CHAPTER VI

CONCLUSION

In the case of heavily doped semiconductors, the main effects are (i) the shift of band edge, and (ii) the band tailing. Stern[10] successfully determined the shift of band edge and Woff[11] showed that this effect arose from the exchange energy of carrier.

The band tailing effect arises from the random arrangement of impurity. For the tail states, the effect of randomness is to make the wave function localize. For describing the effect, the problem of electrical phenomena in disordered structures should be studied. The problem is a statistical one depending on configuration averaging. As shown in chapter I, the disorder effect is treated as a perturbation of ordered system.

In perturbation technique the potential fluctuations produced by impurities above some average values are treated as a perturbation to the periodic potential of the lattice. Therefore the statistical fluctuations of the impurity distribution is neglected. The density of states which is calculated with this technique shows a sharp cut-off in the tail states.

In Kane's method , the electron is required to respond to these fluctuations and the problem is reduced to finding a function describing these fluctuations which is Gaussian.

The tail found by Kane is also Gaussian. As a result Kane's theory overestimates the tail states. This can be traced back to the use of the semiclassical approximation which neglects the kinetic energy of localization. This theory is reasonable for high energy states but not for deep states. However this method is the simplest description of the band shape in disordered semiconductor which gives the roughly approximate value of $\rho(E)$ throughout the band.

In quantum model described earlier Halperin and Lax include the kinetic energy of localization in their method. Their theory is more precise than Kane's theory. They use the minimum counting method for calculating the density of states in the low - energy tail. For the screened Coulomb impurity case and in the Gaussian approximation, the calculation leads to density of states $\rho(E)$ exp(- B(E)) when B(E) vary form $(E)^{1/2}$ to E^2. Over reasonable energy range $\rho(E) \sim \exp(-E^{3/2})$. They concluded that if one is deep enough in the tail, where excited states to be unimportant, the method should be valid. They tested their theory on an exacty solvable one - dimensional Gaussian model and found that the results yield the correct form. However they did not obtain the analytic expression of $\rho(E)$. Thus it is inconvenient to study analytically limiting values. of their expression . Because of the restriction of assumptions, $\rho(E)$ cannot be extended to the intermediate states.

In path integral method, the calculation can be performed analytically because of the introduction of a non-local harmonic trial action (-Chapter IV). Thus one can obtain numerical results of $\rho(E)$ less difficult than Halperin and Lax results. In this approach the solution involves the determination of the appropriate value of z and α (defined in chapter 4). Since the result is written as an analytical expressions, several limiting values of the dimensionless functions can be easily obtained as shown in section 4.7. Furthermore the result of $\rho(E)$ can be extended to intermediate energy. By using Halperin and Lax's limit Sayakanit obtained the same results as Halperin and Lax's results. However Halperin and Lax's limit does not satisfy the variational principle. Sayakanit suggested that one should use the Lloyd and Best variational principle.

In chapter V we used the Lloyd and Best variational principle (which states that z should be determined by maximizing $P(\nu,z)$)in determining the value of z which is finite and depend on ξ. Comparing the present result with the result obtained by using Halperin and Lax's limit, we find that Halperin and Lax give the increasing value of z as ν decreasing and does not depend on ξ. Considering further, in very deep tail region ($\nu>>1$) where the exponential term in the expression of $\rho(E)$ dominate , we have $\rho(E) \sim e^{-B(E)}$. Thus minimizing the term $e^{-B(E)}$ is valid because e^{-x} is convex function obeying the convexity theory. In deep region, the Lloyd and Best variational principle and the variational method of Halperin and Lax in maximizing $\rho(\nu,z)$ give the same asymptotic results. However for the region $\nu \ll 1$, the term $e^{-B(E)}$ is not dominate, and the term $\rho(E)$ becomes more complicate. Thus the variational method of Halperin and Lax is not valid (showing the unphysical region ($\nu \ll 1$)). Thus the Lloyd and Best variational principle appears to be most appropriate for determining the value of z of all regions.

The density of states can be improved further. We have seen in chapter IV, for example that the ground state ($t \rightarrow \infty$) contribution to $\rho(E)$ can be obtained, by keeping only the first term of the series of (4.7.1). The resulting expression is the same as that of Halperin and Lax. We can obtain a better expression by considering the other terms of the series of (4.7.1). In appendix A, we consider the expansion to second order correction.

Similarly the improvement of the density of states at high energy ($t \rightarrow 0$) can be obtained (appendix B). For further study, it would be interesting to evaluate the result presented in the thesis by the numerical calculation on a computer.

Appendix A

From (4.7.1), we have

$$\rho(E) = \frac{1}{16\pi^2 \sqrt{\pi}} \frac{1Q^3}{z\bar{E}_Q} (-i) \int_{-i\infty}^{i\infty} dT (T)^{3/2} \left\{ 8 \sum_{k=0}^{\infty} \binom{-3}{k} (-1)^k e^{-T(2k+3)} \right\}$$

$$\exp\left[\frac{3}{2}\left\{T\left\{2\sum_{k=o}^{\infty}\binom{-1}{k} e^{-2Tk}(-1)^k - 1\right\} - 1\right\} - z^2 \nu T\right] . \exp\left[\frac{1}{2\sqrt{\pi}} \xi z^4 T^2 \int_0^1 dx\right.$$

$$\left. \cdot \int_0^{\infty} dy\, y e^{-y}\left(y + \frac{z^2}{2} J(T,x)\right)^{-3/2}\right] \qquad A.1$$

This expression can be expressed in a closed form by noting that the y - integration can be expressed in terms of a parabolic - cylinder function as

$$\int_0^{\infty} dy\, y e^{-y}\left(y + \frac{z^2}{2} J(T,x)\right)^{-3/2} = 2\sqrt{2} \exp\left[\frac{z^2}{4} J(T,x)\, D_{-3}\, z\sqrt{J(T,x)}\right]$$

Considering the term

$$\int_0^{\infty} dy\, y e^{-y}\left[y + \frac{z^2}{2} J(T,x)\right]^{-3/2} = K$$

Expanding $\left(y + \frac{z^2}{2} j(T,X)\right)^{-3/2}$ by Taylor expansion, we get

$$(y + \frac{z^2}{2} J(T,x))^{-3/2} = \left\{(y + \frac{z^2}{2}) + \frac{z^2}{2}\left\{2e^{-2T} - e^{-2TX} - e^{-2T(1-x)}\right\} + \frac{z^2}{2}\bar{J}(T,x)\right\}^{-3/2}$$

$$= (y + \frac{z^2}{2})^{-3/2} + {}^3/_4\, z^2 \left\{2e^{-2T} - e^{-2TX} - e^{-2T(1-X)} \quad (y+z)\right\}^{-5/2}$$

$$+ \frac{3}{4} z^2 (y+z)^{-5/2} \bar{J}(T,x) \ldots$$

where $J(T,x) = 1 + (2e^{-2T} - e^{-2TX} - e^{-2T(1-x)}), + \bar{J}(T,x)$

Thus when we integrate term by term, we obtain

$$K = \int_0^\infty dy.y.e^{-y}(y + \frac{z^2}{2})^{-3/2} + \int_0^\infty dy.ye^{-y}(y + \frac{z^2}{2})^{-5/2} . \frac{3}{4} z_x^2$$

