CHAPTER III

EXPLICIT DEFINITIONS

In this chapter we study about explicit definitions, the criterion of eliminability, and the criterion of non-creativity.

Some of the material in this chapter is drawn from [4].

3.1 Definition. Let L and L' be two first-order languages such that $L' = L \cup \{ P \}$ where P is a new n-placed relation symbol, and let T be a theory in L. An explicit definition in T is a sentence of the form $(\forall v_1) \ldots (\forall v_n) (P(v_1 \ldots v_n) \longleftrightarrow S)$ where $v_1, \ldots, v_n$ are distinct variables and S is a formula in L such that S has no free variables other than $v_1, \ldots, v_n$.

3.2 The Criterion of Eliminability. Let L and L' be two first-order languages such that $L \subset L'$, and let T be a theory in L. A sentence S in L' satisfies the criterion of eliminability with respect to L if and only if : whenever S_1 is a sentence in L' but not a sentence in L, then there is a sentence S_2 in L such that $T \vdash S \rightarrow (S_1 \longleftrightarrow S_2)$.

We want to show that explicit definitions satisfy the criterion of eliminability.

3.3 Lemma. For any formulas ϕ and ψ,

$$\vdash (\forall v) (\phi \longleftrightarrow \psi) \longrightarrow ((\forall v)\phi \longleftrightarrow (\forall v) \psi).$$

proof. In order to prove $\vdash (\forall v)\ (\phi \longleftrightarrow \psi) \longrightarrow ((\forall v)\ \phi \longleftrightarrow (\forall v)\ \psi)$, we first prove $(\forall v)\ (\phi \longleftrightarrow \psi),\ (\forall v)\phi \vdash (\forall v)\ \psi$.

1.	$(\forall v)\ (\phi \longleftrightarrow \psi)$	hypothesis.
2.	$\phi \longleftrightarrow \psi$	logical axiom (v), 1 by MP.
3.	$(\forall v)\phi$	hypothesis.
4.	ϕ	logical axiom (v), 3 by MP.
5.	ψ	$(\phi \longleftrightarrow \psi) \longrightarrow (\phi \longrightarrow \psi)$, 4 by MP.
6.	$(\forall v)\ \psi$	generalization
7.	$(\forall v)\ (\phi \longleftrightarrow \psi),\ (\forall v)\phi \vdash (\forall v)\psi$	1, 3, 6.

Similarly, we have $(\forall v)\ (\phi \longleftrightarrow \psi),\ (\forall v)\psi \vdash (\forall v)\ \phi$. Since v is not free in $(\forall v)\ \phi$ and $(\forall v)\ \psi$, we get $(\forall v)\ (\phi \longleftrightarrow \psi) \vdash (\forall v)\phi \longleftrightarrow (\forall v)\ \psi$. And since v is not free in $(\forall v)\ (\phi \longleftrightarrow \psi)$, we get $\vdash (\forall v)\ (\phi \longleftrightarrow \psi) \longrightarrow ((\forall v)\ \phi \longleftrightarrow (\forall v)\ \psi)$.

3.4 Lemma. Let S, ψ be formulas and ϕ be a subformula of S. Then $(\forall v_1) \ldots (\forall v_n)\ (\phi \longleftrightarrow \psi) \vdash S \longleftrightarrow S\ [{}^{\phi}_{\psi}]$ where $S\ [{}^{\phi}_{\psi}]$ is a formula obtained from S that replaces every occurrence of subformula ϕ by ψ.

proof. We will prove this lemma by induction on the length of the formula S.

