velocity u is constant at all points. The solution for the continuous point source in a moving medium can be obtained from that for the instantaneous point source if the system of axes be fixed in space instead of moving downwind with the puff. The axis of x is chosen as the direction of the wind, the source of Q $gm.sec^{-1}$ being placed at the origin, as shown in figure 2.4. In the fixed-axes system, the coordinates (x,y,z) of the moving system are to be replaced by (x-ut,y,z). The continuous point source is equivalent to a succession of elementary instantaneous point sources, the concentration at any point being due to the integrated effect of the elementary puffs. At time t' the source emits Qdt' gm of matter, but, because of the wind, the element of air which is at (x,y,z,t) has come from $\left[x-u(t-t'),y,z,t'\right]$. The concentration at (x,y,z,t) due to an instantaneous puff of content Qdt' emitted at time t' is thus, by (2.12).

$$d\chi = \frac{Qdt'}{8\left[\pi K(t-t')\right]^{3/2}} \, e^{-\left[\left\{x-u(t-t')\right\}^2 + y^2 + z^2\right] / 4K(t-t')} \qquad (2.13)$$

The total concentration at (x,y,z,t) in the continuous-point-source cloud is the sum of all such contributions, i.e.equals the integral of (2.13) with respect to t' from t' = 0 to t' = t.

If the source is supposed to be maintained indefinitely, the range of integration is from t'= 0 to t' = ∞ ; in practice this is by far the most important case.

Then the equation (2.13) becomes

$$\chi(x,y,z) = \frac{Q}{8(\pi K)^{3/2}} \int_0^t \frac{e^{-\left[\{x-u(t-t')\}^2 + y^2 + z^2\right]/4K(t-t')}}{(t-t')^{3/2}}\,dt'.$$

The reduction of this formula by evaluating the integral is as follows

$$\chi(x,y,z) = \frac{Q}{8(\pi K)^{3/2}} \int_0^t \frac{e^{-(x^2+y^2+z^2)/4K(t-t') + ux/2K - u^2(t-t')/4K}}{(t-t')^{3/2}}\,dt'$$

$$= \frac{Q\,e^{ux/2K}}{8(\pi K)^{3/2}} \int_0^t \frac{e^{-(x^2+y^2+z^2)/4K(t-t') - u^2(t-t')/4K}}{(t-t')^{3/2}}\,dt',$$

now let $t-t' = v$, $-dt' = dv$,

and $v = t$ as $t' = 0$,

$v = 0$ as $t' = t$.

Thus $$\chi(x,y,z) = \frac{Q\,e^{ux/2K}}{8(\pi K)^{3/2}} \int_0^t \frac{e^{-r^2/4Kv - u^2 v/4K}}{v^{3/2}}\,dv,$$

where $r^2 = x^2 + y^2 + z^2$.

Put $\frac{r^2}{4Kv} = \tau^2$, $v = \frac{r^2}{4K\tau^2}$, $v^{3/2} = \frac{r^3}{8K^{3/2}\tau^3}$,

$-\frac{r^2 dv}{4Kv^2} = 2\tau\,d\tau$, $dv = -\frac{r^2}{2K\tau^3}\,d\tau$.

With limits : $\tau = \infty$ as $v = 0$,

and $\tau = \frac{r}{2\sqrt{Kt}}$ as $v = t$.

Hence $$\chi(x,y,z) = \frac{Q\,e^{ux/2K}}{2\pi^{3/2}Kr} \int_{r/2\sqrt{Kt}}^{\infty} e^{-\tau^2 - u^2 r^2/16K^2\tau^2}\,d\tau$$

$$= \frac{Q\,e^{ux/2K}}{2\pi^{3/2}Kr} \int_0^{\infty} e^{-\tau^2 - u^2 r^2/16K^2\tau^2}\,d\tau,$$

as $t \longrightarrow \infty$,

$$= \frac{Q\, e^{ux/2K}}{2\pi^{3/2} K r} \left(\frac{\sqrt{\pi}}{2}\, e^{-ur/2K} \right)$$

$$= \frac{Q}{4\pi K r}\, e^{-u(r-x)/2K} \quad .$$

Observations of smoke clouds show that, unless the wind is very light, the cloud takes the form of a long thin plume, and interest is centered on concentrations at points not too far removed from the axes of the cloud ($y = z = 0$). In most practical applications $(y^2 + z^2)/\, x^2$ may be regarded as a quantity whose square is negligibly small. In this case,

$$- \frac{u(r-x)}{2K} = - \frac{u}{2K} \left\{ x \left(1 + \frac{y^2 + z^2}{x^2} \right)^{1/2} - x \right\}$$

$$\simeq - \frac{u}{4K} \frac{(y^2 + z^2)}{x}$$

Hence, for all but the lightest winds,

$$\chi(x,y,z) = \frac{Q}{4\pi K r}\, e^{-u(y^2 + z^2)/4Kx} \quad .$$

In practice, this expression is frequently replaced by

$$\chi(x,y,z) = \frac{Q}{4\pi K x}\, e^{-u(y^2 + z^2)/4Kx} \qquad \text{........ (2.14)}$$

without serious error.

Therefore the equation (2.14) will be the same as the equation (2.10) if the values of Q, u and 4K are equal to unity.