$$x\,(2e^{-2T} - e^{-2TX} - e^{-2T(1-X)}) + \int_0^\infty dyye^{-y}\,(y+\frac{z^2}{2})^{-5/2} . \frac{3}{4}z^2\bar{J}(T\ x)$$

$$. \frac{3}{4}z^2\bar{J}(T,x) + \ldots$$

One can evaluate K by noting that

$$\int_0^\infty dyy.e^{-y}(y+\frac{z^2}{2})^{-3/2} = 2^{3/2} e^{1/4\, Z^2} D_{-3}(Z)$$

and $$\int_0^\infty dyye^{-y}(y + z)^{-5/2} = 2^{5/2}\int_0^\infty e^{-y}. dy\ y(2y + z^2)^{-5/2}$$

$$= \frac{2^{5/2}}{3} e^{1/4\, z^2} \frac{1}{z}\left\{D_{-2}(z) - zD_{-3}(z)\right\}$$

Using the recursion formula, the above expression becomes

$$\int_0^\infty dy\, y e^{-y}(y+z)^{-5/2} = 2^{5/2} e^{1/4 z^2} \frac{1}{z} D_{-4}(z)$$

So we get

$$K = 2^{3/2} . e^{1/4 z^2} D_{-3}(z) + 3\sqrt{2}\, e^{1/4 z^2} z D_{-4}(z)(2e^{-2T} - e^{-2Tx} - e^{-2T(1-x)})$$

$$+ 3\sqrt{2} e^{1/4 z^2} z D_{-4}(z)\, \bar{J}(T,x) + \ldots$$

Going further, we evaluate

$$\int_0^1 dx . K = 2^{3/2} . e^{1/4 z^2} D_{-3}(z) + 3\sqrt{2}\, e^{1/4 z^2} z D_{-4}(z) \int_0^1 (2e^{-2T} - e^{-2Tx} - e^{-2T(1-x)}$$

$$+ 3\sqrt{2}\, e^{1/4 z^2} z D_{-4}(z) \int_0^1 \bar{J}(T,x)\, dx + \ldots.$$

$$= 2^{3/2} e^{1/4 z^2} D_{-3}(z) + 3\sqrt{2}\, e^{1/4 z^2} z D_{-4}(z)(2e^{-2T} - \frac{1}{T}) + \ldots$$

By using the above expression, (A.1) can be written as

$$\rho(E) = \frac{1}{16\pi^2\sqrt{\pi}} \frac{1}{z} \frac{Q^3}{E_Q} (-i) \int_{-i\infty}^{i\infty} dT . T^{3/2} \left\{ 8 \sum_{k=o}^{\infty} \binom{-3}{k} (-1)^k e^{-T(2k+3)} \right\} . \exp$$

$$\left[\frac{3}{2}(T \left\{ 2 \sum_{k=o}^{\infty} \binom{-1}{k} e^{-2Tk} (-1)^k - 1 \right\} - 1) - z^2 \sqrt{T} \right] . \exp\left[\frac{1}{2\sqrt{\pi}} \xi z^4 T^2 . (2^{3/2} e^{1/4 z^2} D_{-3}(z) \right.$$

$$\left. + 3\sqrt{2}\, e^{1/4 z^2} z D_{-4}(z)(2e^{-2T} - \frac{1}{T}) + \ldots) \right]$$

$$\rho(E) = \frac{1}{16\pi^2\sqrt{\pi}} \frac{1}{z} \frac{Q^3}{E_Q} (-i) \int_{-i\infty}^{i\infty} dT . T^{3/2} e^{-3/2} . 8 \sum_{k=o}^{\infty} \binom{-3}{k} (-1)^k . \exp$$

$$\left[-T(2k+3) + \frac{3}{2}T \left\{ 2 \sum_{k=o}^{\infty} \binom{-1}{k} e^{-2Tk} (-1)^k - 1 \right\} - z^2 \nu T \right] x$$

$$x \exp\left[\frac{1}{2\sqrt{\pi}} \xi z^4 2^{3/2} e^{3/2} e^{1/4 z^2} D_{-3}(z) T^2 + \right.$$

$$\left. + \frac{2}{2\sqrt{\pi}} \xi . 2 e^{1/4 z^2} z^5 D_{-4}(z) e^{-2T} T^2 \right] x$$

$$\exp\left[- 3\sqrt{2}\, e^{1/4 z^2} z^5 D_{-4}(z) . \frac{1}{2\pi} \xi . T + \ldots \right]$$

By defining

$$N^2 = \sqrt{\frac{2}{\pi}} \xi z^4 e^{1/4 z^2} D_{-3}(z) \quad \text{and } R = \frac{3}{\sqrt{2}\sqrt{\pi}} z^5 e^{1/4 z^2} D_{-4}(z) . \xi \qquad (A.2)$$

$$\rho(E) = \frac{1}{16\pi^2\sqrt{\pi}} \frac{1}{z} \frac{Q^3}{E_Q} (-i) \int_{-i\infty}^{i\infty} dT T^{3/2} e^{-3/2} 8 \sum_{k=o}^{\infty} \binom{-3}{k} (-1)^k . \exp\left[(-(2k+3) \right.$$

$$+ \frac{3}{2} \left\{ 2 \sum_{k=o}^{\infty} \binom{-1}{k} e^{-2Tk} (-1)^k - 1 \right\} - z^2 \nu - R) T$$

$$\left. + (N^2 + 2Re^{-2T}) T^2 + \ldots \right]$$

Since $\left(2 \sum_{k=o}^{\infty} \binom{-1}{k} e^{-2Tk} (-1)^k - 1 \right)$ is equal to

$$1 + 2 \sum_{k=1}^{\infty} \binom{-1}{k} e^{-2Tk} (-1)^k$$

$\rho(E)$ can be written as

$$\rho(E) = \frac{1}{16\pi^2\sqrt{\pi}} \frac{1}{z} \frac{Q^3}{E_Q} (-i) \int_{-i\infty}^{i\infty} dT T^{3/2} e^{-3/2}\, 8 \sum_{k=0}^{\infty} \binom{-3}{k} (-1)^k . \exp\Big[(-(2k+3) + \frac{3}{2}$$

$$-z^2\nu - R)T + T^2N^2\Big] . \exp\Big[3 \sum_{k=1}^{\infty} \binom{-1}{k} e^{-2Tk}(-1)^k T$$

$$+ 2Re^{-2T}T^2 + \ldots\Big]$$

By using the formula $e^x = \sum_{k=0}^{\infty} x^k/k!$, we can write

$$\exp\Big[3 \sum_{k=1}^{\infty} \binom{-1}{k} e^{-2Tk}(-1)^k T + 2Re^{-2T}T^2 + \ldots\ldots\Big]$$

$$= \sum_{k=0}^{\infty} \Big[3 \sum_{k=1}^{\infty} \binom{-1}{k} e^{-2Tk}(-1)^k T + 2Re^{-2T} T^2 + \ldots\Big]^k / k!$$