Suppose S is an atomic formula.

case 1 : S is $t_1 = t_2$ where t_1, t_2 are terms. Since ϕ is a subformula of S, we see that S is ϕ and $S\ [{}^{\phi}_{\psi}]$ is ψ. And Since $\vdash [(\forall v_1) \ldots (\forall v_n)\ (\phi \longleftrightarrow \psi)] \longrightarrow (\phi \longleftrightarrow \psi)$, we get $(\forall v_1) \ldots (\forall v_n)\ (\phi \longleftrightarrow \psi) \vdash S \longleftrightarrow S[{}^{\phi}_{\psi}]$.

case 2 : S is $P(t_1 \ldots t_n)$ where P is an n-placed relation symbol and $t_1, \ldots, t_n$ are terms. Then S is ϕ and S $[{}^{\phi}_{\psi}]$ is ψ. As in case 1, $(\forall v_1) \ldots (\forall v_n)(\phi \longleftrightarrow \psi) \vdash S \longleftrightarrow S[{}^{\phi}_{\psi}]$.

Assume this lemma is true for all formulas S' whose lengths < length of S.

Suppose S is $\sim S'$.

case 1 : If ϕ is a subformula of S', then by induction hypothesis, $(\forall v_1) \ldots (\forall v_n)(\phi \longleftrightarrow \psi) \vdash S' \longleftrightarrow S'[{}^{\phi}_{\psi}]$. Therefore $(\forall v_1) \ldots (\forall v_n)(\phi \longleftrightarrow \psi) \vdash \sim S' \longleftrightarrow \sim S'[{}^{\phi}_{\psi}]$.

case 2 : If ϕ is not a subformula of S', then S is ϕ and S $[{}^{\phi}_{\psi}]$ is ψ. Therefore $(\forall v_1) \ldots (\forall v_n)(\phi \longleftrightarrow \psi) \vdash S \longleftrightarrow S[{}^{\phi}_{\psi}]$.

Suppose S is $S' \wedge S''$.

case 1 : If ϕ is a subformula of S' and S'', then by induction hypothesis, $(\forall v_1) \ldots (\forall v_n)(\phi \longleftrightarrow \psi) \vdash S' \longleftrightarrow S'[{}^{\phi}_{\psi}]$ and $(\forall v_1) \ldots (\forall v_n)(\phi \longleftrightarrow \psi) \vdash S'' \longleftrightarrow S''[{}^{\phi}_{\psi}]$. Therefore $(\forall v_1) \ldots (\forall v_n)(\phi \longleftrightarrow \psi) \vdash S' \wedge S'' \longleftrightarrow S'[{}^{\phi}_{\psi}] \wedge S''[{}^{\phi}_{\psi}]$. Since $S'[{}^{\phi}_{\psi}] \wedge S''[{}^{\phi}_{\psi}]$ is $S' \wedge S''[{}^{\phi}_{\psi}]$, we get $(\forall v_1) \ldots (\forall v_n)(\phi \longleftrightarrow \psi) \vdash S' \wedge S'' \longleftrightarrow S' \wedge S''[{}^{\phi}_{\psi}]$.

case 2 : If ϕ is a subformula of S' but not a subformula of S'', then $(\forall v_1) \ldots (\forall v_n)(\phi \longleftrightarrow \psi) \vdash S' \longleftrightarrow S'[{}^{\phi}_{\psi}]$. Since S'' has no subformula ϕ, it follows that $(\forall v_1) \ldots (\forall v_n)(\phi \longleftrightarrow \psi) \vdash S'' \longleftrightarrow S''[{}^{\phi}_{\psi}]$. Therefore $(\forall v_1) \ldots (\forall v_n)(\phi \longleftrightarrow \psi) \vdash S' \wedge S'' \longleftrightarrow S' \wedge S''[{}^{\phi}_{\psi}]$.

case 3 : If ϕ is a subformula of S'' but not a subformula of S', then similarly to case 2, we get $(\forall v_1)\dots(\forall v_n)\ (\phi \longleftrightarrow \psi) \vdash S' \wedge S'' \longleftrightarrow S' \wedge S''[{}^{\phi}_{\psi}]$.

case 4 : If ϕ is not a subformula of either S' or S'', then ϕ is S and $S[{}^{\phi}_{\psi}]$ is ψ. Therefore $(\forall v_1)\dots(\forall v_n)\ (\phi \longleftrightarrow \psi) \vdash S \longleftrightarrow S[{}^{\phi}_{\psi}]$.