By defing

$$C = \frac{8}{16\pi^2\sqrt{\pi}} \frac{1}{z} \frac{Q^3}{E_Q}$$

We can write

$$\rho(E) = C (-i) \int_{-i\infty}^{i\infty} dT\, T^{3/2} e^{-3/2} \exp\Big[(\frac{3}{2} - z^2\nu - R)T + N^2T^2\Big] \times$$

$$\rho(E) = C\,(-i)\int_{-i\infty}^{i\infty} dT\; T^{3/2} e^{-3/2} \exp\left[(\tfrac{3}{2} - z^2\nu - R)T + N^2T^2\right] \times$$

$$\sum_{k=o}^{\infty} \binom{-3}{k} (-1)^k e^{-(2k+3)T} \qquad \sum_{k=o}^{\infty} \left[3 \sum_{k=o}^{\infty} \binom{-1}{k} e^{-2Tk} (-1)^k T \right.$$

$$\left. + 2Re^{-2T}\, T^2 + \ldots \right]^{k'} / k'\,!$$

$$= C(-i)\int_{-i\infty}^{i\infty} dT . T^{3/2} \exp\left[(\frac{3}{2} - z^2\nu - R)T + N^2T^2\right]_x$$

$$x \sum_{k=o}^{\infty} \sum_{k'=o}^{\infty} \binom{-3}{k} (-1)^k e^{-(2k+3)T} \left[3 \sum_{k=1}^{\infty} \binom{-1}{k} e^{-2Tk} (-1)^k T + 2Re^{-2T} T^2 \right.$$

$$\left. + 2Re^{-2T}T^2 + \ldots \right]^{k'} / k'!$$

Since

$$\sum_{k=o}^{\infty} \sum_{k'=o}^{\infty} \binom{-3}{k} (-1)^k e^{-(2k+3)T} \left[3 \sum_{k=1}^{\infty} \binom{-1}{k} e^{-2Tk} (-1)^{kT} + 2Re^{-2T} T^2 + \ldots \right]^{k'} / k'!$$

$$= e^{-3T} + 12Te^{-5T} + 2T^2Re^{-5T} + \sum_{k=2}^{\infty} \sum_{k'=1}^{\infty} \binom{-3}{k} (-1)^k e^{-(2k+3)T} \; x$$

$$\left[3 \sum_{k=2}^{\infty} \binom{-1}{k} e^{-2Tk} (-1)^k T + 2Re^{-2T} T^2 + \ldots \right]^{k'} / k'!$$

the expression of $\rho(E)$ can be written as

$$\rho(E) = C(-i)\int_{-i\infty}^{i\infty} dT.T^{3/2} e^{-3/2} \exp\left[(\tfrac{3}{2} - z^2\nu - R)T + N^2T^2\right]$$

$$\times\left[e^{-3T} + 12Te^{-5T} + 2T^2Re^{-5T} + \Theta(k,k',T)\right]$$

where

$$\Theta(k,k') = \sum_{k=2}^{\infty}\sum_{k'=1}^{\infty}\binom{-3}{k}(-1)^k e^{-(2k+3)T}$$

$$\left[3\sum_{k=2}^{\infty}\binom{-1}{k}e^{-2Tk}(-1)^k T + 2RT^2e^{-2T} + \ldots\right]^{k'} / k'!$$

We now write

$$\rho(E) = C(-i)\int_{-i\infty}^{i\infty} dTT^{3/2} \exp\left[-3/2 + (-3/2 - z^2\nu - R)T + N^2T^2\right]$$

$$+ c(-i)\,12\int_{-i\infty}^{i\infty} dTT^{5/2} \exp\left[-3/2 + (-7/2 - z^2\nu - R)T + N^2T^2\right]$$

$$+ c(-i)2R\int_{-i\infty}^{i\infty} dTT^{7/2} \exp\left[-3/2 + (-7/2 - z^2\nu - R)T + N^2T^2\right]$$

$$+ c(-i)\int_{-i\infty}^{i\infty} dTT^{3/2}\,\Theta(k,k',T) \exp\left[-3/2 + (3/2 - z^2\nu - R)T + N^2T^2\right] \quad \text{A.3}$$

To consider the ground state contribution to the density of states we let $t \to \infty$

$$\rho(E) \sim \rho_1(E) = c(-i)\int_{-i\infty}^{i\infty} dT\,T^{3/2} \exp\left[N^2T^2 - (3/2 + z^2\nu + R)T \right]$$

Neglecting the term $e^{-3/2}$ and using the formula

$$\int_{-\infty}^{\infty} dt(it)^p \exp(-\beta^2t^2 - iqt) = 2^{-p/2}\sqrt{\pi}\,\beta^{-p-1} \exp(q^2/8\beta^2)D_p(q/\beta\, 2) \qquad A.4$$

we obtains

$$\rho_1(E) = C2^{-3/4}\sqrt{\pi}\,N^{-5/2}\exp\left[-(3/2 + z^2\nu + R)^2/8N^2\right] D_{3/2}((3/2 + z^2\nu + R)/\sqrt{2}\,N)$$

A.5

Since z is vary as E, when $E \to \infty$, z automatic goes to $\to\infty$, we can using the asymptotic relation of parabolic cylinder function, i.e.,

$$D_p(X) \underset{x\to\infty}{\sim} \exp(-1/4X^2)\,X^p + \ldots$$

We can write

$$D_{3/2}(q_1 + R/\sqrt{2}\,N) \sim e^{-(q_1+R)^2/8N^2}\left(\frac{q_1+R}{\sqrt{2}\,N}\right)^{3/2} \qquad A.6$$

Substituting (A.6) into (A.5), we obtain

$$\rho_1(E) = \frac{2^{-3/4}\sqrt{\pi}\,N^{-5/2}}{2^{3/4}\,N^{3/2}} \exp\left[-(q_1 + R)^2/4N^2\right](q_1 + R)^{3/2}$$

$$\rho_1(E) = C\frac{2^{-3/2}}{N^4}\sqrt{\pi}\,(q_1+R)^{3/2}\exp(-(q_1+R)^2/4N^2)) \qquad A.7$$

where $q_1 = \frac{3}{2} + z^2\nu$

The term R arises from the second order expansion of the series solution of $\rho(E)$. For the first order expansion, one neglects R and obtains Sayakanit 's result

$$\rho_1(E) = C2^{-3/2}\cdot\frac{\sqrt{\pi}}{N^4}\,q_1^{3/2}\exp(-q_1^2/4N^2) \qquad A.9$$

Substituting the value of N, q_1 and C in A.9, we get

$$\rho_1(E) = (QE_Q)^3/\xi^2\quad a(\nu,z)\exp(-E_Q^2\,b(\nu,z)/2\xi) \qquad A.10$$

where $$a(\nu,z) = \frac{(q_1)^{3/2}}{8\pi\sqrt{2}\,z^9\exp(z^2/2)(D_{-3}(z))^2}$$

and

$$b(\nu,z) = \frac{\sqrt{\pi}\;q_1^2}{2\sqrt{2}\;z^4\exp(z^2/4)D_{-3}(z)} \qquad A.11$$

The improvement of the tail states

By considering the limit $t \rightarrow \infty$ and keeping only the second order expansion of (A.3), we obtain

$$\rho_2(E) = C(-i)\int_{-i\infty}^{i\infty} dT\,T^{3/2} \exp\left[-(q_1 + R)T + N^2T^2\right]$$

$$+12c(-i)\int_{-i\infty}^{i\infty} dT\,T^{5/2} \exp\left[-(q_2 + R)T + N^2T^2\right]$$

$$+ 2RC(-i)\int_{-i\infty}^{i\infty} dT\,T^{7/2} \exp\left[-(q_2 + R)T + N^2T^2\right] \qquad A.12$$

where $q_2 = q_1 + 2$. The above expression can be evaluate to give :

$$\rho_2(E) = C.2^{-3/4}\sqrt{\pi}\, N^{-5/2} \exp\left[-(q_1 + R)^2/8N^2\right] D_{3/2}((q_1 + R)/\sqrt{2}\, N)$$

$$+ C.2^{-5/4}\sqrt{\pi}\, .12N^{-7/2} \exp\left[-(q_2 + R)^2/8N^2\right] D_{5/2}((q_2+R)/\sqrt{2}N)$$

$$+ C.2R.2^{-7/4}\sqrt{\pi}\, N^{-9/2} \exp\left[-(q_2 + R)^2/8N^2\right] D_{7/2}((q_2 + R)/\sqrt{2}\, N) \qquad A.13$$

Similarly , as with (A.6), we have asymptotic relation

$$D_{3/2}((q_1 + R)/ 2\, N) \sim e^{-(q_1+R)^2/8N^2} . ((q_1+R)/\sqrt{2}\, N)^{3/2} \qquad A.14$$

and

$$D_{7/2}((q_2 + R)/ 2\, N) \sim e^{-(q_2+R)^2/8N^2} .((q_2+R)/\sqrt{2}\, N)^{7/2} \qquad A.15$$

When Eq(A.14) and (A.15) are substituted into (A.13), we obtain

$$\rho_2(E) = C.2^{-3/2}.\sqrt{\pi}\, N^{-4} \exp\left[-(q_1+R)^2/4N^2\right].(q_1+R)^{3/2}$$

$$+ C.12.2^{-5/2}\sqrt{\pi}\, N^{-6} \exp\left[-(q_1+R+2)^2/4N^2\right].(q_1+R+2)^{5/2}$$

$$+ C.2R.2^{-7/2}\sqrt{\pi}\, N^{-8} \exp\left[-(q_1+R+2)^2/4N^2\right].(q_1+R+2)^{7/2}$$

$$= C.2^{-3/2}\sqrt{\pi}\, N^{-4} \exp\left[-(q_1+R)^2/4N^2\right](q_1+R)^{3/2}.$$

$$\left[1 + \frac{6}{N^2}\exp\left[\frac{(q_1+R)^2}{4N^2} - \frac{(q_1+R+2)^2}{4N^2}\right] x \frac{(q_1+R+2)^{5/2}}{(q_1+R)^{3/2}} + \right.$$

$$\left. + \frac{1}{2}\frac{R}{N^4}.\exp\left[-\frac{(q_1+R+2)^2}{4N^2} + \frac{(q_1+R)^2}{4N^2}\right].\frac{(q_1+R+2)^{7/2}}{(q_1+R)^{3/2}}\right]$$

Rearraging the above expression, we get

$$\rho_2(E) = \frac{C.2^{-3/2}\pi}{N^4} \exp\left(-\frac{(q_1+R)^2}{4N^2}\right)(q_1+R)^{3/2}\left[1 + \frac{6}{N^2}\exp\right.$$

$$\left(-\frac{(q_1+1+R)}{N^2}\right).\left(\frac{q_1+R+2}{q_1+R}\right)^{3/2}(q_1+R+2) + \frac{R}{2N^4}.\exp$$

$$\left.\left(-\frac{(q_1+1+R)}{N^2}\right) - \left(\frac{(q_1+R+2)}{q_1+R}\right)^{3/2}.(q_1+R+2)^2\right] \qquad \text{A.16}$$

If we define

$$\rho_o(E) = \frac{C.2^{-3/2}\pi}{N^4}\exp\left[-\frac{(q_1+R)^2}{4N^2}\right](q_1+R)^{3/2} \qquad \text{A.17}$$

and

$$B(\nu,z) = \frac{6}{N^2}\exp\left(\frac{-(q_1+1+R}{N^2}\right)\left(\frac{q_1+R+2}{Q_1+R}\right)^{3/2} \cdot (q_1+R+2)$$

$$+ \frac{R}{2N^4} \cdot \exp\left(-\frac{(q_1+1+R)}{N^2}\right) \cdot \left(\frac{q_1+R+2}{q_1+R}\right)^{3/2} (q_1+R+2)^2 \qquad \text{A.18}$$

(A.16) can be written as

$$\rho_2(E) = \rho_o(E)\,(1 + B(\nu,z))$$

Since when $z \to \infty$, $B(\nu,z) \longrightarrow 0$, we can approximate the above equation as

$$\rho_2(\nu,z) = \rho_o(\nu,z)e^{B(\nu,z)} \qquad \text{A.19}$$

if we write

$$\rho_o(\nu,z) = A(\nu,z)\, e^{-D(\nu,z)} \qquad \text{A.20}$$

$$\text{where } D(\nu,z) = \frac{(q_1+R)^2}{4N^2} = \frac{\left(3/2 + z^2\nu + \frac{3z^5}{\sqrt{2\pi}}\,\xi e^{1/4z^2} D_{-4}(z)\right)^2}{4\sqrt{\frac{2}{\pi}}\;\xi\; z^4 e^{1/4z^2} D_{-3}(z)}$$

$$= \frac{\sqrt{\pi}}{4\sqrt{2}\,\xi\, e^{1/4z^2} D_{-3}(z)}\left[\frac{3}{2z^2} + \nu + \frac{3z^3}{\sqrt{2\pi}}\,\xi\, e^{1/4z^2} D_{-4}(z)\right]^2 \qquad \text{A.21}$$

$$A(\nu,z) = \frac{c \cdot 2^{-3/2}}{N^4} \cdot \sqrt{\pi}\,(q_1+R)^{3/2}$$

$$= \frac{8}{16\pi^2\sqrt{\pi}} \; \frac{1}{z} \; \frac{Q^3}{E_Q} \; \frac{2^{-3/2} \cdot \sqrt{\pi} \cdot 1}{\frac{2}{\pi} \xi^2 \, z^8 e^{1/2z^2} D^2_{-3}(z)}$$

$$\doteq \frac{Q^3 \cdot (3/2 z^2 + \nu + \frac{3z^3}{\sqrt{2\pi}} \cdot \xi e^{1/4 z^2} D_{-4}(z))^{3/2}}{8E_Q \, \xi^2 \sqrt{2}\, \pi \, e^{1/2 z^2} D^2_{-3}(z) \cdot z^6} \qquad \text{A.22}$$

we get

$$\rho_2(\nu,z) = A(\nu,z) e^{-D(\nu,z) + B(\nu,z)} \qquad \text{A.23}$$

The parameter ω introduced in the trial action S_o has not yet been determined. Following Halperin and Lax one may choose z so as to maximize $\rho_2(E)$. When $\xi \longrightarrow o$ the exponential factor will become very sensitive to the choice of z .