Suppose S is $(\forall v)\ S'$.

case 1 : If ϕ is not a subformula of S', then S is ϕ and $S[{}^{\phi}_{\psi}]$ is ψ. Therefore $(\forall v_1)\dots(\forall v_n)\ (\phi \longleftrightarrow \psi) \vdash S \longleftrightarrow S[{}^{\phi}_{\psi}]$.

case 2 : If ϕ is a subformula of S', then by induction hypothesis, $(\forall v_1)\dots(\forall v_n)\ (\phi \longleftrightarrow \psi) \vdash S' \longleftrightarrow S'[{}^{\phi}_{\psi}]$. Thus $(\forall v_1)\dots(\forall v_n)$ $(\phi \longleftrightarrow \psi) \vdash (\forall v)\ (S' \longleftrightarrow S')[{}^{\phi}_{\psi}])$, and by Lemma 3.3, we get $(\forall v_1)\dots(\forall v_n)$ $(\phi \longleftrightarrow \psi) \vdash (\forall v)\ S' \longleftrightarrow (\forall v)\ S'[{}^{\phi}_{\psi}]$.

Hence, this lemma is true for all formulas S.

3.5 Corollary. If S is a sentence, ϕ a subformula of S, and ψ another formula, then $(\forall v_1)\dots(\forall v_n)\ (\phi \longleftrightarrow \psi) \vdash S \longleftrightarrow S[{}^{\phi}_{\psi}]$, where $S[{}^{\phi}_{\psi}]$ is a sentence obtained from S that replaces every occurrence of subformula ϕ by ψ.

proof. Since S is a sentence, we see that S is a formula, and by Lemma 3.4.

3.6 Theorem. Explicit definitions satisfy the criterion of eliminability.

proof. Let L and L' be two first-order languages such that L' =

$L \cup \{P\}$ where P is a new n-placed relation symbol and T be a theory in L. Let $(\forall v_1)\ldots(\forall v_n)\ (P(v_1\ldots v_n) \longleftrightarrow S)$, where S is a formula in L, be an explicit defintion in T.

Let S_1 be any sentence in L' and S_1 is not a sentence in L. Want to show that there is a sentence S_2 in L such that

$$T \vdash (\forall v_1)\ldots(\forall v_n)(P(v_1\ldots v_n) \longleftrightarrow S) \rightarrow (S_1 \longleftrightarrow S_2), \text{ or}$$
$$T \cup \{(\forall v_1)\ldots(\forall v_n)(P(v_1\ldots v_n) \longleftrightarrow S)\} \vdash S_1 \longleftrightarrow S_2.$$

Assume $T \cup \{(\forall v_1)\ldots(\forall v_n)\ (P(v_1\ldots v_n) \longleftrightarrow S\}$. Let S_2 be $S_1[\frac{P(v_1\ldots v_n)}{S}]$. Since S is a formula in L, we get S_2 is a sentence in L. Since $P(v_1\ldots v_n)$ is subformula of S_1, we get $(\forall v_1)\ldots(\forall v_n)$ $(P(v_1\ldots v_n) \longleftrightarrow S) \vdash S_1 \longleftrightarrow S_1[\frac{P(v_1\ldots v_n)}{S}]$. Since $S_1[\frac{P(v_1\ldots v_n)}{S}]$ is S_2, it follows that $(\forall v_1)\ldots(\forall v_n)\ (P(v_1\ldots v_n) \longleftrightarrow S) \vdash S_1 \longleftrightarrow S_2$. Hence $T \cup \{(\forall v_1)\ldots(\forall v_n)\ (P(v_1\ldots v_n) \longleftrightarrow S)\} \vdash S_1 \longleftrightarrow S_2$.