Upon maximizing the exponential term of (A.23),

$$\frac{\partial e^{-D(\nu,z) + B(\nu,z)}}{\partial z} = 0 \qquad \text{A.24}$$

we obtain

$$\frac{\partial D(\nu,z)}{\partial z} = \frac{\partial B(\nu,z)}{\partial z} \qquad \text{A.25}$$

and

$$\frac{\partial B(\nu,z)}{\partial z} = \frac{\delta}{\partial z} \exp\left(\frac{-(q_1+R+1)}{N^2}\right) \cdot \left(\frac{q_1+R+2}{q_1+R}\right) \cdot (q_1+R+2)\left(\frac{6}{N^2} + \frac{R}{2N^4}(q_1+R+2)\right)$$

For convenience let us write

$$I = \exp\left(\frac{-(q_1+R+1)}{N^2}\right) \cdot \left(\frac{q_1+R+2}{q_1+R}\right) \cdot (q_1+R+2)$$

then

$$= \frac{\partial}{\partial z} \frac{P}{N^2} \left(6 + \frac{R}{2N^2}(q_1+R+2)\right) \qquad \text{A.26}$$

By using (A.2), we get

$$\frac{R}{N^2} = \frac{3}{2} za \qquad \text{A.27}$$

where

$$a = \frac{D_{-4}(z)}{D_{-3}(z)} \qquad \text{A.28}$$

So we obtain

$$\frac{\partial B}{\partial z} = I \frac{\partial}{\partial z} \frac{1}{N^2} \left[6 + \frac{3}{4}za(q_1+R+2)\right] + \frac{1}{N^2} \left[6 + \frac{3}{4}za(q_1+R+2)\right] \frac{\partial I}{\partial z}$$

$$= \frac{3I}{N^2} \frac{\partial}{\partial z}\left(\frac{1}{4}za(q_1+R+2)\right) + 3I\left(2 + \frac{1}{4}za(q_1+R+2)\right) \frac{\partial}{\partial z} \frac{1}{N^2}$$

$$+ \frac{3}{N^2} \left(2 + \frac{1}{4}za(q_1+R+2)\right) \frac{\partial I}{\partial z} \qquad \text{A.29}$$

For convenience, let us consider term by term

$$\frac{\partial I}{\partial z} = \frac{\partial}{\partial z}\exp\left(-\frac{(q_1+R+1}{N^2}\right)\cdot\left(\frac{q_1+R+2}{q_1+R}\right)^{3/2}\cdot(q_1+R+2) \qquad A.30$$

$$\frac{d}{dz}\frac{1}{N^2} = -\frac{1}{N^4}N^2\left(\frac{4}{z}-3a\right)$$

$$= -\frac{1}{N^2}\left(\frac{4}{z}-3a\right) \qquad A.31$$

$$\frac{d}{dz}\frac{3}{2}za = \frac{3}{2}a + \frac{3}{2}z^2a - \frac{3}{2}z + \frac{9a^2}{2}z$$

$$= \frac{3}{2}a\left(1 + z^2 - \frac{z}{a} + 3az\right) \qquad A.32$$

where one use $\frac{d}{dz}D_{-4}(z) = \frac{1}{2}zD_{-4}(z) - D_{-3}(z)$

and $\frac{d}{dz}D_{-3}(z) = -\frac{1}{2}zD_{-3}(z) - 3D_{-4}(z)$

$$\frac{d}{dz}(q_1+R+2) = \frac{d}{dz}q_1 + \frac{d}{dz}R$$

$$= \frac{d}{dz}\left(\frac{3}{2}+z^2\nu\right) + \frac{d}{dz}\left(\frac{\xi z^4}{2\sqrt{\pi}}\cdot 3\sqrt{2}\, e^{1/4z^2}D_{-4}(z)\right)$$

$$= 2z\,\nu + R\left(\frac{5}{z} + z - \frac{1}{a}\right) \qquad A.33$$

$$\frac{\partial I}{\partial z} = I\left[-\left(\frac{N^2\left(\frac{d}{dz}q_1 + \frac{dR}{dz}\right) - (q_1+R+1)\frac{d}{dz}N^2}{(N^2)^2}\right)\right.$$

$$\left. + \frac{3}{2}\left(\frac{q_1+R}{q_1+R+2}\right)\left[\frac{(q_1+R)\left(\frac{dq_1}{dz}+\frac{dR}{dz}\right) - (q_1+R+2)\left(\frac{dq_1}{dz}+\frac{dR}{dz}\right)}{(q_1+R)^2}\right]\right] \qquad A.34$$

Substituting (A.31) - (A.33) into (A.34) then we obtain

$$\frac{dI}{dz} = I\left[(2z\nu + R(\frac{5}{z} + z - \frac{1}{a}))\ \frac{q_1+R-3}{(q_1+R)(q_1+R+2)}\right.$$

$$\left. -\frac{3}{2}a(1 + z(z - \frac{1}{a} + 3a)) - \frac{1}{N^2}((q_1+1)(3a - \frac{4}{z}) + 2z\nu)\right] \qquad A.35$$

and

$$\frac{3I}{N^2}\frac{\partial}{\partial z}(\frac{1}{4}za(q_1+R+2)) + 3I(2 + \frac{1}{4}za(q_1+R+2))\frac{\delta}{\partial z}(\frac{1}{N^2})$$

$$= \frac{3}{N^2}\left[2(3a - \frac{4}{z}) + \frac{za}{4}(2z\nu + R(\frac{5}{z} + z - \frac{1}{a}) + (q_1+R+2)(6a - \frac{3}{2} + z - \frac{1}{a}))\right]$$

$$= \frac{3}{N^2}\left[2(3a - \frac{4}{z}) + \frac{za}{4}(2z\nu + (q_1+2)(6a - \frac{3}{z} + z - \frac{1}{a})\right.$$

$$\left. + R(\frac{5}{z} + z - \frac{1}{a} + 6a - \frac{3}{z} + z - \frac{1}{a}))\right]$$

$$= \frac{3}{N^2}\left[2(3a - \frac{4}{z}) + \frac{za}{4}(2z\nu + (q_1+2)(6a - \frac{3}{z} + z - \frac{1}{a})\right.$$

$$\left. + 2R(\frac{1}{z} + z + 3a - \frac{1}{a}))\right] \qquad A.36$$

Substituting (A.35) and (A.36) into (A.29), we get

$$\frac{\partial B}{\partial z} = \frac{3I}{N^2}\left[2(3a - \frac{4}{z}) + \frac{za}{4}(2z\nu + (q_1+2)(6a - \frac{3}{z} + z - \frac{1}{a}) + 2R(\frac{1}{z} + z + 3a - \frac{1}{a}))\right.$$

$$+ (2 + \frac{1}{4}za(q_1+R+2))((2z\nu + R(\frac{5}{z} + z - \frac{1}{a}))\ \frac{(q_1+R-3)}{(q_1+R)(q_1+R+2)}$$

$$\left. - \frac{3}{2}a(1 + z(z - \frac{1}{a} + 3a)) - \frac{1}{N^2}(q_1+1)(3a - \frac{4}{z}) + 2z\nu)\right] \qquad A.37$$

$$\frac{\partial D}{\partial z} = \frac{1}{4} \frac{\partial}{\partial z} \frac{(q_1 + R)^2}{N^2}$$

$$= \frac{1}{4} \left((q_1 + R)^2 \frac{d}{dz}\left(\frac{1}{N^2}\right) + \frac{2}{N^2} (q_1 + R) \frac{\partial}{\partial z} (q_1 + R)\right)$$

$$= \frac{1}{4} \left[-(q_1 + R)^2 \left(\frac{4}{z} - 3a\right) + \frac{2}{N^2}(q_1 + R) \left(2z\nu + R\left(\frac{5}{z} + z - \frac{1}{a}\right)\right) \right]$$

$$= \frac{1}{4}(q_1 + R)^2 \frac{1}{N^2} \left[\frac{2}{(q_1 + R)} \left(2z\nu + R\left(\frac{5}{z} + z - \frac{1}{a}\right)\right) - \frac{4}{z} + 3a \right]$$

$$= D \left[\frac{2}{q_1 + R} \cdot \left(2z\nu + R\left(5/_z + z - \frac{1}{a}\right)\right) - \frac{4}{z} + 3a \right] \qquad \text{A.38}$$

Substituting (A.37) and (A.38) into (A.25) we obtain

$$D \left(\frac{2}{q_1 + R} \cdot \left(2z\nu + R\left(5/_z + z - \frac{1}{a}\right)\right) - \frac{4}{z} + 3a \right) =$$

$$\frac{3I}{N^2} \left[2\left(3a - \frac{4}{z}\right) + \frac{za}{4} \left(2z\nu + (q_1 + 2)\left(6a - \frac{3}{z} + z - \frac{1}{a}\right) \right.\right.$$

$$\left. + 2R\left(\frac{1}{z} + z + 3a - \frac{1}{a}\right)\right)$$

$$+ \left(2 + \frac{1}{4}za\,(q_1 + R + 2)\right) \left(\frac{\left(2z\nu + R\left(\frac{5}{z} + z - \frac{1}{a}\right)\right)(q_1 + R - 3)}{(q_1 + R)(q_1 + R + 2)} \right.$$

$$\left.\left. - \frac{3}{2}a\left(1 + z\left(z - \frac{1}{a} + 3a\right)\right) - \frac{1}{N^2}(q_1 + 1)\left(3a - \frac{4}{z}\right) + 2z \right) \right] \qquad \text{(A.39)}$$

For each value of ν we obtain the value of z by solving (A.39)
Next we replace the value of ν and z into (A.23) and obtain $\rho(\nu)$

In stead of minimizing the exponential term, we can further improve the calculation by maximizing the full expression. That is

$$\frac{\partial \rho_2(\nu,z)}{\partial z} = 0 \qquad \text{A.40}$$

Thus by differentiating (A.23), , we obtain

$$\frac{\partial}{\partial z}\left(A(\nu,z)\, e^{-D(\nu,z) + B(\nu,z)} \right) = 0$$

$$A(\nu,z) \frac{\partial}{\partial z} e^{-D(\nu,z) + B(\nu,z)} + e^{-D(\nu,z) + B(\nu,z)} \frac{\partial}{\partial z} A(\nu,z) = 0$$

$$A(\nu,z) \frac{\partial}{\partial z} (-D(\nu,z) + B(\nu,z)) + \frac{\partial}{\partial z} A(\nu,z) = 0 \qquad \text{A.41}$$

Considering first the expression $\frac{\partial}{\partial z} A(\nu,z)$ where $A(\nu,z)$ is defined by (A.22), we have

$$\frac{\partial}{\partial z} A(\nu,z) = \frac{Q^3}{E_Q \xi^2} \frac{d}{dz} \frac{(\frac{3}{2} z^{-2} + \nu + R)^{3/2}}{8\pi\sqrt{2}\; z^6 \exp(\frac{1}{2}z^2)\, D^2_{-3}(z)}$$

$$= \frac{Q^3}{E_Q \xi^2} \left[\frac{3}{2} \frac{(\frac{3}{2} z^{-2} + \nu + R)^{\frac{1}{2}}}{8\pi\sqrt{2}\; z^6 \exp(z^2/2) D^2_{-3}(z)} \left(\frac{d}{dz}(\tfrac{3}{2} z^{-2}) + \frac{dR}{dz}\right) \right.$$

$$+ \frac{(\frac{3}{2} z^{-2} + \nu + R)^{3/2}}{8\pi\sqrt{2}\exp(z^2/2)} \left(\frac{-z}{z^6 D^2_{-3}(z)} - \frac{-2D^{-3}_{-3}(z) D'_{-3}(z)}{z^6} \right.$$

$$\left.\left. - \frac{6z^{-7}}{D^{-2}_{-3}(z)} \right) \right]$$

$$= -A(\nu,z)\left(\frac{9}{2z^3(\frac{3}{2}z^{-2}+\nu+R)} - \frac{3}{2(\frac{3}{2}z^{-2}+\nu+R)}\frac{dR}{dz} + \frac{6}{z} + z\right.$$

$$\left. + 2\,\frac{(-z/2\,D_{-3}(z) - 3D_{-4}(z))}{D_{-3}(z)}\right)$$

where $\frac{dR}{dz} = R\left(\frac{5}{z} + \frac{z}{2} + \frac{1}{a}\left(\frac{za}{2} - 1\right)\right)$, and $R = \frac{3\xi e^{1/4 z^2}}{\sqrt{2\pi}} z^5 D_{-4}(z)$

$$= -A(\nu,z)\left(\frac{9}{2z^3(3/2\,z^{-2}+\nu+R)} - \frac{3}{2(\frac{3}{2}z^{-2}+\nu+R)}\frac{dR}{dz} + \frac{6}{z} - 6a\right) \quad A.42$$

Eq(A.42) can be substituted into (A.41) to obtain

$$+\frac{\partial}{\partial z}(-D(\nu,z) + B(\nu,z)) = -D\left(\frac{2}{q_1+R}\left(2z\nu + R\left(\frac{5}{z} + z - \frac{1}{a}\right)\right) - \frac{4}{z} + 3a\right)$$

$$+ \frac{3I}{N^2}\left(2\left(3a - \frac{4}{z}\right) + \frac{za}{4}\left(2z\nu + (q_1+R)\left(6a - \frac{3}{z} + z - \frac{1}{a}\right)\right.\right.$$

$$+ 2R\left(\frac{1}{z} + 3a - \frac{1}{a}\right)\Big) + \left(2 + \frac{1}{4}za(q_1+R+2)\right)\left(\left(2z\nu + R\left(\frac{5}{z} + z - \frac{1}{a}\right)\right)\right.$$

$$\frac{(q_1+R-3}{(q_1+R)(q_1+R+2)} - \frac{3}{2}a\left(1 + z\left(z - \frac{1}{a} + 3a\right)\right) -$$

$$-\frac{1}{N^2}(q_1+1)\left(3a - \frac{4}{z}\right) + 2z\nu\Big)\Big) \quad A.43$$

where $\frac{\partial}{\partial z}(-D(\nu,z) + B(\nu,z))$ is obtsined by using (A.37) and (A.38)

Similarly for each value of ν, we obtain the value of z by solving (A.41). Next we replace the value of ν and z into (A.23) and obtain $\rho(\nu)$.