3.7 <u>Corollary</u>. Let L and L' be two first-order languages such that $L \subset L'$ and T be a theory in L. Let S_1 be a sentence in L' but not a sentence in L and S_2 be a sentence in L. If ϕ is a sentence in L' that satisfies the criterion of eliminability and $T \vdash \phi \rightarrow S_2$, then $T \vdash \phi \rightarrow S_1$.

<u>proof</u>. By Theorem 3.6.

3.8 <u>The Criterion of Non-Creativity</u>. Let L and L' be two first-order languages such that $L \subset L'$ and T be a theory in L. A sentence S in L' satisfies the criterion of non-creativity if and only if : for any formula t in L, if $T \vdash S \rightarrow t$ then $T \vdash t$.

3.9 <u>Remark</u>. If a sentence S in L' satisfies the criterion of non-creativity, we say that S is non-creative with respect to theory T.

We want to show that explicit definitions satisfy the criterion of non-creativity.

3.10 <u>Lemma</u>. Let ϕ and ψ be formulas and $P(v_1 \ldots v_n)$ a subformula of either ϕ or ψ and S is another formula such that $(\forall v_1) \ldots (\forall v_n)$ $(P(v_1 \ldots v_n) \longleftrightarrow S)$. Then

(i) $(\phi \longrightarrow \psi)\ [^{P(v_1 \ldots v_n)}_{S}]$ is $\phi[^{P(v_1 \ldots v_n)}_{S}] \longrightarrow \psi[^{P(v_1 \ldots v_n)}_{S}]$,

(ii) $((\forall v)\phi)\ [^{P(v_1 \ldots v_n)}_{S}]$ is $(\forall v)\phi\ [^{P(v_1 \ldots v_n)}_{S}]$,

where $(\phi \longrightarrow \psi)[^{P(v_1 \ldots v_n)}_{S}]$, $\phi[^{P(v_1 \ldots v_n)}_{S}]$ and $\psi[^{P(v_1 \ldots v_n)}_{S}]$ are formulas obtained from $\phi \longrightarrow \psi$, ϕ and ψ with all occurrences of $P(v_1 \ldots v_n)$ replaced by S.

<u>proof</u>. (i) To show $(\phi \longrightarrow \psi)\ [^{P(v_1 \ldots v_n)}_{S}]$ is $\phi[^{P(v_1 \ldots v_n)}_{S}] \longrightarrow \psi[^{P(v_1 \ldots v_n)}_{S}]$.

<u>case 1</u> : $P(v_1 \ldots v_n)$ is a subformula of ϕ but not a subformula of ψ. Therefore in formula $\phi \longrightarrow \psi$, we only substitute S for $P(v_1 \ldots v_n)$ in ϕ and let ψ be the same. Then we get $(\phi \longrightarrow \psi)[^{P(v_1 \ldots v_n)}_{S}]$ is $\phi[^{P(v_1 \ldots v_n)}_{S}] \longrightarrow \psi$.

<u>case 2</u> : $P(v_1 \ldots v_n)$ is a subformula of ϕ but not a subformula of ϕ. Similarly to case 1, we get $(\phi \longrightarrow \psi)\ [^{P(v_1 \ldots v_n)}_{S}]$ is $\phi \longrightarrow \psi[^{P(v_1 \ldots v_n)}_{S}]$.

case 3 : $P(v_1 \ldots v_n)$ is a subformula of both ϕ and ψ. Therefore in formula $\phi \rightarrow \psi$, substitute S for $P(v_1 \ldots v_n)$ in both ϕ and ψ. Then we get $(\phi \longrightarrow \psi) [{}^{P(v_1 \ldots v_n)}_{S}]$ is $\phi[{}^{P(v_1 \ldots v_n)}_{S}] \rightarrow \psi[{}^{P(v_1 \ldots v_n)}_{S}]$.