If we use Lloyd and Best variational principle, we must solve the following equation

$$\frac{dP(\nu,z)}{dz} = 0$$

where

$$\frac{dP(\nu,z)}{dz} = E_Q^2 \int_{\nu}^{\infty} (\nu-\nu') \frac{d\rho_2(\nu',z)}{dz} d\nu'$$

and

$$\frac{d\rho_2(\nu,z)}{dz} = -\Lambda(\nu,z)\Big(\frac{9}{2z^3(\frac{3}{2}z^{-2}+\nu+R)} -$$

$$- \frac{3R}{2(\frac{3}{2}z^{-2}+\nu+R)} \cdot (\frac{5}{z} + \frac{z}{2} + \frac{1}{a}(\frac{za}{2} - 1)) + \frac{6}{z} - 6a)$$

$$+ \Lambda(\nu,z)\Big[-D(\frac{2}{q_1+R}(2z\nu+R(\frac{5}{z}+z-\frac{1}{a})) - \frac{4}{z} + 3a)$$

$$+ \frac{3I}{N^2}(2(3a - \frac{4}{z}) + \frac{za}{4}(2z\nu+(q_1+R)(6a-\frac{3}{z}+z-\frac{1}{a})$$

$$+ 2R(\frac{1}{z} + 3a - \frac{1}{a})) + (2 + \frac{1}{4}za(q_1+R+2))$$

$$((2z\nu+R(\frac{5}{z} + z-\frac{1}{a}))\frac{(q_1+R-3)}{(q_1+R)(q_1+R+2)} - \frac{3}{2}a(1+z(z-\frac{1}{a} + 3a))$$

$$-\frac{1}{N^2}(q_1+1)(3a - \frac{4}{z})+2z\nu)) \Big]$$

Appendix B

The improvement of high energy states

The method presented above can be used to obtain the density of states at high and intermediate energies. For states of high energies one lets $t \to o$. For convenience one replaces α into (4.5.10) where $\alpha = z^2$, and obtains

$$\rho(E) = \frac{1}{16\pi^2} \frac{1}{\pi\sqrt{\alpha}} \frac{Q^3}{E_Q} \int_{-\infty}^{\infty} dt(it)^{-3/2} \left(\frac{t}{\text{sint}}\right)^3 \exp\left[\frac{3}{2}(t \cot t-1) - i\alpha\nu t\right.$$

$$\left. - \frac{\xi\alpha^2 t^2}{2\pi} \int_o^1 dx \int_o^{\infty} dy y e^{-y} (y+i\alpha \sin tx \sin t(1-x))^{-3/2}\right] \qquad \text{B.1}$$

To consider $\rho(E)$ in the limit when t go to zero, we change the following terms in series representation

$$\left(\frac{t}{\sin t}\right)^3 = (t \text{ cosect})^3 = \left(t\left(\frac{1}{t} + \frac{t}{6} + \ldots\right)\right)^3 = 1 + \frac{1}{2}t^2 + \ldots \qquad \text{B.2}$$

$$t \cot t = t\left(\frac{1}{t} - \frac{t}{3} + \ldots\right) = 1 - \frac{t^2}{6} + \ldots \qquad \text{B.3}$$

$$\frac{\text{sint } x \sin t (1-x)}{\sin t} = \left(tx - \frac{t^3}{3}x^3 + \ldots\right)\left(t(1-x) - \frac{t^3}{6}(1-x)^3 + \ldots\right)\left(\frac{1}{t} + \frac{t}{6} + \ldots\right)$$

$$= 0 + tx(1-x) - \frac{t^3}{3}x(1-x) + \ldots \qquad \text{B.4}$$

when $t \to o$, we keep the series upto second-order expansion ,
and then substitute into (B.1) to get

$$\rho(E) = \frac{1}{16\pi^2\sqrt{\pi}} \frac{1}{\sqrt{\alpha}} \frac{Q^3}{E_Q} \int_{-\infty}^{\infty} dt(it)^{-3/2}(1 + \frac{t^2}{2})\exp\left[-\frac{t^2}{4} - i\alpha\nu t - \frac{\xi\alpha^2 t^2}{2\sqrt{\pi}} \int_0^1 dx \int_0^{\infty} dy\, ye^{-y}(y+i\alpha tx(1-x))^{-3/2}\right] \quad \text{B.5}$$

Then we expand $(y + i\alpha tx(1-x))^{-3/2}$ in Binomial series, we obtain

$$(y + i\alpha tx(1-x))^{-3/2} = y^{-3/2} - i\frac{3}{2}\alpha tx(1-x)y^{-5/2} + \ldots \quad \text{B.6}$$

Substituting (B.6) into (B.5) and using the formula

$$\int_0^{\infty} dy\, ye^{-y}\, y^{-3/2} = \sqrt{\pi} \quad \text{B.7}$$

and

$$\int_0^{\infty} dy\, ye^{-y}\, y^{-5/2} = \pi(-3/2) = -2\sqrt{\pi} \quad \text{B.8}$$

we get

$$\rho(E) = \frac{1}{16\pi^2\sqrt{\pi}\sqrt{\alpha}} \frac{Q^3}{E_Q} \int_{-\infty}^{\infty} dt(it)^{-3/2}(1 + \frac{t^2}{2})\ \exp(-\frac{i\alpha^3\xi t^3}{4}) . \exp\left[-i\alpha\nu t - \frac{1}{4}(2\xi\alpha^2 - 1)t^2\right] \quad \text{B.9}$$

When $t \to o$ we can approximate $(1 + \frac{t^2}{2}) \simeq e^{t^2/2}$

and the term $\exp(-\frac{i\alpha^3 \xi t^3}{4}) \sim 1 - \frac{i}{4}\alpha^3 \xi t^3$, so (B.9) becomes

$$\rho(E) = \frac{1}{16\pi^2\sqrt{\pi}} \frac{1}{\sqrt{\alpha}} \frac{Q^3}{E_Q} \int_{-\infty}^{\infty} dt(it)^{-3/2}(1 - \frac{i}{4}\alpha^3 \xi t^3)$$

$$\exp(-i\alpha\nu t - \frac{1}{4}(2\xi\alpha^2 - 1)t^2) \qquad \text{B.10}$$

Since $t \to 0$, $\xi\alpha^2 >> 1$, (B.10) becomes

$$\rho(E) = \frac{1}{16\pi^2\sqrt{\pi}} \frac{1}{\sqrt{\alpha}} \frac{Q^3}{E_Q} \int_{-\infty}^{\infty} dt(it)^{-3/2} \exp\left(-i\alpha\nu t - \frac{\xi\alpha^2}{2} - t^2\right)$$

$$- \frac{1}{16\pi^2\sqrt{\pi}} \frac{1}{\sqrt{\alpha}} \frac{Q^3}{E_Q} \int_{-\infty}^{\infty} dt(it)^{-3/2} \frac{i\alpha^3}{4}\xi t^3 \exp\left[-i\alpha\nu t - (2\xi\alpha^2 - 1)\frac{t^2}{4}\right]$$

$$= \frac{1}{16\pi^2\sqrt{\pi}} \frac{1}{\sqrt{\alpha}} \frac{Q^3}{E_Q} \int_{-\infty}^{\infty} dt(it)^{-3/2} \exp(-i\alpha\nu t - \xi\frac{\alpha^2}{2} t^2)$$

$$+ \int_{-\infty}^{\infty} dt(it)^{3/2} \frac{\alpha^3}{4} \xi \exp(-i\alpha\nu t - (2\xi\alpha^2 - 1)\frac{t^2}{4}) \qquad \text{B.11}$$

By using the formula

$$\int_{-\infty}^{\infty} dt(it)^p \exp(-\beta^2 t^2 - iqt) = 2^{-p/2}\sqrt{\pi}\,\beta^{-p-1} \exp(-q^2/8\beta^2) D_p(q/\beta\sqrt{2})$$

we obtain

$$\rho(E) = \frac{2^{-3/4}}{16\pi^2} \frac{Q^3}{E_Q} \left(\frac{\xi}{2}\right)^{1/4} \exp\left(\frac{-\nu^2}{4\xi}\right) D_{-3/2}\left(\frac{\nu}{\sqrt{\xi}}\right)$$

$$+ \frac{2^{-3/4}}{64\,\pi^2} \frac{Q^3}{E_Q} \alpha^{5/2} \xi \cdot \left(\frac{2\xi\alpha^2-1}{4}\right)^{-5/4} \exp\left(\frac{-\alpha^2 \nu^2}{2(2\xi\alpha^2-1)}\right)$$

$$D_{3/2}\left(\frac{\sqrt{2}\,\alpha\nu}{(2\xi\alpha^2-1)^{1/2}}\right) \qquad B.12$$

we evaluate for the value of α by maximizing $\rho(\nu)$

$$\frac{d\rho(E)}{d\alpha} = 0 \qquad B.13$$

For convenience we differentiates(B.11) and obtain

$$\frac{d\rho(E)}{d\alpha} = \frac{1}{16\pi^2\sqrt{\pi}} \frac{1}{\sqrt{\alpha}} \frac{Q^3}{E_Q} \int_{-\infty}^{\infty} dt\,(it)^{3/2} \cdot \frac{\xi}{4} \cdot \frac{d}{d\alpha}\alpha^{5/2} \cdot \exp\left(-i\alpha\nu t - (2\xi\alpha^2-1)\frac{t^2}{4}\right)$$

$$= \frac{1}{16\pi^2\sqrt{\pi}} \frac{1}{\sqrt{\alpha}} \frac{Q^3}{E_Q} \cdot \frac{\xi}{4} \int_{-\infty}^{\infty} dt\,(it)^{3/2} \cdot \frac{5}{2} \alpha^{3/2} \cdot \exp\left(-i\alpha\nu t - (2\xi\alpha^2-1)\frac{t^2}{4}\right)$$

$$- \int_{-\infty}^{\infty} dt\,(it)^{5/2} \nu\alpha^{5/2} \exp\left(-i\alpha\nu t - (2\xi\alpha^2-1)\frac{t^2}{4}\right)$$

$$+ \int_{-\infty}^{\infty} dt\,(it)^{7/2} \alpha^{7/2} \xi \exp\left(-i\alpha\nu t - (2\xi\alpha^2-1)\frac{t^2}{4}\right)$$

$$= \frac{1}{16\pi^2\sqrt{\pi}} \frac{1}{\sqrt{\alpha}} \frac{Q^3}{E_Q} \cdot \xi/4 \quad \frac{5}{2}\, \alpha^{3/2}\, 2^{7/4}\sqrt{\pi}\, \beta^{-5/4} \exp(-\frac{\alpha^2\nu^2}{2\beta}) D_{3/2}(\frac{\sqrt{2}\,\alpha\nu}{\sqrt{\beta}})$$

$$- \nu\alpha^{5/2}\, 2^{9/4}\sqrt{\pi}\, \beta^{-7/4} \exp(\frac{-\alpha^2\nu^2}{2\beta}) D_{5/2}(\frac{\sqrt{2}\,\alpha\nu}{\sqrt{\beta}})$$

$$+ \xi\alpha^{7/2} \cdot 2^{11/4}\sqrt{\pi}\, \beta^{-9/4} \exp(-\frac{\alpha^2\nu^2}{2\beta}) D_{7/2}(\frac{\sqrt{2}\alpha\nu}{\sqrt{\beta}}) \qquad \text{B.14}$$

where $\beta = 2\alpha^2\xi - 1$.

At the extremum, when we substitute (B.14) into (B.13), we obtain

$$\frac{5}{2}\alpha^{3/2}\, 2^{7/4}\sqrt{\pi}\, \beta^{-5/4} \exp(-\frac{\alpha^2\nu^2}{2\beta}) D_{3/2}(\frac{\sqrt{2}\,\alpha\nu}{\beta}) + \xi\alpha^{7/2}\, 2^{11/4}\sqrt{\pi}\, \beta^{-9/4} \exp(\frac{-\alpha^2\nu^2}{2\beta})$$

$$D_{7/2}(\frac{\sqrt{2}\,\alpha\nu}{\sqrt{\beta}}) = \nu\alpha^{5/2}\, 2^{9/4}\sqrt{\pi}\, \beta^{-7/4} \exp(\frac{-\alpha^2\nu^2}{2\beta}) D_{5/2}(\frac{\sqrt{2}}{\sqrt{\beta}}\alpha\nu)$$

Then by rearranging and dividing by $\xi\alpha^{7/2}\, 2^{11/4}\beta^{-9/4}$, we obtain:

$$\frac{5}{2}\frac{\alpha^{-2}}{\xi}\frac{1}{2}\beta\, D_{3/2}(\frac{\sqrt{2}}{\sqrt{\beta}}\alpha\nu) + D_{7/2}(\frac{\sqrt{2}}{\sqrt{\beta}}\alpha\nu) = \frac{\nu}{\xi\sqrt{2}}\frac{1}{\alpha}\beta\, D_{5/2}(\frac{\sqrt{2}}{\sqrt{\beta}}\alpha\nu)$$

$$-\frac{5}{4}\frac{\beta}{\xi\,\alpha^2} D_{3/2}(\frac{2}{\beta}\alpha\nu) + \frac{\nu}{\sqrt{2}\,\xi} \cdot \frac{1}{\alpha}\sqrt{\beta}\, D_{5/2}(\frac{\sqrt{2}}{\sqrt{\beta}}\alpha\nu) = D_{7/2}(\frac{\sqrt{2}}{\sqrt{\beta}}\alpha\nu) \qquad \text{B.15}$$

For each value of ν we obtain, the value of α which satisfies (B.15). Next we replace the value of ν and α into (B.14) and obtain $\rho(E)$