(ii) To show $((\forall v)\phi) [{}^{P(v_1 \ldots v_n)}_{S}]$ is $(\forall v)\ \phi[{}^{P(v_1 \ldots v_n)}_{S}]$. Since $P(v_1 \ldots v_n)$ is a subformula of ϕ, we have in formula $(\forall v)\phi$, substitute S for $P(v_1 \ldots v_n)$ in ϕ, and so we get $((\forall v)\phi)[{}^{P(v_1 \ldots v_n)}_{S}]$ is $(\forall v)\ \phi[{}^{P(v_1 \ldots v_n)}_{S}]$.

3.11 Theorem. Explicit definitions satisfy the criterion of non-creativity.

proof. Let L and L' be two first-order languages such that $L' = L \cup \{P\}$ where P is a new n-placed relation symbol and T be a theory in L. Let $(\forall v_1) \ldots (\forall v_n)\ (P(v_1 \ldots v_n) \longleftrightarrow S)$, where S is a formula in L, be an explicit definition in T.

To prove this theorem, we must prove that : for any formula t in L, if $T \vdash (\forall v_1) \ldots (\forall v_n)\ (P(v_1 \ldots v_n) \longleftrightarrow S) \rightarrow t$, then $T \vdash t$. In order to prove the above, we prove : if $T \cup \{(\forall v_1) \ldots (\forall v_n)\ (P(v_1 \ldots v_n) \longleftrightarrow S)\} \vdash t$, then $T \vdash t$.

Let t be any formula in L. Assume $T \cup \{(\forall v_1) \ldots (\forall v_n)\ (P(v_1 \ldots v_n) \longleftrightarrow S)\} \vdash t$. Therefore there exists a finite sequence of formulas $\phi_1, \ldots, \phi_n$ such that $\phi_n = t$ and for all i, $1 \leq i \leq n$, ϕ_i is a logical axiom, or $\phi_i \in T$, or $\phi_i = (\forall v_1) \ldots (\forall v_n)(P(v_1 \ldots v_n) \longleftrightarrow S)$, or ϕ_i is a conclusion from ϕ_j, $\phi_j \rightarrow \phi_i$ $(j < i)$ by MP., or ϕ_i is a conclusion from ϕ_j $(j < i)$ by generalization.

We want to show $T \vdash t$. Assume T. Construct a finite sequence of formulas $\phi_1', \ldots, \phi_n'$ as follows ; if ϕ_i has $P(v_1 \ldots v_n)$ as subformula, define ϕ_i' is $\phi_i \left[{}^{P(v_i \ldots v_n)}_{S}\right]$, otherwise define ϕ_i' is ϕ_i .

To show $\phi_1', \ldots, \phi_n'$ can be made into a proof of t from T. To prove this, for each ϕ_i, $1 \leq i \leq n$;

if ϕ_i is a logical axiom, then ϕ_i' is also a logical axiom.

if ϕ_i T, then $\phi_i' = \phi_i \varepsilon T$.

if $\phi_i = (\forall v_1) \ldots (\forall v_n) (P(v_1 \ldots v_n) \longleftrightarrow S)$, then $\phi_i' = (\forall v_1) \ldots (\forall v_n) (S \longleftrightarrow S)$. Since $\vdash S \longleftrightarrow S$, we get $\vdash (\forall v_1) \ldots (\forall v_n) (S \longleftrightarrow S)$.

if ϕ_i is a conclusion from ϕ_j, $\phi_j \rightarrow \phi_i$ $(j < i)$ by MP., then from ϕ_j' and since $(\phi_j \rightarrow \phi_i)'$ is $\phi_j' \rightarrow \phi_i'$, we get ϕ_i' by MP.

if ϕ_i is a conclusion from ϕ_j $(j < i)$ by generalization, then from Lemma 3.10, we get ϕ_i' which comes from ϕ_j' $(j < i)$ by generalization.

Therefore, we get a finite sequence of formulas $\phi_1', \ldots \phi_n'$ which is a proof of t from T.